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Egyptian fractions

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EGYPTIAN FRACTIONS

A Thesis

Presented to the

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In Partial fulfillment

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Master of Arts

in

Mathematics

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ABSTRACT

Egyptian fractions are what we know today as unit fractions that are of the form $\frac{1}{n}$ – with the exception, by the Egyptians, of $\frac{2}{3}$. Egyptian fractions have actually played an important part in mathematics history with its primary roots in number theory.

This paper will trace the history of Egyptian fractions by starting at the time of the Egyptians, working our way to Fibonacci, a geologist named Farey, continued fractions, Diophantine equations, and unsolved problems in number theory.

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I owe a *sincere* indebtedness to all those who have directly or indirectly contributed to this project – the British Museum, MIT press, my advisors, other professors too numerous to mention, fellow students, and friends. Your time, comments, and questions were greatly appreciated.

A special tribute is due to my advisor, Dr. James Okon, who had the patience of Job when it came to squeezing information out of my slow, thickheaded skull. His discussions, time, endless effort, and professionalism was inspiring. A 'thank you' doesn't even begin to tell you how very grateful I am. It has been an honor, a pleasure, and – dare I say it – fun to work with you and be your student.

I would like to thank the Mathematics master students for their time and companionship. I shall miss our Sundays together – BUT NOT THE HOMEWORK!!!

Last, but not least, a special thanks goes to my husband who kept asking me "Did you finish your project yet?" for what has seemed like years to this final completion. Nag, nag, nag. Love ya, babe.

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CHAPTER ONE

In the 1850's a very young lawyer, A. Henry Rhind, went to Thebes due to health reasons. During his stay in Thebes he became interested in Egyptology, excavation, and specialized in tombs. It was there that he obtained a collection of antiquities from the tombs of Shêkh 'Abd el-Kurnah. During the winter of 1858, 25 year old Mr. Rhind, while at a market place in Luxor, Egypt, purchased two important pieces of ancient Egyptian mathematics. These two pieces were said to have been found "...by natives in a room in the ruins of a small building near the Ramesseum" [*G1*]. One of these pieces is what is known today as the Rhind Mathematical Papyrus (RMP) and the other is the Egyptian Mathematical Leather Roll (EMLR). After Rhind's death in 1863, at the ripe old age of 30 by tuberculosis, his collection was passed on to a Mr. David Bremner. Then in 1864 the British Museum Trustees acquired a significant amount of other papyri, along with the RMP and the EMLR.

The EMLR, due to the leather being so brittle, was not unrolled (see Figure 1) for over 60 years after its

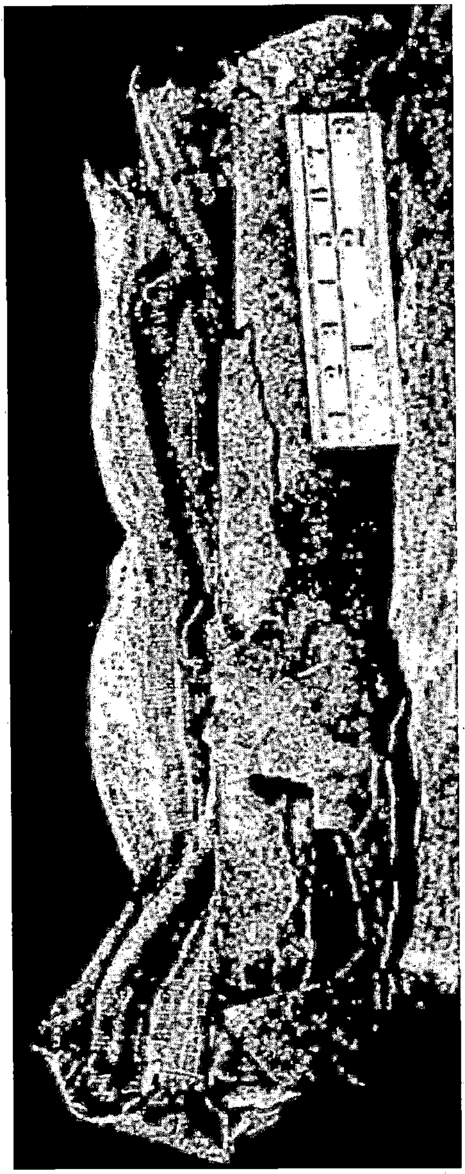

Figure 1. Rolled Egyptian Mathematical Leather Roll at the Time of Discovery

Glanville, S. R. K. (1927). The Mathematical leather roll in the British Museum. Journal of Egyptian Archaeology, 13, 232-239.

known existence due to the fact that the technology did not exist to loosen the coil. The roll was numbered B. M. (British Museum) 10250. Then, in 1926, a new treatment for softening ancient leather was discovered in Berlin. Thanks to Dr. A. Scott and H. R. Hall in 1927, it was finally

> ...possible to reconsider the unrolling of the leather roll. There had been no previous experience of inscribed leather in this condition, for ancient leather has usually 'run' and is altogether in a far more glutinous state than that of the mathematical roll [*Gl*].

At the time of this unprecedented technology, more interest was generated in the chemical process of softening the leather than in the content of the leather roll itself.

> During its unrolling its progress had to be carefully watched and the strong solution applied so as to prevent warping and to ensure uniform uncurling. Finally when almost flat it was pressed between two glass plates and dried in this position [*Sc*, 1927] (see Figure 2).

Although the process was successful in unrolling the leather,

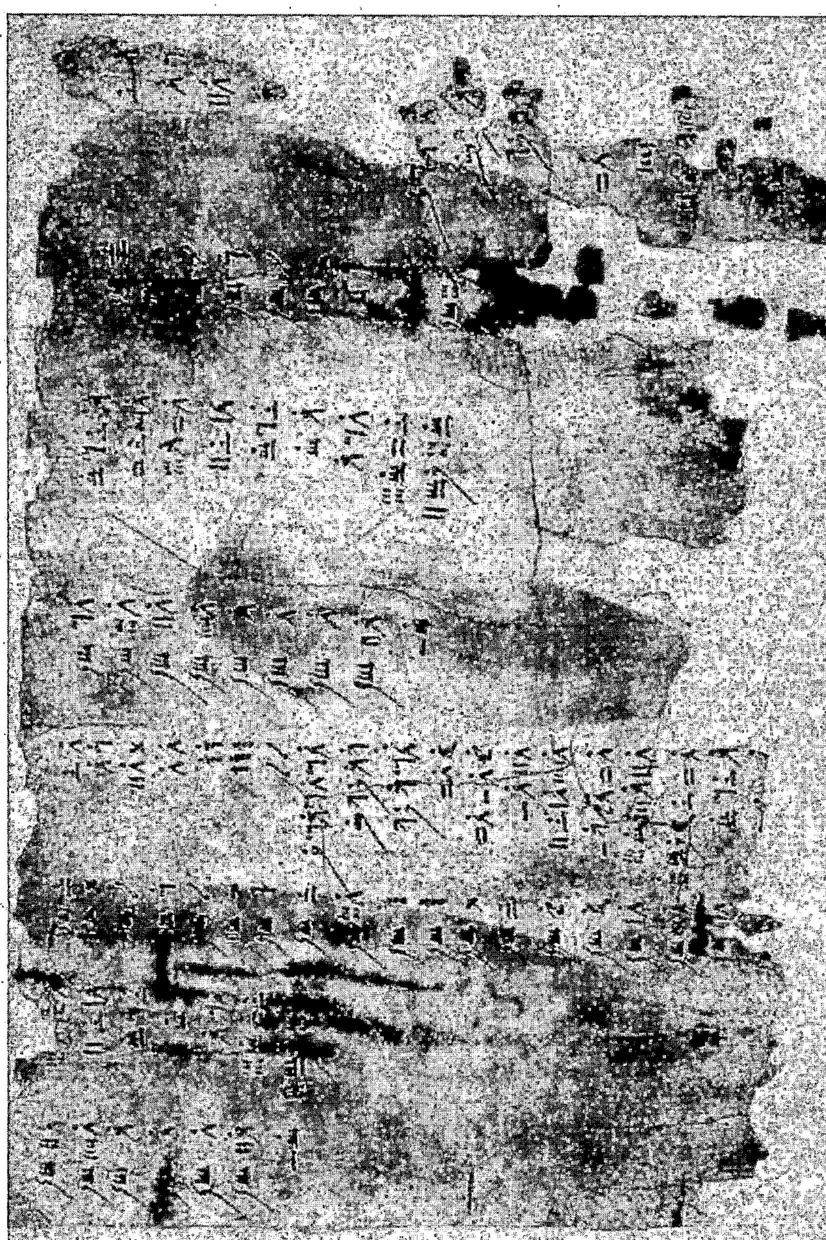

Figure 2. Unrolled Egyptian Mathematical Leather Roll

Gillings, R. J. (1972). Mathematics at the time of the pharaohs. Massachusetts: MIT.

> ...a large number of fragments have been unavoidably broken off one end of the roll since its discovery owing to its extremely brittle state. Many of these fragments were inscribed, and after the unrolling it was possible to fit some of them in their places. A considerable number of smaller fragments still remain unplaced [*G1*].

What is interesting to note is that the EMLR was stored for years in a tin box where it had rolled around and been shaken. This is what had resulted in the ends of the roll being broken. Because of this inadequate storage there are, to date, still quite a bit of fragments that could not be put back, as Glanville pointed out, due to the fact that they could have fit into several different positions on the roll (see Figure 3).

