Essays on Microeconomic Theory

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To my family, for their

unconditional support and love

Declaration

I declare that this thesis consists of approximately 60,000 words.

I certify that Chapter 3 of this thesis was co-authored with Dimitris Papadimitriou and George Vichos. I contributed 50% of the work.

Abstract

This thesis contains three essays in Microeconomic Theory.

Chapter 1 studies the incentives of a seller to voluntarily disclose or sell information about a buyer to a third party. While there are obvious benefits to sharing information with other sellers, there is also an incentive cost which is due to her learning about the buyer through her own trade with him. To study this trade-off we analyse a model in which a buyer interacts sequentially with two sellers, each of whom makes a take-it-or-leave-it offer. The buyer learns his valuation for the good of each seller sequentially but these might be correlated. We model information disclosure using Bayesian persuasion. Chapter 2 provides various extensions of the model presented in Chapter 1.

Chapter 3 provides empirical evidences that demonstrate that the investors of a fund update their opinion on the fund manager's ability faster during bear markets. We build a theoretical model to demonstrate a channel which would result on this empirical observation. We consider a continuum of potential investors who allocate funds in two consecutive periods between a manager and a market index. The manager's alpha, defined as her ability to generate idiosyncratic returns, is her private information and it is either high or low. In each period, the manager receives a private signal on the potential performance of her alpha, and she also obtains some public news on the market's condition.

In Chapter 4 we demonstrate that the relative job security that CEOs enjoy can be partly attributed to the high sophistication of the managerial labour market. To do this we build a theoretical model in which a representative investor proposes a contract to a manager, which also specifies the conditions of his termination. Production is a function of the manager's effort and ability, both of which are his private information. The former is a choice variable, whereas the latter follows a Geometric Brownian motion. The manager's post-termination payoff is generated by an exogenous managerial labour market, and it is equal to his expected ability. The market learns his ability with some given probability, which we interpret as its sophistication. Otherwise, it forms its posterior based on his termination time.

Reading Instructions

Chapters 1, 3, and 4 each correspond to one of the three papers that I wrote during my PhD. Those can be read individually. On the other hand, Chapter 2 contains a collection of extensions of the model presented in Chapter 1, and was originally the online appendix of the corresponding paper. Every effort has been made to make those extensions self-contained, but reading Chapter 1 first is recommended. Unless highly interested on the covered topic the reader may also choose to skip Chapter 2, since it is far more complex and speculative from the rest of the material presented in this thesis.

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Contents

Chapter 1

A Mechanism Design Approach to the Disclosure of Private Client Data

This chapter studies the incentives of a seller to voluntarily disclose or sell information about a buyer to a third party. While there are obvious benefits to sharing information with other sellers, there is also an incentive cost which is due to her learning about the buyer through her own trade with him. To study this trade-off we analyse a model in which a buyer interacts sequentially with two sellers, each of whom makes a take-it-or-leave-it offer. The buyer learns his valuation for the good of each seller sequentially but these might be correlated. In addition, we model information disclosure using Bayesian persuasion, that is we allow the first seller to commit to a disclosure rule which depends on the information she acquires in the first trade. In this setting we fully characterise the first seller's costs and benefits of information sharing. In particular, we show that voluntary information disclosure, or selling of information, is optimal when the correlation between the buyer's valuations for the two goods is not too positively correlated. Also, when information exchange is optimal the buyer benefits from it if his valuations are positively correlated, otherwise he is worse off.

1.1 Introduction

Information exchange between firms is rapidly increasing in both volume and importance. The introduction of new technologies allows firms to cheaply store, analyse, and share data on their clients. In particular, a firm just by interacting with its customers acquires private information on their preferences. This information is valuable to other firms. To give an example, an architect learns her client's willingness to pay to renovate his house. This is valuable information for an interior designer, since the client's preferences for the two services should be highly correlated. By sharing information with the designer the architect might be able to increase her profits. More generally, information exchange is significant in business to business environments. Notably, data brokerage is already a big industry and has recently attracted a lot of attention from both firms, that look to expand their customer base, and regulator.

This chapter poses the following question: "Should a firm share private information about its customers?" One direct way to benefit from disclosing data on its clientèle, such as their purchase history, is to sell it. But even if selling information is not an option a firm can still benefit from disclosure, but in an indirect way. This is because selectively sharing data on its clients could persuade other firms to offer them discounts, the prospect of which would allow the firm to increase its own prices. To see this, suppose that the client of the aforementioned architect opted for a partial renovation only, which was the cheapest option. Then he probably doesn't value that much the interior designer's services. Therefore, by disclosing this information the architect can persuade the designer to offer a discount to her client. In turn this allows the architect to sell to her client both the partial renovation and the prospect of a discount as a bundle of products.

However, information disclosure is associated with an incentive cost. This is because a firm infers its customers' willingness to pay indirectly from their choice of product within its catalogue. But if those are aware that this choice may affect the probability of their getting a discount from a related seller, then this will skew their choices towards cheaper products. In the example above, a client who anticipates a possible discount may skew his choice towards the partial renovation. Therefore, information provision is interwoven with an incentive cost for acquiring it.

To answer the question of optimal disclosure we use a mechanism design approach coupled with Bayesian persuasion. We consider a two period model in which two sellers sequentially interact with a single buyer. The first and second seller make take-it-or-leave-it offers to the buyer in the first and second period, respectively. Each seller can offer one unit of an indivisible good. The buyer's valuation for each of the two goods is either high or low, and evolves stochastically between the two trades. That is when trading with the first seller the buyer does not know his valuation of the good of the second seller. Despite that, the buyer's valuations of the two goods are correlated, so his first period preferences are valuable information to the second seller.

We model information disclosure by using a Bayesian persuasion framework. To be more precise, we assume that the first seller can commit ex ante to a distribution of signals that depends on the buyer's report in her mechanism. The realisation of this signal is observed by the second seller, who uses this information when determining her price.

At this point it is helpful to first consider the case of perfect positive correlation, which implies that the buyer always assigns the same value to the goods of the first and second seller. Hence while trading at period 1, the buyer faces no uncertainty over his valuation of the second good. We show that in this case non-disclosure is always optimal, a result that has been previously stated by Calzolari and Pavan (2006). The reason for the optimality of non-disclosure is twofold. The first is that a period 1 low type buyer assigns no value to the potential second period discount, as his period 2 type will also be low. Hence he obtains zero rents from this trade. The second reason is that the incentive cost of convincing the period 1 high type to be truthful is the highest possible. This is because he knows with certainty that his second period valuation will be high, thus this is when he values the potential discount the most. Consider now the implications of the above intuition for the case of imperfect correlation. It hints that if we increase the probability of a period 1 low type to become high, and decrease the probability of a period 1 high type to remain high, then information disclosure might become optimal.

The main result of our paper is that indeed when the buyer is uncertain about his future valuations information disclosure is optimal for a substantial set of environments. In particular, this uncertainty is a result of imperfect correlation between the valuations of the two sellers. In Propositions 1.2 and 1.3 we characterise all the correlation structures of the buyer's valuations for which information disclosure is optimal. In addition, we derive the corresponding optimal disclosure policy.

We start our analysis with the first best problem of the first seller. In this case the buyer's period 1 valuation is observable by the first seller, but not by the second. This analysis allows us to characterise the benefit of information disclosure abstracting away from any incentive costs. Proposition 1.1 characterises the set of environments for which this benefit is strictly positive. The optimal disclosure policy takes a simple form. When the buyer's types are positively correlated this signal randomises between revealing the high type and pooling it together with the low type. The pooling outcome is the one that creates a discount as discussed in the example above, which provides the benefit of information disclosure. Diametrically, under negative correlation it is the low type that is some times revealed (shown in Proposition 1.3, which focuses on negative correlation).

We next analyse the second best in which the first seller incurs incentive costs for eliciting the buyer's valuation for her good. Proposition 1.2 characterises the set of positive correlation structures for which information disclosure is optimal and shows that this is a non-empty convex subset of the corresponding set characterised in the first best (Proposition 1.1). As we show below this set sometimes includes correlation structures that are arbitrarily close to perfect positive correlation. Proposition 1.3 provides a similar analysis for the case of negative correlation showing that disclosure is optimal whenever it is optimal under the first best. In addition, we show that the optimal disclosure policy, whenever disclosure is optimal, is always the same as in the first best for both positive and negative correlation structures.

We also demonstrate that the buyer might both benefit or lose from information disclosure. While the low types are always indifferent, it is the high types that are influenced by disclosure. In the case of positive correlation, disclosure opens up the possibility of discounts, which are beneficial for the high type buyer. On the other hand, in the negative correlation case, when information disclosure is optimal the optimal mechanism involves lowering the rents of the high type to increase profits.

On a more technical note our analysis proceeds on the following way. First, we derive the buyer's payoff from his second trade. This is positive only if (i) the buyer is a high period 2 type and (ii) the posterior of the second seller is low enough for her to choose to sell to both types. Hence while trading with the first seller the buyer's expected payoff from the second period is a function of (i) the distribution of signals which corresponds to the reported type and (ii) the probability of being a high period 2 type. Thus, when the agent's types are positively correlated information provision will result in additional rents for the high period 1 type, since the low type's distribution will necessarily entail a more frequent discount. We use insights from Gentzkow and Kamenica (2011) to reformulate the seller's information provision problem as a choice of distributions over posteriors. This allows us to use a graphical solution that relies on the concave closure of our objective function. Finally, we find two type depended signal distributions that implement the optimal unconditional distribution of posteriors.

We extend the analysis in several ways. First, we consider the case when the first seller can sell her information to the second seller. We show that the solution of this model is identical to that of the baseline one. In an alternative extension we allow the two sellers to produce a continuous quantity under an isoelastic cost function, which results in a solution comparable to that of the baseline model. In the next section we further extend this model to multi-period contracts, we show how those can incorporate moral hazard, and demonstrate the interplay between information provision and endogenous termination times in this setting. Finally, we provide some sufficient results for non-disclosure to be optimal under continuous types.

My paper is closely related to Calzolari and Pavan (2006), the implications of which the authors have further explored in Calzolari and Pavan (2008, 2009). They also consider a setup where an agent sequentially contracts with two principals, the first of whom can commit on a disclosure rule. However, they restrict their attention to the cases of perfect positive and negative correlation. The implication of this is that at the point of trading with the first seller the buyer knows both his valuations. Under perfect positive correlation, they obtain that privacy is always optimal. However, under perfect negative correlation, and when some additional conditions are satisfied, they establish that the first seller's optimal signal could be informative. In addition, the authors show that an alternative way to make disclosure optimal is to assume externalities. In this paper we examine a conceptually similar model while using a Bayesian persuasion framework, which is a natural setting to consider the case of imperfect correlation. We aim to demonstrate that information exchange can arise under much more natural and economically interesting conditions. Our analysis relies neither on externalities, nor on the extreme assumption of perfect negative correlation. Notably, we show that even close to perfect positive correlation could be enough for some disclosure to be optimal, which implies that the aforementioned result on the optimality of privacy is not robust to stochastic preferences.

Another very relevant paper is that of Dworczak (2016*a*), where a seller auctions an object and the winner's payoff depends on both his type and a generic aftermarket. Crucially, the aftermarket related payoff depends on the public posterior on the winner's type. This introduces an information design aspect to the auctioneer's problem. The author identifies a class of mechanisms, called cutoff mechanism, that are implementable regardless of the aftermarket's form. He subsequently identifies the optimal mechanism within this class, which for the single agent model has no information disclosure. On an accompanying paper Dworczak (2016*b*), provides sufficient condition for a cutoff mechanism to be optimal. Among other results he shows that if the seller only acts as an information intermediary, that is she buys information from the bidders and sells it to the aftermarket, then information provision is never implementable.

Our paper presents an interplay of mechanism and information design. In some sense it is related to the literature of dynamic mechanism design, since we allow for the buyer's preferences to evolve stochastically over time (for example see Pavan et al. (2014); Garrett and Pavan (2012); Eső and Szentes (2017) and the references therein). In particular, the dynamic extension of our model, which can be found in the next chapter, follows closely Battaglini (2005). He considers a firm contracting with a consumer, whose private valuation of the firm's product is either high or low and evolves stochastically according to an exogenous Markov matrix.

Nevertheless, in contrast to this literature in our model the first seller can only indirectly affect the choices of the second period, that is of seller two. To be more precise, her only tool is the disclosure policy she commits on. Hence the first seller engages in Bayesian persuasion with the second on the buyer's reported type. In that sense our analysis is related to the literature of information design (for example see Inostroza and Pavan (2017) and the references therein). In particular, the solution method that we use is related to the work of

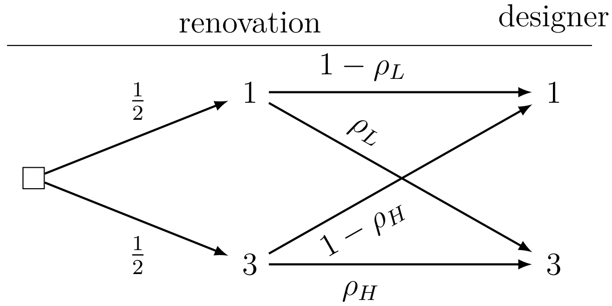
Gentzkow and Kamenica (2011), who provide a framework for solving a vast class of information design problems. In a more recent paper, Ely (2017) considers a dynamic information design problem where the designer receives signals on the underline state overtime. Roesler and Szentes (2017) revisit the canonical bilateral trade model. They assume that the buyer is uncertain about her valuation of the product, but receives a signal on it. They derive the buyer-optimal distribution of signals and show that this generates efficient trade.

The rest of the paper proceeds as follows. Section 1.2 provides an example that further clarifies the potential benefits and costs of information provision, and how those depend on the stochasticity of the buyer's preferences. Section 1.3 formally defines our baseline model. Section 1.4 provides the analysis and the corresponding results. Section 1.5 considers two natural extensions of our model. The first allows for the seller to sell information, while the second for the buyer to know both his types at the first period. Finally, section 1.6 discusses the implications of the model and concludes.

1.2 Example

To fix ideas consider the following illustrative example. An architect makes a take-it-or-leave-it offer to a buyer for the renovation of his house. Suppose the architect has zero cost of production and that a high type buyer values the architect's services at $\theta_H = 3$, while a low type at $\theta_L = 1$. Assume throughout that the buyer's type (call it renovation type) is his private information, and that its two realisations are equiprobable. Standard argumentation shows that the architect's optimal pricing strategy is to only sell to the high type by setting price $\overline{p} = 3$.

Next we expand the space of interactions by assuming that the buyer can not only hire an architect, but also an interior designer. The latter will also make a take-it-or-leave-it offer to the buyer, but only after the architect has made hers, and has zero production cost. Despite that, the buyer can opt to hire the interior designer without hiring the architect and visa versa. Again, we set the buyer's valuation of the designer's services (call it design type) to be either $\theta_H = 3$ or $\theta_L = 1$. The buyer's valuations for the two services will not necessarily be the same, and even he will only learn the latter when the interior designer presents her product to him. Thus the buyer's uncertainty over his preferences is resolved sequentially. In particular, assume that a high renovation type remains a high design type with probability ρ_H, whereas a low renovation type turns into a high design type with probability ρ_L. The architect is able to connect the buyer with the interior designer. For simplicity, we assume that in the absence of such a connection the designer's posterior on the buyer's design type is $(\rho_H + \rho_L)/2$, which equals the public prior.



Through her interaction with the buyer, the architect acquires valuable private information on his preferences. This is because the buyer's willingness to pay $\overline{p} = 3$ for the renovation reveals him as a high type. This information is valuable to the designer because the buyer's renovation and designer types are correlated. Hence the designer could use the buyer's purchase history to decide between offering the high and low price, that is $\overline{p} = 3$ and $\underline{p} = 1$, respectively.

This begs the question "Is it possible for the architect to increase her expected revenue by using her private information on the buyer's preferences?". It turns out that the answer depends a lot on the degree of uncertainty that the buyer has on his own preferences, that is ρ_L and ρ_H. To demonstrate this suppose

$$\frac{\rho_H + \rho_L}{2} > \frac{1}{3}$$

Since $\theta_L/\theta_H = 1/3$ we infer that the designer's optimal pricing policy, in the absence of any communication with the architect, is to set $\overline{p} = 3$ and only sell to the high design type. Thus there is some room for the architect to attempt to persuade the designer, in the Bayesian sense, to offer a discount to the buyer. This could ultimately be beneficial for the architect, as she could charge the buyer for this discount[1].

To demonstrate this fix $(\rho_H, \rho_L) = (1/2, 1/4)$ and suppose that the architect is able to commit in advance to both the buyer and the interior designer that she will reveal a high renovation type with probability $g_H = 1/2$. Henceforth, whenever she sends a buyer to the designer without revealing him as a high renovation type the latter's posterior on the

[1]Note that the architect could not be benefitted by just establishing a connection and not providing any information on the buyer's preferences. In addition, this would be true even if the buyer could only hire the interior designer through the architect. This is because the net benefit of this connection for the buyer would be zero, since a low design type will not buy at $\overline{p} = 3$ and even a high type will be indifferent between buying or not.

former's design type to be high is

$$\frac{1-g_H}{1+(1-g_H)} \cdot (\rho_H - \rho_L) + \rho_L \;=\; \frac{1}{3}$$

Thus, not revealing the buyer as a high renovation type achieves a discount, since the designer's optimal price becomes $\underline{p} = 1$. The benefit of this signal for the architect is that she is able to charge the low house type for this discount with price

$$p_L \;=\; \rho_L \cdot (\overline{p} - \underline{p}) \;=\; \rho_L \cdot (\theta_H - \theta_L)$$

whereas when she was not engaging in bayesian persuasion her interaction with the low renovation type had zero value, since such a type would not buy from her at her previously optimal price $\overline{p} = 3$. It is worth mentioning that this benefit exists only under imperfect correlation as otherwise $\rho_L = 0$.

However, this informative signal generates an incentive cost[2], which will decrease the price that the architect can charge to the high renovation type p_H below θ_H. This is because the high type is tempted to behave like a low type since in this case he gets the design discount with probability 1, instead of $1 - g_H = 1/2$. Hence the maximum difference between p_H and p_L has to be $\theta_H - \rho_H(\theta_H - \theta_L)/2$, otherwise even the high type would opt to get the guaranteed design discount instead of the uncertain one together with the renovation. As a result, the usage of the signal will decrease the price charged to the high renovation type by

$$\frac{\rho_H}{2}(\theta_H - \theta_L) - p_L \;=\; \left(\frac{\rho_H}{2} - \rho_L\right)(\theta_H - \theta_L)\,,$$

Interestingly, the benefit of information provision is associated to the architect's trade with the low renovation type, whereas the cost to her trade with the high type. Therefore to find the net impact that the usage of the informative signal has on the architect's revenue multiply both the benefit and the cost by 1/2, which is the probability of facing each type, and subtract the latter from the former

$$\frac{1}{2}\,\rho_L(\theta_H - \theta_L) - \frac{1}{2}\left(\frac{\rho_H}{2} - \rho_L\right)(\theta_H - \theta_L) \;=\; \left(\rho_L - \frac{\rho_H}{4}\right)(\theta_H - \theta_L)$$

But under the assumed transitioning probabilities $\rho_L - \rho_H/4 = 1/16$, which is positive. As a result, the usage of the informative signal is beneficial for the architect.

The provided expression demonstrates the role of ρ_H and ρ_L, which effectively capture

[2]The discussion here ignores the incentive compatibility constrain of the low type, but it is easy to check that it holds in all the cases that are considered.

the relative importance that each renovation type assigns to the design discount. If ρ_H is too high relatively to ρ_L, then the incentive cost outweighs the benefit of information provision. In particular, consider the case of perfect positive correlation $(\rho_H, \rho_L) = (1, 0)$. We can show that the interior designer's posterior when the architect does not reveal the buyer as a high renovation type remains 1/3. Hence the discount is still achieved with probability 1/2 for the high type, and 1 for the low type. Diametrically to the previous example the net impact on the architect's revenue is negative as $\rho_L - \rho_H/4 = -1/4$. Hence she is better off when not using the informative signal. Indeed, we show that this holds for any disclosure policy, that is for any distribution of signals.

The above discussion implicitly assumes that the buyer's renovation and design types are positively correlated, but what happens if $\rho_L > \rho_H$? Again information provision could be beneficial, for example consider the case of perfect negative correlation $(\rho_H, \rho_L) = (0, 1)$. However, suppose that the architect instead of revealing the high renovation type with probability $g_H = 1/2$, reveals the low one with probability $g_L = 1/2$. Hence when the architect does not reveal the buyer as a low renovation type the designer's posterior on facing a high design type is 1/3. Thus the discount is achieved with probability 1 for the high renovation type, and 1/2 for the low one. The high renovation type does not care at all about the signal, since his valuation of the interior designer's services is always low. On the other hand, the low renovation type always has a high valuation for the designer's work. As a result, the possibility of obtaining the discount allows the architect to increase the price charge to him by $(\theta_H - \theta_L)/2$. Therefore, the architect is better off when using the informative signal.

To sum up the above three examples hint that information provision is decreasing in correlation. Under perfect positive correlation the cost of incentive compatibility, of the high renovation type that is, outweighs the potential benefit, which is selling the discount to the low type. However, under imperfect positive correlation is was shown that the benefit dominates and information provision becomes optimal. Finally, under perfect negative correlation the incentive cost disappears and again information provision is optimal, however the nature of the signal that achieves that changes. The subsequent analysis will demonstrate that to some extent this intuition is relevant even when the architect is able to optimally design a mechanism to transmit information to the interior designer.

1.3 Model

Consider a two period model $t \in \{1, 2\}$, in which a buyer trades sequentially with sellers S_1 and S_2. Each seller supplies an indivisible good. In period 1 S_1 makes a take-it-or-leave-it offer to the buyer, while in period 2 it is S_2 that makes an offer. The sellers and the buyer are risk neutral, and all outside options are normalised to zero. Let p_t denote the price charged by S_t, and q_t the corresponding probability of supplying the good. Then the buyer's payoff from trading in period t is

$$\theta_t \, q_t - p_t$$

where $\theta_t \in \{\theta_L, \theta_H\}$, $\theta_H > \theta_L > 0$, is the value the buyer assigns to each seller's product, and the public prior on θ_1 is $\mu_0 = \Pr(\theta_1 = \theta_H)$. We allow the buyer's type θ_t to be imperfectly correlated across sellers

$$\begin{aligned} \rho_H &= \Pr(\theta_2 = \theta_H \,|\, \theta_1 = \theta_H) \\ \rho_L &= \Pr(\theta_2 = \theta_H \,|\, \theta_1 = \theta_L\,) \end{aligned}$$

Crucially, even the buyer himself will not know θ_2 before contracting with the second seller. Therefore, the model allows for some sequential resolution of uncertainty on the buyer's stochastic preferences, as in the literature of Dynamic Mechanism Design. To simplify the exposition we will initially assume that $\rho_H \geq \rho_L$, that is a period 1 high buyer type is more likely to remain so in period 2, than a low type to become one. Nevertheless, the diametrically opposite case is also considered in a separated subsection. For simplicity S_2 is only allowed to make an offer to the buyer if he accepted S_1's offer, however the analysis does not rely on this restriction.

The interaction between S_1 and the buyer will be private. Hence his report in S_1's mechanism and the outcome of the trade will not be directly observable by S_2. Nevertheless, S_1 will be able to credible convey additional information to S_2 by committing ex ante to a signal $s \in S$ with distribution $g(s \,|\, \widehat{\theta}_1)$, which is conditioned on the buyer's reported type $\widehat{\theta}_1$. It is easy to argue that the revelation principle applies in this setting. Hence in period 1 S_1 offers to the buyer a mechanism

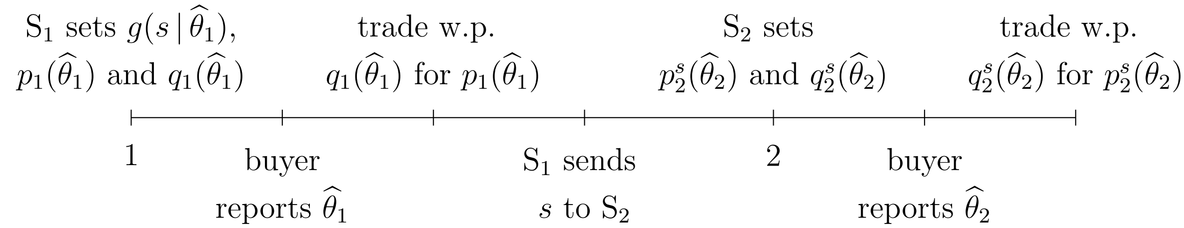
$$\left\{\, p_1(\widehat{\theta}_1),\, q_1(\widehat{\theta}_1),\, g(s \,|\, \widehat{\theta}_1) \,\right\},$$

which specifies the price and probability of trade, and the signal's conditional distribution, respectively. Therefore the mechanism design problem of S_1 includes an information design aspect. To be more precise, S_1 is engaging in Bayesian Persuasion with S_2 about the agent's report $\widehat{\theta}_1$. In period 2, and after having received signal s, S_2 makes an offer

$$\left\{ p_2(\widehat{\theta}_2, s),\, q_2(\widehat{\theta}_2, s) \right\},$$

which depends on the signal realisation s, because this affects S_2's posterior belief on θ_1 and as a result on θ_2.

To sum up, the timing of the model is as follows. At the beginning of period 1, S_1 publicly commits to a distribution $g(s \,|\, \widehat{\theta}_1)$, and offers a corresponding mechanism to the buyer. This includes $g(s \,|\, \widehat{\theta}_1)$, as well as a choice over quantities and prices. The buyer reports $\widehat{\theta}_1$ and trades with S_1. Then the public signal s is realised and observed by S_2. At the beginning of period 2, S_2 makes a new offer to the buyer. Subsequently, the buyer reports $\widehat{\theta}_2$ and trades with S_2, at which point the game ends.



1.4 Analysis

Section 1.4.1 solves S_2's payoff maximisation problem and derives the buyer's payoff from his second trade. Subsequently, those are used to describe S_1's information provision problem. To facilitate the exposition, and because it is an interesting question on its own right, section 1.4.2 derives the solution of this problem under the assumption that the buyer's type is directly observable by S_1. This is equivalent to the first best solution of S_1's payoff maximisation problem. Section 1.4.3 reverts to the original setup, where the buyer's type is his private information, and compares its solution to the first best. Finally, 1.4.4 considers the case of negative correlation.

1.4.1 The buyer's post contractual payoff

We solve S_2's payoff maximisation problem, and derive the buyer's expected payoff from his trade with her. Let $\mu_1^s = \Pr(\theta_1 = \theta_H \,|\, s)$ denote S_2's posterior belief on the buyer's initial type after receiving s, and $\mu_2^s = \Pr(\theta_2 = \theta_H \,|\, s)$ the corresponding posterior on θ_2. Those two are connected according to

$$\mu_2^s \;=\; \mu_1^s\,\rho_H + (1-\mu_1^s)\,\rho_L$$

S_2's problem is quite standard and a more detailed treatment can be found in the appendix. Essentially, this can be reduced to a decision between setting a high price $\overline{p} = \theta_H$, or a low price $\underline{p} = \theta_L$. In the first case only the high type buys her product, while in the second both. When her posterior on θ_2, μ_2^s, is relatively high she opts for the high price, otherwise for the low one. The cutoff in which she is indifferent between the two pricing policies is the ratio θ_L/θ_H. It will be convenient to express this in terms of the realisations of the period 1 posterior μ_1^s. Hence define

$$\mu^* \;=\; \frac{\theta_L/\theta_H - \rho_L}{\rho_H - \rho_L}$$

and note that under positive correlation[3] $\mu^* \in [0,1]$ if and only if $\rho_L \le \theta_L/\theta_H \le \rho_H$. The following lemma characterises the buyer's payoff, which is the only result needed to proceed with S_1's payoff maximisation problem.

Lemma 1.1. *The payoff of a low buyer type under S_2 is equal to zero, while that of the high one equals*

$$Q(\mu_1^s) \;=\; \begin{cases} \theta_H - \theta_L & ,\ \textit{if}\ \ \mu_1^s \le \mu^* \\ 0 & ,\ \textit{if}\ \ \mu_1^s > \mu^* \end{cases} \tag{1.1}$$

Proof. In Appendix A.1. □

The payoff of a low buyer type in period 2 is always equal to zero, as he captures no rents. Conversely, the high type's payoff is positive, but only if the posterior μ_1^s is low enough for S_2 to opt to serve both types.

Hereafter, the buyer's expected continuation payoff at the end of period 1 will be referred to as his *post contractual* payoff. It follows from the above lemma that for a period 1 low buyer type, that reported $\widehat{\theta}_1$ in S_1's mechanism, this is equal to $\rho_L \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\widehat{\theta}_1\big]$, while for a high type this is $\rho_H \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\widehat{\theta}_1\big]$.

1.4.2 The first best contract of Seller 1

Next, we solve S_1's payoff maximisation problem under the assumption that if the buyer opts to participate in S_1's contract, then his type is automatically reveal to her, but not to S_2. This is essentially S_1's first best contract. Hence the analysis of this subsection

[3] In the subsection where we consider negative correlation the corresponding statement is: $\mu^* \in [0,1]$ if and only if $\rho_H \le \theta_L/\theta_H \le \rho_L$.

considers the potential benefit of information provision, without the associated incentive cost. Nonetheless, this is not only a theoretical exercise. For example, an insurance firm could ask a potential client to undergo a health examination. Therefore, in this subsection we will only consider the buyer's individual rationality constrains. Hence S_1 solves

$$\begin{aligned} \max_{p_L, p_H, q_L, q_H, g} \quad & \mu_0 p_H + (1-\mu_0)\, p_L \\ \text{s.t. } (\text{IR}_L) \quad & \theta_L\, q_L - p_L + \rho_L\, \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_L\right] \geq 0 \\ (\text{IR}_H) \quad & \theta_H q_H - p_H + \rho_H \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_H\right] \geq 0 \end{aligned} \tag{$\mathcal{P}_f$}$$

where we write $\{p_1(\theta_L), p_1(\theta_H), q_1(\theta_L), q_1(\theta_H)\}$ as $\{p_L, p_H, q_L, q_H\}$, in order to maintain a compact notation. Both of the individual rationality constrains need to bind, as otherwise S_1 could increase p_L or p_H. Hence we can use the binding (IR_L) and (IR_H) to substitute the prices in S_1's objective function and obtain the unconstrained problem

$$\max_{q_L, q_H, g} \left\{ \begin{array}{c} \mu_0\, \theta_H\, q_H \;+\; (1-\mu_0)\, \theta_L\, q_L \\ +\mu_0 \rho_H\, \mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_H\big] + (1-\mu_0)\rho_L \mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_L\big] \end{array} \right\} \tag{$\mathcal{P}'_f$}$$

The first line of the above objective function is quite standard. It represents the surplus generated from the trade of period 1. This is optimised by supplying both types with probability one. The second line represents the additional surplus that S_1 captures from the buyer by controlling the flow of information to S_2, as well as access to her.

On the rest of this subsection we focus on S_1's information provision problem in the first best, of which the choice variable is the distribution $g(s \,|\, \theta_1)$ and the objective function the second line of ($\mathcal{P}'_f$).

$$\max_{g} \left\{ \mu_0 \rho_H\, \mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_H\big] + (1-\mu_0)\rho_L \mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_L\big] \right\} \tag{$\mathcal{G}_f$}$$

The solution approach follows closely Gentzkow and Kamenica (2011). First, we rewrite the objective function of ($\mathcal{G}_f$) as an expectation that uses the unconditional distribution of signal s. This is the ex ante probability of signal s to be realised and it is not conditioned on θ_1. Abusing notation we will denote this by

$$g(s) \;=\; \mu_0\, g(s \,|\, \theta_H) + (1-\mu_0)\, g(s \,|\, \theta_L)\,.$$

Second, we argue that instead of using the conditional distribution over signal realisations $g(s \,|\, \theta_1)$ as choice variables, S_1 can equivalently use the unconditional one over posteriors $\tilde{g}(\mu)$ with the addition of one constrain. Third, a graphical argument based on the concave closure of the reformulated objective function is used to derive the optimal $\tilde{g}(\mu)$. Finally, we identify a conditional distribution over signals that implements the optimal unconditional one over posteriors.

Lemma 1.2. *In the first best, S_1's information provision problem equivalently becomes*

$$\max_{g} \ \mathbb{E}_g[J_f(\mu_1^s)] \tag{$\mathcal{G}_f$}$$

where its point-wise value $J_f(\mu_1^s)$ is

$$J_f(\mu_1^s) \ = \ Q(\mu_1^s) \cdot [\, \mu_1^s \cdot (\rho_H - \rho_L) + \rho_L \,] \tag{1.2}$$

Proof. In Appendix A.1. □

GK show that any Bayes plausible distribution over posteriors $\tilde{g}(\mu)$ can be expressed as a conditional distribution over signals $g(s \,|\, \theta_1)$ and visa versa. A distribution over posteriors $\tilde{g}(\mu)$ is called Bayes plausible if the expected value of the posterior μ is equal to the prior μ_0. Then S_1's information provision problem equivalently becomes

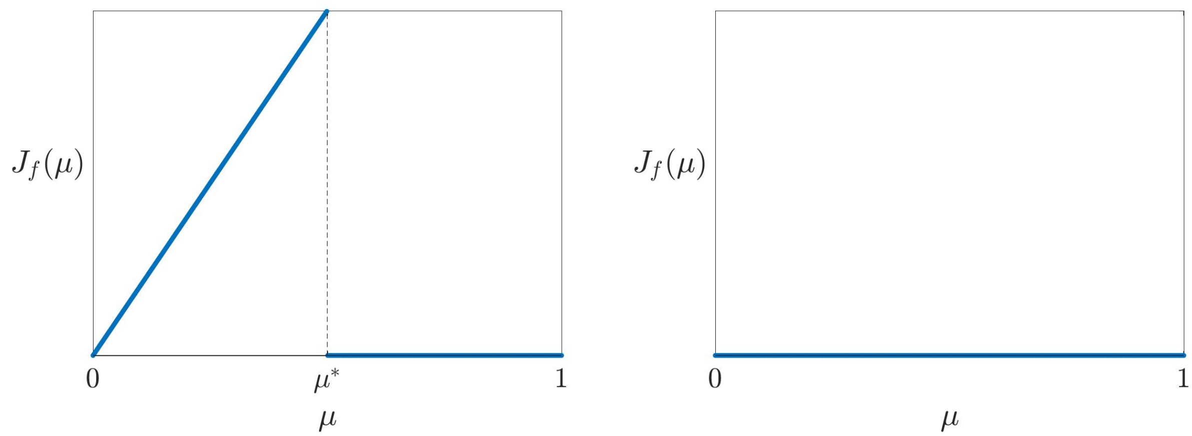
$$\max_{\tilde{g}} \ \mathbb{E}_{\tilde{g}}\big[J_f(\mu)\big] \quad \text{s.t.} \ \ \mathbb{E}_{\tilde{g}}\,[\,\mu\,] \ = \ \mu_0 \tag{$\mathcal{G}_f'$}$$

To solve this it is important to characterise the graph of J_f, which has three possible cases. Figures (1.1a) and (1.1b) demonstrate two of them. In particular, Figure (1.1a) is relevant when $\rho_L \leq \theta_L/\theta_H < \rho_H$, which as argued in the previous section is the only case where information provision can have an effect on prices.

To make this more clear consider the alternative shown in Figure (1.1b), for which $\theta_L/\theta_H < \rho_L \leq \rho_H$ has been assumed. In this case even if the buyer is revealed as a low period 1 type the probability of him to be a high type in the second period ρ_L is sufficiently high for S_2 to charge $\overline{p} = \theta_H$. As a result, the buyer's post contractual payoff is zero, irrespectively of the realisation of μ. Diametrically, if $\rho_L \leq \rho_H \leq \theta_L/\theta_H$ then even if the buyer is revealed as a high period 1 type, S_1 will still charge the low price. Hence again information provision will have no impact on the optimal pricing strategy of S_2. Therefore a sufficient and necessary condition for information provision to have any impact is that

$$\rho_L \ \leq \ \frac{\theta_L}{\theta_H} \ < \ \rho_H \tag{1.3}$$

But this is only a necessary condition for information provision to strictly dominate non-disclosure. To fully solve ($\mathcal{G}_f'$) an optimality argument based on the concave closure of $J(\mu)$



(a) $\rho_L < \theta_L/\theta_H$ (b) $\rho_L \geq \theta_L/\theta_H$

Figure 1.1: Two cases of the graph of J_f. The inequalities $\rho_L < (\geq)\, \theta_L/\theta_H$ are equivalent to $0 < (\geq)\, \mu^*$.

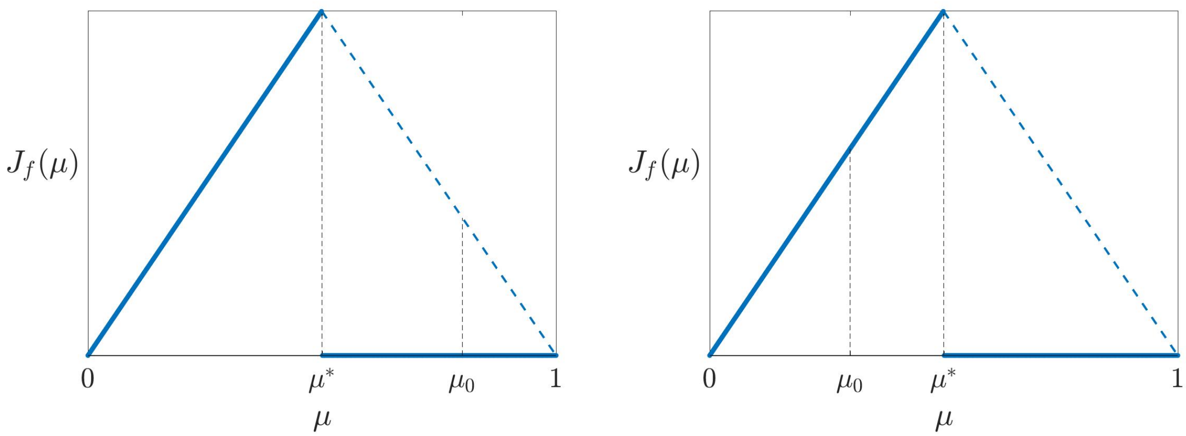
will be used. This will be denoted by $\mathcal{J}_f(\mu)$ and defined as

$$\mathcal{J}_f(\mu) \;=\; \sup\big\{z \,|\, (\mu, z) \in \text{co}(J_f)\big\},$$

where $\text{co}(J_f)$ denotes the convex hull of the graph of J_f. Thus for a given $\mu \in [0,1]$, $\mathcal{J}_f(\mu)$ is the highest value that can be achieved on the vertical line that passes through μ by using any linear combination of points that are below the graph of J_f. This implies that $\mathcal{J}_f(\mu) \geq J_f(\mu)$, however the inequality could be strict. Figure (1.2) plots $\mathcal{J}_f$ as a dashed line whenever it is strictly bigger than J_f.

Next we want to explain how $\mathcal{J}_f$ looks like and how it is derived. Assume throughout this discussion that (1.3) holds, so that information provision can have an impact on prices. First we argue that if $\mu \leq \mu^*$, then there is not a linear combination of points of J_f that achieve something above $J_f(\mu)$. Since J_f is stepwise linear it suffices to only consider two posterior realisations $\mu^- \leq \mu \leq \mu^+$. If $\mu^* < \mu^+$, then the linear combination of $J_f(\mu^-)$ and $J_f(\mu^+)$ will always be below $J_f(\mu)$. If instead $\mu^+ \leq \mu^*$, then the linear combination will be equal to $J_f(\mu)$. Therefore it has to be that $\mathcal{J}_f(\mu) = J_f(\mu)$. Suppose instead that $\mu > \mu^*$, then any choice of μ^- and μ^+ such that $\mu^- \leq \mu^* < \mu < \mu^+$ will provide a linear combination higher than $J_f(\mu) = 0$. But it is always optimal to increase both μ^- and μ^+ to their maximum values, which are μ^* and 1, respectively. Therefore

$$\mathcal{J}_f(\mu) \;=\; \begin{cases} J_f(\mu) & ,\text{ if } \;\mu \leq \mu^* \\ J_f(\mu^*) \cdot \frac{\mu - \mu^*}{1 - \mu^*} & ,\text{ if } \;\mu \geq \mu^* \end{cases}$$



(a) $\mu_0 > \mu^*$ (b) $\mu_0 < \mu^*$

Figure 1.2: Two cases of the graph of J_f for which information provision can have an impact on price setting.

The optimality argument that we use to solve S_1's information provision problem is based on the following observation: $\tilde{g}(\mu)$ solves ($\mathcal{G}_f'$) if and only if $\mathbb{E}_{\tilde{g}}[J_f(\mu)] = \mathcal{J}_f(\mu_0)$. This follows from noting that $\mathcal{J}_f(\mu_0)$ represents the maximum value that can be achieved on the vertical axis passing through μ_0 by taking linear combinations of J_f. But by its definition $\tilde{g}(\mu)$ is a set of linear combinations of J_f, and the Bayes plausibility constrain requires that those will give a value on the same vertical axis. Therefore, we can conclude that an informative signal does strictly better than no information provision if and only if $\mathcal{J}_f(\mu_0) > J_f(\mu_0)$, which is the case when $\mu_0 > \mu^*$. For a graphical illustration of this also refer to Figure (1.2). Before providing the main result of this subsection, we introduce the following definition.

Definition 1. An informative signal distribution $g(s \mid \theta_1)$ *strictly solves* S_1's payoff maximisation problem if it is part of one of its solutions (p_L, p_H, q_L, q_H, g), and there is no solution that uses no information provision.

Our aim is to characterise the set of parameters which can support an informative signal that strictly dominates no information provision. Using the above definition we can rule out cases where even though information provision is optimal, it is also inconsequential. An example of such a case would be when $\rho_L > \theta_L/\theta_H$, because even if S_1 reveals the buyer as a low period 1 type S_2 will still offer the high price.

Proposition 1.1. *In the first best, an informative signal strictly solves* S_1*'s payoff maximisation problem* ($\mathcal{P}_f$) *iff*

$$\rho_L \; < \; \frac{\theta_L}{\theta_H} \; < \; \rho_H \mu_0 + \rho_L(1 - \mu_0) \tag{1.4}$$

If this holds, then an optimal signal is $s \in \{\underline{s}, \overline{s}\}$ *with distribution*

$$g_f(\underline{s} \,|\, \theta_L) = 1, \qquad \textit{and} \qquad g_f(\underline{s} \,|\, \theta_H) = \frac{1-\mu_0}{\mu_0} \frac{\frac{\theta_L}{\theta_H} - \rho_L}{\rho_H - \frac{\theta_L}{\theta_H}} \tag{1.5}$$

In addition, it is optimal to supply both types with probability one.

Proof. In Appendix A.1. □

In the appendix we combine the necessary and sufficient condition for information provision to have impact on S_2's pricing policy (1.3) together with $\mu_0 > \mu^*$ to obtain (1.4). Subsequently, we show that g_f implements a randomisation between posteriors μ^* and 1, which we argued above that is the optimal distribution of posteriors when (1.4) holds.

When the optimal signal is informative it has a straightforward interpretation, which is that with probability $g_f(\overline{s} \,|\, \theta_H)$ S_1 reveals the high period 1 type, and otherwise says nothing. Essentially, S_1 is attempting to convince S_2 to some times offer a discount to the buyer and subsequently charges the buyer for the expected benefit of this discount. To do this she creates two signals in one of which she pools some period 1 high types with the corresponding low types. For such an informative signal to be beneficial two conditions are required. First the probability of a low period 1 type to become high period 2 type has to be low enough for S_2 to be persuaded to some times offer this discount. Second, it has to be that in the absence of persuasion S_2 would charge the high price, as otherwise the buyer would already be getting the best price possible and S_1 would have no reason to interfere.

An interesting implication of the above analysis is that information provision can be optimal even when the buyer's type is perfectly correlated across the two sellers, that is $\rho_H = 1 - \rho_L = 1$, which will be shown to not be true in S_1's second best contract. In the next section we will demonstrate that the reason why this is possible only in the first best is that the benefit of information provision can be obtained without the associated cost that the incentive compatibility constrain of the high type creates.

1.4.3 The second best contract of Seller 1

Next we analyse the second best, where θ_1 is the buyer's private information. We will demonstrate that S_1's information provision problem can be manipulated in a way that allows us to use the same solution method as in the first best. S_1 solves

$$\max_{p_1(\widehat{\theta}_1),\, q_1(\widehat{\theta}_1),\, g(s\,|\,\widehat{\theta}_1)} \left\{ \mu_0\, p_1(\widehat{\theta}_H) + (1-\mu_0)\, p_1(\widehat{\theta}_L) \right\} \quad \text{s.t. } (\mathrm{IR}_L\,),\ \ (\mathrm{IR}_H)\,,$$

$$\begin{aligned}(\mathrm{IC}_L\,)\ \ &\theta_L\, q_1(\widehat{\theta}_L) + \rho_L\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_L\right] - p_1(\widehat{\theta}_L\,)\\ &\qquad\qquad \geq\ \theta_L\, q_1(\widehat{\theta}_H) + \rho_L\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_H\right] - p_1(\widehat{\theta}_H)\\ (\mathrm{IC}_H)\ \ &\theta_H q_1(\widehat{\theta}_H) + \rho_H \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_H\right] - p_1(\widehat{\theta}_H)\\ &\qquad\qquad \geq\ \theta_H\, q_1(\widehat{\theta}_L) + \rho_H \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_L\right] - p_1(\widehat{\theta}_L)\end{aligned} \tag{$\mathcal{P}$}$$

where the individual rationality constraints, (IR_L) and (IR_H), are as in the previous subsection. It is important to underline that the buyer's report $\widehat{\theta}_1$ affects not only the probability and price of trade in period 1, but also the distribution of the signal s. Therefore the two events on which the expectations above are conditioned are $\widehat{\theta}_1 = \widehat{\theta}_L$ and $\widehat{\theta}_1 = \widehat{\theta}_H$, which we have shortened to the reported type. On the other hand, the probabilities of becoming a period 2 high type, ρ_L and ρ_H, are only a function of the buyer's period 1 type and remain the same on both sides of the above inequalities.

To maintain a compact notation we will hereafter use θ_L and θ_H to also denote the reports $\widehat{\theta}_L$ and $\widehat{\theta}_H$, respectively, and write $\{p_L, p_H, q_L, q_H\}$ instead of

$$\{p_1(\widehat{\theta}_L), p_1(\widehat{\theta}_H), q_1(\widehat{\theta}_L), q_1(\widehat{\theta}_H)\}.$$

Similarly to the first best, it is convenient to reduce the number of constrains by substituting the transfers p_L and p_H.

Lemma 1.3. *In the second best, S_1's payoff maximisation problem equivalently becomes*

$$\max_{q_L, q_H, g} \left\{ \begin{array}{c} \mu_0\, \theta_H q_H\, + (\theta_L - \mu_0\theta_H)\, q_L \\ + \mu_0\rho_H\, \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_H\big] + (1-\mu_0)\rho_L \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_L\big] \\ - \mu_0(\rho_H - \rho_L)\mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_L\big] \end{array} \right\} \tag{$\mathcal{P}'$}$$

$$s.t. \quad (\theta_H - \theta_L)\,(q_H - q_L)\ \geq\ (\rho_H - \rho_L)\Big(\mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_L\big] - \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_H\big]\Big) \tag{$\mathcal{P}_c$}$$

Proof. In Appendix A.1. □

The proof invokes the standard arguments used in mechanism design with binary type space. First, we argue that (IR_L) and (IC_H) have to bind. Second, we use the two equations to obtain transfers $\{p_L, p_H\}$ as functions of polices $\{q_L, q_H, g\}$. Substituting those in S_1's objective function and in (IC_L) gives the objective function of $(\mathcal{P}')$ and its constrain $(\mathcal{P}_c)$, respectively.

S_1's objective function is quite similar with the first best, however its value is reduced because of the rents captured by the period 1 high type. Those consists of two parts, the first of which is generated from the trade of S_1's product. This appears on the first line of the objective function of ($\mathcal{P}'$) reducing the marginal benefit from trading with the low type q_L. The second part of the buyer's rents in period 1 are due to his post contractual payoff. To be more specific, those are generated from the fact that when $\rho_H > \rho_L$ the value that each type assigns to the possibility of obtaining a discount from S_2 is different, and part of what S_1 sells is this discount.

It follows from ($\mathcal{P}'$) that it is always optimal to supply the high type with probability one, as increasing q_H not only increases the objective function of ($\mathcal{P}'$) but also loosens ($\mathcal{P}_c$). The same is not true for the probability of supplying the low type q_L. If ($\mathcal{P}_c$) was ignored, then the point-wise optimal q_L would be

$$\bar{q}_L = \begin{cases} 1 & \text{, if} \quad \mu_0 \leq \theta_L/\theta_H \\ 0 & \text{, if} \quad \mu_0 \geq \theta_L/\theta_H \end{cases} \tag{1.6}$$

However, we will shortly demonstrate that the above will not always be implementable together with an informative signal.

In the rest of the analysis we derive the point-wise optimal signal distribution $g(s \,|\, \theta_1)$. In other words we solve S_1's information provision problem while ignoring the constrain ($\mathcal{P}_c$). The proof of the proposition of this subsection, which can be found in the appendix, shows that whenever ($\mathcal{P}_c$) binds S_1 prefers to decrease q_L instead of altering the point-wise optimal signal distribution $g(s \,|\, \theta_1)$. Define the information provision problem of S_1 as

$$\max_g \left\{ \begin{array}{c} \rho_L \cdot \mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_L\big] \\ -\mu_0\rho_H \cdot \Big(\mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_L\big] - \mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_H\big]\Big) \end{array} \right\} \tag{$\mathcal{G}$}$$

the objective function of which follows from gathering terms on the last two lines of the objective function of ($\mathcal{P}'$).

To better understand the incentives of S_1 to provide information remember that in the first best this is done in order to create a discount for the buyer. Suppose then that in the absence of information provision S_2 opts for the high price. Then the buyer's rents from the second period are zero, henceforth the objective function of ($\mathcal{G}$) would be zero. Is it possible for S_1 to do better? The answer depends on the relative size of ρ_L and ρ_H. An informative signal could achieve strictly positive $\mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_L\big]$ and $\mathbb{E}_g\big[Q(\mu_1^s) \,|\, \theta_H\big]$, but it is not obvious what impact this would have on S_1's payoff.

Remark 1. For any choice of $g(s \mid \theta_1)$:

$$\mathbb{E}_g\big[Q(\mu_1^s) \mid \theta_H\big] \;\leq\; \mathbb{E}_g\big[Q(\mu_1^s) \mid \theta_L\big] \tag{1.7}$$

and the inequality is strict if $g(s \mid \theta_1)$ generates an impact on S_2's price.

Proof. It is without loss of generality to consider a signal $s_1, ..., s_n$ that induces progressively higher posteriors $\mu_1^{s_1} < ... < \mu_1^{s_2}$. But this implies strict monotone likelihood ratio dominance

$$\frac{g(s_n \mid \theta_H)}{g(s_n \mid \theta_L)} \;>\; \frac{g(s_{n'} \mid \theta_H)}{g(s_{n'} \mid \theta_L)} \quad \Leftrightarrow \quad n \;>\; n' \,,$$

which in turn implies that the CDF conditioned on θ_H strictly first order stochastically dominates the one conditioned on θ_L :

$$G(s \mid \theta_H) \;<\; G(s \mid \theta_L) \quad \text{for all} \quad s \;<\; s_n$$

But $Q(\mu_1^s)$ is non-increasing, which implies (1.7). To show that it holds with strict inequality when the signal generates an impact on prices, note that in this case $G(s \mid \theta_L)$ will have strictly more mass than $G(s \mid \theta_H)$ on the posteriors below μ^*. □

The above remark brings to the front the tension between the benefit of information provision and its potential cost. Its benefit is that it some times persuades S_2 to provide a discount to the buyer. In the first best, S_1 could capture all the expected benefit of this discount. On the other hand, in the second best not only is she not able to capture the expected benefit of the high period 1 type, but she actually has to leave some additional rents to him in order to not pretend to be a low one. This is because any informative signal will induce a better distribution for the low type, since S_2 is persuaded to offer the discount exactly when she assigns a higher probability of facing a buyer whose valuation of her product is low. We can use the above discussion to derive the equivalent of Theorem 1 of Calzolari and Pavan (2006) within our framework.

Remark 2. Suppose that the buyer's type is perfectly correlated across sellers, that is $\rho_L = 0$ and $\rho_H = 1$, then no information provision is optimal.

Proof. Follows trivially from noting that under perfect correlation ($\mathcal{G}$) becomes

$$\max_g \quad -\mu_0 \cdot \Big(\mathbb{E}_g\left[Q(\mu_1^s) \mid \theta_L\right] \;-\; \mathbb{E}_g\left[Q(\mu_1^s) \mid \theta_H\right]\Big)$$

the objective function of which is always non-positive, as argued in Remark 1. But no information provision gives always at least zero, thus it is optimal. □

Similarly to CP we showed that under perfect positive correlation privacy is optimal. As argued above when $\rho_L = 0$ the potential benefit of information provision becomes zero, while the associated incentive cost is the highest possible. Hence when we restrict ourselves to the region of positive correlation, some degree of stochasticity on the buyer's preferences is a necessary condition for information provision to be optimal.

We move on to deriving the solution of S_1's information provision problem ($\mathcal{G}$) under any imperfect positive correlation, that is under the restriction that $\rho_L \leq \rho_H$. As in the first best, to solve ($\mathcal{G}$) we start by rewriting its objective function as an expectation that uses the unconditional distribution $g(s)$.

Lemma 1.4. *In the second best, S_1's information provision problem equivalently becomes*

$$\max_{g} \ \mathbb{E}_g[J(\mu_1^s)] \tag{$\mathcal{G}$}$$

where its point-wise value $J(\mu_1^s)$ is

$$J(\mu_1^s) \ = \ \frac{\rho_H - \rho_L}{1-\mu_0} \cdot Q(\mu_1^s) \cdot \left(\mu_1^s - \frac{\mu_0 \rho_H - \rho_L}{\rho_H - \rho_L}\right) \tag{1.8}$$

Proof. In Appendix A.1. □

S_1's point-wise post contractual payoff J has two components. The first represents the surplus she apprehends from both types through managing their access to S_2 and transmitting information to her. The second represents the rents captured by the period 1 high type. The latter results on $J_f(\mu_1^s) > J(\mu_1^s)$ on the region of posteriors which achieve a discount. Crucially, for those posteriors $J(\mu)$ could even be negative. Under no information provision, i.e. when $\Pr(\mu_1^s = \mu_0) = 1$, S_1's post contractual payoff is equal to $\rho_L \cdot Q(\mu_0)$. In this case, S_1 essentially charges both period 1 types the post contractual payoff of the low one. Therefore this is her benefit from selling access to S_2. However, as we argued in the previous subsection whenever

$$\mu_0 \, \rho_H + (1-\mu_0) \, \rho_H \ > \ \frac{\theta_L}{\theta_H}$$

holds, this implies $Q(\mu_0) = 0$ because S_2 will charge $\overline{p} = \theta_H$ for her good. Thus, the buyer captures no surplus from the second period and the same is true for S_1. As argued in the initial example, depending on ρ_L and ρ_H it is possible for S_1 to do better by some times creating a discount, which is achieved by providing an informative signal to S_2. Effectively, S_1 engages in Bayesian persuasion with S_2, and the underline state variable is the buyer's period 1 reported type, which he is incentivised to truthfully report in S_1's mechanism.

Similarly to before, we express S_1's information provision problem as a choice of distributions over posteriors $\tilde{g}(\mu)$. Hence it equivalently becomes

$$\max_{\tilde{g}} \; \mathbb{E}_{\tilde{g}}\big[J(\mu)\big] \quad \text{s.t.} \;\; \mathbb{E}_{\tilde{g}}[\,\mu\,] = \mu_0 \tag{$\mathcal{G}'$}$$

We solve this by invoking the optimality condition $\mathbb{E}_{\tilde{g}}\big[J(\mu)\big] = \mathcal{J}(\mu_0)$, where

$$\mathcal{J}(\mu) \;=\; \sup\big\{z \,|\, (\mu, z) \in \text{co}(J)\big\}$$

denotes the concave closure of J. To obtain the functional form of $\mathcal{J}$, the shape of J needs to be characterised. Throughout the following analysis maintain the supposition that $\rho_L \leq \theta_L/\theta_H < \rho_H$, so that information provision has an impact on prices. Then the graph of J has two possible cases depending on the relative size of $\mu_0\rho_H$ and θ_L/θ_H. First assume that $\mu_0\rho_H < \theta_L/\theta_H$, in which case J looks very similar to J_f. A representative graph is given in Figure (1.3a). J is linear and increasing for posteriors in $[0, \mu^*]$, strictly positive at μ^*, and equals zero for posterios in $(\mu^*, \mu]$. This means that the analysis of the first best extends to this subcase of the second best. In particular, the concave closure of J is

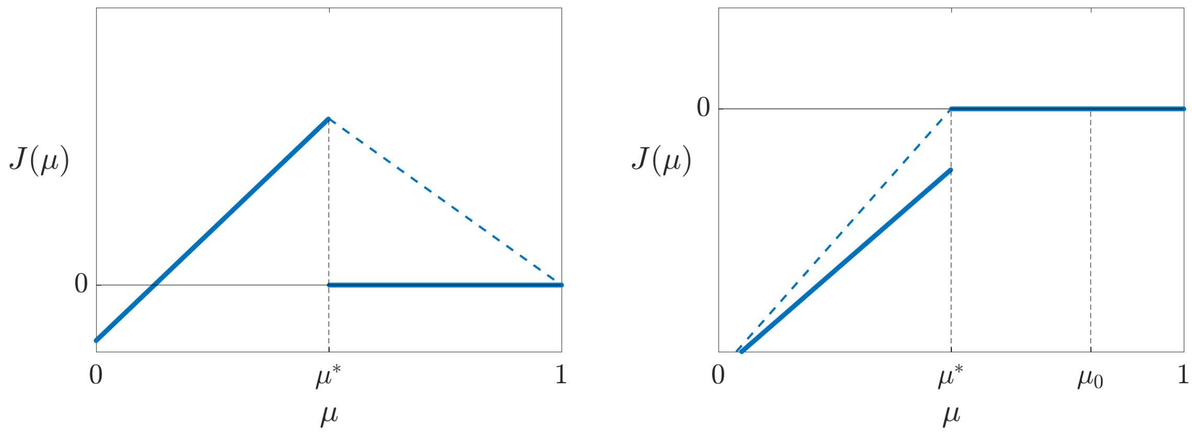
$$\mathcal{J}(\mu) \;=\; \begin{cases} J(\mu) & , \text{if} \quad \mu \leq \mu^* \\ J(\mu^*) \cdot \frac{\mu-\mu^*}{1-\mu^*} & , \text{if} \quad \mu \geq \mu^* \end{cases}$$

from which we infer that information provision is strictly optimal when

$$\rho_L \;<\; \frac{\theta_L}{\theta_H} \;< \rho_H\,\mu_0 + \rho_L\,(1-\mu_0)$$

On the other hand, if $\mu_0\rho_H \geq \theta_L/\theta_H$ then J reaches μ^* while still being non-positive, as shown in Figure (1.3b). Hence, it never becomes strictly positive. Simple algebra gives that in this case $\mu_0 \geq \mu^*$, hence no information provision is optimal. Intuitively, S_1 can always achieve at least zero under non-disclosure, hence there is no benefit from creating an informative signal.