As to the leather roll itself, the EMLR was described as having a pale cream color where the material is of an unidentified animal skin that has many hairs and roots, which can be seen with the naked eye. "The process used to originally preserve the skin is unknown, but the experiments made so far seem to indicate that

Figure 3. Unplaced Fragments of the Egyptian Mathematical Leather Roll

Glanville, S. R. K. (1927). The Mathematical leather roll in the British Museum. Journal of Egyptian Archaeology, 13, 232-239.

it was not by means of 'tanning' as we understand this term" [*Sc*, 1927]. The roll measures $17\frac{3}{8}$ by $10\frac{1}{4}$ inches at a maximum and the writing on the leather is still very clear and distinct except for certain areas that have a stain of some type which is very noticeable.

The EMLR contains a set of *26* equalities of unit fractions in duplicate. It was due to this duplication that it was possible to reconstruct the right-hand column where most of the damage had occurred on the roll with the fragments that were found. "The disappointment of archaeologists over the contents was not shared by historians of mathematics, who found it of great interest" [*Gi*, 1978]. (Most scientists of the time had assumed that it had to be of some importance because leather was more expensive than papyrus at the time of its conception.) As to why there was a second copy of the first two columns, there remains no answer.

> Its real mathematical interest lies in discovering what would have been the use of such a table to the person armed with it, and further what was its relation, if any, to the Rhind Papyrus with which it was discovered [*Gl*].

The RMP is dated in the 33rd year of the Hyksos period during the reign of King Aauserrē (Apophis) who ruled around the middle of the sixteenth century B.C.. It was written by a scribe by the name of Ahmes (or A'h-mose or Ahmosè) who wrote the innermost part first working downwards and from right to left. It is also because of this scribe that the Rhind Mathematical Papyrus is also known as the Ahmes Papyrus. Ahmes also mentions in the papyrus that he

> ...is copying earlier work written down in the reign of King Ny-maat-re (Nymare). This was the throne name of Ammenemes III, who was the sixth king of the Twelfth Dynasty and reigned during the second half of the nineteenth century BC" [*Ro*].

As it exists today, the RMP

> ...consists of two pieces separately mounted between sheets of plate glass and numbered 10057 and 10058. These two pieces once formed a single roll and were probably separated in modern times by an unskilful [sic] unroller [*Pe*]...

and *not* by Ahmes as once thought for the purpose of carrying convenience. Due to its age it was also in a brittle state and had some areas more fragmented than others.

Fragments of the papyrus were found in the possession of the New York Historical Society from a Mr. Edwin Smith who was also a collector around the time of Mr. A. H. Rhind. No one quite knows for certain how he happened to have some of the most important fragments of the papyrus but T. E. Peet, in 1923, does give a very interesting possibility:

> It seems likely that the dates March 17/62 and Dec. 10/63, written on the paper mounts by Edwin Smith, represent the time when he acquired the fragments; at least they were in his hands by those dates.
>
> If these two last dates are indeed the dates of acquisition, and it is not easy to see what else they could be, it would seem that the native finders of the roll attempted to open it in or before 1858, when they sold the main portion to Mr. Rhind, but that they kept back the fragments and sold them to Mr. Smith in two installments [sic] on the dates given (p. 2).

The total length and width as an entire roll is around 18 feet long and 13 inches high (approximately 543 cm by 33 cm, respectively). The roll, however, is not one entire piece of papyrus. It was actually formed of fourteen sheets of papyrus, approximately 40 cm wide by

33 cm high, gummed together to make one long, continuous roll. Another interesting feature of the papyrus is that it was made out of two layers, unlike the paper that we are familiar with today, from the *Cyperus papyrus* plant.

To make the sheet of papyrus, one layer of the plant was laid at 90 degrees to the second layer and then hammered so that the plant fibers exploded and formed a natural "glue" which made it very strong. It is because of these two different layers the RMP is labeled the *recto* and the *verso*. The *recto* is the surface in which the fibers of the papyrus would run horizontally and was the innermost part of the papyrus when rolled. The *verso* is the other side of the papyrus with the fibers of the papyrus running vertically.

The papyrus labeled 10057 (see figure 4), in its present length is 319 cm by 33 cm high. The recto of 10057 has problems 1 through 60. The verso of 10057 is mainly blank with the exception of problems 86 and 87, which are mathematically irrelevant because they do not

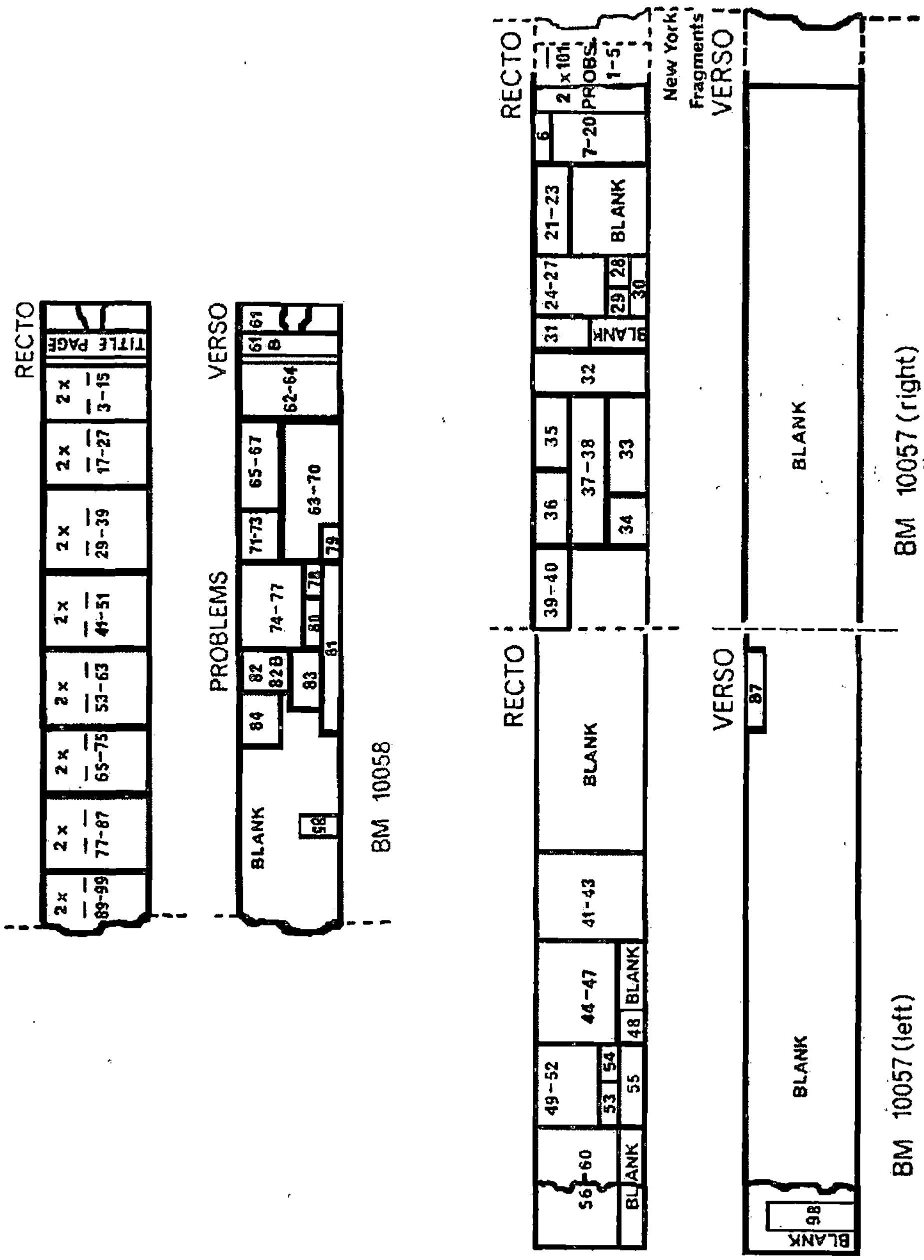

Figure 4. Rhind Mathematical Papyrus Layout

Robins, G., & Shute, C. (1987). The Rhind mathematical papyrus: An ancient Egyptian text. New York: Dover.

contain anything mathematical. Problem 86 consists of "...a patch of three strips pasted on it in ancient times; they have on them part of a list of accounts from another papyrus" [*Ro*] and number 87 contains a particular year of an unknown king's reign.

The papyrus labeled 10058 (see Figure 4), in its present length is 206 cm by 33 cm high. Aside from the title, date and name of the scribe Ahmes, the recto of 10058 is solely dedicated to the $\frac{2}{n}$ table where n is odd and $3 \le n \le 101$. The verso of 10058 contains problems 61 through 85, but number 85 has been determined to be some type of doodle written upside down.

What exactly are these problems on the RMP? Problems 1 through 40 deal with basic arithmetic; 41-47 deal with volume of cubical and rectangular containers; 48-55 deal with area; 56-60 deal with slope of angles; 61-84 deal with miscellaneous arithmetic problems. (See [*Ch*] for further details.)

Another interesting feature with the papyrus is that two colors of ink were used – black and red. The red was "...employed for headings, and also in order to bring into prominence certain figures in the problems" [*Pe*]. The red ink also helped in marking the start of one problem from another.