The connection between the first and second best solution of S_1's information provision problem is very similar to the corresponding ones of its optimal supply problem. In the first best of the latter S_1 supplies her product to both types with probability one, whereas in its second best the low types gets the product only if its virtual type $\theta_L - \mu_0\theta_H$ is positive. Interestingly, in S_1's information provision problem a 'virtual type' also appears. This is because in ($\mathcal{G}$), $\mathbb{E}_g[Q(\mu_1^s)\,|\,\theta_L]$ is multiplied by $\rho_L - \mu_0\rho_H$, whereas in ($\mathcal{G}_f$) by $(1-\mu_0)\rho_L$. However, in this case information could be supplied even if this 'virtual type' was negative.



(a) $\mu_0\rho_H < \frac{\theta_L}{\theta_H}$ (b) $\mu_0\rho_H \geq \frac{\theta_L}{\theta_H}$

Figure 1.3: A representative graph of J. The dashed line denotes its concave closure, when this is above J. For the second graph we can show that $\mu_0 \geq \mu^*$ always holds.

This is because the decision to transmit information cannot be taken independently for each type. Hence the additional incentive to increase $\mu_0\rho_H\mathbb{E}_g[Q(\mu_1^s)\,|\,\theta_H]$ could skew S_1's decision towards an informative signal.

Proposition 1.2. *In the second best, an informative signal strictly solves S_1's payoff maximisation problem* ($\mathcal{P}$) *iff*

$$\max\{\rho_L, \rho_H\mu_0\} \;<\; \frac{\theta_L}{\theta_H} \;<\; \rho_H\mu_0 + \rho_L(1-\mu_0) \tag{1.9}$$

If this holds, then an optimal signal is $s \in \{\underline{s}, \overline{s}\}$ with distribution

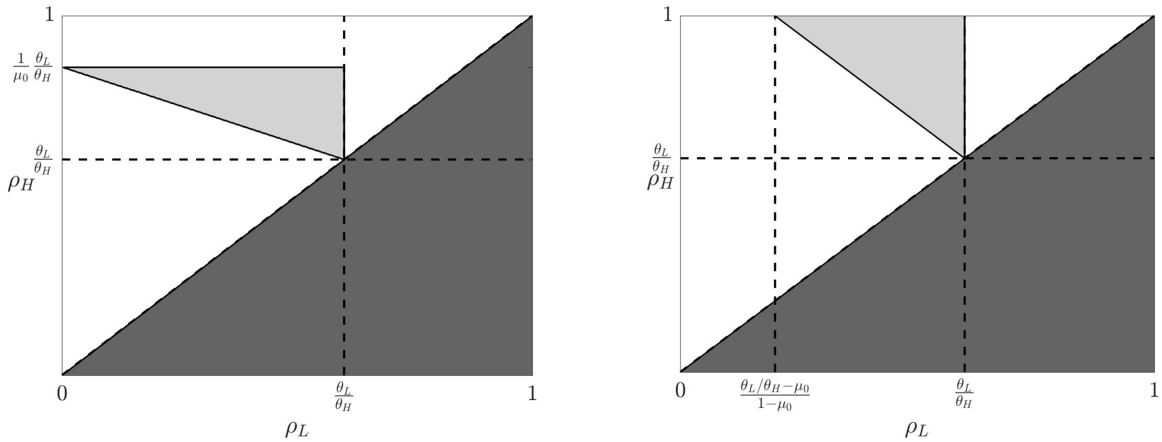
$$g^*\left(\underline{s}\,|\,\theta_L\right) \;=\; 1 \quad \textit{and} \quad g^*\left(\underline{s}\,|\,\theta_H\right) \;=\; \frac{1-\mu_0}{\mu_0}\,\frac{\frac{\theta_L}{\theta_H}-\rho_L}{\rho_H-\frac{\theta_L}{\theta_H}} \tag{1.10}$$

In addition, the high type is always supplied $q_H^* = 1$. Under no information provision the low type's optimal supply schedule is the point-wise optimal (1.6)*. However, under information provision it becomes*

$$q_L^* \;=\; \begin{cases} 1-(\rho_H-\rho_L)[1-g^*(\underline{s}\,|\,\theta_H)] & , \textit{ if } \;\mu_0 < \theta_L/\theta_H \\ 0 & , \textit{ if } \;\mu_0 \geq \theta_L/\theta_H \end{cases} \tag{1.11}$$

Proof. In Appendix A.1. □

There are a few interesting observations to make. First, the necessary and sufficient condition for information provision to be strictly optimal in the second best (1.9) defines a



(a) $\mu_0 > \theta_L/\theta_H$ (b) $\mu_0 < \theta_L/\theta_H$

Figure 1.4: The light grey area is the set of points for which information provision is strictly optimal in the second best. The dark grey is the area of negative correlation.

convex sets of transitioning probabilities ρ_L and ρ_H. Figures (1.4a) and (1.4b) demonstrate its triangular shape. Its vertical side is the axis passing from θ_L/θ_H due to the restriction that $\rho_L < \theta_L/\theta_H$. Its diagonal is representing the second inequality of (1.9), which can equivalently be written as

$$\rho_H > \frac{\theta_L}{\theta_H}\frac{1}{\mu_0} - \rho_L \cdot \frac{1-\mu_0}{\mu_0} \tag{1.12}$$

from which we can see that the bottom corner of this triangle will always by at $(\frac{\theta_L}{\theta_H}, \frac{\theta_L}{\theta_H})$.

Second, we can note that (1.9) is obtained by imposing $\rho_H\mu_0 < \theta_L/\theta_H$ on the corresponding condition of the first best (1.4). In Figure (1.4a), which assumes that $\mu_0 > \theta_L/\theta_H$ this is represented by the top side of the triangle. In particular, the corresponding set of the first best would also include all the area above it. On the other hand, if $\mu_0 < \theta_L/\theta_H$, then this implies $\rho_H\mu_0 < \theta_L/\theta_H$ and as a result the set defined by the first and second best conditions is the same, as shown in Figure (1.4b). Therefore, in the second best information provision is strictly optimal for a smaller set of transitioning probabilities, but only if $\mu_0 > \theta_L/\theta_H$.

Third, whenever information provision is strictly optimal in the second best the optimal distribution over posteriors is the same with that of the first best. This is a randomisation between the biggest reputation that still persuades S_2 to offer the low price, which is μ^*, and one. However, this does not mean that S_1's payoff is the same, since her second best payoff will be strictly smaller.

Forth, we want to consider the welfare implications of our results. In particular, we would like to see how information provision affects the buyer's welfare. To do this we compare his payoff to that of an alternative model where information provision would not be possible.

We restrict our attention to when information provision is optimal for S_1, in the original model, as otherwise we know that the buyer's payoff would be the same.

Corollary 1.1. *Suppose that information provision is optimal for S_1. Then the payoff of a low period 1 type is the same regardless of if information provision is possible or not. However, a high period 1 type is strictly better off when information provision is possible.*

Proof. In Appendix A.1. □

To further elaborate on this result note that introducing the possibility of information exchange between S_1 and S_2 leaves the payoff of a low period 1 type buyer unchanged, as his individual rationality constrain binds. However, the payoff a high type increases, since he captures additional rents. This result relies on the fact that the buyer is perfectly aware of when and how his purchase history from the first seller is shared with the second. This is exactly what legislations such as the European General Data Protection Regulation aim to achieve.

1.4.4 Negative correlation

Next we consider the case of negative correlation, that is we solve S_1's payoff maximisation problem in the second best ($\mathcal{P}$) under the assumption that $\rho_L > \rho_H$. In this case, the buyer's payoff from his second trade is

$$Q^{-}(\mu_1^s) \;=\; \mathbb{1}\{\mu_1^s \geq \mu^*\} \cdot (\theta_H - \theta_L)$$

since in contrast to the case of positive correlation S_2 is convinced to offer the low price when her posterior on the period 1 type is relatively high.

Our main difficulty is that when an informative signal is provided it is not obvious which period 1 type is really the 'high' type. This is because it could be that a period 1 low type is better off than a period 1 high type due to the former's post contractual payoff being higher than the latter's. Despite that, we can show that the representation of Lemma 1.3 can be extended to the case of negative correlation with the addition of one constrain.

Lemma 1.5. *In the second best and under negative correlation, S_1's payoff maximisation problem ($\mathcal{P}$) equivalently becomes ($\mathcal{P}'$) subject to ($\mathcal{P}_c$) and*

$$(\theta_H - \theta_L) \cdot q_L \;\geq\; (\rho_L - \rho_H) \cdot \mathbb{E}_g\big[Q^{-}(\mu_1^s) \,|\, \theta_L\big] \tag{$\mathcal{P}_h$}$$

Proof. In Appendix A.1. □

The proof shows that there are two sets of potential solutions for ($\mathcal{P}$). In the first it is the (IR_L) and (IC_H) that bind, whereas in the second it is the (IR_H) and (IC_L). Nevertheless, when we restrict attention in the second set we obtain a maximum that is on the boundary of the first set. Henceforth, it is without loss to assume that the (IR_L) and (IC_H) bind. This is the premise on which the proof of Lemma 1.3 relies, which is why its result extends here. However, in this setting it is not necessarily true that the individual rationality constrain of the high type will be satisfied, hence we need to introduce the constrain ($\mathcal{P}_h$).

Proposition 1.3. *In the second best and under negative correlation, an informative signal strictly solves* S_1*'s payoff maximisation problem* ($\mathcal{P}$) *iff*

$$\rho_H \; < \; \frac{\theta_L}{\theta_H} \; < \; \rho_H \mu_0 + \rho_L(1-\mu_0) \tag{1.13}$$

If this holds, then an optimal signal is $s \in \{\underline{s}, \overline{s}\}$ *with distribution*

$$g^-\left(\underline{s} \,|\, \theta_H\right) \, = \, 1 \quad \textit{and} \quad g^-\left(\underline{s} \,|\, \theta_L\right) \, = \, \frac{\mu_0}{1-\mu_0} \frac{\frac{\theta_L}{\theta_H} - \rho_H}{\rho_L - \frac{\theta_L}{\theta_H}} \tag{1.14}$$

Moreover, the above signal also strictly solves S_1*'s first best payoff maximisation problem* ($\mathcal{P}_f$) *under negative correlation.*

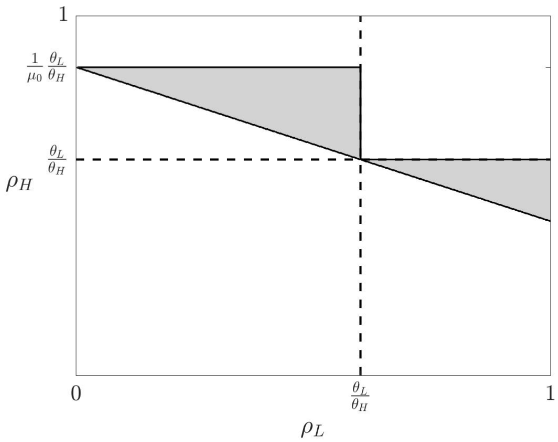
In addition, the high type is always supplied $q_H^- = 1$. *Under no information provision the low type's optimal supply schedule is the point-wise optimal* (1.6). *However, under information provision it becomes*

$$q_L^- \; = \; \begin{cases} 1 - (\rho_L - \rho_H)[1 - g^-(\underline{s} \,|\, \theta_L)] & , \textit{if} \;\; \mu_0 < \theta_L/\theta_H \\ (\rho_L - \rho_H) g^-(\underline{s} \,|\, \theta_L) & , \textit{if} \;\; \mu_0 \geq \theta_L/\theta_H \end{cases}$$

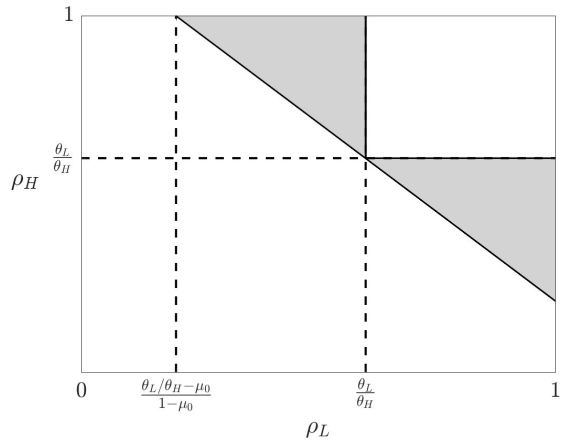
Proof. In Appendix A.1. □

First, note that under negative correlation the informative signal implements a randomisation between posteriors 0 and μ^*. This is the opposite of that of positive correlation, which was randomising between μ^* and 1. The reason is that under negative correlation S_2 is persuaded to offer the discount for high realisations of μ_1^s instead of the low ones, because a high period 1 type is less likely to have a high valuation for her product.

Second, under negative correlation the same informative signal strictly solves both the first and second best. This is because we showed in Lemma 1.5 that S_1 still solves ($\mathcal{P}'$) subject to ($\mathcal{P}_c$). But we know from the previous subsection that a necessary and sufficient condition for the incentive cost to be low enough for information provision to be beneficial



(a) $\mu_0 > \theta_L/\theta_H$



(b) $\mu_0 < \theta_L/\theta_H$

Figure 1.5: The light grey area is the set of points for which information provision is optimal. Both positive and negative correlation are considered.

for S_1 is that $\rho_H\mu_0 < \theta_L/\theta_H$. But this is implied by the necessary condition for information provision to have an impact on prices, which is

$$\rho_H \; < \; \frac{\theta_L}{\theta_H} \; < \; \rho_L \tag{1.15}$$

Figures (1.5a) and (1.5b) provide the full set of transitioning probabilities for which information provision is optimal. These are obtained by extending the diagonals of the north-west triangles, which continue to represent (1.12), and bounding the remain area from above with the θ_L/θ_H horizontal axis, which represents the $\rho_H < \theta_L/\theta_H$ restriction. As explain before, the south-east triangles are the same for both the first and second best payoff maximisation problems.

Notably, if the slope of the diagonal was negative enough, then the south-east corner of the box would be included in the shared area. This point represents the case of perfect negative correlation, which was also covered in Proposition 2 from Calzolari and Pavan (2006). In the setup considered here, the equivalent of this result follows as a corollary of Proposition 1.3.

Remark 3. Suppose that the buyer's type is perfectly negatively correlated across sellers, that is $\rho_L = 1$ and $\rho_H = 0$. Then an informative signal strictly solves S_1's payoff maximisation problem ($\mathcal{P}$) iff

$$\frac{\theta_L}{\theta_H} \; < \; 1 - \mu_0 \tag{1.16}$$

Diametrically, to the case of perfect positive correlation, which was considered in Remark

2, here we obtain that an informative signal could be optimal even if the buyer's preferences were non-stochastic. This is because under perfect negative correlation the period 1 high type assigns zero value to the second trade, since his second period type will always be low. As a result, S_1 can create an informative signal while paying zero extra rents. Hence, S_1's information provision problem in the second best is identical to that of the first best. But in the first best an informative signal could be optimal even if the buyer's preferences were non-stochastic, which explains the above result.

Finally, we want to consider the effect of information provision on the buyer's welfare under negative correlation. As in the previous subsection, we compare the buyer's welfare in our model to an alternative one where information provision would not be possible.

Corollary 1.2. *Suppose that information provision is optimal for S_1. Then the payoff of a low period 1 type is the same regardless of if information provision is possible or not. The same is true for a high type if $\mu_0 \geq \theta_L/\theta_H$. However if $\mu_0 < \theta_L/\theta_H$, then a high type is strictly worse off when information provision is possible.*

Proof. In Appendix A.1. □

Similarly to the previous subsection, the payoff of the low period 1 type is always zero, since his individual rationality constrain binds. The crucial difference, compared to the case of positive correlation, is that the high type is worse off when information provision is possible. This is because there is a disagreement between the two types on the relative value they assign to the trade of the first and second period. The high period 1 type prioritises the first period, whereas the low the second. Therefore, when using an informative signal S_1 can exploit this misalignment to reduce the rents captured by the high type. Despite that, S_1 is only exploiting this benefit of information provision when an informative signal is already optimal in the first best. That is reducing the high type's rents is not the primary reason she discloses information, but she still enjoys it as an indirect effect. Finally, we should point out that S_1 does not achieve her first best payoff, since the low type is not served with probability one.

1.5 Extensions

1.5.1 Selling information

Throughout the previous section we assumed that S_1 could only indirectly profit from providing information to S_2. That is by charging the buyer for the discounts the informative signal could generate. However, information disclosure also generates an expected benefit for S_2, since she can use this to adjust her price and sell more frequently. Therefore, S_1 could capture part of this benefit by charging S_2 for the signal she provides. Let $\gamma \in [0,1]$ denote the proportion that S_1 captures from the expected benefit that information provision creates for S_2. Thus, setting $\gamma = 0$ would collapse the extended model to the baseline one, which allocates all the bargaining power to S_2. Diametrically, setting $\gamma = 1$ would give all the bargaining power to S_1, who would capture from S_2 all the expected benefit of her signal.

Proposition 1.4. *The solution of the extended model, where S_1 can directly profit from selling information to S_2, is identical to that of the baseline one under both positive and negative correlation and for all $\gamma \in [0,1]$.*

Proof. In Appendix A.2. □

The proof demonstrates that even though the existence of direct selling motives is beneficial for S_1, and it affects her incentives to provide information, this effect is not strong enough to alter the distribution of her optimal signal. The approach that we use in the proof is identical to that of the previous section. Henceforth, our characterisation of the subsets of priors and transitioning probabilities for which an informative signal is strictly optimal, and the distribution of this signal, is robust to direct selling motives.

Hence the welfare implications of our baseline model are still relevant. Interestingly, our analysis hints that regulations aiming at restricting information exchange between sellers should not focus on banning or reducing monetary transactions. This is because allowing the first seller to directly benefit from selling information does not affect her optimal signal. More importantly, we have already demonstrated that information exchange is not necessarily adverse for the consumers.

1.5.2 Static imperfect correlation

The main analysis assumes that the buyer's type evolves dynamically between the two offers. Hence the buyer only learns his second period type when trading with S_2. However, we could imagine an alternative model where the buyer would knew both his types in the first period. Hence in this case the buyer's type would be static, but it would have two imperfectly correlated components. This would effectively result on a four element type space $(\theta_1, \theta_2) \in \big\{(\theta_L, \theta_L), (\theta_H, \theta_L), (\theta_L, \theta_H), (\theta_H, \theta_H)\big\}$. We focus mainly on the case that we deemed the most reasonable by assuming that

$$\max\left\{\rho_L \,,\, \frac{\theta_L}{\theta_H}\right\} \;<\; \rho_H \tag{1.17}$$

which incorporates not only positive correlation, but also that if a buyer is revealed as a high period 1 type, then this suffices for the second seller to charge the high price.

Proposition 1.5. *Suppose that the buyer knows both his types when trading with the first seller, and that (1.17) holds. Then no information provision is optimal. However, if (1.17) does not hold then there are parameters for which information provision becomes strictly optimal.*

Proof. In Appendix A.2. □

Our proof proceeds as follows. First, we argue in a way similar to Remark 1 that it is not possible for the seller to transmit information on the buyer's second type. To understand this note that if this was the case the signal distribution of (θ_H, θ_L) would first order stochastically dominate that of (θ_H, θ_H). Hence the latter's second period payoff is higher under the former's distribution. In addition, the first period preferences of (θ_H, θ_H) are fully aligned with those of (θ_H, θ_L), since their valuation of the first good is the same. Hence S_1 does not have a tool at her disposal to separate them when providing information on the second period type. The same is true for (θ_L, θ_H) and (θ_L, θ_L)

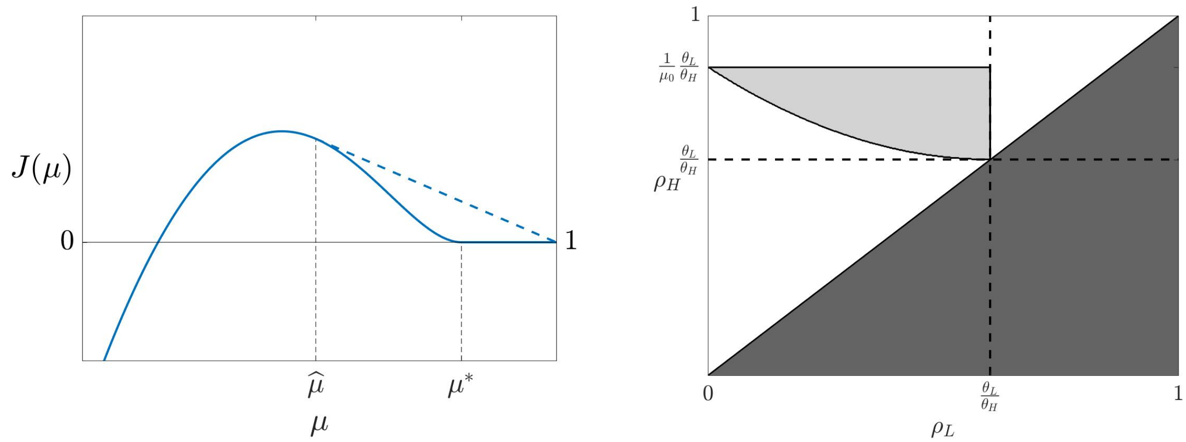
Second, suppose that S_1 tries to include all types in her contract. Then she cannot charge (θ_L, θ_H) extra for the possibility of a discount, since he could pretend to be (θ_L, θ_L) who assigns zero value to this. But similarly to the baseline model selling this discount to the low type is the only potential benefit of information provision, hence non-disclosure is optimal. There is an alternative way for S_1 to potentially increase her payoff, which is to exclude (θ_L, θ_L) from her contract. However, in that case it is only the high period 1 type that may assign the low valuation to the second seller's good. But because of (1.17) even if the first seller reveals the high period 1 type, this will still not be enough to generate a discount. Hence, we conclude that no information provision is optimal.

However, information provision may be optimal if (1.17) does not hold and the prior on the period 1 type is high enough for the first seller to not want to supply it. This is because in that case revealing a high type generates a discount, and the cost of excluding (θ_L, θ_L) from the contract is zero.

1.5.3 Isoelastic cost and other extensions

In an alternative extension we allow for each seller to supply a continuous quantity of a good[4] under an isoelastic cost function. We restrict attention to the case of positive correlation, for which we obtain similar results to the baseline model. The analysis is similar to that of

[4] We can equivalently interpreted this as producing a single good and choosing its quality level.



(a) J in the 2nd best

(b) Area of information provision

Figure 1.6: The above plots consider the case where q_t is produced under an isoelastic cost. On the right plot, the light grey area is the set of points for which information provision is optimal. Only positive correlation is considered.

the previous section, but due to its size it has been moved to Appendix A.3. Figure (1.6a) gives the the graph of J in the second best, which is the equivalent of Figure (1.3a) from the baseline model. In this cases the concave closure is built by finding the tangency point $\widehat{\mu}$. The set of transitioning probabilities for which information provision is optimal is as shown in (1.6b), which closely resembles (1.4a).

In the next chapter we further extend the above model by allowing the interaction of the first seller with the buyer to span over multiply periods. Since the buyer's payoff evolves stochastically during his contract under S_1, this extension is similar to the model of Battaglini (2005). However, the aforementioned paper does not include a second seller, and as a result the corresponding information provision problem does not exist. We show that S_1's preferred signal is informative for almost any choice of correlations across periods and sellers, with the noteworthy exemption of perfect positive correlation.

In a different section of the next chapter we show that the above multi-period model can also incorporated moral hazard. Therefore, it can be used to describe labour contracts. In this case, the information exchange is between employers and on the inferred ability of an employee. In a yet different section we provide some preliminary results on endogenous termination times and how the incentives of the employer to terminate her contract with the employee are affected by the fact that the second employer can use this termination time as a signal. A similar setup has been considered in Garrett and Pavan (2012), but instead of the employee being offered a new contract they assume a constant continuation value.

In the final section of the next chapter we allow for the buyer's type to be continuous.

We will not be able to derive the optimal signal for a generic distribution on the buyer's valuation and its evolution, however we will extend Calzolari and Pavan (2006) by providing some sufficient conditions for no disclosure to be optimal even under stochastic preferences.

1.6 Conclusions

In this paper we considered the following question: "Should a seller disclose private data on her clients?" We argued that selectively disclosing some data has two benefits. First, the seller can charge other firms for this information. Second, she can persuade those other firms to offer discounts to her customers, which she can subsequently use to increase her own prices. However, information disclosure entails an incentive cost, since it skews the choices of the seller's customers towards cheaper options. Our main contribution is to show that in a natural economic setting in which the buyer is uncertain about his future valuations information disclosure is optimal for a substantial set of environments.

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Chapter 2

Models of Mechanism Design with Bayesian Persuasion

This chapter considers four distinct extensions of the model presented in Chapter 1. Each is covered in a different section. To facilitate the reading we separate the analysis from the proofs. In the first section we allow the buyer's interaction with S_1 to span over multiple periods. At the termination of this contract S_2 makes her take-it-or-leave it offer to the buyer, which we continue to assume that it is for a single trade in one period. We show that the latter's preferred signal is informative for almost any choice of correlations across periods and sellers, and we characterise this signal. A noteworthy exemption is when the buyer's type is static, in which case no disclosure is optimal.

The second section considers an alternative extension, which allows for the buyer's type to be continuous. We will not be able to derive the optimal signal for a generic distribution on the buyer's valuation and its evolution, however we will extend Calzolari and Pavan (2006) by providing some sufficient conditions for no disclosure to be optimal in our setup.

The third section shows that our analysis is still valid when moral hazard is introduced. Since this setting is mostly related to the labour market we will interpret the agent as an employee and the two principals as two employers. Hence the communication between the two employers has the natural interpretation of a reference letter.

The last section allows the termination time of the first contract to be specified by it, and provides some preliminary results on the optimal termination time.

2.1 Multi-period contracts

In this section we allow for S_1's interaction with the buyer to span over multiply periods, and explore the implications of this extension on her information provision problem. Assume that $t \in \{0, .., \infty\}$ and S_1's contract with the buyer is exogenously[1] terminated at the end of each period with probability $1 - \gamma \in (0, 1)$. Let $\tau \in \{0, .., \infty\}$ denote this exogenous termination time. Then the buyer's type θ_t evolves under S_1 according to

$$\begin{aligned} \varphi_H &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau > t) \\ \varphi_L &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau > t) \end{aligned}$$

whereas the transitioning probabilities across sellers are as in the baseline setup

$$\begin{aligned} \rho_H &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau = t) \\ \rho_L &= \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau = t) \end{aligned}$$

and we continue to denote by $\mu_0 = \Pr(\theta_0 = \theta_H)$ the initial public prior.

For the sake of simplicity it will be assumed that S_2 trades with the buyer only once in period $\tau + 1$, and then the game ends[2]. We will also assume that S_1 controls access to S_2, in the sense that the latter will make an offer only if the former's parternship with the buyer was exogenously terminated. However, we have argued in Appendix A.3 that our results go through even without this assumption. We maintain the restriction that $\rho_H \geq \max\{\rho_L, \theta_L/\theta_H\}$, and add that $\varphi_H \geq \varphi_L$. Both principals can fully commit to their contracts. Conversely, the buyer will not be able to do so for any period in advance. As a result, the contract proposed by S_1 can essentially be interpreted as a series of history dependent single-period offers.

Hence, the timing on the multi-period model is as follows. At the beginning of period 1, S_1 publicly commits on a distribution $g_t(s \,|\, \widehat{\theta}^t)$, where $\widehat{\theta}^t$ denotes the history of reported types up to and including $\widehat{\theta}_t$, and offers to the buyer contract

$$\Big\{ p_t(\widehat{\theta}^t),\, q_t(\widehat{\theta}^t),\, g_t(s \,|\, \widehat{\theta}^t) \Big\}_{t=0}^{\infty}$$

which also includes a series of one period history depended prices and quantities. Subsequently, and at the beginning of each of the following periods, the buyer decides between either trading with S_1, which entails reporting his type, or terminating their partnership. Even if the buyer accept S_1's offer their contract may still end, after the quantity $q_t(\theta^t)$ has been traded, due to the exogenous termination probability $1 - \gamma$. If the contract is termi-

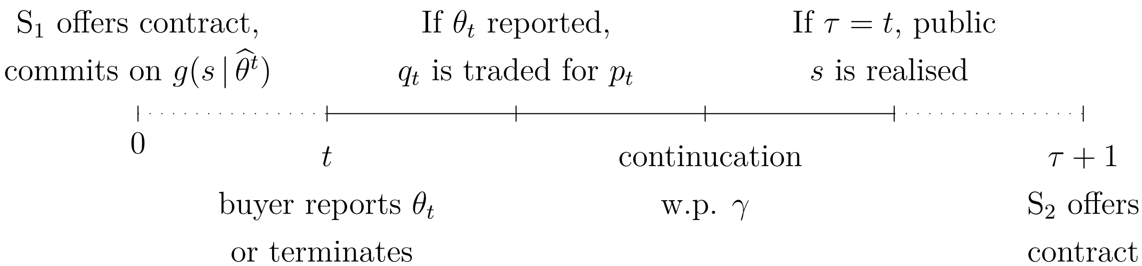
[1]This assumption is imposed mainly in order to simplify the notation. Even if termination was allowed to be endogenous, for any given choice of termination time S_1's information provision problem can be solved using the approach introduces in this section.

[2]Equivalently, it could be assumed that the buyer's type is static under S_2. Dropping this restriction would not significantly alter the results of the main analysis, but it allows some useful closed-form expressions to be derived and it makes the connection between this and the two period model easier.

nated at $\tau = t$ the s is realised and observed by S_2. Finally at $\tau + 1$, the period after its termination, S_2 offers to the buyer

$$\left\{ p(\widehat{\theta}_{\tau+1}, s, \tau),\ q(\widehat{\theta}_{\tau+1}, s, \tau) \right\}$$

which is a single period contract that utilises both the realised termination time τ and the signal s, and subsequently the game ends.



This section proceeds as follows. In subsection 2.1.1 we solve for S_2's optimal contract and characterise the buyer's post contractual payoff in this setting. In subsection 2.1.2 we derive a representation of S_1's payoff that does not depend on transfers and provide a generic sufficient condition for implementation. Next, in subsection 2.1.3 we show how the part of S_1's payoff that is related to the buyer's subsequent contract with S_2 can be reformulated so that her information design problem can be approached in a way similar to the literature of Bayesian Persuasion, and we characterise its solution.

2.1.1 The buyer's post-contractual payoff

S_2's payoff maximisation problem is solved, and the buyer's expected payoff from trading with her is derived. For realised termination $\tau = t$, let $\mu_t^s = \Pr(\theta_t = \theta_H \mid s)$ denote S_2's posterior belief on the buyer's type in period t, and $\beta_t^s = \Pr(\theta_{t+1} = \theta_H \mid s)$ the corresponding posterior on θ_{t+1}. Those two are connected according to

$$\beta_t^s \;=\; \mu_t^s\,\rho_H \,+\, \left(1 - \mu_t^s\right)\rho_L$$

The problem itself is quite standard and its treatment can be found in the appendix. The following lemma characterises the buyer's payoff, which is the only result needed to proceed with S_1's payoff maximisation problem.

Lemma 2.1.1. *The payoff of a low buyer type under S_2 is equal to zero, while that of the* *high one equals*

$$B(\beta_t^s) \; = \; \begin{cases} b^{1+\epsilon} \cdot (\theta_H - \theta_L) \cdot \left(\frac{\theta_L - \beta_t^s \theta_H}{1-\beta_t^s}\right)^{\epsilon} & , \; \text{if} \quad \beta_t^s \leq \theta_L/\theta_H \\ 0 & , \; \text{if} \quad \beta_t^s \geq \theta_L/\theta_H \end{cases} \tag{2.1.1}$$

Also, on the subset of posteriors $[0, \frac{\theta_L}{\theta_H})$ *it is decreasing, and strictly concave (convex) for*

$$\beta_t^s \; < (>) \; \frac{\theta_L}{\theta_H} + \frac{1-\epsilon}{2}\left(1 - \frac{\theta_L}{\theta_H}\right).$$

Proof. In Appendix B.1. □

The payoff of a low buyer type in period 2 is always equal to zero, as it captures no rents. Conversely, the high type's payoff is positive, but only if the posterior β_t^s is low enough for S_2 to opt to serve both types. Indeed,we show in the proof that the rents he captures are proportional to the quantity bought by the low type. As β_t^s increases, a distortion on the low type's supplied quantity has smaller effect on S_2's expected payoff. Hence it becomes cheaper for her to increase this distortion in order to tighten the incentive compatibility constrain of the high type and decrease his rents. As a result, both the quantity supplied to the low type, and the high type's rents are decreasing in β_t^s.

2.1.2 Payoff equivalence

S_1's payoff maximisation problem is

$$\max_{p,q,g} \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} \gamma^t \delta^t \cdot \Big(p_t(\theta^t) - c\big[q_t(\theta^t)\big] \Big) \right] \quad \text{subject to IR}(\theta^t) \text{ and IC}(\theta^t) \tag{$\mathcal{P}$}$$

where $\text{IR}(\theta^t)$ and $\text{IC}(\theta^t)$ refer to the individual rationality and incentive compatibility constrains of a θ^t buyer type. To make notation more compact three special cases of θ^t will be defined. First, let $\theta_L^t = \{\theta^{t-1}, \theta_L\}$ and $\theta_H^t = \{\theta^{t-1}, \theta_H\}$ denote a history such that the buyer's type in period t is low and high, respectively. In addition, for given generic θ^{t-1} and $t' \geq t$ let

$$L_t^{t'} \; = \; \{\theta^{t-1}, \theta_L, .., \theta_L\}\,, \tag{2.1.2}$$

denote a history such that the buyer's type has been low for all periods after, and including, period t. Also, whenever $t = 0$ simply write $L^{t'}$.

The proof of the subsequent proposition, which follows closely Battaglini (2005), demonstrates that the information rents captured by a period t high type θ_H^t are closely related to the histories $\{L_t^{t'}\}_{t'>t}$. In particular, when the implementation constrains, which will be provided shortly, do not bind the information rents captured by a period t high type are given by

$$U_t^H(\theta^{t-1}) \;\equiv\; \sum_{t'=t}^{\infty} \big[\gamma\delta(\rho_H - \rho_L)\big]^{t'-t} \cdot \Big\{ (\theta_H - \theta_L) q_t(L_t^{t'}) \\ + \delta(1-\gamma)(\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_{t'}^s) \,|\, L_t^{t'}\big] \Big) \Big\}$$

where $\beta_t^s = (\rho_H - \rho_L)\Pr(\theta_t = \theta_H \,|\, \tau = t, s) + \rho_L$. Then ($\mathcal{P}$) simplifies to the following problem, which only depends on policies (q, g) and not on transfers p.

Proposition 2.1.1. *Suppose that a solution of*

$$\max_{q,g} \;\; \Bigg\{ \mathbb{E}_\theta \Bigg[\sum_{t=0}^{\infty} \gamma^t \delta^t \cdot \Big(\theta_t \, q_t(\theta^t) - c\big[q_t(\theta^t)\big] \\ + \delta(1-\gamma) \Pr(\theta_{t+1} = \theta_H | \, \tau = t, \theta_t) \mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta^t\big] \Big) \Bigg] \\ - \mu_0 \sum_{t=0}^{\infty} \gamma^t \delta^t (\varphi_H - \varphi_L)^t \Big[(\theta_H - \theta_L) q_t(L^t) + \delta(1-\gamma)(\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_t^s) \,|\, L^t\big] \Big) \Big] \Bigg\} \quad (\mathcal{P}')$$

satisfies

$$(\theta_H - \theta_L)\Big[q_t(\theta_H^t) - q_t(\theta_L^t)\Big] + (\varphi_H - \varphi_L)\gamma\delta\Big[U_{t+1}^H(\theta_H^t) - U_{t+1}^H(\theta_L^t)\Big] \\ \geq \;\; (\rho_H - \rho_L)(1-\gamma)\delta\Big[\mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_L^t\big] - \mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_H^t\big]\Big] \quad (\mathcal{P}_c)$$

Then those policies are also a solution to ($\mathcal{P}$) *and there exists a contract that implements them.*

Proof. In Appendix B.1. □

The proof of the above proposition follows closely Battaglini (2005). ($\mathcal{P}_c$) is obtain by solving a relaxed version of ($\mathcal{P}$) where its downward slopping constrains IC(θ_L^t) are ignored, and showing that for this relaxed problem the upward slopping ones IC(θ_H^t) have to bind on its maximum. This allows the derivation of an expression for the period 0 high type's expected payments that only depends on the policies (q, g). The same can be done for the period 0 low type by using his individual rationality constrain. Substituting those expected payment in S_1's payoff gives ($\mathcal{P}'$). Henceforth, if the policies that solve the relaxed problem ($\mathcal{P}'$) satisfy the ignored downward slopping constrains IC(θ_L^t), then those also solve ($\mathcal{P}$). To check this, the derived expression for the expected payments is substituted in IC(θ_L^t), which gives ($\mathcal{P}_c$).

The expectation over the first summation in ($\mathcal{P}'$) denotes the ex-ante total surplus of S_1's partnership with the buyer, which includes the latter's post contractual payoff. Moreover, as shown in the proof of the proposition, the second summation denotes the rents captured by a period 1 high type. Note that even though a period t low type has some probability of becoming a high type, and as a result acquiring some rents in the future periods, S_1 can charge him in advance for those. Therefore, the buyer manages to capture positive rents only while he has never been a low type in the past. Interestingly, those rents are only related to the worst possible history $L^t = \{\theta_L, .., \theta_L\}$, which will create all the inefficiency of the multi-period contract.

For the model provided it can be shown that if the buyer's continuation value was not type depended, i.e. $\rho_H = \rho_L$, then the solution of ($\mathcal{P}'$) would always be implementable. However, this will not generically be true when $\rho_H > \rho_L$, since the high type has an additional incentive to misreport when the signal is informative. This is because the low type's signal will generically result in lower posteriors, which is beneficial for the buyer.

Corollary 2.1.1 (Implementation)**.** *The point-wise optimal level of production is*

$$q_t^*(\theta^t) = \begin{cases} (\theta_t)^\epsilon & , \ \theta^t \neq L^t \\ (\xi_t)^\epsilon & , \ \theta^t = L^t \end{cases} , \quad \textit{where} \ \ \xi_t = \max\left\{0\,,\, \theta_L - \frac{\mu_0(\theta_H - \theta_L)}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t\right\} \tag{2.1.3}$$

and this is implementable for any choice of signal distribution g if

$$\left(\frac{\theta_H}{\theta_L}\right)^\epsilon \ \geq \ 1 + \delta\,(\rho_H - \rho_L)\, b^{1+\epsilon} \tag{2.1.4}$$

Proof. In Appendix B.1. □

The relaxed implementation condition (2.1.4) is identical to that of the two period model, and will hereafter be assumed to hold.

2.1.3 Information provision

This section solves S_1's information provision problem. Her optimal signal distribution g_t will be derived for any realisation of $\tau \in \{0, .., \infty\}$. Hence considering only the part of ($\mathcal{P}'$) that is affected by the signal s on realised $\tau = t$, and ignoring the discount factor δ^{t+1} that multiplies it, gives

$$\max_g \left\{ \sum_{\theta^t} \left[\Pr(\theta^t, \tau = t) \sum_s \Pr(\theta_{t+1} = \theta_H | \, \tau = t, \theta_t) B(\beta_t^s) g_t(s \,|\, \theta^t) \right] \right.$$
$$\left. - \mu_0 \Pr(\tau = t)(\varphi_H - \varphi_L)^t (\rho_H - \rho_L) \sum_s B(\beta_t^s) g_t(s \,|\, L^t) \right\} \quad (\mathcal{G}_t)$$

The first line of $(\mathcal{G}_t)$ represents the expected rents from the buyer's contract with S_2, which are captured by S_1 through the individual rationality and incentive compatibility constrains of her own contract. On the other hand, the second line corresponds to the rents captured by the buyer in his first contract. Those are generated from the fact that whenever the signal s is informative the high buyer types θ_H^t have a stronger incentive to misreport, because their continuation payoff $B(\beta)$ is multiplied by ρ_H instead of ρ_L.

Next we transform $(\mathcal{G}_t)$ into an equivalent problem that will only depend on the posteriors $\Pr(\theta_t = \theta_H \,|\, \tau = t)$ and $\Pr(\theta^t = L^t \,|\, \tau = t)$. Introduce the following notation

$$\begin{aligned} \mu_t &= \Pr(\theta_t = \theta_H \,|\, \tau = t), \quad & \mu_t^s &= \Pr(\theta_t = \theta_H \,|\, s, \tau = t) \\ \lambda_t &= \Pr(\theta^t = L^t \,|\, \tau = t), \quad & \lambda_t^s &= \Pr(\theta^t = L^t \,|\, s, \tau = t) \end{aligned}$$

The interim posterior beliefs on the first column only use the information provided by the termination time τ, whereas the posteriors on the second column also depend on the signal s; μ_t^s is the posterior on $\theta_t = \theta_H$, while λ_t^s on $\theta^t = L^t$. Note that the first event only depends on the contemporaneous θ_t, while the second on the whole history θ^t. Finally, abusing notation let $g_t(s)$ denote the probability of sending signal s after a contract being terminated at time $\tau = t$, that is

$$g_t(s) = \sum_{\theta_t} \Pr(\theta^t \,|\, \tau = t) \, g_t(s \,|\, \theta^t)$$

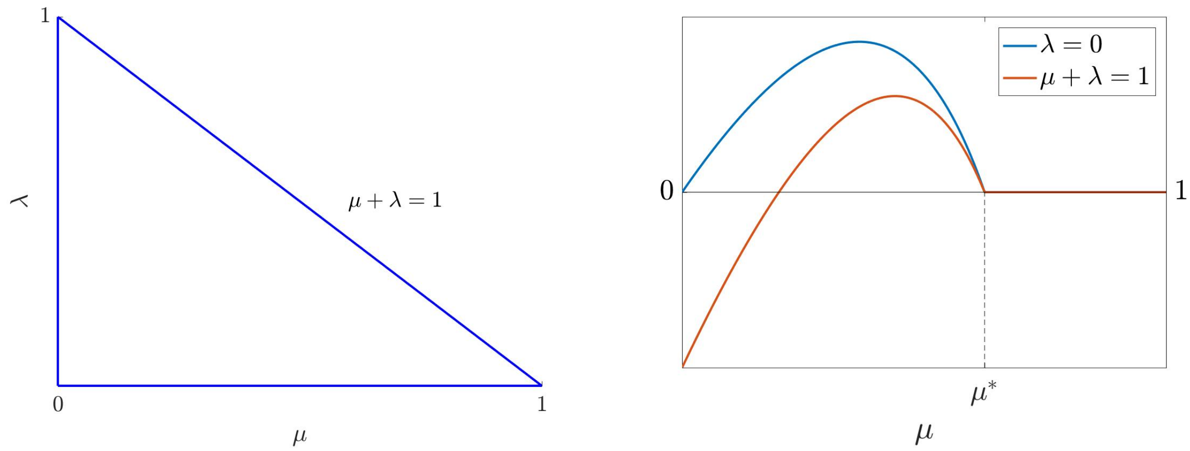
To proceed we provide the following Lemma, which generalises Lemma A.3.6 of Section A.3.2.

Lemma 2.1.2. *S_1's information provision problem in period t equivalently becomes*

$$\max_g \ \mathbb{E}_g[J_t(\mu_t^s, \lambda_t^s)] \quad (\mathcal{G}_t)$$

where its point-wise value $J_t(\mu_t^s, \lambda_t^s)$ is

$$J_t(\mu_t^s, \lambda_t^s) = B(\beta_t^s)\big(\beta_t^s - \lambda_t^s \, \psi_t\big), \quad \textit{and} \quad \psi_t \equiv \frac{\mu_0}{1-\mu_0} \left(\frac{\varphi_H - \varphi_L}{1 - \varphi_L} \right)^t (\rho_H - \rho_L) \quad (2.1.5)$$



(a) Domain (b) Boundaries

Figure 2.1: The domain of $J_t(\mu, \lambda)$, for $t > 1$, and its value on the two boundaries.

Proof. In Appendix B.1. □

Similarly to the previous section, the information provision problem of S_1 is reformulated by considering (μ_t^s, λ_t^s) as the underline random variables, instead of the signal s, with joint distribution $\tilde{g}_t$. Hence, S_1 equivalently solves

$$\max_{\tilde{g}_t} \ \mathbb{E}_{\tilde{g}_t}\big[J_t(\mu, \lambda)\big] \quad \text{s.t. } \mathbb{E}_{\tilde{g}_t}[\mu] = \mu_t\,, \quad \mathbb{E}_{\tilde{g}_t}[\lambda] = \lambda_t\,, \qquad \mu, \lambda \in [0,1]\,, \text{ and } \mu + \lambda \leq 1. \tag{$\mathcal{G}'_t$}$$

The constrains ensure that the joint distribution $\tilde{g}_t(\mu, \lambda)$ is Bayes plausible. Note that it is not enough for μ and λ to be probabilities, that is to be in $[0, 1]$, as they represent mutually exclusive events. Hence, their sum has to be less than one. The inequality can be strict as the history L^t does not necessarily cover all θ^t such that $\theta_t = \theta_L$. Therefore $\Pr(\theta_t \neq \theta_H \cap \theta^t \neq L^t)$, which is the complement of $\Pr(\theta_t = \theta_H \cup \theta^t = L^t) = \mu_t^s + \lambda_t^s$, can be strictly positive. Equivalently, it is possible that $\mu_t^s + \lambda_t^s < 1$.

The domain of $J(\mu, \lambda)$ is a right-angled triangle, of which each of its legs has length one. A representative graph of it on the sides where $(\mu+\lambda = 1)$ and $(\lambda = 0)$ is given in Figure 2.1. Those two sides are connected by straight lines, as $J(\mu, \lambda)$ is linear in λ. It will be convenient to define $J_t(\mu, \lambda)$ as a function of of μ only on the two aforementioned boundaries. Hence, for $i \in \{f, t\}$ let

$$\bar{J}_i(\mu) \ = \ \zeta_i\, B(\beta)(\beta - \Psi_i)\,, \quad \text{where} \quad \begin{cases} \zeta_f = 1 & \Psi_f \ = 0 \\ \zeta_t \ = 1 + \frac{\psi_t}{\rho_H - \rho_L} & \Psi_t \ = \frac{\psi_t \rho_H}{\rho_H - \rho_L + \psi_t} \end{cases}$$

and note that $\beta = \mu\rho_H + (1-\mu)\rho_L$ as those two posteriors are always connecting through this linear relationship. Lemma B.1.2, which due to its size has being moved, together with the rest of the proofs, to Section B.1 shows that $\bar{J}_f(\mu)$ represents the $(\lambda = 0)$ boundary, whereas $\bar{J}_t(\mu)$ the $(\mu + \lambda = 1)$ one. Interestingly, the latter's functional form is very similar to that of $J(\mu)$, derived in Section A.3.3, and that of the former is identical to $J_f(\mu)$, which was derived in Section A.3.2. This is because in the multi-period setup $J_f(\mu)$ corresponds to the representation of S_1's first best post contractual payoff.

To clarify the last claim suppose that S_1 could commit on (p, q, g) before learning the buyer's type, as in $(\mathcal{P})$, but his type was directly observed by her. Then instead of $J_t(\mu, \lambda)$, S_1 information provision problem would have $\bar{J}_f(\mu)$ as an objective function. This is because the rents paid by S_1 are multiplied by the probability of L^t to occur. Therefore $\lambda_t = 0$ corresponds to the case where no rents are paid, which is the same as ignoring the incentive compatibility constrains in $(\mathcal{P})$. In addition, for $\varphi_H < 1$ it is ease to show that $\bar{J}_f(\mu)$ is the limit of $J_t(\mu, \lambda)$ as t goes to infinity, which shows that S_1's information provision problem converges to its first-best solution after long contracting periods.

To solve S_1's information provision problem, first the concave closure of $J_t(\mu, \lambda)$ needs to be characterised. This will be denoted by $\mathcal{J}_t(\mu, \lambda)$ and defined as

$$\mathcal{J}_t(\mu, \lambda) \;=\; \sup\{z \,|\, (\mu, \lambda, z) \in \mathrm{co}(J_t)\},$$

where $\mathrm{co}(J_t)$ denotes the convex hull of the graph of $J_t(\mu, \lambda)$ on D.

In addition, for $i \in \{f, t\}$ let $\bar{\mathcal{J}}_i(\mu)$ denote the value of $\mathcal{J}_t(\mu, \lambda)$ on the subsets $(\lambda = 0)$ and $(\mu + \lambda = 1)$, respectively, which is also the concave closure of the corresponding $\bar{J}_i(\mu)$. The following proposition characterises $\mathcal{J}_t(\mu, \lambda)$. To facilitate its exposition, a new point needs to be introduced. For $i \in \{f, t\}$ and $\Psi_i < \frac{\theta_L}{\theta_H} < \rho_H$, let $\hat{\mu}_i$ be the unique solution of

$$\bar{J}_i(\hat{\mu}_i) + \bar{J}_i'(\hat{\mu}_i)(1 - \hat{\mu}_i) \;=\; 0\,, \tag{2.1.6}$$

if this exists, and zero otherwise. The functional form of $\hat{\mu}_i$ can be found in the proof of the following proposition in, but it is not copied here, due to its size.

Proposition 2.1.2. *For any interior point $\mathcal{J}_t(\mu, \lambda) > J_t(\mu, \lambda)$.*

- *On the boundary $(\mu = 0)$: $\mathcal{J}_t(0, \lambda) = \bar{J}_t(0, \lambda)$.*
- *On the boundary $(\lambda = 0)$, and when $\Psi_t < \frac{\theta_L}{\theta_H}$ also on $(\mu + \lambda = 1)$:*

$$\bar{\mathcal{J}}_i(\mu) \;=\; \begin{cases} \bar{J}_i(\mu) & ,\ \textit{for}\ \ \mu \,\leq\, \hat{\mu}_i \\ \bar{J}_i(\hat{\mu}_i) + \bar{J}_i'(\hat{\mu}_i)(\mu - \hat{\mu}_i) & ,\ \textit{for}\ \ \mu \,\geq\, \hat{\mu}_i \end{cases} \tag{2.1.7}$$

- *On the boundary* $(\mu+\lambda=1)$*, if* $\Psi_t \geq \frac{\theta_L}{\theta_H}$ *and* $\epsilon \geq 1$*, then* $\mathcal{J}_t(\mu) = \bar{J}_t(\mu)$*.*

Finally, $\hat{\mu}_i$ *is non-decreasing in* Ψ_i*, and strictly increasing when it is positive.*

Proof. In Appendix B.1. □

The case where $\Psi_t \geq \theta_L/\theta_H$ and $\epsilon < 1$ can also easily be described, however it is omitted to make the statement of the above proposition more compact. Whenever the interim posteriors (μ_t, λ_t) are on one of the two boundaries, the optimal signal follows trivially from the corresponding bullet point. In particular, for the boundaries $(\lambda = 0)$ and $(\mu + \lambda = 1)$ the analysis is almost identical to that of Sections A.3.2 and A.3.3, respectively, hence it is omitted. On $(\mu = 0)$ any signal is optimal, informative or not, because $\mathcal{J}_t(0,\lambda)$ is linear on λ.

In addition, the above characterisation can be used to derive the optimal signal under static types

$$\varphi_H = 1 - \varphi_L = \rho_H = 1 - \rho_L = 1\,,$$

since in this case it has to be that $\mu_t + \lambda_t = 1$ for all periods $t \geq 0$. Non surprisingly the corresponding result of Section A.3.3, on the optimality of no information provision under static types, can be extended to the general model.

Corollary 2.1.2 (Privacy under Static Types)**.** *Suppose that the buyer's type is static, then no information provision is optimal on all periods* $t \geq 0$*.*

Proof. In Appendix B.1. □

Unfortunately, it is not as ease to describe the optimal signal when (μ_t, λ_t) is an interior point, as the functional form of $\mathcal{J}_t(\mu,\lambda)$ on those point is harder to obtain. However, it is possible to provide the following characterisation.

Proposition 2.1.3. *For any interior point* (μ,λ) *the value of* $\mathcal{J}_t(\mu,\lambda)$ *is a linear combination between* $\bar{\mathcal{J}}_f(\mu') = \mathcal{J}_t(\mu', 0)$ *and* $\bar{\mathcal{J}}_t(\mu'') = \mathcal{J}_t(\mu'', 1-\mu'')$*, where*

$$\mu' \;=\; \mu - \frac{\lambda}{x}\,, \quad \text{and} \quad \mu'' \;=\; \frac{1-\lambda+x\mu}{1+x}\,. \tag{2.1.8}$$

The values of $\mathcal{J}_t(\mu,\lambda)$ *and* x *are given by the solution of*

$$\begin{aligned} \mathcal{J}_t(\mu,\lambda) \;&=\; \max_x \left\{ (1-\mu-\lambda)\,\frac{\bar{\mathcal{J}}_f(\mu')}{1-\mu'} + \lambda\,\frac{\bar{\mathcal{J}}_t(\mu'')}{1-\mu''} \right\} \\ &\quad s.t \quad x \in \left(-\infty\,,\, -\frac{1-\lambda}{\mu}\right] \cup \left[\frac{\lambda}{\mu}\,,\, +\infty\right) \end{aligned} \tag{2.1.9}$$

Proof. In Appendix B.1. □

The proof relies on the fact that for any interior point (μ, λ) there exist two corresponding boundary points $(\mu_1, 0)$ and $(\mu_2, 1-\mu_2)$, along with an appropriate weight ω, such that $J_t(\mu, \lambda)$ can be written as a linear combination of those two points, that is

$$J_t(\mu, \lambda) \;=\; \omega J_t(\mu_1, 0) + (1-\omega) J_t(\mu_2, 1-\mu_2)\,.$$

This makes it possible to write the concave closure $\mathcal{J}_t(\mu, \lambda)$ as a linear combination of its value on points of the boundaries $(\mu = 0)$ and $(\mu + \lambda = 1)$ exclusively. Therefore, $\mathcal{J}_t(\mu, \lambda)$ can be expressed as a linear combination of $\mathcal{J}_t(\mu', 0)$ and $\mathcal{J}_t(\mu'', 1-\mu'')$, the inputs of which belong in one of those two subsets of D. Hence, to find $\mathcal{J}_t(\mu, \lambda)$ it suffices to pick those two points by maximising the value of their linear combination, or equivalently pick the slope x of the line that connects the interior point (μ, λ) with the two boundaries. After some algebra it can be shown that this problem is equivalent to (2.1.9).

An immediate implication of Proposition 2.1.3 is that the optimal signal may need to use more than two, but no more than four possible realisations $s \in \{s'_0, s'_1, s''_0, s''_1\}$. This is because each of the values $\bar{\mathcal{J}}_f(\mu')$ and $\bar{\mathcal{J}}_t(\mu'')$ may require an additional linear combination on the corresponding boundary to be reached, similarly to the simple model of Section A.3.

Intuitively, S_1 engages in two distinct randomisations. First, she randomises over the boundary on which the posteriors will be, where $\bar{\mathcal{J}}_f$ denotes the first best, and $\bar{\mathcal{J}}_t$ the worst possible second best. This only affects the posterior $\lambda^s_t = \Pr(\theta^t = L^t)$. Subsequently, she randomises over the posterior $\mu^s_t = \Pr(\theta_t = \theta_H)$. It is noteworthy that S_2 only cares about μ^s_t, therefore the first randomisation has no effect on the buyer's expected post contractual payoff. In particular, in the first best S_1 would only provide information on the buyer's last reported type θ_t, and not on the history L^t. Despite that, in the second best the first randomisation over posteriors λ^s_t is used as a way to reduce the expected rents of a period 1 high type. Because those rents are mostly generated from the first periods, as t increases $\mathcal{J}_t(\mu, \lambda)$ converges to $\bar{\mathcal{J}}_f(\mu)$ and the effect of λ^s_t on S_1's post contractual payoff becomes negligible. Hence, her information provision problem converges to the first best, covered in Section A.3.2.

A closed-form representation of $\mathcal{J}_t(\mu, \lambda)$ is hard to obtain, since (2.1.9) may not have a corresponding closed-form solution, however the subsequent corollary identifies a case where this is possible.

Corollary 2.1.3. *Suppose that $\hat{\mu}_t = 0$, then*

$$\mathcal{J}_t(\mu, \lambda) \;=\; (1-\mu)\bar{J}_f(0) - \lambda\,\psi_t\,, \quad \textit{for all} \quad (\mu, \lambda) \in D\,. \tag{2.1.10}$$

Proof. As $\Psi_t > 0 = \Psi_f$, it follows from Proposition 2.1.2 that $\hat{\mu}_t = 0$ implies $\hat{\mu}_f = 0$. Hence substituting in (2.1.7) gives that for all $\mu \in [0,1]$

$$\bar{\mathcal{J}}_f(\mu) \;=\; \bar{J}_f(\mu)(1-\mu) \quad \text{and} \quad \bar{\mathcal{J}}_t(\mu) \;=\; \bar{J}_t(\mu)(1-\mu)\,.$$

Then substituting the above in the objective function of (2.1.9) gives

$$\mathcal{J}_t(\mu,\lambda) \;=\; (1-\mu-\lambda)\,\bar{J}_f(0) + \lambda\,\bar{J}_t(0)$$

Finally, to obtain (2.1.10) substitute that $\bar{J}_t(0) = \bar{J}_f(0) - \psi_t$. □

Therefore, if $\mathcal{J}_t(\mu,\lambda)$ is linear on its boundaries, the same property holds on its interior. Because of this any randomisation between the two boundaries will define a corresponding optimal signal.

2.1.4 Special cases

This section considers S_1's information in two special cases. The first assumes that $\varphi_H = 1$, which implies that once the buyer becomes a high type he remains one. This specification is of interest because it describes the dynamics of information provision for addictive products, those that once the buyer develops a taste for them he becomes a loyal costumer. The second considers the generic case $\varphi_H < 1$ for very large t, that is it looks at the optimal information provision that follows the termination of contracts with very long duration.