Now that we have these two documents – the EMLR and the RMP – what exactly is their importance? Many mathematical historians view the significance of the EMLR and the RMP as unquestionable and unprecedented. They give us "...our principle source of knowledge as to how the Egyptians counted, reckoned and measured..." [*Ne*]. Still others believe that they made negligible contributions to our mathematical knowledge and that their system was clumsy and cumbersome. The debate of their importance in math history goes on although they will go so far as to agree that

> ...the Rhind Papyrus, though elementary, is a respectable mathematical accomplishment, proffering problems some of which the average intelligent man of the modern world – 38 centuries more intelligent, perhaps, than A'h-mosè – would have trouble solving [*Ne*].

Another discussion that still reigns today is the relationship between the two documents. The EMLR and the RMP "...have always been considered as related to one another in some way, but no general agreement in this respect has been accepted with any certainty" [*Gi*, 1975]. One reason that they might be related somehow was brought up in 1927 by Scott:

> The roll is said to have been found with the great Rhind papyrus, and this account is confirmed by the fact that the writing is of the same period as that of the papyrus, and that the contents are of a character found in it (p 57).

Another reasoning was brought up by R. J. Gillings in 1975 who had the view that

> ...in ancient Egypt, the EMLR stood roughly in a similar relation to the RMP, and particularly the Recto thereof, as, say, a modern log book would stand in relation to a mathematical text book, that is, an adjunct to it, for simplifying and reducing calculations, albeit not by any means as detailed or as thorough" (p 159).

Clearly no one knows for sure if these two documents were meant to be used together or are related somehow. This is just another question among many that will remain unanswered.

CHAPTER TWO

Egyptian fractions are what we know of today as unit fractions, which are of the form $\frac{1}{n}$. However, there is one exception that the Egyptians had and that exception is the number $\frac{2}{3}$. No one knows why this is the only special case. Some believe that it was just more practical for the problems that they had to solve back then.

As for why they used unit fractions, one could point out that the lay people of Egypt were a simple people. It was easier to see a "fair share" by cutting and dividing things, such as loaves of bread or land, into uniform pieces for everyone involved. For example:

> $\frac{6}{10} = \frac{1}{2} + \frac{1}{10}$ may be more cumbersome than $\frac{3}{5}$, but in some sense the actual division is easier to accomplish this way. If we divide 5 loaves in half and the sixth one into tenths, then give each man $\frac{1}{2} + \frac{1}{10}$ of a loaf, it is then clear to all concerned that everyone has the same portion of bread [*Ka*].

Another train of thought is that they may have used unit fractions because it may have been easier to compare fractions in unit form than in the conventional way we use fractions. For example, let us compare $\frac{3}{4}$ to $\frac{4}{5}$. In unit fractions it is easy to see which fraction is bigger (without using a calculator or converting the fractions to decimals). By converting the fractions to unit fractions it is easier to see that:

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

$$\frac{4}{5} = \frac{1}{2} + \frac{1}{4} + \frac{1}{20}$$

Thus we can see that $\frac{4}{5}$ is larger than $\frac{3}{4}$ by $\frac{1}{20}$. This was not a bad system to use considering the times.

Now the scribe, Ahmes, had written answers to problems in the Rhind or Ahmes Papyrus as whole numbers plus a sum of progressively smaller unit fractions. He did not use any unit fraction more than once in his answer. No one knows why the Egyptians did not repeat a fraction (e.g. $\frac{2}{9} = \frac{1}{9} + \frac{1}{9}$), although there are some who believe that it represented an estimated error of the original fraction.

As mentioned earlier in Chapter 1, the largest table in the Rhind Papyrus is the table $\frac{2}{n}$ where n is odd and $3 \le n \le 101$ (see Table 1). By looking at $\frac{2}{n}$ fractions we begin to notice a pattern emerging: $\frac{2}{n} = \frac{1}{q} + \frac{1}{nq}$. Thus we have an algorithm that can be used to find $\frac{2}{n}$. The Egyptians expressed $\frac{2}{n}$ as a sum of two unit fractions using the following formula $\frac{2}{n} = \frac{1}{\frac{(n+1)}{2}} + \frac{1}{\frac{[n(n+1)]}{2}}$ which gives us $\frac{2}{n} = \frac{2}{(n+1)} + \frac{2}{[n(n+1)]}$ where $n \neq 0$. But notice that this is just the same formula for

$\frac{1}{n} = \frac{1}{(n+1)} + \frac{1}{[n(n+1)]}$ where $n \neq 0$.

There are two different ways to solve for Egyptian fractions. One way is to find a minimum amount of unit fractions for a specific fraction, and the other is to find Egyptian fractions by finding unit fractions that give us the smallest possible denominator. However,

Table 1. Complete $\frac{2}{n}$ Table of the Rhind Mathematical Papyrus

Peet, T. E. (1923). The Rhind mathematical papyrus: British Museum 10057 and 10058. London: Hodder & Stoughton.

2/3	2/3				**2/53**	1/30 + 1/318 + 1/795		
2/5	1/3 + 1/15				**2/55**	1/30 + 1/320		
2/7	1/4 + 1/28				**2/57**	1/38 + 1/114		
2/9	1/6 + 1/18				**2/59**	1/36 + 1/236 + 1/531		
2/11	1/6 + 1/66				**2/61**	1/40 + 1/244 + 1/488 + 1/810		
2/13	1/8 + 1/52 + 1/104				**2/63**	1/42 + 1/126		
2/15	1/10 + 1/30				**2/65**	1/39 + 1/195		
2/17	1/12 + 1/51 + 1/68				**2/67**	1/40 + 1/335 + 1/730		
2/19	1/12 + 1/76 + 1/114				**2/69**	1/46 + 1/138		
2/21	1/14 + 1/42				**2/71**	1/40 + 1/568 + 1/710		
2/23	1/12 + 1/276				**2/73**	1/60 + 1/219 + 1/292 + 1/365		
2/25	1/15 + 1/75				**2/75**	1/50 + 1/150		
2/27	1/18 + 1/54				**2/77**	1/44 + 1/308		
2/29	1/24 + 1/58 + 1/174 + 1/232				**2/79**	1/60 + 1/237 + 1/316 + 1/790		
2/31	1/20 + 1/124 + 1/155				**2/81**	1/54 + 1/162		
2/33	1/22 + 1/66				**2/83**	1/60 + 1/332 + 1/415 + 1/498		
2/35	1/30 + 1/42				**2/85**	1/51 + 1/255		
2/37	1/24 + 1/111 + 1/296				**2/87**	1/58 + 1/174		
2/39	1/26 + 1/78				**2/89**	1/60 + 1/356 + 1/534 + 1/890		
2/41	1/24 + 1/246 + 1/328				**2/91**	1/70 + 1/130		
2/43	1/42 + 1/86 + 1/129 + 1/301				**2/93**	1/62 + 1/186		
2/45	1/30 + 1/90				**2/95**	1/60 + 1/380 + 1/570		
2/47	1/30 + 1/141 + 1/470				**2/97**	1/56 + 1/679 + 1/776		
2/49	1/28 + 1/196				**2/99**	1/66 + 1/198		
2/51	1/34 + 1/102				**2/101**	1/101 + 1/202 + 1/303 + 1/606		

there is no known algorithm for getting unit fractions that have a minimum amount of terms *and* the smallest possible denominators at this time.

What is interesting to note about Egyptian fractions is that they were used for over 2000 years in the Mediterranean basin with almost no change in the system of mathematics. No one seems to know why this happened.

> The static character of Egyptian culture, the blight that fell upon Egyptian science around the middle of the second millennium, has often been emphasized but never adequately explained. Religious and political factor undoubtedly played a part in turning a dynamic society into one of stone [*Ne*].

With all the questions about the Egyptians and their fractions, it is unlikely that this subject will go away quietly in the mathematical world.

CHAPTER THREE

In 1909 when James J. Sylvester [*Sy*] wrote "The Collected Mathematical Papers of James Joseph Sylvester" he had included in his book a method for expressing any rational number that was between zero and one as a sum of unit fractions. What he obviously did not know, since he did not mention it in his multiple volume book, was that in 1202 Fibonacci, in his book Liber Abaci, had produced an "algorithm" to write any fraction $0 < \frac{a}{b} < 1$, as a sum of unit fractions. (What is not known is if Fibonacci had a proof that his method worked all the time.) Because of this, the Fibonacci Algorithm is sometimes noted as the Fibonacci-Sylvester Algorithm.

In order to use the Fibonacci-Sylvester Algorithm we must state a few Lemma's to get us started properly.

Lemma 1. [*Bu*, 2]. *Archimedean Property*. If a and b are any positive integers, then there exists a positive integer n such that $na \geq b$.

Proof. See [*Bu*, 2]. □

Lemma 2. If a and b are any positive integers such that $\frac{a}{b} < 1$, then there exists a positive integer n such that $\frac{1}{n} \leq \frac{a}{b} < \frac{1}{n-1}$.

Proof. By Lemma 1 there exists such a positive integer m such that $b \leq am$. Let n be the least positive integer such that $na \geq b$. Then the first inequality $\frac{1}{n} \leq \frac{a}{b}$ is satisfied. By our choice of n, $(n-1)a < b$ and the second inequality, $\frac{a}{b} < \frac{1}{n-1}$, is satisfied. Thus $\frac{1}{n} \leq \frac{a}{b} < \frac{1}{n-1}$. □

Algorithm 3. The Fibonacci-Sylvester Algorithm. Let $f_0 = \frac{a_0}{b_0}$ be a rational number in lowest terms such that $0 < \frac{a_0}{b_0} < 1$. We can obtain a sequence of fractions as follows:

1) Let $n_1 \geq 2$ be a positive integer such that

$$\frac{1}{n_1} \leq f_0 < \frac{1}{n_1 - 1}.$$

2) Let $f_1 = f_0 - \frac{1}{n_1}$.