Addictive products

To solve S_1's information provision problem under the restriction that $\varphi_H = 1$ note that for an buyer that is terminated in period t there are only two possible events; either $\theta_t = \theta_H$, or $\theta^t = L^t$. As a result it has to be that $\mu + \lambda = 1$. Hence, $J_t(\mu,\lambda) = \bar{J}_t(\mu)$. In addition, simple algebra shows that $\varphi_H = 1$ implies $\bar{J}_t(\mu) = \bar{J}_0(\mu)$ for all $t \geq 0$. This is identical to the objective function of S_1's information provision problem under single period contracts (A.3.14). However, the Bayes plausibility restriction is different since the prior on the buyer's type is μ_t, instead of μ_0, where for generic φ_L the former is given by

$$\mu_t \;=\; \mu_{t-1}(\varphi_H - \varphi_L) + \varphi_L \quad \Rightarrow \quad \mu_t \;=\; \mu_0(\varphi_H - \varphi_L)^t + \varphi_L\,\frac{1-(\varphi_H-\varphi_L)^t}{1-(\varphi_H-\varphi_L)}$$

As a result, under the restriction that $\varphi_H = 1$ the above becomes

$$\mu_t \;=\; 1 - (1-\mu_0)(1-\varphi_L)^t\,,$$

and S_1 equivalently solves

$$\max_{\tilde{g}_t} \ \mathbb{E}\big[\bar{J}_0(\mu)\big] \quad \text{s.t.} \ \ \mathbb{E}[\mu] = \mu_t \, , \quad \mu \in [0,1]. \tag{2.1.11}$$

As argued in the previous section the effect of μ_0 on S_1's information provision problem is not clear, because it affects both the buyer's expected post contractual payoff and the information rents that a high type captures. Nevertheless, an increase on the posterior μ_t unambiguously bends S_1's optimal signal towards informativeness.

Proposition 2.1.4. *An informative signal is strictly optimal if and only if*

$$\max\big\{\rho_L, \rho_H \mu_0\big\} \ < \ \frac{\theta_L}{\theta_H} \quad \textit{and} \quad \mu_t \ > \ \hat{\mu}_0 \, , \tag{2.1.12}$$

in which case S_1's optimal signal $s \in \{\underline{s}, \overline{s}\}$ has distribution

$$g_t^*\left(\underline{s} \,|\, \theta_L\right) \ = \ 1 \, , \qquad \textit{and} \qquad g_t^*\left(\underline{s} \,|\, \theta_H\right) \ = \ \frac{1-\mu_t}{\mu_t} \frac{\hat{\mu}_0}{1-\hat{\mu}_0}. \tag{2.1.13}$$

Also, if an informative signal is strictly optimal for some t, then it remains so for all $t' \geq t$.

Proof. For $\rho_H \mu_0 < \theta_L/\theta_H$ the proof is identical to that of Proposition A.3.2. If $\rho_H \mu_0 \geq \theta_L/\theta_H$, then as argued in Section A.3.3 $\bar{J}_0(\mu)$ is flat for $\mu \geq \mu^*$ and negative below it. But then $\varphi_H = 1$ implies $\mu_t \geq \mu_0 \geq \mu^*$, hence $\mathcal{J}_0(\mu_t) = \bar{J}_0(\mu_t)$ for all $t \geq 0$ and information provision is never strictly optimal. The last statement follows trivially from noting that $\mu_{t+1} \geq \mu_t$. □

Intuitively, an increase in μ_t only affects S_1's post contractual payoff as it appears in the first best. This is because the rents captured by the high type depend on his initial reputation μ_0, but not on μ_t. To understand this note that a period 1 high type reveals all his private information in period 1, hence he captures no more rents in the subsequent periods. In addition, those are all the rents that S_1 pays, since if a period t high type had been a low one before, then she would have charged him for his future expected rents at this point.

Therefore, an increase in μ_t moves the buyer's post contractual payoff towards its flat part, which has the same effect on S_1's post contractual payoff, net of the rents paid to the period 1 high type. Hence, when μ_t becomes big enough S_1 may choose to randomise between revealing the high type and not. Since $\varphi_H = 1$ implies $\mu_{t+1} \geq \mu_t$, the above can also be restated in terms of the duration of the contract. That is the longer this duration is, the more likely S_1 becomes to provide an informative signal to S_2.

Long contracts

Next the restriction $\varphi_H = 1$ is dropped, but $t \to \infty$ is imposed instead. In particular, it will be assumed that $0 < \varphi_L \leq \varphi_H < 1$. In this case S_2's posterior on an buyer whose contract has not been terminated in period t is

$$\lim_{t \Rightarrow \infty} \mu_t \;=\; \mu_\infty \;=\; \frac{\varphi_L}{1 - (\varphi_H - \varphi_L)}\,,$$

In addition, it is ease to show that

$$\lim_{t \to \infty} J_t(\mu, \lambda) \;=\; \bar{J}_f(\mu)\,, \quad \text{and} \quad \lim_{t \to \infty} \Pr(\theta^t = L^t) \;=\; 0,$$

Hence the objective function of S_1's information provision problem becomes identical to that of single period contracts under the first best. However, the buyer's interim reputation is not his initial one μ_0, but its limit μ_∞.

Proposition 2.1.5. *For contracts of long duration, $t \to \infty$, if $\mu_\infty \leq \hat{\mu}_f$, then no information provision is optimal. In the opposite case, S_1's optimal signal $s \in \{\underline{s}, \overline{s}\}$, is informative and has distribution*

$$g_f\left(\underline{s} \,|\, \theta_L\right) \;=\; 1\,, \qquad \textit{and} \qquad g_f\left(\underline{s} \,|\, \theta_H\right) \;=\; \frac{1 - \mu_\infty}{\mu_\infty} \frac{\hat{\mu}_f}{1 - \hat{\mu}_f}. \tag{2.1.14}$$

Proof. Identical to that of Proposition A.3.1. □

As a result, both S_1's information provision problem and its solution are identical with that of the first best of the period t probem. The reason for that is that the amount of rents that S_1 pays for periods long in the future tends to zero. This is because, as argued before, those rents are all captured by a perpetually high type. However, the probability of facing such a type tends to zero as t goes to infinity. As a result, the expected amount of rents paid in period t also tends to zero, and the corresponding information provision problem becomes identical to the first best.

2.2 Continuous types

In this section we expand the model studied in Appendix A.3 to allow for continuous types. Hence we consider again a two period model $t \in \{1, 2\}$, where in period 1 the buyer is offered a contract by S_1 and in period 2 by S_2. Instead of the binary type space of the previous sections, assume that θ_1 is distributed according to the continuously differentiable cumulative distribution function (CDF) $F_1(\theta_1)$ supported on $[\underline{\theta}_1, \overline{\theta}_1]$. Let $f_1(\theta_1) > 0$ denote the corresponding density. The inverse hazard rate of $F_1(\theta_1)$ is denoted by

$$\mu_1(\theta_1) \;=\; \frac{1 - F_1(\theta_1)}{f_1(\theta_1)}\,,$$

and similarly to most of the literature it is assumed that this is non-decreasing in θ_1. For the period 2 buyer type θ_2 suppose that its continuously differentiable CDF is $F_2(\theta_2 \,|\, \theta_1)$, with corresponding density $f_2(\theta_2 \,|\, \theta_1) > 0$ and support $[\underline{\theta}_2, \overline{\theta}_2]$. In addition, to capture some notion of positive correlation across periods assume that if $\tilde{\theta}_1 > \theta_1$, then $F_2(\cdot \,|\, \tilde{\theta}_1)$ first order stochastically dominates (FOSD) $F_2(\cdot \,|\, \theta_1)$. Finally, to simplify the exposition the support S of the signal s is restricted to be finite[3].

Similarly to the binary type specification, S_2's payoff maximisation problem is solved for any posterior $F_2^s(\theta_2)$. Let the corresponding inverse hazard rate be denoted by

$$\mu_2^s(\theta_2) \;=\; \frac{1 - F_2^s(\theta_2)}{f_2^s(\theta_2)}$$

Lemma 2.2.1. *The payoff of a θ_2 buyer type from his contract with S_2 is*

$$V_2^s(\theta_2) \;=\; b \int_{\underline{\theta}_2}^{\theta_2} q_2^s(x)\,\mathrm{d}x\,, \tag{2.2.1}$$

where $q_2^s(\theta_1)$ is the solution of S_2's payoff maximisation problem

$$\max_{q_2} \int_{\underline{\theta}_2}^{\overline{\theta}_2} \Big\{ b\, q_2(\theta_2)\big[\theta_2 - \mu_2^s(\theta_2)\big] - c[q_2(\theta_2)] \Big\}\, \mathrm{d}F_2^s(\theta_2), \qquad \text{subject to} \;\; q_2(\theta_2) \;\; \text{being non-decreasing.} \tag{2.2.2}$$

Proof. In Appendix B.2. □

Next, the focus is turned to S_1's payoff maximisation problem, part of which is the choice of $g(s \,|\, \theta_1)$. For a given signal realisation s, let the expected payoff a θ_1 buyer type from his contract under S_2 be denoted by

$$\bar{V}_2^s(\theta_1) \;=\; \mathbb{E}_{\theta_1}\big[V_2^s(\theta_2) \,|\, \theta_1, s\big]$$

and note that this is not a function of the reported $\hat{\theta}_1$. Nevertheless, this influences the

[3]This rules out perfect revelation of θ_1, however the statements that will be made on the optimality of no information provision would also hold if such a signal was allowed. More generically, the subsequent analysis holds for any CDF $G(s \,|\, \theta_1)$, with support $[\underline{s}, \overline{s}]$, that induces integrable posteriors.

buyer's post contractual payoff $\mathbb{E}_g\left[\bar{V}_2^s(\theta_1)\,|\,\hat{\theta}_1\right]$, since the distribution of s potentially depends on $\hat{\theta}_1$. To connect this with the analysis of the binary type space, I first look at S_1's information provision problem under the first best.

Proposition 2.2.1. *Suppose that the buyer's type is static and that there are some types that are not supplied by S_2 under no information provision, equivalently $\mu_1(\underline{\theta}_1) > \underline{\theta}_1$. Then S_1's first best contract entails some information provision.*

Proof. In Appendix B.2. □

The proof follows from a simple observation, whenever some buyer types are not contracted by S_2 this is inefficient for S_1. This is because they do not capture any rents from S_2's contract. The reason why S_2 would choose to exclude some buyer types from her contract is because the asymmetry of information can potentially be significant enough so that supplying them would be non-profitable. But then S_1 can reduce this asymmetry by reporting whenever the buyer's type belongs in this set of excluded types. This does not affect the rents captured by the higher types, since the excluded types would not be part of their contract to begin with. However, some of the revealed low types will be supplied a positive quantity, since the asymmetry of information will be reduced.

Depending on the value of ϵ similar arguments can be used even if all buyer types were contracted by S_2, however in this case it would be the convexity of her supply schedule that would be make randomisation between revealing information and not profitable for S_1. Note that even though the above argument is given for static types, its underline intuition is still relevant under dynamic ones. However, in this case more structure needs to be imposed on $F_2(\theta_2\,|\,\theta_1)$, as for example is done later in this section when the second best is discussed.

Now, the focus of the analysis is switched back to the second best under S_1. The payoff of a period 1 buyer of type θ_1 when reporting $\hat{\theta}_1$ is

$$\widehat{V}_1(\hat{\theta}_1, \theta_1) \;=\; \theta_1 q_1(\hat{\theta}_1) - p_1(\hat{\theta}_1) + \mathbb{E}_g\left[\bar{V}_2^s(\theta_1)\,|\,\hat{\theta}_1\right].$$

Let the buyer's payoff under truthful reporting be $V_1(\theta_1) = \widehat{V}_1(\theta_1, \theta_1)$. Then S_1's revenue maximisation problem is

$$\begin{aligned} &\max_{p_1, q_1, g} \int_{\underline{\theta}_1}^{\overline{\theta}_1} \Big\{ p_1(\theta_1) - c[q_1(\theta_1)] \Big\} \,\mathrm{d}F_1(\theta_1)\,, \\ &\text{subject to}\;\; V_1(\theta_1) \;=\; \max_{\hat{\theta}_1} \widehat{V}_1(\hat{\theta}_1, \theta_1) \end{aligned} \tag{2.2.3}$$

Let $F_1^s(x) \equiv \Pr(\theta_1 \leq x\,|\,s)$ denote the posterior CDF on θ_1 after a signal realisation $s \in S$, with corresponding inverse hazard rate

$$\mu_1^s(\theta_1) = \frac{1 - F_1^s(\theta_0)}{f_1^s(\theta_1)}$$

Lemma 2.2.2. *S_1's information provision problem is*

$$\max_g \; \mathbb{E}_g \left[\int_{\underline{\theta}_1}^{\bar{\theta}_1} \left\{ \mu_1^s(\theta_1) - \mu_1(\theta_1) \right\} \frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1} \, \mathrm{d}F_1^s(\theta_1) \right] \tag{2.2.4}$$

A sufficient condition for $g(s \,|\, \theta_1)$ to be implementable along with the point-wise optimal choice of production $q_1^*(\theta_1) = \max\left\{0 \,,\, \theta_1 - \mu_1(\theta_1)\right\}^\epsilon$ is that

$$q_1(\hat{\theta}_1) + \mathbb{E}_g \left[\frac{\mathrm{d}\bar{V}_2(\theta_1)}{\mathrm{d}\theta_1} \,\middle|\, \hat{\theta}_1 \right] \tag{2.2.5}$$

is non-decreasing in $\hat{\theta}_1$. This always holds under no information provision.

Proof. In Appendix B.2. □

Next, the optimal deterministic signalling structure is derived. That is the optimal among the ones where S_1 does not randomise. On this subset of distributions $g(s \,|\, \theta_1)$, she is restricted to choosing a partition $\{\Theta_1^s\}_{s \in S}$ of $[\underline{\theta}_1, \overline{\theta}_1]$ and reporting to S_2 the set of this partition on which θ_1 belongs. Hence the realised signal s satisfies $\theta_1 \in \Theta_1^s$. Interestingly, it is relatively ease to show that under such a signalling structure $\mu_1^s(\theta_1) \le \mu_1(\theta_1)$, which combined with (2.2.4) gives the following.

Proposition 2.2.2. *Suppose that S_1 is restricted to using deterministic signals, then no information provision is optimal.*

Proof. In Appendix B.2. □

The value of this proposition is twofold. First, it is ease to imagine scenarios where S_1 will not be able to credible randomise, but she can still reveal some information on the buyer's types. Such cases can be modelled by restricting attention to deterministic signals. In addition, the above result demonstrates that if S_1 cannot credible randomise, then there are no benefits from establishing some other more restrictive device of communication. The second reason why the above result is important is related to the technical complexity of calculating the optimal signal without this restriction.

In the remaining of this section, a case is presented for which S_1's optimal signal can be derived. Hence drop the restriction of deterministic signals, and instead impose the following specification on $F_2(\theta_2 \,|\, \theta_1)$. For the type of period 2 assume that

$$\begin{cases} \theta_2 \;=\; \theta_1 & \text{, with probability} \quad \rho \\ \theta_2 \;\sim\; F_1(\cdot) & \text{, with probability} \;\; 1-\rho \end{cases}$$

This structure allows for both perfect correlation, which is equivalent to static types, and full independence. Intuitively, somebody would expect that information provision is optimal for interior values of ρ, as in the binary type specification, however the opposite is true.

Proposition 2.2.3. *Suppose that the buyer's type is redrawn with probability ρ, then no information provision is optimal for all $\rho \in [0,1]$.*

Proof. In Appendix B.2. □

The above proposition demonstrates that just imperfect correlation is not enough for information provision to be optimal. Its proof underlines a novel result, which builds on previous work from Calzolari and Pavan (2006). This is that if in the absence of information provision S_2 opts for the same quantities that S_1 would if she was integrated with her, then no information provision is optimal. In some sense, disclosure has value only if it moves the policy choices away from those that maximise the total surplus of both S_1 and S_2. As shown in the appendix this is not true under the above specification of $F_2(\theta_2 \,|\, \theta_1)$.

2.3 Moral Hazard, Employment Contracts, and References

In this section we consider an alternative version of our mutli-period model, in which we allow for both moral hazard and endogenous termination. Continue to assume that $t \in \{0, ..., \infty\}$. Also because this setting is more often associated with the labour market we will switch to having two principals P_a and P_b (she) interacting with a single agent (he). All three of them are risk neutral, and discount the future with $\delta \in (0,1)$. At time zero P_a proposes a contract to the agent. If accepted, it lasts up to a termination time τ. The next period, i.e. $\tau + 1$, P_a switches to her outside option and the agent receives a new offer from P_b. For simplicity it is assumed that P_b approaches the agent only if he first entered a contract with P_a. All outside options are normalised to zero. If an agent is in a contract with one of the principals in period t, then the value of his production is

$$y_t^a \;=\; \theta_t \, e_t \quad \text{and} \quad y_t^b \;=\; b \, \theta_t \, e_t$$

for P_a and P_b, respectively. On the other hand, the agent's period payoff is

$$w - c(e_t), \quad \text{where} \quad c(e_t) \; = \; \frac{(e_t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}$$

and $e_t \geq 0$ is the effort level, which the agent chooses privately. The agent's ability $\theta_t \in \{\theta_L, \theta_H\}$, $0 < \theta_L < \theta_H$ is also his private information. The public prior in period 0 is $\mu_0 = \Pr(\theta_0 = \theta_H) \in (0,1)$. During the agent's employment under P_a his type θ_t evolves stochastically according to

$$\begin{aligned} \varphi_H \; &= \; \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau > t) \\ \varphi_L \; &= \; \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau > t) \end{aligned}$$

In contrast, θ_t will be assumed to be perfectly sticky[4], that is $\Pr(\theta_{t+1} = \theta_t \,|\, t > \tau) = 1$. However, the agent's type will be allowed to evolve between the two principals according to

$$\begin{aligned} \rho_H \; &= \; \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_H \,,\, \tau = t) \\ \rho_L \; &= \; \Pr(\theta_{t+1} = \theta_H \,|\, \theta_t = \theta_L \,,\, \tau = t) \end{aligned}$$

Similarly to before we assume that $\varphi_H \geq \varphi_L$ and $\rho_H \geq \rho_L$.

We allow for P_a's partnership with the agent to be terminated in two possible ways. The first possibility is that it is exogenously severed, which is assumed to occur with probability $1-\gamma$ at the end of each period. The second possibility is that it is terminated endogenously, that is the contract offered by P_a specifies its termination after a certain set of histories. Let τ_γ denote the exogenous termination time and τ_a the endogenous one. Then the realised termination time is

$$\tau \; = \; \min\{\tau_\gamma, \tau_a\}$$

Two distinct cases will be considered for τ_a. The first will restrict P_a to offering a contract that sets $\tau_a = \infty$. This type of contract can only be terminated from the exogenous τ_γ, and have two interesting subcases. For $\gamma = 0$ it becomes a fixed term contract, as it is always terminated at the end of period 0. On the other hand, for $\gamma > 0$ it resembles a tenure contract, because it is only halted due to exogenous circumstances. Finally, the second case that will be considered will allow P_a to commit ex-ante on a history depended τ_a, which will result on flexible contracts. Define the probability of endogenous continuation

$$f_t(I_t) \; = \; \Pr(\tau_a > t \,|\, \tau_a > t-1 \,,\, I_t),$$

[4] This restriction is imposed mainly to facilitate the exposition. The main results would not be affected if it was dropped.

where I_t denotes the information set available to P_a at the end of period t. That is $f_t(I_t)$ is the probability that the agent's contract with P_a will continue on $t+1$, endogenously at least, given that it was active on t. Note that if the agent is paired with P_a In period t, that implies that the contract was not terminated at the end of the previous period, which is why the probability is conditioned on $\tau_a > t-1$. As a result the probably of continuation is $\gamma f_t(I_t)$.

The interaction between P_a and the agent will be partly private. On the one hand, P_b will learn the realised termination time τ and the contract offered by P_b. On the other, she will observe neither the realised production $\{y_t^a\}_{t=0}^{\tau}$, nor the agent's reports to P_a's mechanism[5]. However, P_a will be assumed to be able to credible convey additional information to P_b by committing ex ante on a signal $s \in S$ with distribution $g_t(s \,|\, I_\tau)$.

Both principals can fully commit on their contracts, and the agent is not subject to limited liability. To avoid unnecessary complexity it will be assumed that even though the agent cannot commit on not leaving P_a's contract on $t < \tau$, if he chooses to do so the latter can make sure that he will not receive an offer from P_b[6] . Let the history of realised types up to t be denoted by $\theta^t = \{\theta_0, .., \theta_t\}$. It is ease to show that the revelation principle applies in this setting, thus let $\hat{\theta}^t$ denote the history of the agent's reported types. At time zero P_a offers to the agent the following contract

$$\{\, w_t^a(\hat{\theta}^t, y_t),\, e_t^a(\hat{\theta}^t),\, f_t(\hat{\theta}^t),\, g_t(s \,|\, \hat{\theta}^t) \,\}_{t=0}^{\infty},$$

which specifies his compensation, the recommended effort level, the contract's termination time, and the signal's conditional distribution, respectively. After the agent's employment under P_a has being terminated, and only if he accepted her proposal, P_b offers contract

$$\{\, w^b(\hat{\theta}_{\tau+1}, \tau, s, y_t^b),\, e^b(\hat{\theta}_{\tau+1}, \tau, s) \,\}.$$

where $\hat{\theta}_{\tau+1}$ is the agent's report on his valuation of P_b's product. In this case, it is without loss to consider a static contract, because θ_t does not fluctuate for $t \geq \tau + 1$. Moreover, this depends on the termination time τ and the realised signal s because both affect P_b's posterior.

[5]Even if P_a's mechanism was private, it would be without loss to focus on equilibria where she credible reveals it to P_b.

[6]Alternatively, the contract would also have to specify the information provided on such an even, even though it would never happen on path. P_a would find it optimal to always allocate some non-zero probability of revealing a high type, which is the agent's least preferred signal, so that she can punish early departures. This would also increase the agent's outside option, which would decreases P_a payoff by a fixed constant. However, it would not alternate her optimal contract, as long as this was implementable.

2.3.1 Payoff Equivalence and Implementation

In this subsection, first the agent's payoff under P_b is derived. Second, a representation of P_a's payoff is obtained that does not depend on wages. Third, a generic sufficient condition for implementation is provided. Forth, it is shown how the part of P_a's payoff that is related to the agent's subsequent contract with P_b can be reformulated so that the information design problem of P_a can be approach in a way similar to the literature on Bayesian Persuasion.

The agent's post-contractual payoff

P_b is facing a simple static mechanism design problem with binary types. For realised termination time t let $\beta_t^s = \Pr(\theta_{t+1} = \theta_H \,|\, \tau = t\,, s)$ denote her posterior belief on θ_{t+1}, which is connected to her posterior on θ_t according to

$$\beta_t^s \;=\; \rho_H \Pr(\theta_t = \theta_H \,|\, \tau = t\,, s) + \rho_L \Pr(\theta_t = \theta_L \,|\, \tau = t\,, s).$$

P_b payoff maximisation problem is quite standard and its treatment can be found in the appendix. The following lemma characterises the agent's continuation payoff, which is the only result relevant to P_b's problem. To simplify its statement define the following two parameters

$$\kappa = \left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}} \qquad \text{and} \qquad K \;=\; \frac{1-\kappa}{1-\delta}\,\frac{(b\cdot\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}}.$$

Lemma 2.3.1. *The total discounted payoff of a low agent type under P_b is always equal to zero, while that of the high one is equal to*

$$B(\beta_t^s) \;=\; K \cdot \left(\frac{1-\beta_t^s}{1-\beta_t^s\,\kappa}\right)^{1+\epsilon} \tag{2.3.1}$$

which is a strictly decreasing function.

Proof. In Appendix B.3. □

The continuation payoff of a low agent type is always equal to zero, as he captures no rents. In contrast, the high type's payoff is positive, but decreasing and for $\beta_t^s = 1$ it actually becomes zero. Intuitively, the more likely a high type becomes, the less a distortion on the low type's production affects P_b's expected payoff. Hence it becomes cheaper for P_b to use this distortion to incentivise the truthful reporting of θ_H.

Payoff Equivalence

The revelation principle applies for P_a's mechanism design problem. Moreover, using the reported type, she can construct a perfect estimate of the agent's choice of effort. Hence, any misalignment between this estimate and the recommended effort can be punished strongly enough for the agent to have to mask it. As a result, a history of reports $\hat{\theta}^t$ implies choice of effort

$$\hat{e}_t^a(\hat{\theta}^t, \theta_t) \;=\; e_t^a(\hat{\theta}^t) \cdot \frac{\hat{\theta}_t}{\theta_t}$$

Hereafter, y_t will be dropped from the on path wage $w(\hat{\theta}^t, y_t)$, because this will not depend on it. Let the probability that the contract will endogenously continue up to t', conditional on not have being endogenously terminated at $t-1$, and the history $\theta^{t'}$ be denoted by

$$f_t^{t'}(\theta^{t'}) \;=\; \begin{cases} 1 & , \text{for } \; t' < t \\ \Pr(\tau_a > t' \,|\, \tau_a > t-1\,,\, \theta^{t'}) & , \text{for } \; t' \geq t \end{cases}$$

A special case of this is $t' = t$, where it becomes the probability of endogenous continuation $f_t(\theta^t)$. Using the above notation P_a's payoff maximisation problem becomes

$$\max_{w,e,f,g} \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1}) \gamma^t \delta^t \Big(\theta_t \, e_t^a(\theta^t) - w_t^a(\theta^t) \Big) \right] \qquad (\mathcal{P})$$
$$\text{subject to IR}(\theta^t) \text{ and IC}(\theta^t),$$

where $\mathrm{IR}(\theta^t)$ and $\mathrm{IC}(\theta^t)$ refer to the individual rationality and incentive compatibility constrains, respectively, of a θ^t agent type. Note that continuing up to period t only depends on history θ^{t+1}, as the decision to not terminate the contract is taken at the end of each period, with the last relevant devision being in period t.

To make notation more compact three special cases of θ^t will be defined. First, let $\theta_L^t = \{\theta^{t-1}, \theta_L\}$ and $\theta_H^t = \{\theta^{t-1}, \theta_H\}$ denote a history such that the buyer's type in period t is low and high, respectively. In addition, for given generic θ^{t-1} and $t' \geq t$ let

$$L_t^{t'} \;=\; \{\theta^{t-1}, \theta_L, .., \theta_L\}\,, \qquad (2.3.2)$$

denote a history such that the buyer's type has been low for all periods after, and including, period t. Also, whenever $t = 0$ simply write $L^{t'}$.

The proof of the subsequent proposition follows closely Battaglini (2005) and it is almost identical to that of Proposition 2.1.1, demonstrates that the information rents captured by a period t high type θ_H^t are closely related to the histories $\{L_t^{t'}\}_{t'>t}$. In particular, when the implementation constrains, which will be provided shortly, do not bind the information rents captured by a period t high type are given by

$$U_t^H(\theta^{t-1}) = \sum_{t'=t}^{\infty} f_t^{t'-1}(L_t^{t'-1})[\gamma\delta(\varphi_H - \varphi_L)]^{t'-t}\Bigg\{\frac{e_t^a(L_t^{t'})^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}(1-\kappa) + [1 - f_{t'}(L_t^{t'})\gamma](\rho_H - \rho_L)\delta\mathbb{E}_g\big[B(\beta_{t'}^s) \,|\, L_t^{t'}\big]\Big)\Bigg\}$$

Then ($\mathcal{P}$) simplifies to the following problem, which only depends on policies (e_t^a, f_t, g_t) and not on the price p_t^a, paid by the agent to P_a.

Proposition 2.3.1. *Suppose that a solution of*

$$\max_{e,f,g} \Bigg\{ \mathbb{E}_\theta\Bigg[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t\Big(\theta_t e_t^a(\theta^t) - \frac{e_t^a(\theta^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} + [1 - f_t(\theta^t)\gamma]\delta\Pr(\theta_{t+1} = \theta_H |\, \tau = t, \theta_t)\mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta^t\big]\Big)\Bigg] - \mu_0\sum_{t=0}^{\infty} f_0^{t-1}(L^{t-1})\gamma^t\delta^t(\varphi_H - \varphi_L)^t\Bigg[\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}(1-\kappa) + \delta[1 - f_t(L^t)](\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_t^s) \,|\, L^t\big]\Big)\Bigg]\Bigg\} \tag{$\mathcal{P}'$}$$

satisfies

$$\frac{e_t^a(\theta_H^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{1}{\kappa} - 1\right) - \frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}(1-\kappa) + (\varphi_H - \varphi_L)\gamma\delta\Big[f_t(\theta_H^t)U_{t+1}^H(\theta_H^t) - f_t(\theta_L^t)U_{t+1}^H(\theta_L^t)\Big] \geq (\rho_H - \rho_L)\delta\Big[\big[1 - f_t(\theta_L^t)\gamma\big]\mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_L^t\big] - \big[1 - f_t(\theta_H^t)\gamma\big]\mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_H^t\big]\Big] \tag{$\mathcal{P}_c$}$$

Then those policies are also a solution to ($\mathcal{P}$) *and there exists a contract that implements them.*

Proof. In Appendix B.3. □

The above representation is obtain by ignoring the downward slopping IC constrains, i.e. IC(θ^{t-1}, θ_L), of ($\mathcal{P}$) and showing that in this case the upward slopping ones, that is IC(θ^{t-1}, θ_H), bind. This allow the derivation of an expression for the period 0 high type's wages that only depends on the policies (e, τ, g). The same can be done for the period 0 low type by using its individual rationality constrains. Substituting those in P_a's payoff gives ($\mathcal{P}'$). As a result, whenever its solution satisfies the ignored constrains this is also a solution to ($\mathcal{P}$). In order to check that the IC(θ^{t-1}, θ_L) constrains are indeed satisfied the derived expression for the wages is substituted, which allows me to obtain ($\mathcal{P}_c$).

This approach is similar to that used in Battaglini (2005) and most of the literature of Dynamic Mechanism Design with continuous types. If the agent's post-contractual payoff was a constant or zero, then the solution of ($\mathcal{P}'$) would always be implementable. However, this will no generically be true here because the high type has an additional incentive to pretend to be a low type, as the signal of the latter will generically result in lower posteriors, which is beneficial to her.

Nevertheless, we will show that when the production technology of P_a is sufficiently more efficient from that of P_b then the implementation constrains will be satisfied. However, our sufficient condition for implementation will be relevant only for the following familly of endogenous termination policies.

Definition 2. Call a termination policy *non-decreasing* if for every $t \geq 0$ and realised paths θ^t and $\tilde{\theta}^t$:

$$\theta_{t'} \geq \tilde{\theta}_{t'} \text{ for all } t' \leq t \quad \Rightarrow \quad f_t(\theta^t) \geq f_t(\tilde{\theta}^t)\,.$$

That is a termination policy is non-decreasing if whenever a history θ^t is weakly better than an alternative one $\tilde{\theta}^t$, on each $t' \leq t$, the probability of continuation of the former is higher than that of the latter on all periods up to t. This simple requires that an agent with 'better' history of types is allocated by the contract a higher probability of continuation.

Corollary 2.3.1 (Point-wise optimal effort)**.** *The point-wise optimal level of effort is*

$$e_t^*(\theta^t) = \begin{cases} (\theta_t)^\epsilon & , \text{ if } \theta^t \neq L^t \\ (\theta_L)^\epsilon/\xi_t & , \text{ if } \theta^t = L^t \end{cases} , \quad \text{where} \quad \xi_t = 1 + \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t \left(1 - \frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \tag{2.3.3}$$

In addition, if either (i) termination is only exogenous, or (ii) the endogenous termination policy is non-decreasing, then a sufficient condition for the point-wise optimal level of effort to be implementable under any information provision policy is that

$$\frac{\theta_H^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}} - 1\right) - \frac{\theta_L^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(1 - \frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \geq \frac{\delta}{1-\delta}(\rho_H - \rho_L)(1-\kappa)\,\frac{(b\,\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}} \tag{2.3.4}$$

When $\epsilon = 1$ the above condition simplifies to

$$\frac{1-\kappa}{\kappa} \geq \frac{\delta}{1-\delta} \cdot (\rho_H - \rho_L) \cdot b^2 \theta_L^2 \tag{2.3.5}$$

which is always satisfied for b small enough.

Proof. In Appendix B.3. □

Most of the main results of this specification will be presented for the $\epsilon = 1$ case, which allows us to derive then in closed-form. As a result, the implementation condition that will be used for the rest of the analysis will be (2.3.5).

Information Provision

This subsection provides some results on the information provision problem of P_a. It will be shown that for the general case it is difficult to write its solution in some ease and meaningful way, however a characterisation of the optimal signal will be provided. More descriptive solution will be derived in the next subsections, where some interesting sub-cases of the model are considered.

P_a's optimal signal will be characterised for any given distribution of the termination time τ, $f = \{f_t\}_{t=0}^{\infty}$, and on each of its possible realisation $t \in \{0, .., \infty\}$. Hence considering only the part of ($\mathcal{P}'$) that is affected by the signal s on realised $\tau = t$ gives

$$\max_{g_t} \left\{ \sum_{\theta^t} \left[\Pr(\theta^t) \Pr(\tau = t \,|\, \theta^t) \sum_s \Pr(\theta_{t+1} = \theta_H | \, \tau = t, \theta_t) B(\beta_t^s) g_t(s \,|\, \theta^t) \right] \right.$$
$$\left. - \, \delta^{t+1} \mu_0 \Pr(\tau = t \,|\, L^t)(\varphi_H - \varphi_L)^t (\rho_H - \rho_L) \sum_s B(\beta_t^s) g_t(s \,|\, L^t) \right\} \qquad (\mathcal{G})$$

The first line of ($\mathcal{G}$) represents the expected information rents from the agent's contract with P_b. Those are captured by P_a through the agent's individual rationality constrains. On the other hand, the second line corresponds to the rents captured by the agent in P_a's contract. Those are due to the fact that whenever the signal s is informative high types have an extra incentive to pretend to be low types, as their continuation payoff $B(\beta)$ is decreasing in their reputation.

Next, ($\mathcal{G}$) is transformed into an equivalent problem that will only take as inputs two posterior beliefs on θ^t, for an agent whose contract was terminated on t. Introduce the following notation

$$\begin{aligned} \eta_t &= \Pr(\theta_t = \theta_H \,|\, \tau = t), \quad & \eta_t^s &= \Pr(\theta_t = \theta_H \,|\, s, \tau = t) \\ \lambda_t &= \Pr(\theta^t = L^t \,|\, \tau = t), \quad & \lambda_t^s &= \Pr(\theta^t = L^t \,|\, s, \tau = t) \end{aligned}$$

The interim posterior beliefs on the first column only use the information provided by the termination time τ, and those will be the inputs of the equivalent transformation of ($\mathcal{G}$). In contrast, the posteriors on the second column also depend on the signal s; η_t^s is the posterior on $\theta_t = \theta_H$, while λ_t^s on $\theta^t = L^t$. Note that the first event only depends on the contemporaneous θ_t, while the second on the whole history θ^t. Moreover, P_b's posterior on $\theta_{\tau+1}$ and θ_τ are connected according to

$$\beta_t^s \;=\; \rho_H \eta_t^s + \rho_L(1-\eta_t^s).$$

In the subsequent analysis the underline choice variable will always be the distribution of η_t^s, because this is the one influenced by the signal s, however to facilitate the exposition in many case the results will be presented in terms of β_t^s. Finally, abusing notation let $g_t(s)$ denote the probability of sending signal s after a contract termination at time $\tau = t$, that is

$$g_t(s) \;=\; \sum_{\theta_t} \Pr(\theta^t \,|\, \tau = t) g_t(s \,|\, \theta^t)$$

The rest of the analysis will be based on the following transformation.

Lemma 2.3.2. *P_a's information provision problem in period t equivalently becomes*

$$\max_{g_t} \;\; \mathbb{E}_{g_t}[J_t(\eta_t^s, \lambda_t^s)] \tag{$\mathcal{G}$}$$

where its point-wise value $J_t(\eta_t^s, \lambda_t^s)$ is

$$J_t(\eta_t^s, \lambda_t^s) \;=\; B(\beta_t^s)\big(\beta_t^s - \psi_t \lambda_t^s\big), \quad \textit{and} \quad \psi_t \;=\; \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t (\rho_H - \rho_L)\,. \tag{2.3.6}$$

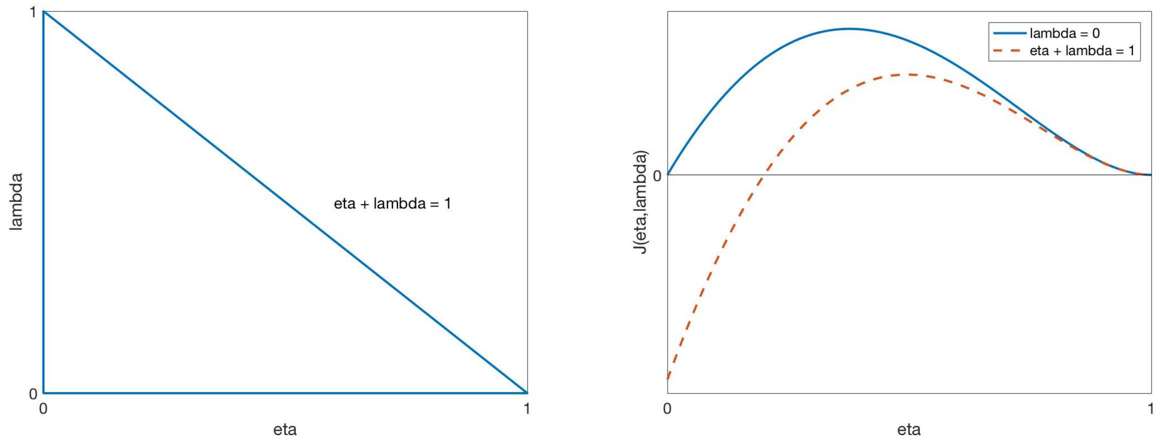
Proof. Identical to that of Lemma 2.1.2 □

Similarly to the previous sections, the information provision problem of P_a is reformulated by considering (η_t^s, λ_t^s) as the underline random variables, instead of the signal s, with joint distribution $\tilde{g}_t$. Hence, P_a equivalently solves

$$\begin{aligned} \max_{\tilde{g}_t} \;\; \mathbb{E}_{\tilde{g}_t}\big[J_t(\eta,\lambda)\big] \quad \text{s.t. } & \mathbb{E}_{\tilde{g}_t}[\eta] = \eta_t\,, \quad \mathbb{E}_{\tilde{g}_t}[\lambda] = \lambda_t\,, \\ & \eta, \lambda \in [0,1]\,, \text{ and } \eta + \lambda \le 1. \end{aligned} \tag{$\mathcal{G}_t'$}$$

The constrains ensure that the joint distribution $\tilde{g}_t(\eta, \lambda)$ is Bayes plausible.

The domain of $J_t(\eta, \lambda)$ is a right-angled triangle, of which each of its legs has length one. A representative graph of it on the sides where $(\eta + \lambda = 1)$ and $(\lambda = 0)$ is given in plot (2.2b). Those two sides are connected by straight lines, as $J_t(\eta, \lambda)$ is linear in λ. Note that in plot (2.2b) the functional form of J_t on both sides is initially increasing and concave, and subsequently changes to decreasing and convex. The next lemma will show that this is a



(a) Domain (b) Corner values

Figure 2.2: The domain and the corner values of $J_t(\eta, \lambda)$.

generic representation of those two sides, and will provide a characterisation of $J_t(\eta, \lambda)$ on the rest of its domain. Define functions

$$\begin{aligned} \beta^*(\psi) &= \frac{1}{2\kappa}\left[2 + \epsilon(1-\kappa) - \sqrt{1-\kappa}\sqrt{4(1+\epsilon) + \epsilon^2(1-\kappa) - \psi 4\kappa(1+\epsilon)}\right] \\ \beta^{**}(\psi) &= 1 - \frac{\epsilon(1-\kappa)(1-\psi)}{2(1-\kappa\psi) + (1-\kappa)\epsilon} \end{aligned}$$

for generic input ψ. It will be convenient to define $J_t(\eta, \lambda)$ as a function of of η only on the two aforementioned boundaries. Hence, for $i \in \{f, t\}$ let

$$\bar{J}_i(\eta) = \zeta_i \, B(\beta)(\beta - \Psi_i)\,, \quad \text{where} \quad \begin{cases} \zeta_f = 1 & \Psi_f = 0 \\ \zeta_t = 1 + \frac{\psi_t}{\rho_H - \rho_L} & \Psi_t = \frac{\psi_t \rho_H}{\rho_H - \rho_L + \psi_t} \end{cases}$$

and note that $\beta = \eta\rho_H + (1-\eta)\rho_L$ as those two posteriors are always connecting through this linear relationship. Lemma 2.3.3 shows that $\bar{J}_f(\eta)$ represents the $(\lambda = 0)$ boundary, whereas $\bar{J}_t(\eta)$ the $(\eta + \lambda = 1)$.

Lemma 2.3.3. *$J_t(\eta, \lambda)$ is neither concave, nor convex on any of the interior points of its domain.*

- *On the boundary $(\eta = 0)$ it is linear and decreasing on λ*
- *On the boundary $(\lambda = 0)$ it satisfies $\bar{J}_f(\eta) = J_t(\eta, 0)$*
- *On the boundary $(\eta + \lambda = 1)$ it satisfies $\bar{J}_t(\eta) = J_t(\eta, 1 - \eta)$*

Moreover, $J_i(\eta)$ is increasing (concave) for

$$\eta \;\leq\; \frac{\beta^*(\Psi_i) - \rho_L}{\rho_H - \rho_L} \quad \left(\eta \;\leq\; \frac{\beta^{**}(\Psi_i) - \rho_L}{\rho_H - \rho_L}\right), \tag{2.3.7}$$

and decreasing (convex) otherwise. Finally, $0 < \beta^*(\Psi_i) \leq \beta^{**}(\Psi_i) < 1$.

Proof. In Appendix B.3. □

To solve P$_a$'s information provision problem, first the concave closure of $J_t(\eta, \lambda)$ needs to be characterised. This will be denoted by $\mathcal{J}_t(\eta, \lambda)$ and defined as

$$\mathcal{J}_t(\eta, \lambda) \;=\; \sup\{z \,|\, (\eta, \lambda, z) \in co(J_t)\},$$

where $co(J_t)$ denotes the convex hull of the graph of J_t. In addition, for $i \in \{f, t\}$ let $\bar{\mathcal{J}}_i(\eta)$ denote the value of $\mathcal{J}_t(\eta, \lambda)$ on the boundaries ($\lambda = 0$) and ($\eta + \lambda = 1$), respectively, which is also the concave closure of the corresponding $\bar{J}_i(\eta)$. The following proposition characterises $\mathcal{J}_t(\eta, \lambda)$. To facilitate its exposition, a new point needs to be introduced. For $i \in \{f, t\}$ and $\Psi_i < \frac{\theta_L}{\theta_H} < \rho_H$, let $\hat{\eta}_i$ be the unique solution of

$$\bar{J}_i(\hat{\eta}_i) + \bar{J}_i'(\hat{\eta}_i)(1 - \hat{\eta}_i) \;=\; 0\,, \tag{2.3.8}$$

if this exists, and zero otherwise. The functional form of $\hat{\eta}_i$ can be found in the proof of the following proposition in, but it is not copied here, due to its size.

Proposition 2.3.2. *For any interior point $\mathcal{J}_t(\eta, \lambda) > J_t(\eta, \lambda)$. On the boundary ($\eta = 0$): $\mathcal{J}_t = J_t$. On the boundaries ($\eta + \lambda = 1$) and ($\lambda = 0$):*

- *If $\rho_H \leq b^{**}(\Psi_i)$, then $\mathcal{J}_i = J_i$*
- *Otherwise, there exists $\hat{\eta}_i \in (0, 1)$ such that*

$$\bar{\mathcal{J}}_i(\eta) \;=\; \begin{cases} J_i(\eta) & ,\ \textit{for}\ \ \eta \leq \hat{\eta}_i \\ J_i(\hat{\eta}_i) + J_i'(\hat{\eta}_i)(\eta - \hat{\eta}_i) & ,\ \textit{for}\ \ \eta \geq \hat{\eta}_i \end{cases}. \tag{2.3.9}$$

where $\hat{\eta}_i = (\hat{\beta}_i - \rho_L)/(\rho_H - \rho_L)$ and $\hat{\beta}_i \in \big(\beta^*(\Psi_i), \beta^{**}(\Psi_i)\big)$. In addition, for $\epsilon = 1$

$$\hat{\beta}_i \;=\; 1 - \frac{(1-\kappa)^2(\rho_H - \Psi_i)}{2 - (3\rho_H + \Psi_i)\kappa + (\rho_H - \Psi_i + 2\rho_H\Psi_i)\kappa^2}. \tag{2.3.10}$$

Proof. The statement for the interior points follows from noting that J_t is never concave on any of them. The one for the boundary $(\eta = 0)$ follows because the function is linear with respect to λ.

Finally, for the boundaries $(\lambda = 0)$ and $(\eta + \lambda = 1)$ note that if $\rho_H \leq \beta^{**}(\Psi_i)$, then $J_i(\eta)$ does not have a convex part. Otherwise, it is convex on $\eta > \frac{\beta^{**}(\Psi_i)-\rho_L}{\rho_H-\rho_L}$. Hence, there is a unique line that connects $\eta = 1$ with some $\hat{\eta}_i$ between the maximum of J_i and the point on which its concavity changes. This is defined as the solution of

$$J_i(1) \;=\; J_i(\hat{\eta}_i) + J_i'(\hat{\eta}_i)(1-\hat{\eta}_i)\,. \tag{2.3.11}$$

For $\epsilon = 1$, $\hat{\eta}_i$ can be given in closed form. To solve this let $\widetilde{J}_i(\beta) \;=\; J_i[(\beta - \rho_L)/(\rho_H - \rho_L)]$, that is define a function that has β as the underline variable instead of η. Then the graph of $\tilde{J}_i(\beta)$ on $[\rho_L, \rho_H]$ is equal to that of $J_i(\eta)$ on $[0, 1]$. Hence, solve

$$\widetilde{J}_i(\rho_H) \;=\; \widetilde{J}_i(\hat{\beta}_i) + \widetilde{J}_i'(\hat{\beta}_i)(\rho_H - \hat{\beta}_i)\,, \tag{2.3.12}$$

which can be solved to obtain (2.3.10). □

Therefore, we are ready to state our first result.

Corollary 2.3.2. *Suppose that (η_t, λ_t) are interior points, then there is always information provision. On the boundary $(\eta = 0)$ no information provision is optimal. On the boundaries $(\lambda = 0)$ and $(\eta + \lambda = 1)$*

- *If $\rho_H \leq b^{**}(\Psi_i)$, no information provision is optimal*
- *Otherwise if $\eta_t > \hat{\eta}_i$, then any optimal signal must randomise between $\hat{\eta}_i$ and 1.*

2.3.2 Exogenous Termination

This section considers two distinct specifications. The first one solves the information provision problem of P_a under the assumption that $\gamma = 0$ and finds a simple sufficient condition for implementation. Imposing $\gamma = 0$ restricts the probability of the contract to continue after period 0 to zero. This is equivalent to considering a situation where P_a is exogenously restricted to offer contracts of fix time length, for example when the task to be completed by the agent is not recurring.

The second specification that will be considered will assume that $\gamma > 0$, but will restrict P_a to offering tenure contracts. That is in this case the contract will only be terminated because of the exogenous τ_γ. For the sake of the exposition assume that $\epsilon = 1$, for both of the above specifications.

Fix term contracts

Here we solve ($\mathcal{G}_t'$) for the case where $\gamma = 0$. In particular, we only need to solve it for $t = 0$, since the contract will always be terminated at the end of period 0. Thus, it has to be that $\eta + \lambda = 1$, because $\theta_0 = \theta_H$ and $\theta_0 = \theta_L$ are the only two possible histories in period 0. Hence

$$J_0(\eta, \lambda) \;=\; J_0(\eta, 1-\eta) \;=\; \bar{J}_0(\eta)$$

Hence hereafter we will simple write $J_0(\eta)$. In addition, some algebra gives that $\psi_0 = \mu_0 \rho_H$. Hence, substituting in the expression of $\bar{J}_0(\eta)$ gives that

$$J_0(\eta) \;=\; B(\beta) \cdot \frac{\beta - \mu_0 \rho_H}{1 - \mu_0}$$

As a result, ($\mathcal{G}_t'$) reduces to

$$\max_{\tilde{g}_0} \; \mathbb{E}_{\tilde{g}_0}\big[J_0(\eta)\big] \quad \text{s.t.} \;\; \mathbb{E}_{\tilde{g}_0}[\eta] = \mu_0\,, \quad \eta \in [0,1]. \tag{2.3.13}$$

In particular, the optimal signal needs to satisfy $\mathbb{E}_{\tilde{g}_0}\big[J_0(\eta)\big] = \mathcal{J}_0(\mu_0)$, where $\mathcal{J}_0$ is the concave closure of J_0. Hence the concave closure of $J_0(\eta)$ needs to be characterised, in order to solve (2.3.13). However, this has already be done in Proposition 2.3.2. Let $\beta_0^{**} = \beta^{**}(\mu_0 \rho_H)$ and remember that we have shown that if $\rho_H \leq \beta_0^{**}$, then $\mathcal{J}_0 = J_0$. Otherwise, there exists $\hat{\eta}_0 = (\hat{\beta}_0 - \rho_L)/(\rho_H - \rho_L)$ such that

$$\mathcal{J}_0(\eta) \;=\; \begin{cases} J_0(\eta) & , \text{ for } \;\; \eta \leq \hat{\eta}_0 \\ J_0(\hat{\eta}_i) + J_0'(\hat{\eta}_0)(\eta - \hat{\eta}_0) & , \text{ for } \;\; \eta \geq \hat{\eta}_0 \end{cases}, \tag{2.3.14}$$

$$\hat{\beta}_0 \;=\; 1 - \frac{(1-\kappa)^2(1-\mu_0)\rho_H}{2 - (3+\mu_0)\rho_H \kappa + (1 - \mu_0 + 2\mu_0 \rho_H)\rho_H \kappa^2}. \tag{2.3.15}$$

P_b's posterior on θ_0 is not affected by the termination time, because this is not correlated with the agent's type. Hence, the optimal signal structure has to satisfy $\mathbb{E}_{\tilde{g}_0}[J_0(\eta)] = \mathcal{J}_0(\mu_0)$, which implies the following necessary and sufficient condition for information provision to be optimal in (2.3.13)

$$\rho_H \;\geq\; \beta_0^{**} \quad \text{and} \quad \mu_0(\rho_H - \rho_L) + \rho_L \;\geq\; \hat{\beta}_0 \tag{2.3.16}$$

The first inequality ensures that the convex part of $J_0(\eta)$ exists, and the second that μ_0 is big enough for the prior on θ_1 to be on this convex part.

Proposition 2.3.3 (Fixed Term Contracts)**.** *An informative signal strictly solves* (2.3.13)

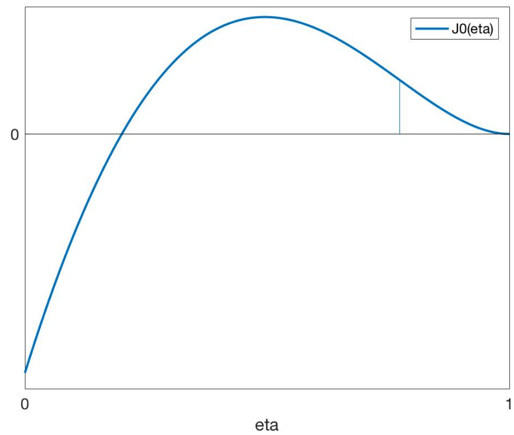


Figure 2.3: $J_0(\eta)$

if and only if

$$\begin{aligned} \rho_H &> 1 - \frac{(1-\kappa)(1-\mu_0\rho_H)}{2(1-\kappa\mu_0\rho_H)+(1-\kappa)} \qquad \textit{and} \\ \mu_0(\rho_H-\rho_L)+\rho_L &\geq 1 - \frac{(1-\kappa)^2(1-\mu_0)\rho_H}{2-(3+\mu_0)\rho_H\kappa+(1-\mu_0+2\mu_0\rho_H)\rho_H\kappa^2} \end{aligned} \tag{2.3.17}$$

The higher ρ_H is the less binding the first line of (2.3.17) becomes. Also, suppose that the first line is satisfied, then the higher either ρ_H, or ρ_L is, the less the second line of (2.3.17) binds. When this is satisfied the optimal signal $s \in \{s_L, s_H\}$ has distribution

$$g_0(s_L \,|\, \theta_L) \;=\; 1 \quad \textit{and} \quad g_0(s_L \,|\, \theta_H) \;=\; \frac{1-\mu_0}{\mu_0}\frac{\hat{\eta}_0}{1-\hat{\eta}_0}. \tag{2.3.18}$$

Proof of Proposition 2.3.3. Condition (2.3.17) follows by substituting the functional forms of β_0^{**} and $\hat{\beta}_0$ in (2.3.16). Next, the statements on its dependence on ρ_L and ρ_H is proven. For the first line re-write inequality as

$$\rho_H[2(1-\kappa\mu_0\rho_H)+(1-\kappa)] \;\geq\; 2(1-\kappa\mu_0\rho_H)+(1-\kappa)-(1-\kappa)(1-\mu_0\rho_H).$$

Differentiate both sides with respect to ρ_H, and subtract the derivative of the right hand side from that of the left one to obtain

$$\begin{aligned} 2(1-\kappa\mu_0\rho_H)+(1-\kappa)-2\kappa\mu_0\rho_H+2\kappa\mu_0-\mu_0(1-\kappa) \;&=\; \\ 2(1-\kappa\mu_0\rho_H)+(1-\mu_0)(1-\kappa)+2\kappa\mu_0(1-\rho_H) \;&\geq\; 0. \end{aligned}$$

As a result the more ρ_H increases the less binding this inequality becomes. To prove the same for the second line calculate

$$\frac{\partial \hat{\beta}_0}{\partial \rho_H} = \frac{-(1-\kappa)^2(1-\mu_0)2(1-\mu_0\rho_H^2\kappa^2)}{2-(3+\mu_0)\rho_H\kappa+(1-\mu_0+2\mu_0\rho_H)\rho_H\kappa^2}$$

which is negative for $\rho_H > \beta_0^{**}$. This is because in this case $\hat{\beta}_0 < \beta_0^{**} < 1$, which implies that the denominator above have to be positive. Hence if $\rho_H > \hat{\beta}_0^{**}$, as ρ_H increases the left hand side increasing and the right hand side decreases.

Finally, the optimal signal is obtained by the following argumentation. Suppose that $\rho_H \leq \beta_0^{**}$, then $J_0(\eta)$ is concave for all $\eta \in [0,1]$, hence an uninformative signal is optimal. Instead suppose that $\rho_H > \beta_0^{**}$, then the convex hull of J in linear on $[\hat{\eta}_0, 1]$ and strictly concave everywhere else. Hence if $\mu_0 \leq \hat{\eta}_0$, then no information provision is still optimal. If $\mu_0 > \hat{\eta}_0$ then the optimal signal randomises between posteriors $\hat{\eta}_0$ and 1. Let s_H be the signal that fully reveals the high type. Then the probability of sending s_L is obtained by solving

$$\hat{\eta}_0 = \frac{\mu_0 g_0(s_L \mid \theta_H)}{\mu_0 g_0(s_L|\theta_H) + 1 - \mu_0}. \tag{2.3.19}$$

□

To understand this result note that $B(\beta)$, which represents the agent's information rents from his contract with P_b, is convex[7]. This provides an incentive towards information provision for P_a, because she captures the agent's continuation value through his individual rationality constrain. However, she also has to pay information rents to the high type that are proportional to his continuation value, which create an incentive towards the opposite direction. For high μ_0 the information rents that P_a pays are low enough for the first incentive to dominate, while the opposite is true for low μ_0.

The optimal signal, under information provision, has a realisation s_H that reveals the high type, and one s_L that is always sent for the low type, but also some times for the high type. However, the proposed optimal signal may not always be implementable, as under the point-wise optimal level of effort the implementation constrain (B.3.4) needs to hold. However, this is implied by (2.3.5), hence our solution is relevant for at least a subset of parameters.

A case of special interest is the following one.

Example 2.3.1 (Privacy)**.** Suppose that $\rho_L = 0$ and $\rho_H = 1$. Then no information provision is optimal and implementable.

[7]Even though depending on the ϵ this may not always be true for all its domain, it is still a generic result that $B(b)$ is convex for high b, which suffices to obtain the kind of results provided above.

Proof. For $(\rho_L, \rho_H) = (0, 1)$ the second line of (2.3.17), which is a necessary condition for information provision, is satisfied, while the first becomes

$$\mu_0 \ \geq \ 1 - \frac{(1-\kappa)^2(1-\mu_0)}{2-(3+\mu_0)\kappa+(1-\mu_0+2\mu_0)\kappa^2} \ \Leftrightarrow \ \frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa^2} \ \geq \ 1$$

If the denominator is negative, then this cannot hold. If it is positive, then

$$\frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa^2} \ < \ \frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa} \ = \ \frac{(1-\kappa)^2}{2(1-\kappa)} \ = \ \frac{1-\kappa}{2} \ < 1,$$

hence again the above inequality cannot hold. □

Proposition 2.3.3 gives that the necessary and sufficient condition for information provision (2.3.17) binds less as ρ_H increases. Hence, if privacy is optimal for $(\rho_L, \rho_H) = (0, 1)$, then the same is true for any $\rho_H < 1$ and $\rho_L = 0$. Hence, in order to break the no information provision result, as identified in Calzolari and Pavan (2006)[8], it has to be that an initially low type has at least some positive probability of being a high type under P_b.

Example 2.3.2 (Information Provision)**.** Assume that $\rho_L > 1/2$ and $\kappa \Rightarrow 0$, then information provision is optimal and implementable.