3) For $i \geq 1$ repeat steps 1 and 2 with f_{i+1} and n_{i+1} replacing f_i and n_i respectively, until we get N such that $f_N = 0$.

Theorem 4. The Fibonacci-Sylvester Algorithm produces a finite sequence of rational numbers $f_0, f_1, f_2, \ldots, f_N$ between zero and one having decreasing numerators and $f_N = 0$. Thus we end up with the Egyptian fraction representation of $f_0 = \frac{a_0}{b_0} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_{N-1}} + \frac{1}{n_N}$, where $\frac{1}{n_i} = f_{i-1} - f_i$ and $i \geq 1$.

Proof. Using the fact that the fractions f_i are in reduced form, we will show that when we have two consecutive fractions, $f_i = \frac{a_i}{b_i}$ and $f_{i+1} = \frac{a_{i+1}}{b_{i+1}}$, then

$a_i > a_{i+1}$. We note that it is sufficient to consider one case, $i = 0$. Let $f_0 = \frac{a_0}{b_0}$ where $0 < \frac{a_0}{b_0} < 1$ and let n_1 be a positive integer such that $\frac{1}{n_1} \le \frac{a_0}{b_0} < \frac{1}{n_1 - 1}$. Since $f_0 = \frac{a_0}{b_0}$ then $f_1 = \frac{a_0}{b_0} - \frac{1}{n_1}$ which implies that $f_1 = \frac{a_0 n_1 - b_0}{b_0 n_1}$. Then $a_0 n_1 - b_0 \ge a_1$ and $b_0 n_1 \ge b_1$.

Case (1). $\frac{a_1}{b_1} = 0$.

If $f_1 = \frac{a_0}{b_0} - \frac{1}{n_1} = 0$ then $a_0 > a_1 = 0$, $f_0 = \frac{1}{n_1}$ and we are finished.

Case (2). $\frac{a_1}{b_1} \ne 0$.

Since $\frac{a_0}{b_0} < \frac{1}{n_1 - 1}$, we have $a_0(n_1 - 1) < b_0$ so $a_0 n_1 - b_0 < a_0$. Thus $a_1 \le a_0 n_1 - b_0 < a_0$. So $a_0 > a_1$.

Thus by repetition of the Fibonacci-Sylvester Algorithm, we get $a_0 > a_1 > a_2 > \ldots > a_k$ where each a_k is a positive integer. Thus we know that $f_0 > f_1 > \ldots$ eventually terminates and $f_0 = \frac{a_0}{b_0}$, $f_1 = \frac{a_0}{b_0} - \frac{1}{n_1}$,

$f_2 = \frac{a_1}{b_1} - \frac{1}{n_2} = \frac{a_0}{b_0} - \sum_{i=1}^{2} \frac{1}{n_i}, \ldots, f_N = \frac{a_0}{b_0} - \sum_{i=1}^{N} \frac{1}{n_i} = 0$. So we get the Egyptian fraction representation

$$\frac{a_0}{b_0} = \frac{1}{n_1} + \frac{1}{n_2} + \ldots + \frac{1}{n_{N-1}} + \frac{1}{n_N}. \quad \square$$

Here's how the algorithm works. Think of a fraction that you want to express as an Egyptian fraction, say $\frac{a_0}{b_0} = \frac{7}{9} = f_0$. We need to find the largest unit fraction not larger than $\frac{7}{9}$ as by Lemma 2. We get $\frac{1}{2} \leq \frac{7}{9} < \frac{1}{1}$. So $\frac{1}{2} = \frac{1}{n_1}$. Now subtract to get

$f_1 = f_0 - \frac{1}{n_1} = \frac{7}{9} - \frac{1}{2} = \frac{5}{18} = \frac{a_1}{b_1}$. Since $\frac{5}{18} \neq 0$ we repeat the process. So we need to find the largest unit fraction not larger than $\frac{5}{18}$ which is $\frac{1}{4} = \frac{1}{n_2}$. Again, we subtract the two fractions $f_2 = f_1 - \frac{1}{n_2} = \frac{5}{18} - \frac{1}{4}$ $= \frac{1}{36} = \frac{a_2}{b_2}$. Since $\frac{1}{36} \neq 0$ we repeat the process yet again. Now we need to find the largest unit fraction not larger than $\frac{1}{36}$ which, in this case, is $\frac{1}{36}$. Therefore $f_3 = f_2 - \frac{1}{n_3} = \frac{1}{36} - \frac{1}{36} = 0$ and we are finished. Thus by the Fibonacci-Sylvester Algorithm we see that $\frac{7}{9}$ has the Egyptian fraction representation $\frac{7}{9} = \frac{1}{2} + \frac{1}{4} + \frac{1}{36}$ and that $f_0 = \frac{7}{9} > f_1 = \frac{5}{18} > f_2 = \frac{1}{36} > f_3 = 0$.

An interesting note with this method is that, unlike the Egyptians, this method doesn't consider $\frac{2}{3}$ as part of the "unit" fractions.

CHAPTER FOUR

John Farey is more famous for what he didn't know than what he knew. He was actually a British geologist, not a mathematician. He studied mathematics, drawing, and surveying in England at Halifax, Yorkshire. Most of the geology that he acquired was taught to him by William Smith, known as the founder of English geology, who "...made the revolutionary discovery that fossils can be used to identify the succession of layers or strata of rocks" [*Fe*]. It is because of Farey's tenacity in referring to Smith's name in many of his published articles for various scientific magazines that Smith's name was more recognizable in geological circles. All in all Farey published 60 various scientific papers ranging from geology, music, physics, to mathematics.

In 1816 he published an article in the *Philosophical Magazine* called *On a Curious Property of Vulgar Fractions* [*Fa*] (see Figure 5). It was there that he noticed a "curious property" while scrutinizing the tables made by Henry Goodwin in his article *Complete Decimal Quotients*.

On a curious Property of vulgar Fractions.
By Mr. J. Farey, *Sen. To Mr. Tilloch*

Sir.—On examining lately, some very curious and elaborate Tables of "Complete decimal Quotients," calculated by Henry Goodwyn, Esq. of Blackheath, of which he has printed a copious specimen, for private circulation among curious and practical calculators, preparatory to the printing of the whole of these useful Tables, if sufficient encouragement, either public or individual, should appear to warrant such a step: I was fortunate while so doing, to deduce from them the following general property; viz.

If all the possible vulgar fractions of different values, whose greatest denominator (when in their lowest terms) does not exceed any given number, be arranged in the order of their values, or quotients; then if both the numerator and the denominator of any fraction therein, be added to the numerator and the denominator, respectively, of the fraction next but one to it (on either side), the sums will give the fraction next to it; although, perhaps, not in its lowest terms.

For example, if 5 be the greatest denominator given; then are all the possible fractions, when arranged, $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$; taking $\frac{1}{3}$, as the given fraction, we have $\frac{1+1}{5+3} = \frac{2}{8} = \frac{1}{4}$ the next smaller fraction than $\frac{1}{3}$; or $\frac{1+1}{3+2} = \frac{2}{5}$, the next larger fraction to $\frac{1}{3}$. Again, if 99 be the largest denominator, then, in a part of the arranged Table, we should have $\frac{15}{52}$, $\frac{28}{97}$, $\frac{13}{45}$, $\frac{24}{83}$, $\frac{11}{38}$, &c.; and if the third of these fractions be given, we have $\frac{15+13}{52+45} = \frac{28}{97}$ the second: or $\frac{13+11}{45+38} = \frac{24}{83}$ the fourth of them: and so in all the other cases.

I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of any easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers; and am

Sir,
Your obedient humble servant,

J. Farey.
Howland-street.

Figure 5. Farey's Letter

Bruckheimer, M., & Arcavi, A. (1995). Farey series and Pick's area theorem. Mathematical Intelligencer, 17 (4), 64-67.

It is because he pointed out this "curious property" that he is known more in the mathematical world and nearly forgotten in the geological one.

What was this finding that Farey noticed? Take two irreducible fractions between zero and one whose denominators don't exceed a certain number. If you add the two numerators together and the two denominators together, you get another fraction that falls between the two called the *mediant*.

Definition 1. The *mediant* is defined as the reduced form of the fraction $\frac{a+a'}{b+b'}$ where $\frac{a}{b}$ and $\frac{a'}{b'}$ are fractions between 0 and 1. (The *mediant* may not be *exactly* halfway between the two consecutive fractions, but it will belong somewhere between the two of them.)

Before we begin to delve more into Farey fractions, we need to define just exactly what a Farey sequence is.

Definition 2. Arranged in increasing order, a Farey sequence of order N is defined as

$$F_N = \left\{ \frac{a}{b} \;\middle|\; 0 \leq \frac{a}{b} \leq 1,\ \gcd(a, b) = 1,\ N \geq b \right\}.$$

To construct the table of a Farey sequence, F_n, where $n = 1, 2, 3, \ldots$, and the denominators in F_n cannot exceed n, we perform the following procedure. The first row, $n = 1$, which is designated F_1, automatically has the fractions $F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$. The second row, $n = 2$, which is designated F_2, has the fractions $\frac{0}{1}, \frac{1}{1}$ and the *mediant* $\frac{0+1}{1+1} = \frac{1}{2}$. So the row $F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$. In general, given F_n, we form the *mediants* of adjacent fractions in F_n. Those *mediants* whose denominators are less that or equal to $n + 1$ are added to F_n to form F_{n+1}. So to get F_3 we first copy all the fractions of the F_2 row in its order. The second step is to insert the *mediants* $\frac{0+1}{1+2} = \frac{1}{3}$ between $\frac{0}{1}$ and $\frac{1}{2}$, and $\frac{1+1}{2+1} = \frac{2}{3}$ between $\frac{1}{2}$ and $\frac{1}{1}$. So this new row would look like $F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$. The Farey sequence table would look like Table 2. If you

Table 2. Example of Farey Sequence Table

F(1)	0/1																												1/1
F(2)	0/1														1/2														1/1
F(3)	0/1									1/3					1/2					2/3									1/1
F(4)	0/1							1/4		1/3					1/2					2/3		3/4							1/1
F(5)	0/1					1/5		1/4		1/3		2/5			1/2			3/5		2/3		3/4		4/5					1/1
F(6)	0/1				1/6	1/5		1/4		1/3		2/5			1/2			3/5		2/3		3/4		4/5	5/6				1/1
F(7)	0/1			1/7	1/6	1/5		1/4	2/7	1/3		2/5	3/7		1/2		4/7	3/5		2/3	5/7	3/4		4/5	5/6	6/7			1/1
F(8)	0/1		1/8	1/7	1/6	1/5		1/4	2/7	1/3	3/8	2/5	3/7		1/2		4/7	3/5	5/8	2/3	5/7	3/4		4/5	5/6	6/7	7/8		1/1
F(9)	0/1	1/9	1/8	1/7	1/6	1/5	2/9	1/4	2/7	1/3	3/8	2/5	3/7	4/9	1/2	5/9	4/7	3/5	5/8	2/3	5/7	3/4	7/9	4/5	5/6	6/7	7/8	8/9	1/1

look closely at the table you will notice that a pattern emerges. Except for the row F_1, each F_n row, by symmetry, has the term $\frac{1}{2}$ as its middle term and has an odd number of terms. You will also notice that each row, from left to right, has irreducible fractions arranged in increasing order and that if $\frac{a}{b}$ is in the nth row then $b \leq n$.

This property that Farey noted was actually noticed and proved by C. Haros in 1802. However, it seems that Cauchy $[Ca]$ didn't realize that it was Haros' finding, read Farey's statement, supplied the proof later that same year in 1816, and gave the credit to Farey by naming the theorem after him. (For more details see $[Br]$.)

Lemma 5. Let $\frac{a}{b} \in F_n$ be reduced fraction. Then there exists a reduced fraction $\frac{a'}{b'} \in F_n$ such that $(b, b') = 1$, $a'b - ab' = 1$ and $\frac{a}{b} < \frac{a'}{b'}$ are adjacent in F_n.

Proof. The Diophantine equation $by - ax = 1$ has an infinite number of solutions (x, y) with $x > 0$, $y > 0$. This is because if (x_0, y_0) is a particular solution, then $x = x_0 + bt$, $y = y_0 + at$ is also a solution for all $t \in \mathbb{Z}$. If we choose a solution (x_0, y_0) then, by the Archimedean property, there exists a T such that $bT > n - x_0$. Thus $S = \{t \mid x_0 + bt \leq n\}$ is a bounded set of integers. Therefore S has a maximum element, t_0. By our choice of t_0, $b' = x_0 + bt \leq n$. However we need $x_0 + bt_0 > 0$. So if $x_0 + bt_0 \leq 0$ then $x_0 + b(t_0 + 1) \leq b \leq n$, but $(t_0 + 1) \in S$ is a contradiction since t_0 is maximal. Thus $0 < b' \leq n$. Now let $a' = y_0 + at_0$ and suppose $a' \leq 0$. So $y_0 + at_0 \leq 0$ and then $by_0 + bat_0 \leq 0$ which means that, by the use of $by_0 - ax_0 = 1$, $1 + ax_0 + abt_0 \leq 0$ so that $1 + a(x_0 + bt_0) \leq 0$ which is a contradiction. Now we need $a' = y_0 + bt_0 \leq b'$. If $a' > b'$ then $y_0 + at_0 > x_0 + bt_0$. So $by_0 + abt_0 > bx_0 + b^2t_0$ which means that $1 + ax_0 + abt_0 > bx_0 + b^2t_0$, by the use of $by_0 - ax_0 = 1$, so $1 > bx_0 - ax_0 + b^2t_0 - abt_0$. However

$1 > x_0(b-a) + t_0 b(b-a)$ which means that $1 > (x_0 + t_0 b)(b-a)$ is a contradiction because $x_0 + t_0 b \geq 1$ and $(b-a) \geq 1$ which means that $1 > 1 \cdot 1$. Thus $0 < a' \leq b' \leq n$ and $a'b - ab' = 1$. Now we have $\gcd(b, b') = 1$, $\gcd(a', b') = 1$, and $\frac{a'}{b'} \in F_n$. So it remains to show that $\frac{a}{b}, \frac{a'}{b'}$ are adjacent fractions in F_n.

Let us suppose there exists f, g where $\frac{f}{g} \in F_n$ such that $\frac{f}{g} > \frac{a}{b}$, then $\frac{f}{g} - \frac{a}{b} > 0$. We will show that $\frac{f}{g} \geq \frac{a'}{b'}$. Let $\frac{f}{g} - \frac{a}{b} = \frac{fb - ag}{bg} = \frac{m}{bg}$ where $m \geq 1$. To show that $x = b'm$, $y = a'm$ is a solution to $by - ax = m$, we recall $a'b - ab' = 1$ so $(ma')b - a(mb') = m$. So if $y = a'm$, $x = b'm$, then x, y is a solution to $by - ax = m$. Since (g, f) is also a solution to $by - ax = m$, $g \in \{b'm + bt \mid t \in \mathbb{Z}\}$. Now $mb' + b \geq b' + b > n$. Since $g < n$, $g = b'm + bt$ with $t \leq 0$. So $g \leq mb'$. Since

$m > 0$, $\frac{1}{g} \geq \frac{1}{mb'}$ and $\frac{m}{gb} \geq \frac{1}{bb'}$. Now

$\frac{f}{g} - \frac{a}{b} = \frac{m}{gb} \geq \frac{1}{bb'} = \frac{a'}{b'} - \frac{a}{b}$ and so $\frac{f}{g} \geq \frac{a'}{b'}$. Thus $\frac{a}{b}, \frac{a'}{b'}$ are adjacent fractions in F_n. □

Property 1. If $\frac{a}{b}$ and $\frac{c}{d}$ are two reduced adjacent terms of F_n where $\frac{a}{b} < \frac{c}{d}$, then $bc - ad = 1$. In particular,

$\frac{c}{d} = \frac{a}{b} + \frac{1}{bd}$.

Proof. As in the proof of the previous lemma, there exists a reduced fraction $\frac{a'}{b'} \in F_n$ such that $\frac{a}{b} < \frac{a'}{b'}$ are adjacent and $a'b - ab' = 1$. Thus $\frac{a'}{b'} = \frac{c}{d}$. Since $\frac{a'}{b'}$, $\frac{c}{d}$ are reduced, then $a' = c$, $b' = d$. □

Property 2. Let $\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}$ be reduced adjacent fractions belonging in F_n. Then $\frac{a'}{b'} = \frac{a + a''}{b + b''}$.

Proof. From Property 1, $a'b - ab' = 1$ and $a''b' - a'b'' = 1$. Thus $a'(b + b'') = a'b + a'b'' = 1 + ab' + a''b' - 1 = b'(a + a'')$. □

The next theorem is immediate from Property 1.

Theorem 6. *(Cauchy)*. If $\frac{a}{b} < \frac{a'}{b'}$ reduced consecutive fractions in F_n then $\frac{a'}{b'} - \frac{a}{b} = \frac{1}{bb'}$.

Remark. If $\frac{a}{b}, \frac{a'}{b'}$ are reduced fractions such that $\frac{a'}{b'} - \frac{a}{b} = \frac{1}{bb'}$ then $\frac{a}{b}, \frac{a'}{b'}$ are consecutive in F_n where $n = \max\{b, b'\}$.

Proof. We see that $a'b - ab' = 1$. So, in any solution x, y to $bx - ay = 1$, $y \in \{b' + tb \mid t \in \mathbb{Z}\}$. Clearly b' is a maximum solution such that $b' \leq n = \max\{b, b'\}$ since $b + b' > n$. Then by the proof of Lemma 5, $\frac{a}{b}, \frac{a'}{b'}$ are adjacent in F_n. □

Algorithm 7. Farey Sequence Algorithm. Let $f_0 = \frac{a}{b} \in F_b$ be a positive rational number in lowest terms. We can obtain a finite sequence of fractions using the Farey table as follows:

a) Given a nonzero fraction $f_n = \frac{a_n}{b_n} \in F_{b_n}$ we define $f_{n+1} < f_n$ to be the fraction in F_{b_n} adjacent to f_n. Note that $f_n = f_{n+1} + \frac{1}{b_n b_{n+1}}$ and, since $b_{n+1} \le b_n$, $a_{n+1} < a_n$.

b) Repeat until we get $a_{n+1} = 0$.