Proof. As in the proof of the previous example $\rho_H = 1$ gives that the second line of (2.3.17) is satisfied, while the first becomes

$$\rho_L \ \geq \ 1 - \frac{(1-\kappa)^2}{2-(3+\mu_0)\kappa+(1+\mu_0)\kappa^2},$$

the right hand side of which goes to $1/2$ as $\kappa \Rightarrow 0$. Moreover, it has already being argued, using (2.3.5), that for $\kappa \Rightarrow 0$ information provision is implementable. □

Tenure Contracts

Skill Accomolation

The first part of this subsection assumes that $\varphi_H = 1$, that is it assumes that the agent can only become better as time passes. Define recursively the posterior belief of a type that is employed In period t as

$$\mu_t \ = \ \mu_{t-1}(\varphi_H - \varphi_L) + \varphi_L \quad \Rightarrow \quad \mu_t \ = \ \mu_0(\varphi_H - \varphi_L)^t + \varphi_L \frac{1-(\varphi_H - \varphi_L)^t}{1-(\varphi_H - \varphi_L)}$$

[8] The reason why the result does not hold in this setting is because the agent's type is not perfectly correlated across employments. The cited paper identifies a few other reasons.

and note that $\mu_{t+1} > \mu_t$. As the termination time is not correlated with the agent's type, the posterior reputation of an agent that is terminated on $t+1$ is $\eta_t = \mu_t$. To solve P_a information provision problem note that for an agent that is terminated on $t+1$ there are only two possible events, either $\theta_t = \theta_H$, or $\theta^t = L^t$. As a result it has to be that $\eta + \lambda = 1$. Hence, similarly to before

$$J_0(\eta, \lambda) \;=\; J_0(\eta, 1-\eta) \;=\; \bar{J}_0(\eta)$$

In addition, algebra identical to that of the previous section shows that $\varphi_H = 1$ implies $\Psi_t = \mu_0 \rho_H$. Hence, substituting in the expression of $\bar{J}_t(\eta)$ gives that

$$J_t(\eta) \;=\; J_0(\eta) \;=\; B(b) \cdot \frac{b - \mu_0 \rho_H}{1 - \mu_0}$$

Hence, P_a information provision problem becomes

$$\max_{\tilde{g}_t} \; \mathbb{E}_{\tilde{g}_t}\big[J_0(\eta)\big] \quad \text{s.t.} \;\; \mathbb{E}[\eta] = \mu_t \,, \quad \eta \in [0,1]. \tag{2.3.20}$$

In particular, the optimal signal needs to satisfy $\mathbb{E}_{\tilde{g}_t}\big[J_0(\eta)\big] = \mathcal{J}_0(\mu_t)$. This is identical to P_a's problem under fixed term contracts, however now instead of μ_0, the restriction on the distribution of posteriors is that $\mathbb{E}_{\tilde{g}_t}[\eta] = \mu_t$. Hence, even if μ_0 is such that no information provision is optimal In period 0, it is still possible for this result to be reversed as time progresses.

Proposition 2.3.4 (Skill Accomulation)**.** *An informative signal strictly solves (2.3.20) if and only if*

$$\begin{aligned} \mu_t(\rho_H - \rho_L) + \rho_L \;&\geq\; 1 - \frac{(1-\kappa)^2(1-\mu_0)\rho_H}{2 - (3+\mu_0)\rho_H\kappa + (1 - \mu_0 + 2\mu_0\rho_H)\rho_H\kappa^2} \\ \textit{and} \qquad \rho_H \;&>\; 1 - \frac{\epsilon(1-\kappa)(1-\psi)}{2(1-\kappa\psi) + (1-\kappa)\epsilon} \end{aligned} \tag{2.3.21}$$

The higher ρ_H is the less binding the second line of (2.3.21) becomes. Also, suppose that the second line is satisfied, then the higher either ρ_H, ρ_L, or μ_t is, the less the first line of (2.3.21) binds. When this is satisfied the optimal signal $s \in \{s_L, s_H\}$ has distribution

$$g_t(s_L \,|\, \theta_L) \;=\; 1 \quad \textit{and} \quad g_t(s_L \,|\, \theta_H) \;=\; \frac{1-\mu_t}{\mu_t}\frac{\hat{\eta}_0}{1-\hat{\eta}_0}. \tag{2.3.22}$$

Proof. The second line of (2.3.21) and the right hand side of its first line follows from the functional form of J_0, hence they are not affected by the evolution of beliefs over time. In contrast, the passage of time affects the left hand side of the first line, which represent the prior belief of a type that is terminated on $t+1$. Similarly, $\hat{\eta}_0$ is only a function of the functional form of J_0 so it is not affected by μ_t. $\square$

Long Contracts

Next drop the assumption that $\varphi_H = 1$. That is assume that $0 \leq \varphi_L \leq \varphi_H < 1$, but only consider the information provision problem of P_a as $t \to \infty$. In this case P_b's posterior on an buyer whose contract has not been terminated in period t is

$$\lim_{t\to\infty} \mu_t \;=\; \mu_\infty \;=\; \frac{\varphi_L}{1-(\varphi_H - \varphi_L)}\,,$$

To state the condition for information provision on the limit, note that Proposition 2.3.2 has already defined $\hat{\eta}_f = \frac{\hat{\beta}_f - \rho_L}{\rho_H - \rho_L}$ and

$$\hat{\beta}_f \;=\; 1 - \frac{(1-\kappa)^2 \rho_H}{2 - 3\rho_H\kappa + \rho_H \kappa^2}\,.$$

Proposition 2.3.5 (Long Contracts)**.** *Information provision is strictly optimal on the steady state of* ($\mathcal{G}_t'$)*, under tenure contracts, if and only if*

$$\begin{aligned} \mu_\infty(\rho_H - \rho_L) + \rho_L \;&\geq\; 1 - \frac{(1-\kappa)^2\rho_H}{2 - 3\rho_H\kappa + \rho_H\kappa^2} \\ \textit{and} \qquad \rho_H \;&>\; \frac{2}{3-\kappa} \end{aligned} \tag{2.3.23}$$

The higher ρ_H is the less binding the second line of (2.3.23) *becomes. Also, suppose that the second line is satisfied, then the higher either ρ_H, ρ_L, or μ_∞ is, the less the first line of* (2.3.23) *binds. When this is satisfied the optimal signal $s \in \{s_L, s_H\}$ has distribution*

$$g_\infty(s_L \,|\, \theta_L) \;=\; 1 \quad \textit{and} \quad g_\infty(s_L \,|\, \theta_H) \;=\; \frac{1-\mu_\infty}{\mu_\infty}\frac{\hat{\eta}_\infty}{1-\hat{\eta}_\infty}. \tag{2.3.24}$$

Proof. As already stated J_∞ is the limit of J_t for $t \to \infty$. Hence, the relevant condition for information provision can be obtained by substituting the points already derived in the previous section. In particular, the necessary and sufficient condition for J_∞ to have a convex part is $\rho_H > \beta^{**}(0)$, which gives the second line of (2.3.23). When this is satisfied for the prior to be big enough to be on the linear part it has to be that $\mu_\infty(\rho_H - \rho_L) + \rho_L \geq \hat{\beta}_f$, where the functional form of $\hat{\beta}_f$ is given in Proposition 2.3.2.

All of the statement regarding (2.3.23) and when it is more binding follow from simple differentiation. The statement on the optimal signal under information provision follows noting that this has to randomise between $\hat{\eta}_f$ and one, while the expected value of the posterior needs to remain μ_∞. □

Finally, note that because $\gamma > 0$ is a non-decreasing termination policy it follows that (2.3.5) is a sufficient condition for implementation. Moreover, this always holds for $\kappa \Rightarrow 0$, which turns (2.3.23) into

$$\mu_\infty(\rho_H - \rho_L) + \rho_L \;\geq\; 1 - \frac{\rho_H}{2} \quad \text{and} \quad \rho_H \;>\; \frac{2}{3} \tag{2.3.25}$$

Example 2.3.3 (Information Provision)**.** Let $\kappa \to 0$, $\rho_H \to 1$, then information provision is optimal on the steady state, and any solution is implementable on all the path, if and only if

$$\mu_\infty(1 - \rho_L) + \rho_L \;\geq\; \frac{1}{2},$$

which is satisfied for $\rho_L \geq 1/2$, and not satisfied for $\rho_L = \varphi_L \to 0$.

2.4 Endogenous termination

The model cosidered here is that introduced in Section 2.3. Here, we allow P_a to commit in advance on some probability of terminating a contract after the realisation of a certain history of reported types. To make the results more tractable assume throughout this section that $\varphi_H = 1$, which as argued before in the beginning of the subsection on tenure contracts implies that $\mathcal{J}_t(\eta, \lambda) = J_0(\eta)$. Moreover, $\xi_t = \xi_0$. To maintain the notation as light as possible, let

$$u_H \;=\; \frac{\theta_H^{1+\epsilon}}{1+\epsilon} \quad \text{and} \quad u_l \;=\; \frac{\theta_L^{1+\epsilon}}{\xi_0^{\epsilon}(1+\epsilon)}\,, \tag{2.4.1}$$

represent the point-wise optimal flow payoffs of P_a from a high and low type, respectively. Note that once a low type turns high it remains so, hence those two are the only relevant possibilities in terms of flow payoffs. Moreover, let

$$f_t \;=\; \Pr(\tau_a > t+1 \,|\, \tau_a > t) \quad \text{and} \quad x_t(\theta^t) \;=\; \Pr(\tau_a > t+1 \,|\, \tau_a > t, \theta^t) \tag{2.4.2}$$

As a result, once the point-wise optimal effort and signal have being substituted in ($\mathcal{P}'$), this obtain the following recursive representation

$$V_t(\mu_t) \;=\; \max_{x(\theta^t)\in[0,1]} \; \mu_t u_H + (1-\mu_t)u_l + \delta\gamma f_t V_{t+1}(\mu_{t+1}) + \delta(1-\gamma f_t)\mathcal{J}_0(\eta_t), \tag{2.4.3}$$

where note that the past history of a high type does not matter, while there is only one history on which the agent is a low type in t, that is L^t. Hence, it is without loss to condition $x_t(\theta^t)$ only on the current type and time, hence write x_t^H and x_t^l and note that

$$f_t \;=\; \mu_t x_t^H + (1-\mu_t)x_t^l\,, \qquad \eta_t \;=\; \frac{\mu_t(1-\gamma x_t^H)}{1-\gamma f_t}\,, \qquad \mu_{t+1} \;=\; \frac{\mu_t x_t^H + (1-\mu_t)x_t^l\varphi_L}{f_t}\,.$$

The treatment of the above problem is quite tedious and can be found in Section B.4 . To reduce the number of case that need to be considered the following assumption is imposed.

Assumption 1. $\mathcal{J}_0$ is twice continuously differentiable and concave. Both u_H and u_l are positive. Also,

$$u_H \;>\; (1-\delta)\mathcal{J}_0(1) \qquad \text{and} \qquad \frac{u_H}{1-\delta\gamma} + \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1) \;>\; \mathcal{J}_0(0) + \mathcal{J}_0'(0). \tag{2.4.4}$$

The main result which characterises the solution of P_a optimal stopping problem is given below.

Proposition 2.4.1. *Continuing a high type, with probability one, is always strictly optimal. Stopping a low type, with probability one, for every $\mu_t \in [0,1]$ is strictly suboptimal if*

$$\frac{\varphi_L\,u_H}{1-\delta\gamma} + (1-\varphi_L)u_l - \frac{1-\delta}{1-\delta\gamma}\big[1-\delta\gamma(1-\varphi_L)\big]\mathcal{J}_0(1) + \big[1-\delta(1-\varphi_L)\big]\mathcal{J}_0'(1) \;>\; 0 \tag{2.4.5}$$

is satisfied. In contrast, if it holds in the reversed direction and

$$\frac{\varphi_L\,u_H}{1-\delta\gamma} + (1-\varphi_L)u_l + \delta\left[(1-\gamma)\frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma} + \gamma(1-\varphi_L)\right]\mathcal{J}_0(1) - \delta(1-\varphi_L)\mathcal{J}_0'(1) \;<\; \mathcal{J}_0(0) \tag{2.4.6}$$

is satisfied, then stopping a low type, with probability one, is optimal for all $\mu_t \in [0,1]$. Otherwise, there exists $\tilde{\mu}$ such that stopping a low type, with probability one, is optimal if and only if $\mu_t > \tilde{\mu}$.

Whenever the low type is not stopped, with probability one, the reputation of both a terminated η_t and non-terminated μ_{t+1} agent increases over time.

Example 2.4.1. Let $\rho_L = \mu_0\rho_H$ and $\rho_H = 1$. Then for u_L and φ_L small enough the optimal contract has the low type to be fired with probability one, one for some interior $\tilde{\mu}$.

Proof. Under the above choice of parameters $\mathcal{J}_0(0) = \mathcal{J}_0(1) = 0$ and $\mathcal{J}_0'(1) < 0$. As a result neither of the above two inequalities hold when

$$\begin{aligned} \frac{\varphi_L u_H}{1-\delta\gamma} + (1-\varphi_L)u_l &< -\mathcal{J}_0'(1)[1-\delta(1-\varphi_L)] \\ \frac{\varphi_L u_H}{1-\delta\gamma} + (1-\varphi_L)u_l &> \delta(1-\varphi_L)\mathcal{J}_0'(1) \end{aligned}$$

The second inequality always hold. The first inequality holds if the left hand side is small enough, which is true when φ_L and u_l and small enough. □

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Chapter 3

The Effect of Market Conditions and Career Concerns in the Fund Industry

A continuum of potential investors allocate funds in two consecutive periods between a manager and a market index. The manager's alpha, defined as her ability to generate idiosyncratic returns, is her private information and it is either high or low. In each period, the manager receives a private signal on the potential performance of her alpha, and she also obtains some public news on the market's condition. The investors observe her decision to either follow a market neutral strategy, or an index tracking one. It is shown that the latter always results in a loss on reputation, which is also reflected on its fund flows. This loss is smaller in bull markets, when investors expect more managers to use high beta strategies. As a result, a manager's performance in bull market is less informative about her ability than in bear markets, because a high beta strategy does not rely on it. We empirically verify that flows of funds that follow high beta strategies are less responsive to the fund's performance from those that follow market neutral strategies.

3.1 Introduction

In recent years, there has been a growing concern in the financial markets about the role of various financial intermediaries such as mutual funds and hedge funds, as the proportion of the institutional ownership of equities has sharply increased and the Global Assets under Management are estimated to exceed $100 trillion by 2020[1]. The managers of these funds are competing with each other, but also with alternative investment vehicles such as market index funds or ETFs, to attract new investors. One of the ways in which they differenti-

[1] This is according to a research by PWC.

ate themselves is through their investment strategy. In particular, managers often signal their confidence by choosing strategies that are highly idiosyncratic[2], and importantly their incentive to pick these strategies fluctuates with the general market conditions.

Our first contribution is to build a model in which a manager's investment decision provides an imperfect signal on her ability to generate idiosyncratic returns. To be more precise, the manager will skew her investment choice towards a strategy with low exposure to the market in order to signal her confidence. A highly skilled manager is more likely to invest in her idiosyncratic project, since this will deliver on average superior returns. The investors cannot observe directly the manager's ability, but because of the above they will associate an idiosyncratic strategy with a competent manager; in turn, this will endow such a strategy with a reputational benefit. This asymmetry of information between the manager and her potential investors is the main driving force behind the results of this paper.

Our second contribution is to demonstrate that the signalling value of investing in a low beta strategy depends on the market conditions. Managers have a dual objective; they want to maximize their contemporaneous returns but also their perceived reputation. The better the market (*bull*) is, the more the managers face a trade-off between these two objectives, and the less the investors penalize managers for choosing a high beta strategy. Consequently, there is an interaction between managers' career concerns and market conditions.

To analyse the above interactions we consider a two period model in which there is a continuum of investors and a single fund manager. Each investor chooses between investing his wealth through the manager, or directly in the market index, and this choice is affected by an investor specific stochastic preference shock. The manager's utility is a function of the fees she collects, which are an exogenous proportion of her fund's assets under management (AUM) at the end of each period. After the investors have allocated their funds, the manager publicly chooses between a high or low beta investment strategy. We model the manager's ability as the ex ante expected return of her idiosyncratic strategy, which is either high or low. In each of the two periods, and before picking an investment strategy, the manager also receives a private signal on the contemporaneous profitability of her idiosyncratic project. Both her ability and this signal are her private information, and she uses them to form her final estimate of the profitability of her contemporaneous idiosyncratic strategy. As a result, a high type manager is more likely to form a high estimate, but this is not always the case.

To model market conditions, we assume that the manager also receives a signal on the market's contemporaneous return. This signal is eventually revealed to the investors, but only after they have made their own investment choice. In some sense, we allow for them

[2]For example, a recent article in Financial Times explains how institutional investors are turning to alternative investments in recent years.

to eventually understand the market conditions under which the manager acted. However, at this point it will be too late for them to use this information to trade on their own[3]. In section 3.3.4, we extend our setting by allowing two managers to coexist in the market, in order to study how the competition is affected by market conditions. We focus mainly on the first period, since in the second the manager's investment choice is not affected by her reputational concerns. In fact, the second period is introduced in order to create those concerns.

For our first result, we analyse a refinement of the perfect Bayesian Equilibrium, which we call *monotonic equilibrium* and we prove that this always exists. The only additional restriction that this refinement is imposing is that the manager's reputation is non-decreasing on her performance. In addition, under mild parametric restrictions we demonstrate that the monotonic equilibrium is unique.

Our second result is to demonstrate that investing in an idiosyncratic strategy carries a reputational benefit. This is because, the cut-off of the high manager type is smaller than that of the low. In other words the high type is more receptive to the idea of adopting a low beta strategy. Intuitively, the manager's choice is affected by two incentives. On the one hand, she wants to increase her reputation, which skews her preferences towards idiosyncratic investments. On the other hand, she cares about the realised return of her strategy, since her fees depend on it. Hence, for a relatively low private signal even a high type may opt to forfeit the reputational benefit, because investing in the market will generate higher returns, and as a result more fees. Therefore, the investment strategy is informative but it does not fully reveal the manager's ability, which is a realistic representation of the fund industry.

Our third and most important result is to show that the reputational benefit of investing in the idiosyncratic project is decreasing in the market conditions. In particular, we prove that the expected sensitivity of reputation to performance is higher in bear markets than in bull markets. This is because investors understand the dual objective of managers and the fact that a manager is more likely to invest in the market when the market conditions are good, and thus update their beliefs less aggressively when this is the case; instead, in bad times any change in fund's performance is much more likely to be attributed to the ability of the manager.

We use the above results to discuss the competition between funds, in terms of their sizes, and its fluctuation depending on market conditions. We predict that the likelihood of changes in the ranking of the funds, measured by assets under management, is hump shaped on the market return, but is also higher during bear markets than during bull markets, due to the higher informativeness of performance; we also find some empirical evidence supporting

[3]In other words, manager has a superior market-timing ability compared to an investor.

this prediction. This is in line with the common perception that the industry only rearranges its interaction with its investors during crises.

Finally, as an extension to our model, we study the case where investors cannot observe the managers' investment decision. In this scenario, we assume that the investors cannot observe if the manager had invested on the market or their idiosyncratic portfolio, and we conclude that, under this assumption, the conditions for the existence of a monotonic equilibrium cannot be satisfied.

Academic research in financial intermediaries has so far mainly focused on establishing various empirical results about their structure, returns, flows, managers' skill and many other characteristics; there have been far fewer theoretical papers. One of the seminal papers about mutual funds is from Berk and Green (2004); they construct a benchmark rational model in which the lack of persistence of outperformance, is not due to lack of superior skill by active managers, but is explained by the competition between funds and reallocation of investors' capital between them.

Our paper aims to contribute to various strands of literature that we outline below. First, it is related to many papers that study how managers' concerns about their reputation affect their investment behaviour. Chen (2015) examines the risk taking behaviour of a manager who privately knows his ability and shows that in this model investing in the risky project always makes manager's reputation higher, thus leading to overinvestment in such risky projects. Dasgupta and Prat (2008) study the reputational concerns of managers, and show how they may lead to herding and can explain some market anomalies; their focus though is mainly on the asset pricing implications of this behaviour. Similarly, Guerrieri and Kondor (2012) build a general equilibrium model of delegated portfolio management to study the asset pricing implications of career concerns; they find that as investors update their beliefs about managers, these concerns lead to a reputational premium, which can change signs depending on the economic conditions. Moreover, Malliaris and Yan (2015) show that career concerns induce a preference over the skewness of their strategy returns, while Hu et al. (2011) present a model of fund industry in which managers alter their risk-taking behaviour based on their past performance and show that this relationship is U-shaped. Huang et al. (2012) on the other hand, build a theoretical framework to show how investors are rationally learning about the managers' skills, and test their predictions about the fund flow-performance relationship empirically; however, they do not take into account any strategic behaviour by the fund managers.

The paper most relevant to our work is that by Franzoni and Schmalz (2017). In their work, they study the relationship between the fund to performance sensitivity and an aggregate risk factor and they find that this is hump shaped. They also build a theoretical model in which investors update their beliefs about the managers' skills while they also learn about the fund's exposure to the market. The second inference in extreme markets is noisier for two reasons. The first is idiosyncratic risk and the second is that investors who are uncertain about risk loadings cannot perfectly adjust fund returns for the contribution of aggregate risk realizations. As a result it becomes harder for investors to judge the managers and update their beliefs, and this is what drives the documented result. The theory we propose differs from that of Franzoni and Schmalz (2017) because their model describes the fund's loading on aggregate risk (β) as a preset fund specific exposure, whereas our model gives the ability for managers to strategically choose their investment decision. Also we further investigate how this investment decision will affect the managers' decision if it is observable by the investors or not. Moreover the data source considered for their paper is the CPRSP Mutual Fund Database which is different from the Morningstar CISDM which we use for the empirical part, making it difficult to compare our results. Although the implementation and the structure of their model is completely different to ours and does not imply the same predictions we are making, we conclude that the aggregate risk realizations matter for mutual fund investors and managers.

Another strand of literature in which we contribute to is the empirical research on the fund flows and characteristics. It is well documented that mutual fund investors chase past returns, Ippolito (1992) and Warther (1995) present empirical evidence supporting our predictions. Sirri and Tufano (1998) show that the flow-performance relationship is convex, and asymmetrically so on the positive side of returns. Furthermore, Chevalier and Ellison (1997), show that managers engage in window dressing their portfolios. More recently, Wahal and Wang (2011) study the competition between funds, by looking at the effect of the entry of new mutual funds on fees, flows and equilibrium prices. Finally, Ma (2013) provides a very comprehensive survey of empirical findings concerning the relation between mutual fund flows and performance.

The rest of the paper is organized as follows. In section 3.2, we introduced our theoretical framework and our equilibrium. Section 3.3 proves its existence, identifies a condition under which this is unique, and presents our theoretical predictions. In particular, section 3.3.4 discusses the implications of adding a second manager. Subsequently, section 3.4 presents our empirical results. Section 3.5 considers an alternatively model where the investment decision is unobservable. Finally, section 3.6 concludes.

3.2 The Model

3.2.1 Setup

This is a two period model $t \in \{1, 2\}$. There is one fund manager (she) and a continuum of investors (he) of measure one, who collectively form the market. The manager discounts the future with $\delta \in (0, 1]$.

At the beginning of period t, each investor decides how to invests a unit of wealth. At the end of period t, he consumes all the wealth that this investment generated. The investor is restricted to a binary decision. First, he can opt to allocate all his wealth in an index tracking strategy. This has the same returns as the market portfolio, which is given by

$$m_t \sim \mathcal{N}(\mu \,, \sigma_m^2) \tag{3.1}$$

Second, he can choose to invest all his wealth in the manager's fund[4]. For each unit of wealth invested with the manager let $R_t = \exp(r_t)$ denote its value at the end of this period, where

$$r_t \;=\; (1 - \beta_t) \cdot a_t \;+\; \beta_t \cdot m_t \tag{3.2}$$

is the fund's return. This has two components, one of which is the market return m_t. The second is given by

$$a_t \sim \mathcal{N}(\alpha, \sigma^2\,) \tag{3.3}$$

which represents the market neutral component of the manager's investment strategy[5]. Adhering to the fund industry's convention, the manager's ability to create idiosyncratic profits is called alpha, and is represented by $\alpha \in \{L, H\}$ where $L < H$. The manager's ability is her private information. The investors share the public prior $\pi = \mathbb{P}(\alpha = H)$.

Finally, β_t represents the fund's exposure to the market. This is publicly chosen by the manager after the investors have allocated their wealth. For simplicity we assume that $\beta_t \in \{0, 1\}$. Note that the model's beta β_t despite its relevance to the corresponding variable of the CAPM model, is not the same variable. Rather the former represents a deterministic

[4]Our underlining intuition is that most of the market participants take a rule of thumb approach to their investment through intermediaries. For example, they set apart 5% of their wealth and then they decide if they should invest this amount to a fund.

[5]For example think of a long/short equity fund that invests $(1 - \beta_t)$ of its assets on a market neutral portfolio and β_t on the S&P 500 index. For the most part we refrain from giving a specific interpretation of the components of the fund's return r_t, or which part of its investment strategy they represent. Our framework relies on the simple intuition that some of the return generated by the manager stems from her own ability and some from factor loading. In fact m_t could represent any such factor, and for some funds other choices would be more sensible. For example, a macro fund is more related to the risk-free interest rate than to the equity markets.

investment decision, whereas the latter its estimate.

In addition, before making her investment decision β_t, but after the investors have allocated their wealth, the manager receives two signals

$$s_t \sim \mathcal{N}(a_t\,,\,\nu^2\,) \qquad \text{and} \qquad s_t^m \sim \mathcal{N}(m_t, \nu_m^2). \tag{3.4}$$

On the one hand, s_t is private and it is associated to the manager's contemporaneous confidence on her alpha[6]. On the other hand, s_t^m is public but it only becomes available after the investors have committed their capital to the manager's fund. This market signal is considered to be the standard piece of information that most institutional participants receive on the market's condition.

For simplicity, we assume that the manager's fees are exogenously set to a given percentage $f_t \in [0, 1]$ of her asset under management (AUM) at the end of t[7]. Even though we do not allow for incentive fees, the plain managerial fees f_t we consider suffice to create direct incentives for the manager to perform in t, as her period income per dollar invested is $f_t R_t$.

Two more important assumptions have been made. First, that the manager's investment decision is binary. In particular, it allows for either investing all of the fund's assets in the manager's idiosyncratic strategy a_t, or all in the market m_t. Second, that this decision is observable by the rest of the market participants. The former assumption is imposed mainly to make the model more tractable. We speculate that altering it to allow for $\beta_t \in \{\underline{b}, \overline{b}\}$, where $\underline{b} < \overline{b}$, would not affect our results qualitatively[8]. Regarding the latter assumption, it appears to be reasonable for long investment horizons. This is because the fund's exposure to the market can be ex-ante approximately inferred, either by estimating a multi-factor regression, or by looking at its past portfolio composition, which in many cases is public.

[6]This could reflect the fact that her idiosyncratic strategy has some seasonality that she is able to partly predict. Another interpretation is that the strategy itself changes across periods, in which case α represents the manager's latent ability to come up with new ideas to beat the market.

[7]Endogenizing the choice of fees is left for future research. The complexity of allowing an endogenous choice is that the fees would then serve as a signalling device for the managers' ability, thus making the equilibrium much harder to find.

[8]A possibility that we exclude and is worth mentioning is that of a manager that bets against the market. In particular, in strong bear markets most funds would prefer to short the market portfolio, instead of adopting a strategy that is neutral to it. This would have a significant impact on our analysis. Despite that, it is ignored both to facilitate the exposition and because funds that systematically hold big negative positions are not that common.

3.2.2 Payoffs

Investors are risk-neutral, however each one's decision is influenced by an exogenous preference shock

$$z_t^j \sim \operatorname{Exp}(\lambda)\,, \quad \text{where} \quad j \in \big[0,1\big] \tag{3.5}$$

stands for the shock on investor's j preferences at period t. Hence, his payoff from investing in $i \in \{1, m\}$ is

$$v(i, z_t^j) = \begin{cases} \exp(z_t^j - \bar{z}) \cdot (1 - f_t) \cdot R_t & , i = 1 \\ \exp(m_t) & , i = m \end{cases} \tag{3.6}$$

where $\bar{z} > 0$ is a constant that we introduce to ensure that under the lowest preference shock $z_t^j = 0$ the investor would opt for the market instead.

There is a plethora of ways to interpret this shock, a valid one being that each investor values specific fund characteristics, for example the fund's classification with regards to its investment strategy, its portfolio composition, leverage, etc. An alternative one would be that he is influenced by interpersonal relationships, network effects, word of mouth, or other forms of private information. Our analysis will be silent as to what generates this shock.

Furthermore, note that because R_t comes from a log-normal distribution, we could adopt a CRRA utility function for the investor without altering his decision significantly. However, we opt not to do so in order to maintain our expressions as compact as possible. On the other hand, it will be assumed that the manager has log preferences. In particular, if A_t stands for the AUM the fund in the beginning of t, then manager's payoff at t is $\log\big(A_t f_t R_t\big)$. Again we speculate that most of our results would not be significantly different if a generic CRRA was used instead of log, however it turns out that this is the most convenient functional form to work with.

3.2.3 Timing

To sum up, the timing in our model is as follows. In each period $t \in \{1, 2\}$, first the preference shock z_t^j, $j \in [0, 1]$, is realised and then the investors decide how to allocate their wealth. Second, the manager receives the private and public signals s_t and s_t^m, respectively. Third, the investment decision β_t is made by the manager, R_t is realised, and both become public. Forth, the fund's AUM is divided between the manager and her investors, according to the fee f_t, and is consumed immediately. Finally, we assume that the investors that are active in the second period observe the public variables of the first period before allocating their wealth. Importantly, they know (R_1, β_1, s_1^m) and use them to update their beliefs on the manager's ability α.

3.2.4 Monotonic equilibrium

We call an equilibrium of our model *perfect Bayesian* (PBE), if all market participants use Bayes' rule to update their beliefs on α, whenever possible, and choose their actions in order to maximise their expected discounted payoff in each point they are taking an action. It is easy to demonstrate that we have multiple equilibria, which is a common setback for this type of models. For this reason we will further refine the set of equilibria using the following definition.