Remark. Since $a_0 > a_1 > a_2 > \ldots$, Algorithm 7 must terminate after a finite number of steps. If $f_{n+1} = 0$ then $\frac{a_0}{b_0} = \frac{1}{b_0 b_1} + \ldots + \frac{1}{b_{n-1} b_n}$.

Let's look at an example of this algorithm. Say we picked the fraction $f_0 = \frac{7}{9} \in F_9$. Now we pick $\frac{3}{4} < \frac{7}{9}$ adjacent to $\frac{7}{9}$ in F_9. We note that $\frac{7}{9} = \frac{1}{4 \cdot 9} + \frac{3}{4}$. Next we choose $\frac{2}{3} < \frac{3}{4}$ adjacent to $\frac{3}{4}$ in F_4. Now $\frac{3}{4} = \frac{1}{3 \cdot 4} + \frac{2}{3}$ and $\frac{7}{9} = \frac{1}{4 \cdot 9} + \frac{1}{3 \cdot 4} + \frac{2}{3}$. Next $\frac{1}{2} < \frac{2}{3}$ is adjacent to $\frac{2}{3} \in F_3$. This gives $\frac{2}{3} = \frac{1}{2 \cdot 3} + \frac{1}{2}$. So $\frac{7}{9} = \frac{1}{4 \cdot 9} + \frac{1}{3 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2}$. Finally, $\frac{0}{1}$ is adjacent to $\frac{1}{2} \in F_2$, so $\frac{7}{9} = \frac{1}{4 \cdot 9} + \frac{1}{3 \cdot 4} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2} + \frac{0}{1}$. Here we observe that our numerator is 0 and, thus, we are finished.

The Farey sequence algorithm yields the Egyptian fraction representation of $\frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36}$.

How do we know how many fractions are in each F_n row? With the help of the Euler phi-function we can determine this. Recalling that, for $n > 0$, $\phi(n)$ is the number of positive integers less than n and relatively prime to n. We note that $F_n = F_{n-1} \cup \left\{\frac{a}{n} \mid \gcd(a, n) = 1\right\}$ and the cardinality of $\left\{\frac{a}{n} \mid \gcd(a, n) = 1\right\}$ is $\phi(n)$. So

$Card(F_n) = Card(F_{n-1}) + \phi(n)$ for $n \geq 2$. Also, we note that the $Card(F_1) = 1 + \phi(1)$ so we can use induction to show that the $Card(F_n) = 1 + \sum_{k=1}^{n} \phi(k)$.

Proof. We proceed by induction on n.

a) First consider $n = 1$. Since the F_1 row always contains the two entries $\frac{0}{1}$, $\frac{1}{1}$, and $\phi(1) = 1$ then $F_1 = 1 + \phi(1)$.

b) Now assume $n \geq 1$ and $Card(F_n) = 1 + \sum_{k=1}^{n} \phi(k)$.

Then, as noted above,

$Card(F_{n+1}) = Card(F_n) + \phi(n + 1)$

$= 1 + \sum_{k=1}^{n} \phi(k) + \phi(n + 1) = 1 + \sum_{k=1}^{n+1} \phi(k)$. Thus

$Card(F_n) = 1 + \sum_{k=1}^{n} \phi(k)$ for all $n \geq 1$.

Although it seems unlikely that anyone ever heard of, let alone used, Farey fractions, they actually have been used in mathematics. The most notorious place in math that Farey fractions have been utilized has been in the area of fractals. They have been used in defining a subtree of the Stern-Brocot Tree [*Gr*] and in the Mandelbrot Set [*De*]. However, Farey fractions are not without their problems.

The main disadvantage with the Farey method when trying to find Egyptian fractions is that the procedure can become *very* long. This is because you first must build the Farey table to F_b and then follow the procedures set forth in Algorithm 7.

CHAPTER FIVE

It seems, in the history of mathematics, that Fibonacci was the first person to use continued fractions. His notation, however, of continued fractions is not as we know it today. He used, what I consider to be, a string of fractions to represent a condensed notation for continued fractions. For example, $\frac{2\ 3\ 4}{3\ 5\ 9}$,

in Fibonacci's notation, represented $\dfrac{4 + \dfrac{3 + \frac{2}{3}}{5}}{9}$.

Nowadays we tend to write continued fractions in descending order instead of ascending order. The same example, by contemporary standard, is written as

$0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{7}}}}$. This type of example is known as

a finite simple continued fraction.

Definition 3. Finite Continued Fraction. A fraction of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

where $a_0, a_1, \ldots, a_n$ are real numbers, all of which except possibly a_0 are positive. The numbers $a_1, a_2, \ldots, a_n$ are the *partial denominators* of this fraction. Such a fraction is called *simple* if all of the a_i are integers [*Bu*]. For simplicity we can denote the fraction as $[a_0; a_1, a_2, \ldots, a_n]$ [*Kh*].

Definition 4. Let $[a_0; a_1, a_2, \ldots, a_n]$ be a finite continued fraction. From the partial denominators of the continued fractions we can form the fractions $C_0 = [a_0]$, $C_1 = [a_0; a_1]$, $C_2 = [a_0; a_1, a_2], \ldots,$ $C_n = [a_0; a_1, a_2, \ldots, a_n]$. These continued fractions, where $C_k = [a_0; a_1, a_2, \ldots, a_n]$ is made by cutting off the expansion after the kth partial denominator, a_k, are called the kth *convergents* of the given continued fraction where $0 \leq k \leq n$ [*Bu*].

Note that we can also rewrite the form C_k as $C_k = \left[a_0; a_1, a_2, \ldots, a_{k-1} + \dfrac{1}{a_k}\right]$ and that we can establish formulas for the convergents in such a manner that will facilitate the computation with continued fractions.

Theorem 8. [*Bu*, 279]. Any rational number can be written as a finite simple continued fraction.

Proof. Let $s = \dfrac{a}{b}$ where s is a reduced rational number and $b > 0$. We proceed by induction on b. If $b = 1$ then $a = (a - 1) + 1 = (a - 1) + \dfrac{1}{1}$. Now assume that $b > 1$ and each fraction $\dfrac{c}{d}$ with $0 < d < b$ has a continued fraction representation. Then by applying the Division Algorithm we have $a = q \cdot b + r$ where

$0 \le r < b$. If $r = 0$ then $\frac{a}{b} = q$ and this gives us $b = 1$ which is a contradiction. So $r \ne 0$ and $\frac{a}{b} = q + \frac{r}{b} = q + \frac{1}{\frac{b}{r}}$ where $0 \le r < b$. By the inductive hypothesis $\frac{b}{r} = [a_1, \ldots, a_N]$ where there exists an $a_i \in N$. Thus $\frac{a}{b} = [q; a_1, a_2, \ldots, a_N]$. □

Definition 5. Let us define positive integers p_k, q_k where $k = 0, 1, 2, \ldots, n$ and $a_i \in N$, where a_i are the partial denominators of the continued fraction $[a_0; a_1, a_2, \ldots, a_k]$, as:

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 & q_1 &= a_1 \\ p_2 &= a_2 p_1 + p_0 & q_2 &= a_2 q_1 + q_0 \\ &\vdots & &\vdots \\ p_k &= a_k p_{k-1} + p_{k-2} & q_k &= a_k q_{k-1} + q_{k-2} \end{aligned}$$

[Bu].

Theorem 9. $[Bu, 288]$. (a) The convergents with even subscripts form a strictly increasing sequence; that is,

$$C_0 < C_2 < C_4 < \ldots$$

(b) The convergents with odd subscripts form a decreasing sequence; that is

$$C_1 > C_3 > C_5 > \ldots$$

(c) Each convergent with an odd subscript is greater than every convergent with an even subscript.

Theorem 10. $[Bu, 284]$. The kth convergent of the simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ has the value $C_k = \frac{p_k}{q_k}$ where $0 \leq k \leq n$ (see Definition 4 for defining p_k, q_k).

Proof. We proceed by induction on k.

a) If $k = 0$. $C_0 = \frac{p_0}{q_0} = a_0$ and we are done.

b) Now assume $C_k = \frac{p_k}{q_k}$. Then this means that

$$C_{k+1} = [a_0; a_1, a_2, \ldots, a_k, a_{k+1}] = \left[a_0; a_1, a_2, \ldots, a_k + \frac{1}{a_{k+1}}\right]$$

$$= \frac{\left(a_k + \frac{1}{a_{k+1}}\right)p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right)q_{k-1} + q_{k-2}} = \frac{a_k p_{k-1} + \left(\frac{1}{a_{k+1}}\right)p_{k-1} + p_{k-2}}{a_k q_{k-1} + \left(\frac{1}{a_{k+1}}\right)q_{k-1} + q_{k-2}}$$

$$= \frac{(a_k p_{k-1} + p_{k-2}) + \left(\frac{1}{a_{k+1}}\right)p_{k-1}}{(a_k q_{k-1} + q_{k-2}) + \left(\frac{1}{a_{k+1}}\right)q_{k-1}}.$$ Then by using

substitution where $p_k = a_k p_{k-1} + p_{k-2}$ and

$q_k = a_k q_{k-1} + q_{k-2}$ we get the following:

$$\frac{p_k + \left(\frac{1}{a_{k+1}}\right)p_{k-1}}{q_k + \left(\frac{1}{a_{k+1}}\right)q_{k-1}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}} = C_{k+1}. \quad \square$$

Theorem 11. [Bu, 285]. If $C_k = \frac{p_k}{q_k}$ is the kth convergent of the finite simple continued fraction $[a_0; a_1, a_2, \ldots, a_n]$ then $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$ where $1 \le k \le n$.

Proof. We proceed by induction on k.

a) When $k = 1$. Since $p_0 = a_0$, $q_0 = 1$, $p_1 = a_1 a_0 + 1$ and $q_1 = a_1$, then

$$p_1 q_0 - p_0 q_1 = (a_1 a_0 + 1) \cdot 1 - a_0 a_1 = 1.$$

b) When $k > 1$. If $p_k = a_k p_{k-1} + p_{k-2}$ and $q_k = a_k q_{k-1} + q_{k-2}$, then, $p_{k+1} = a_{k+1} p_k + p_{k-1}$ and $q_{k+1} = a_{k+1} q_k + q_{k-1}$. So $p_{k+1} q_k - q_{k+1} p_k$

$$= \left((a_{k+1})(p_k) + p_{k-1}\right) q_k - \left((a_{k+1})(q_k) + q_{k-1}\right) p_k$$

$$= a_{k+1} p_k q_k + p_{k-1} q_k - a_{k+1} q_k p_k - q_{k-1} p_k$$

$$= p_{k-1} q_k - q_{k-1} p_k = -1\left(p_k q_{k-1} - q_k p_{k-1}\right) = (-1)^k. \quad \square$$

To explain how the process works, let us use the fraction $\frac{7}{9}$ and write it in finite continued fraction form. Using the Euclidean Algorithm we come up with

$$9 = \underline{1} \cdot 7 + 2$$

$$7 = \underline{3} \cdot 2 + 1$$

$$2 = \underline{2} \cdot 1 + 0$$

Thus, with the underlined information we found from above, we can easily form the *partial denominators* of the continued fractions as $\frac{7}{9} = [0; 1, 3, 2]$. The fraction

$\frac{7}{9}$ is obviously a finite fraction and it is less than 1 so we can easily write it in continued fraction form due to the Euclidean Algorithm. Thus

$$\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}} \quad \text{or} \quad 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

which helps in finding the convergents to $\frac{7}{9}$ which are $\left[\frac{0}{1}; \frac{1}{1}, \frac{3}{4}, \frac{7}{9}\right]$.

What is interesting to note is that Farey fractions are linked to continued fractions. By the remark following Theorem 6 (see Chapter 4), if we take any two successive convergents we can see that they are consecutive Farey fractions. For example, by using any of the convergents of $\frac{7}{9}$, say $C_2 = \frac{3}{4}$ and $C_3 = \frac{7}{9}$, and utilizing the properties employed by Cauchy (see Theorem 6 in Chapter 4) we would see that $4 \cdot 7 - 3 \cdot 9 = 1$ and that $\frac{28}{36} - \frac{27}{36} = \frac{1}{36} = \frac{1}{b \cdot b'}$. This is also true with *any* of the convergents that we so choose.

By a quick examination of the Farey table (see page 30) we would have noticed that $\frac{3}{4}$ and $\frac{7}{9}$ are, indeed, consecutive fractions in F_9.

However continued fractions are not at all what they are cracked up to be. Besides being a tedious exercise in algebraic manipulation,

> The main disadvantage of continued fractions, and it is a grave disadvantage, is that no one has figured out how to perform the arithmetic operations of addition, subtraction, and multiplication with them. Their importance, which is considerable, has been in the theory and practice of approximation [*Be*].

CHAPTER SIX

A Diophantine equation, in its simplest form, is the linear equation $ax + by = c$ where a, b, and c belong to the set of integers and where both $a, b \neq 0$. When integers are substituted for x and y that make the statement true, these integers are called solutions for the equation. We will give two approaches to solving Diophantine equations — one using Farey fractions and one using continued fractions.

Theorem 12. Let $ax + by = c$ be a Diophantine equation where $0 < a < b$ and let $d = \gcd(a, b)$. Let $\frac{p}{q}$ be the fraction in $F_{b/d}$ preceding $\frac{a/d}{b/d}$. If $d \mid c$, then

$$x = \frac{c}{d} \cdot q, \quad y = \frac{-c}{d} \cdot p$$ is a solution to $ax + by = c$.

Proof. (Farey Fractions). By Theorem 6, $\frac{a}{b} - \frac{p}{q} = \frac{1}{\frac{b \cdot q}{d}}$ so $\frac{a}{d} \cdot q - \frac{b}{d} \cdot p = 1$. Then $a \cdot \frac{(c \cdot q)}{d} + b \cdot \frac{(-c \cdot p)}{d} = c$. Thus $x = \frac{c}{d} \cdot q$, $y = \frac{-c}{d} \cdot p$ is a solution to $ax + by = c$. □

Recall from Definitions 3 and 5 (pages 40 and 43, respectively) that p_k, q_k are positive integers where $k = 0, 1, 2, \ldots, n$ and $a_i = N$ where a_i are the partial denominators of the continued fraction $[a_0; a_1, a_2, \ldots, a_n]$.

Theorem 13. Let $ax + by = c$ be a Diophantine equation where $0 < a < b$ and let $d = \gcd(a, b)$. Let $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_n]$ be the continued fraction expansion [*Bu*].

a) If n is *even* then $x = c \cdot q_{n-1}$ and $y = -c \cdot p_{n-1}$ are solutions for the equation $ax + by = c$.

b) If n is *odd* then $y = c \cdot p_{n-1}$ and $x = -c \cdot q_{n-1}$ are solutions to the equation $ax + by = c$.

Proof. *(Continued fractions).* Let $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_n]$. Then, using Theorem 11, $p_n q_{n-1} - q_n p_{n-1} = (-1)^n$, and $\frac{p_n}{q_n} = \frac{a}{b}$. Thus, if n is even, $x = -c \cdot q_{n-1}$, $y = c \cdot p_{n-1}$ is a solution to $ax + by = c$ and, if n is odd, $x = c \cdot q_{n-1}$, $y = -c \cdot p_{n-1}$ solves $ax + by = c$. □

In the proof of Theorem 13 when n is even we found $a \cdot q_{n-1} - b \cdot p_{n-1} = 1$, and $b \cdot p_{n-1} - a \cdot q_{n-1} = 1$ if n is odd. Thus, by Theorem 6, $\frac{p_{n-1}}{q_{n-1}}, \frac{a}{b}$ are adjacent fractions in F_n where $n = \max\{q_{n-1}, b\}$.

Let us look at an example of a Diophantine equation such as $56x + 72y = 40$. So $\gcd(56, 72) = 8$ and, since $8 \mid 40$, then $7x + 9y = 5$. By using the Euclidean Algorithm we can find a solution to $7x + 9y = 5$ by using $\frac{7}{9}$ which yields

$$9 = \underline{1} \cdot 7 + 2$$

$$7 = \underline{3} \cdot 2 + 1$$

$$2 = \underline{2} \cdot 1 + 0$$

Utilizing the underlined information we can form the *partial quotients* of the continued fraction as $\frac{7}{9} = [0; 1, 3, 2]$. Using our previous knowledge of continued fractions, $\frac{7}{9} = 0 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$ which helps in determining

the convergents which are $\left[\frac{0}{1}; \frac{3}{4}, \frac{7}{9}, \frac{1}{1}\right]$. Having a

knowledge of Farey fractions we notice that $\frac{7}{9} = \frac{a}{b}$ means

we are in row $F_9 = \left\{\frac{0}{1}, \ldots, \frac{1}{2}, \ldots \frac{3}{4}, \frac{7}{9}, \ldots, \frac{1}{1}\right\}$. By looking

at F_9 we see that $\frac{p}{q} = \frac{3}{4} < \frac{7}{9}$ are adjacent fractions. So,

by Theorem 12, $x = 5 \cdot 4 = 20$, $y = 5 \cdot (-3) = -15$ is a

solution because $7(20) + 9(-15) = 5$ and also to

$56 \cdot 20 + 72(-15) = 40$. (For an example of continued

fractions see Chapter 5.)

Now so far we have only discussed the linear Diophantine equation such as the example explained above. However, there are many other types of Diophantine equations.

The next type of Diophantine equation that will be examined is when $\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. We will explore two different variations dealing with this equation.

Algorithm 14. Modified Egyptian Fraction Algorithm.

Let $\frac{a_0}{b_0} = \frac{3}{n} = f_0$ be a positive rational number such that $0 \le \frac{a_0}{b_0} < 1$, where $\frac{3}{n}$ is not in reduced form. We can obtain a finite sequence as follows:

1) Let $\frac{3}{n} = \frac{1}{n} + \frac{2}{n}$ where $\frac{1}{n} = \frac{1}{n_1}$ and $n \ge 4$.

2) If *n is odd*

then let $\frac{2}{n} = \frac{2}{(n+1)} + \frac{2}{[n(n+1)]} = \frac{1}{q} + \frac{1}{nq}$ where

$\frac{1}{q} = \frac{1}{n_2}$, $\frac{1}{nq} = \frac{1}{n_3}$ are in reduced fraction form.

If *n is even*

then $\frac{2}{n} = \frac{1}{m}$ and $\frac{1}{m} = \frac{1}{(m+1)} + \frac{1}{m(m+1)} = \frac{1}{n_2} + \frac{1}{n_3}$.

3) Then $\frac{3}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$ gives us

the Egyptian fraction representation.

Here's how the algorithm works. Think of a fraction $\frac{3}{n}$ that you want to express as three distinct Egyptian fractions, where $n \geq 4$ and where n can be *any* number, say $\frac{a_0}{b_0} = \frac{3}{13} = f_0$. We will automatically come up with $\frac{1}{n_1} = \frac{1}{13}$ and $\frac{2}{n} = \frac{2}{13}$. Using $\frac{2}{13} = \frac{2}{(13+1)} + \frac{2}{[13(13+1)]} = \frac{1}{7} + \frac{1}{7 \cdot 13}$ $= \frac{1}{n_2} + \frac{1}{n_3}$ we get the Egyptian Fraction representation $\frac{3}{13} = \frac{1}{13} + \frac{1}{7} + \frac{1}{91}$.

Granted, some of these $\frac{3}{n}$ numbers can be written as a unit fraction, especially if 3 divides n, and some of these $\frac{3}{n}$ can be written as the sum of two distinct unit fractions instead of three unit fractions when n is even. But what is interesting to notice is that fractions such as $\frac{3}{5}$, $\frac{3}{11}$ and $\frac{3}{17}$, if you look at the Farey table, follow a unit fraction. This assures the reader of a two-term unit fraction representation $\frac{3}{b'} = \frac{1}{b} + \frac{1}{bb'}$ (see

Algorithm 2 in Chapter 4 to find the unit fractions).

However, just because a unit fraction does not precede $\frac{3}{n}$ does not mean $\frac{3}{n}$ does not have a two-term representation. Another well-known method used in finding a two-term expansion of $\frac{3}{n}$ is when $n \equiv 2 \pmod 3$ (see [Ep]).

Let's turn our attention back to $\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Theorem 15. Let $n \geq 5$ be an odd integer not divisible by 3. Then there exists distinct positive odd integers a, b, and c such that $\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ [Ha].

Proof. Since n is odd and is not divisible by 3, we have two cases: when $n \equiv 1 \pmod 6$ and $n \equiv 5 \pmod 6$ [Ha].

Case (1). $n \equiv 1 \pmod 6$.

Let $n \equiv 1 \pmod 6$ where $\frac{3}{n} = \frac{3}{(1 + 6x)} = f_0$.

Proof of Case (1). Using the Fibonacci-Sylvester Algorithm (see Algorithm 3 in Chapter 3) we find that $\frac{1}{1 + 2x} \leq \frac{3}{1 + 6x} < \frac{1}{2x}$ and so we get

$\frac{3}{(1+6x)} = \frac{1}{(1+2x)} + \frac{2}{(1+8x+12x^2)}$. Using Algorithm 3 once again we find that $\frac{1}{1+4x+6x^2} \le \frac{2}{1+8x+12x^2} < \frac{1}{4x+6x^2}$ so $\frac{2}{(1+8x+12x^2)} = \frac{1}{(1+4x+6x^2)} + \frac{1}{(1+8x+12x^2)(1+4x+6x^2)}$.

Thus we get the unit fraction representation of

$$\frac{3}{1+6x} = \frac{1}{(1+2x)} + \frac{1}{(1+4x+6x^2)} + \frac{1}{(1+4x+6x^2)(1+8x+12x^2)}$$

where the denominators are odd. This completes the proof of *Case (1)*.

Case (2). $n \equiv 5 \pmod 6$.

Let $n \equiv 5 \pmod 6$ where $\frac{3}{n} = \frac{3}{(5+6x)}$.

Proof of Case (2). If $n \equiv 5 \pmod 6$ then $n \equiv 2 \pmod 3$.

If $n \equiv 2 \pmod 3$ then $n = 2 \cdot 3^0 + a_1 \cdot 3^1 + a_2 \cdot 3^2 + \ldots + a_N \cdot 3^N$ for some $0 \le a_i \le 2$, $N > 0$. Let r be the first subscript such that $a_r < 2$. If $2 = a_1 = a_2 = \ldots = a_N$ then $r = N + 1$. If we denote a_r as α then

$n \equiv (2 \cdot 3^0 + 2 \cdot 3^1 + \ldots + 2 \cdot 3^{r-1} + \alpha \cdot 3^r) \pmod{3^{r+1}}$ with $\alpha = 0$ or 1 which means that we get two cases.

Case (2i). $\alpha = 0$.

Proof of Case (2i). Let $\alpha = 0$ then, using

$$n \equiv \left(2 \cdot 3^0 + 2 \cdot 3^1 + \ldots + 2 \cdot 3^{r-1} + 0 \cdot 3^r\right)\left(\bmod 3^{r+1}\right),$$

$$n \equiv (3 - 1)\left(3^0 + 3^1 + \ldots + 3^{r-1}\right)\left(\bmod 3^{r+1}\right) \equiv \left(3^r - 1\right)\left(\bmod 3^{r+1}\right).$$

Then $3^r n \equiv \left(3^{2r} - 3^r\right)\left(\bmod 3^{r+1}\right) \equiv -3^r \left(\bmod 3^{r+1}\right)$. Now since $n \equiv \left(3^r - 1\right)\left(\bmod 3^{r+1}\right)$ and $3^r n \equiv -3^r \left(\bmod 3^{r+1}\right)$ then $3^r n + n \equiv -1\left(\bmod 3^{r+1}\right)$.

Since $\left(3^r + 1\right) n$ is even, there exists an odd integer u such that $\left(3^r + 1\right) n = 3^{r+1}u - 1$. Let $t = 3^r u$. so $\left(3^r + 1\right) n = 3t - 1$. Now $\frac{3}{n} - \frac{1}{nt} = \frac{3t - 1}{nt} = \frac{\left(3^r + 1\right) n}{nt}$ $= \frac{\left(3^r + 1\right)}{t} = \frac{\left(3^r + 1\right)}{3^r u} = \frac{1}{u} + \frac{1}{t}$. So $\frac{3}{n} = \frac{1}{nt} + \frac{1}{u} + \frac{1}{t}$ where $u < t < nt$ and n, t, u are odd. This completes the proof of *Case (2i)*.

Case (2ii). $\alpha = 1$.

Proof of Case (2ii). Let $\alpha = 1$ then, using

$$n \equiv \left(2 \cdot 3^0 + 2 \cdot 3^1 + \ldots + 2 \cdot 3^{r-1} + 1 \cdot 3^r\right)\left(\bmod 3^{r+1}\right),$$

$$n \equiv \left(2 \cdot \frac{3^r - 1}{2} + 3^r\right)\left(\text{mod } 3^{r+1}\right) \equiv \left(2 \cdot 3^r - 1\right)\left(\text{mod } 3^{r+1}\right)$$

$$\equiv \left(3 \cdot 3^r - 3^r - 1\right)\left(\text{mod } 3^{r+1}\right) \equiv \left(-3^r - 1\right)\left(\text{mod } 3^{r+1}\right).$$ So

$$n + 3^r + 1 \equiv 0\left(\text{mod } 3^{r+1}\right).$$

Since $n + 3^r + 1$ is odd there exists an odd integer u such that $n + 3^r + 1 = 3^{r+1}u$. Let $t = 3^r u$. Then $n + 3^r + 1 = 3t$. Now $\frac{3}{n} - \frac{1}{t} = \frac{3t - n}{nt} = \frac{3^r + 1}{nt} = \frac{3^r + 1}{n \cdot 3^r u}$

$= \frac{1}{nu} + \frac{1}{n \cdot 3^r u} = \frac{1}{nu} + \frac{1}{nt}$. So $\frac{3}{n} = \frac{1}{t} + \frac{1}{nu} + \frac{1}{nt}$ where n, nu, nt are distinct odd integers. □

CHAPTER SEVEN

In this chapter we will lightly discuss a few open-ended problems in the field of number theory as it relates to the Diophantine equations $\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ and $\frac{5}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Erdös and Straus [*Er*], in 1950, conjectured that $\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where n is an integer and $n > 1$, is solvable for a three-term unit fraction for positive integers a, b, and c. Many people have worked on the problem. (For details see [*Mo*].) Straus had verified the equation for $n < 5000$, Bernstein [*Be*] verified it for $n < 8000$, Shapiro verified it for $n < 20,000$, Obláth [*Ob*] verified it for $n < 106,128$, Yamamoto [*Ya*] verified the equation for $n < 10^7$, and Nicola Franceschine [*Fr*] verified the equation for $n < 10^8$. "Others have worked on more general versions of this problem including

Schinzel, Sierpínski, Sedlácek, Palamà, Stewart, Webb, Breusch, Graham, and Vaughan" [*Ep*]. To date the $\frac{4}{n}$ problem still remains unsolved.

In 1951 Sierpínski [*Si*] made the conjecture that $\frac{5}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, where n is an integer and $n > 2$, is also solvable for a three-term unit fraction where a, b, and c are positive integers. A few people have worked on this problem, namely Sierpínski who proved it for $n \leq 1000$, Kiss [*Ki*] who proved it for $n \leq 10,000$, Palamà [*Pa*] who verified the equation for all $n \leq 922,321$ and Stewart [*St*] who then verified it for $n \leq 1,057,438,801$. Others who have worked on the problem for general cases are Schinzel, Sedláček, Palâma, Stewart who collaborated with Webb, Vaughan, and Graham. (For more details see [*Mo*].) As of yet, the problem of $\frac{5}{n}$ has not been solved. Not much information was as forthcoming on this conjecture as there was on $\frac{4}{n}$.

Although these conjectures have been solved for large values of *n*, no one, to date, has been able to prove them for every positive integer *n*. As mentioned earlier in Chapter Two, there is also no known algorithm for finding unit fractions that have a minimum amount of terms *and* the smallest possible denominator.

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