Definition 3. Call a PBE a *monotonic equilibrium* if the manager's reputation, for a given choice of investment strategy, is non-decreasing on her performance.

Therefore, the only requirement that our refinement imposes is that the manager's reputation is not penalised by the fact that she delivers good returns for her investors. The above definition implies that there exists φ_0 and φ_1 such that the public posterior on the manager's ability is given by

$$\begin{aligned} \varphi_0 &= \mathbb{P}(\alpha = H \,|\, r_1, s_1^m, \beta_1), \quad \text{for } \beta_1 = 0 \\ \varphi_1 &= \mathbb{P}(\alpha = H \,|\, r_1, s_1^m, \beta_1), \quad \text{for } \beta_1 = 1 \end{aligned} \tag{3.7}$$

We separate the posteriors that follow each choice of β_1 because those will turn out to have different functional forms.

3.3 Analysis

We begin our analysis by first discussing the manager's optimal investment strategy in the second period and how this affects her career concerns in the first period. Second, we characterise the monotonic equilibrium and prove its existence and uniqueness. Third, we present our results on the baseline model with the single manager. Forth, we discuss the implications of adding a second manager.

3.3.1 Investment and AUM in the second period

Here we provide a description of how we solve for the manager's investment decision in the second period and the corresponding AUM that this implies. The interested reader can find a more detailed analysis in Appendix C.2.

In the second period the manager faces no career concerns. Hence, the objective of her investment decision is to maximise the expected fees she collects at the end of this period.

Because those fees are proportional to her fund's AUM at the end of the second period, and we have assumed log preferences, the manager's payoff maximisation problem simplifies to

$$\max_{\beta_2 \in \{0,1\}} \mathbb{E}\big[\log(R_2) \,\big|\, \beta_2,\, \alpha,\, s_2,\, s_2^m\big]$$

When opting for her idiosyncratic strategy $\beta_2 = 0$ the above expectation uses the manager's ability α and private signal s_2, whereas the index tracking strategy $\beta_2 = 1$ depends only the market signal s_2^m. Since we have assumed that the returns and the corresponding signals are log-normally distributed we can calculate the above expectation for each choice in closed form. This gives that the manager's optimal second period strategy is to invest in her idiosyncratic project if and only if $s_2 \geq c(\alpha, s_2^m)$ where

$$c(\alpha, s_2^m) \;=\; \frac{\psi_m}{\psi} \cdot s_2^m + \frac{1-\psi_m}{\psi} \cdot \mu - \frac{1-\psi}{\psi} \cdot \alpha \tag{3.1}$$

The constants ψ and ψ_m are the weights that the Bayesian updating gives to the signals s_2 and s_2^m, respectively, and their functional form can be found in Appendix C.2. Given the above cut-off strategy we can calculate the expected terminal value of one unit of wealth that is invested by the manager. For a high and low type we will denote those by u_2^H and u_2^L, respectively. Therefore, for given posterior reputation φ, and while ignoring the preference shock z, the expected payoff of an investor that opts for the manager is given by

$$[1 - f_2] \cdot [\varphi \cdot u_2^H \;+\; (1-\varphi)\, u_2^L]$$

This together with the assumed preference shock allows us to calculate the assets of the second period in closed-form. Those are given by

$$A_2(\varphi) \;=\; \Big(e^{-(\mu + \bar{z} + \sigma_m^2/2)} \cdot [1 - f_2] \cdot [\varphi \cdot u_2^H \;+\; (1-\varphi) \cdot u_2^L]\Big)^{\lambda} \tag{3.2}$$

which is an increasing function of the manager's reputation φ. One thing we can note is that as long as $\lambda > 1$, the assets under management are a convex function of the reputation φ. This is a result that has been widely documented in the relevant empirical literature, in slightly different forms.

3.3.2 Existence and uniqueness of the monotonic equilibrium

In this section we demonstrate that the monotonic equilibrium exists and under mild conditions it is unique. First, we want to understand the manager's incentives in the first period.

Her expected discounted payoff at this point is

$$\mathbb{E}_R\Big[\log\left[R_1 f_1\, A_1(\pi)\right] \,+\, \delta\cdot\log\left[R_2 f_2 A_2(\varphi_\beta)\right] \;\Big|\; s^m,\, s,\, \beta\,,\alpha\Big]$$

where $A_1(\pi)$ is the equilibrium allocation of AUM in the first period, which has a functional form similar to that of $A_2(\varphi_\beta)$.

Hereafter, the focus of the paper shifts to the interactions of the first period. As a result, in order to make our formulas more compact, the time subscript t is dropped, whenever this does not create an ambiguity. Using the properties of the natural logarithm we simplify the manager's payoff maximisation problem in period 1 to

$$\max_{\beta\in\{0,1\}}\ \mathbb{E}_r\Big[r \,+\, \delta\cdot\lambda\cdot\left[\varphi_\beta(r,s^m)\cdot(u^H-u^L)\,+\,u^L\,\right] \;\Big|\; s^m,\, s,\, \beta\,,\alpha\Big] \tag{3.3}$$

Therefore, the manager cares both about her returns in the first period r, but also on how those affect her posterior reputation $\varphi_\beta(r,s^m)$. This reputation is important because it affects the amount of AUM that the manager will manage to gather in the beginning of the second period.

First, we want to offer a characterisation of the monotonic equilibrium.

Lemma 3.1. *In any monotonic equilibrium the high and low type invest in their idiosyncratic strategy if and only if*

$$s\,\geq\,h(s^m)\quad\textit{and}\quad s\,\geq\,l(s^m)\,, \tag{3.4}$$

respectively, where

$$l(s^m)-h(s^m)\;=\;\frac{1-\psi}{\psi}\cdot(H-L) \tag{3.5}$$

Proof. In Appendix C.1. □

Hence the more confident the manager becomes on her alpha, the more likely she is to use her idiosyncratic strategy, instead of the index tracking one. In addition, the fact that the high type's cutoff is lower captures the fact that a competent manager uses her idiosyncratic investment strategy relatively more often.

Second, we want to calculate the manager's posterior reputation after each investment decision as a fuction of her performance.

Lemma 3.2 (Posteriors)**.** *In any monotonic equilibrium the manager's posterior reputation* *in the beginning of the second period, if she invested on her alpha $\beta = 0$ in the first, is*

$$\varphi_0(r, s^m) \;=\; \left(1 + \frac{1-\pi}{\pi} \cdot \rho(r) \cdot \frac{\Phi\left(\frac{r - l(s^m)(1+\psi) + L\psi}{\nu\sqrt{1+\psi}}\right)}{\Phi\left(\frac{r - h(s^m)(1+\psi) + H\psi}{\nu\sqrt{1+\psi}}\right)}\right)^{-1}, \tag{3.6}$$

where

$$\rho(r) \;=\; \exp\left(\frac{-2(H-L)\,r + H^2 - L^2}{2\nu^2\psi(1+\psi)}\right).$$

On the other hand, if she invested in the market $\beta = 1$ then this becomes

$$\varphi_1(s^m) \;=\; \left(1 + \frac{1-\pi}{\pi} \cdot \frac{\Phi\left(\frac{l(s^m) - L}{\nu}\right)}{\Phi\left(\frac{h(s^m) - H}{\nu}\right)}\right)^{-1} \tag{3.7}$$

Proof. In Appendix C.1. □

The investors form their posterior belief on the manager's ability by observing her investment decision β and the realised return r. Note that when using her idiosyncratic investment strategy the manager's performance r is generated by her alpha. Hence, in this case the realisation r carries additional information on the manager's ability. On the other hand, when using the index tracking strategy r is equal to the market's return m, which carries no additional information on the manager's ability. This is why φ_0 is a function of r, but φ_1 is not.

Using the above two lemmas, we prove the main result of this part.

Proposition 3.1. *A monotonic equilibrium always exists. Moreover, a sufficient condition for it to be unique is that*

$$\delta \cdot \lambda \cdot (H - L) \;\leq\; \psi^2 \cdot \nu^2 \tag{3.8}$$

Proof. In Appendix C.1. □

We believe that (3.8) is satisfied for a wide range of parametric specifications that we would consider natural given the economic setting we study. This translates into two requirements. First, that the difference between the ability of the two types is not too big. Second, that the precision of the signal s is neither so small that it becomes irrelevant, nor so big that the manager's ex-ante ability α becomes irrelevant instead.

3.3.3 Results

Here, we present some important properties of the unique monotonic equilibrium. We assume throughout that (3.8) holds. To maintain the notation as light as possible keep using $\varphi_0(r, s^m)$ and $\varphi_1(s^m)$ to refer to the equilibrium reputations, which are obtained after substituting the corresponding values for $h(s^m)$ and $l(s^m)$.

Proposition 3.2 (Point-wise dominance)**.** *There is a strict reputational benefit for the manager from investing in her alpha, that is*

$$\varphi_0(r, s^m) \;>\; \varphi_1(s^m), \quad \textit{for all} \quad r, s^m \in \mathbb{R}. \tag{3.9}$$

Proof. In Appendix C.1. □

We already know that in every monotonic equilibrium $\varphi_0(r, s^m)$ is increasing in r, in other words high performance is beneficial for the manager's reputation. The proof demonstrates the result by taking the limit of the left hand side to minus infinity and showing that even there the inequality holds. Hence the equilibrium difference between the cutoffs used by the high and low type is such that the investors' inference on the manager's type relies relatively more on her choice of strategy than on the subsequent performance of her fund.

This may seem counterintuitive at first, but it has a very simple explanation. In the appendix we show that for a monotonic equilibrium to also be rational the difference between the equilibrium cutoffs $l(s^m)$ and $h(s^m)$ cannot be too large. If that was the case, then a low type would have to be so confident in order to invest in her alpha that a very bad performance, under the low beta strategy, would be associated with a high type. An immediate consequence of which would be that the manager's reputation would be non-monotonic on her performance. But those are exactly the type of equilibria that appear to be the less realistic.

The above claim is the most challenging one to verify in the data. This is because for each fund we never observe the counter-factual, that is how the fund's flow would look like if it had chosen a lower, or higher beta strategy. Moreover, the simplifying assumption $\beta \in \{0, 1\}$ makes this result stronger than what an alternative model, where the two betas are closer to each other, would give. Despite that, we can verify empirically that up to a certain extend a low beta strategy creates enough signalling value to counter the effect of a low subsequent performance.

As a direct consequence of point-wise dominance, we can now get the following interesting proposition, which characterises the effect of the manager's career concerns on her investment behaviour.

Proposition 3.3 (Investment Behaviour)**.** *The equilibrium cutoffs $h(s^m)$ and $l(s^m)$ are decreasing in the discount factor δ. Moreover, there is overinvestment in the manager's idiosyncratic project, that is*

$$h(s^m) \ \leq \ c(H, s^m) \qquad \textit{and} \qquad l(s^m) \ \leq \ c(L, s^m). \tag{3.10}$$

Proof. In Appendix C.1. □

The proof is a simple application of the implicit function theorem on equation (C.1.17), the solution of which is shown in the proof of Proposition 3.1 to be $h(s^m)$. The corresponding result for $l(s^m)$ is obtained by invoking the fact that in every monotonic equilibrium those two cutoffs are connected through a linear relationship, which was again demonstrated in the above proof.

We use the term *"over-investment"* to describe the fact that the managers invests in her idiosyncratic strategy more often than in the absence of career concerns. In other words, over-investment exists when the managers "lower her standards" with regards to her private signal, i.e. she lowers the confidence level required for her to choose the idiosyncratic investment. Note that the manager's optimal cutoff, in the absence of career concerns, corresponds to that already derived from for the second period in (3.1). This is because it is generated by the inefficiency in the investment decision that the manager's career concerns create, which is connected to the underlying parameter δ.

The above proposition demonstrates that there is a bias towards active management in the financial intermediation industry, which is due to its inherent informational asymmetries. To be more precise, we expect managers to get on average less exposure to the market than what would maximise the fund's expected return. Moreover, this action is associated with competence and it is rewarded with an increase in the funds AUM. Hence, our model provides a theoretical justification for this well documented fact.

Next, we want to see how this bias depends on the unobserved, to the econometricians, market signal s^m and the manager's prior reputation π.

Proposition 3.4. *The cutoffs $h(s^m)$ and $l(s^m)$ are increasing in the market signal s^m. In addition, there exist lower bounds $\bar{s}^m$ and $\bar{\pi}$ such that for every (s^m, π) such that $s^m \geq \bar{s}^m$ and $\pi \geq \bar{\pi}$ both cutoffs $h(s^m)$ and $l(s^m)$ are increasing functions of the manager's prior reputation π.*

Proof. The proof of the first statement is similar to that of Proposition 3.3. The proof of the second follows from Lemma C.1.4, which found in Appendix C.1. □

The first statement is a very intuitive result. The better the manager expects the market portfolio to perform, the more eager she becomes to invest in it, which translates into higher equilibrium cutoffs.

The crucial implication of the proposition's second statement is that the bias created from the signalling value, of investing in the idiosyncratic strategy, is decreasing in the manager's prior reputation. This is because the equilibrium cutoffs are bounded above by the expected return maximising cutoff $c(\alpha, s^m)$, hence the more π increases the closer they get to it.

A caveat of this result is that it only holds for a manager that is already relatively recognized in the market, in particular it is shown in the appendix that we need at least $\pi > 1/2$. Intuitively, the closer the prior is to either zero or one, the less it is affected by the actions of the manager. To make this more concrete, think of the extreme case where $\pi \to 1$, in which case it is very difficult for the investors to change their opinion about her ability, as they already know it with almost total certainty. Hence, there is a corresponding result that can be stated for managers of very low reputation. Even though in our model we allow for funds of small size to stay active, in reality most of them would either shut down, or would not even be reported in most datasets, hence we focus just on funds with reputation greater than a 1/2. [9].

Another interesting feature of the presented specification is that it provides a better understanding on how the sensitivity of the fund's asset flows to its performance depend on the market conditions. Let $\varphi(r, s^m, \beta)$ stand for the manager's reputation in either of the two cases and call $\mathrm{d}\varphi/\mathrm{d}r$ its *sensitivity* with respect to her performance.

Proposition 3.5. *The conditional probability that the manager has invested in the market portfolio $\mathbb{P}(\beta = 1 \,|\, m)$ is increasing in its contemporaneous performance m.*

In addition, for a sufficiently reputable manager the conditional expected sensitivity of the manager's reputation with respect to her performance, i.e. $\mathbb{E}_{s^m}[\mathrm{d}\varphi/\mathrm{d}r \,|\, m]$, is decreasing in m.

Proof. In Appendix C.1. □

When markets are expected to perform well, the manager's direct incentives outweigh those of career concerns. Hence we know from Proposition 3.4 that she is more likely to give up the reputational benefit of following a low beta strategy. But high beta strategies carry no information with respect to the manager's ability, Hence, even though as noted in Proposition 3.2 investing in low beta always has a reputational benefit, this benefit is less pronounced in good markets. Therefore investors are expected to be based more on manager's performance

[9] Despite that we hope to test empirically if we can obtain a corresponding result for the flows of small fund.

to update their belief about the ability of the manager, when markets are bear than when markets are bull. This result is also supported by the empirical evidence we provide in section 3.4.

3.3.4 Discussion on the competition between funds

It follows from the previous discussion that managers will be judged much more strictly on their performance in bear markets than in bull markets. This in turn has some implications for the relative ranking of the various funds with respect to their reputation, or equivalently their AUM.

To study this we extend our model by allowing a second manager to operate in the market. We formally define the investor's preference shock in this case and derive the corresponding AUMs of the two funds in Appendix C.2. In fact, the whole analysis of this paper and all our results remain unchanged with the addition of a second manager. The reason is that manager's utility is such that it is only a function $\varphi_\beta(r, s^m) \cdot (u^H - u^L) + u^L$ and is independent of the number of managers that exist in the model[10].

Our main aim is to study the likelihood of a change in the rank of managers, in terms of investors' beliefs about their ability and relate that to the market conditions. In what follows, we explain why this effect is not monotonic in m_t[11].

In the Appendix it is shown that:

$$\mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \;=\; \mathbb{P}(\varphi_0^1 > \varphi_1^2 \,|\, s^m)\, \mathbb{P}(0,1 \,|\, s^m) + \mathbb{P}(\varphi_0^1 > \varphi_0^2 \,|\, s^m)\, \mathbb{P}(0,0 \,|\, s^m), \tag{3.11}$$

What this equation suggests is that the ranks of managers can change through two possible scenarios. In the first scenario, with probability $\mathbb{P}(0,1 \,|\, s^m)$, one of the two managers invests in his idiosyncratic portfolio and the other follows the market; this probability goes to zero for both very large and very small s^m, as then both managers invest in the market or both invest in their own project.In turn, this makes the first term of equation (25) to be hump-shaped in s^m. Under this scenario, manager 1 has a reputational benefit from choosing $\beta = 0$ (see Proposition 2) which then makes it possible for his ex-post reputation to be higher than that of manager 2 (despite his initial disadvantage, in terms of the priors π^1, π^2); clearly the smaller is the distance between their prior reputations, $\pi^2 - \pi^1$, the larger will be this likelihood.

In the second scenario, with probability $\mathbb{P}(0,0 \,|\, s^m)$ both managers invest in their own

[10] In particular, equation (39) and thus the determination of the cutoffs l and h will remain the same.

[11] Note, we always condition on s^m as we know that all investors observe this market signal.

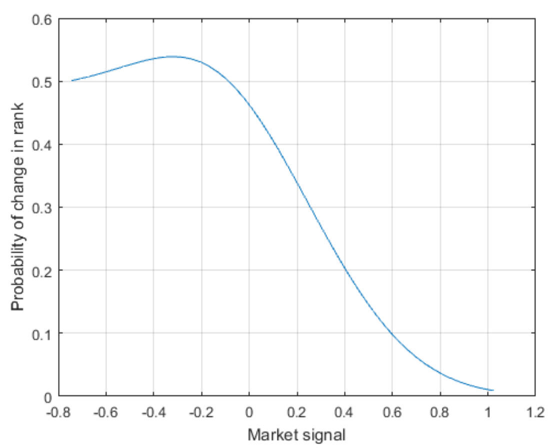
project and manager one receives a much higher return than the other, thus overcoming the effect of the initial prior reputations; in other words, since $\pi^1 < \pi^2$, in order for the posterior reputations to have the opposite order, what needs to happen is that the realized return of manager 1 is much higher than that of 2. This is clearly not possible if they both invest in the market. However, when they both invest in their idiosyncratic project this can happen either because one is luckier than the other, or simply because manager one has high skill and manager two has low skill. This scenario is less likely to occur as the market conditions get better since $\mathbb{P}(0, 0 \,|\, s^m)$ is decreasing in s^m, as we can see from Proposition 5. Moreover, we can get the following remark:

Remark 4. The likelihood of a change in the ranks of managers is higher in a very bad market, than in a very good market. That is:

$$\lim_{s^m \to -\infty} \mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) > \lim_{s^m \to +\infty} \mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \tag{3.12}$$

The proof of this remark is quite simple. As the market becomes really good, the probability of a manager investing in his own project goes to zero, and hence from (25) we see that the probability of a rank change will tend to zero. In contrast, for a very negative market signal, this probability is strictly negative, since $\mathbb{P}(0, 0|s^m) = 1$ and $\mathbb{P}(\varphi_0^1 > \varphi_0^2 \,|\, s^m) > 0$[12];

From the above analysis, it is clear that the overall effect does not have to be monotonic in s^m. Hence we use simulations to illustrate the properties of the probability of interest as a function of the market signal, confirming also the observation in the aforementioned remark[13].



[12]This probability is always strictly positive, since we know that $\varphi_0^1(r^1, s^m) \to 1$ as $r^1 \to +\infty$ and $s^m \to -\infty$, or intuitively the return of manager 1 may be much larger than that of manager 2 when they invest in their own projects (either because one has high skill and the other has low or because one is just luckier than the other) and hence this can always lead in a change of ranks.

[13]For this simulation we set the parameters as: $\pi^1 = 0.6$,$\pi^2 = 0.601$, $\alpha^H = 0.16$, $\alpha^L = 0.1$,$\sigma = \nu = 0.35$, $f^1 = f^2 = 0.01$, $\sigma_m \nu_m = 0.25$, $\lambda_1 = \lambda_2 = 0.8$ and $\delta = 0.5$.

On the y-axis we have the probability of change in rank, and on the x-axis the corresponding market signal. As it can been seen from the graph the total effect is hump-shaped in s^m, it is decreasing as the market signal becomes relatively large and also it is smaller when market conditions are good compared to when they are bad.

In the next section, we find empirical evidence supporting our results. This is done by constructing divisions in which each fund is allocated in accordance with their AUM. Subsequently, we calculate the proportion of funds that changed division from the beginning of each period to its end. Approximately, this measures the probability to which the above proposition refers.

3.4 Empirics

3.4.1 Data

The data used in this study comes from Morningstar CISDM database. The time span of our sample is from January 1994 to December 2015. To mitigate survivorship bias we include defunct funds in the sample. We have created a larger group of strategies to accumulate the Morningstar's categories. All fund returns have been converted to USD (U.S dollars) using the exchange rates of each period separately. Observations of performance or assets under management, with more than 30 missing values, have been deleted. All observations are monthly. Our main variable of interest is flows, which gives the proportional in and out flows of the fund with respect to its assets under management. For the market return we consider the S&P 500 and as fund excess returns, the difference of the funds return with the market. In particular, we use the corresponding Fama-French market factor obtained from the WRDS (or from Keneth French's website at Darmouth). We also examine the relationship of alpha and beta of a fund as well as their relationship to the flows.

3.4.2 Empirical Evidence

The purpose of this section is to empirically test some of the assumptions as well as the results of our model and show that our model can be empirically supported by data. Throughout this section we will use for simplicity the CAPM alpha and beta, calculated using a 32 months period (which we will define in this section as *one period*)[14]. Moreover we will to the log of the assets of a fund lagged by one period, simply as the fund's *assets*. First of all, our model assumes that investors get a signal about the market (s^m) before everyone else

[14]We have also performed robustness checked using the 4-factor alphas and betas.

does. This would imply some form of market-timing [15]. We first run the following panel regression, with fixed effects:

$$\text{Beta}_t = \lambda_0 + \lambda_1 r_{m,t} + \lambda_2 \text{Assets}_{t-1} + \lambda_3 \text{Age}_t + d_i + \varepsilon_t$$

where $r_{m,t}$ is the period market return (described above) and d_i corresponds to the fixed effects dummy (although the subscript i for the fund has been suppressed in the rest of the variables). The results are shown below:

Table 3.1: Estimation results : Beta on Market Return.

The baseline model we run is summarized by beta $\sim r_m$ + assets + controls.

Variable	**Coefficient**	**(Std. Err.)**
r_m	**0.03256***	(0.01372)
assets	0.01467**	(0.00508)
age	0.00502**	(0.00144)
Intercept	0.02430	(0.08400)

Significance levels : † : 10% $*$: 5% $**$: 1%

The positive and significant coefficient in front of the market return supports our model assumption (as well as with the prediction of Proposition 3.3 about over-investment), in the sense that it indicates that when markets are bull, it is more likely that managers choose to get higher exposure to the market. This is consistent with what we would observe if indeed managers had market-timing abilities.

Another result of our model is that in equilibrium $l > h$. Given the definition of the cutoff equilibrium strategies described in (5), this leads to: $P(\beta = 1|L) > P(\beta = 1|H)$. If this is the case, we would expect to see in data that funds with higher alpha, have on average lower betas, i.e they choose to invest on their idiosyncratic project since they benefit both from potential higher returns thanks to their superior alpha as well as from signalling their skill. Indeed this is the case. We are using the following cross-sectional baseline model, for the last date in our data, December 2015[16]:

$$\text{Alpha}_t = \lambda_0 + \lambda_1 \text{Beta}_t + \lambda_2 \text{Assets}_{t-1} + \Lambda_3 \text{Controls} + \varepsilon_t,$$

where controls include the age and the strategy of the fund. As shown in Table 2 the coefficient of interest is negative, suggesting that more skilled managers pick a high beta less often.

Even more importantly, we want to test the second implication of Proposition 3.5. That

[15] In the empirical literature there have been studies both in favour as well as against this finding.
[16] We only include funds that report US dollars as their base currency.

Table 3.2: Cross-sectional Regression of Alphas on Betas and controls, t = 12/2015.The baseline model we run is summarized by alpha $\sim$ beta + assets + controls.

Variable	**Coefficient**	**(Std. Err.)**
beta	**-0.00958****	(0.00085)
assets	0.00006	(0.00016)
age	0.00001	(0.00005)
strategy	0.00003	(0.00009)
Intercept	0.00130	(0.00284)

Significance levels : † : 10% $*$: 5% $**$: 1%

is, we want to test whether the data suggest that the sensitivity of flows to performance is higher when beta is 0, or consequently is higher when markets are bear than when they are bull. We will measure the fund flows, as in Sirri and Tufano (1998):

$$\text{Flows}_t = \frac{TNA_t - (1 + R_t)TNA_{t-1}}{TNA_{t-1}}$$

where TNA is the total net assets and R is the return of the fund. We will use the simple return of the fund, r_i, as the measure of performance, as in Clifford et al. (2013). We think that this is the most appropriate measure of performance to test the predictions of our model. The following two tables[17] verify the above finding, and support our predictions[18]. First regression is a cross-sectional one for December 2015.

$$\text{AvFlows}_t = \lambda_1 r_{i,t} \cdot \text{Bigbeta}_t + \lambda_2 \text{Assets}_{t-1} + \Lambda_3 \text{Controls} + \varepsilon_t$$

where *AvFlows* is the average flows of the previous period, Bigbeta $= 1_{\{\beta \geq 0.3\}}$, $r_{i,t}$ is the fund's period return and controls include the age, the strategy and the bigbeta dummy of the fund (the intercept λ_0 is just suppressed in the above equation).

The second table we are presenting is a panel regression with *fixed effects*, where we regress flows on the interaction of annual fund's performance and market return, including the usual controls. That is, our baseline model is:

$$\text{AvFlows}_t = \lambda_1 r_{i,t} \cdot r_{m,t} + \lambda_2 \text{Assets}_{t-1} + \Lambda_3 \text{Controls} + d_i + \varepsilon_t$$

where controls include the fund's beta and the period return of the market and of the fund itself.

[17]Since in our model, the funds only select between $\beta = \{0, 1\}$, thus making the implicit assumption that there is no short-selling of the market, we will exclude all observation with negative β, which are anyway less than 15% of our sample.

[18]This result was only recently documented empirically in a paper by Franzoni and Schmalz (2013).

Table 3.3: Flows on Performance and Beta, $t = 12/2015$

Variable	**Coefficient**	**(Std. Err.)**
r_i·Bigbeta	**-0.12510****	(0.03433)
Bigbeta	0.01204	(0.01522)
assets	-0.01437**	(0.00389)
strategy	0.00352	(0.00231)
age	-0.00133	(0.00120)
Intercept	0.25846**	(0.06906)

Significance levels : † : 10% $*$: 5% $**$: 1%

Table 3.4: Flows on the interaction of Fund Performance and Market Return

Variable	**Coefficient**	**(Std. Err.)**
$r_i \cdot r_m$	**-0.15297****	(0.03697)
beta	0.00423	(0.00648)
r_i	0.07538**	(0.02036)
r_m	0.02828**	(0.00955)
assets	-0.02718**	(0.00224)
Intercept	0.47264**	(0.03964)

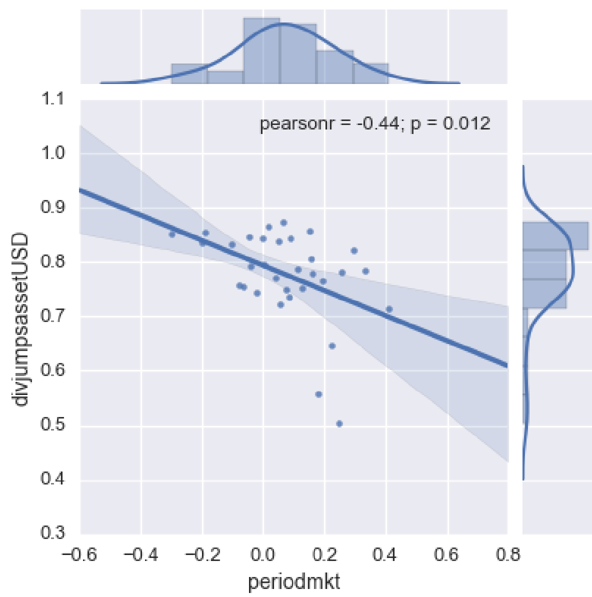
Significance levels : † : 10% $*$: 5% $**$: 1%

In both cases we can see that the coefficient of interest is significantly *negative*. The interpretation of these two regressions is the following: the first one shows that funds with higher beta are not judged so much on their performance; that is the higher the beta, the less important is the flow performance relationship. The second table, on the other hand supports the statement that the sensitivity of fund flows to performance depends on the state of the market and more specifically it is decreasing on the market return. Under the predictions of our model, these two results are almost equivalent, and we indeed get that the coefficient in both cases is negative and significant, thus supporting one of our main results as well.

Finally, we want to provide some empirical evidence relevant to the discussion in on the competition of funds. Namely, we find support for Remark 1, by demonstrating that the probability of changes in the ranking of funds, with respect to their AUM, is higher under adverse market conditions. To achieve this a new variable is constructed. First, the sample is separated in periods of eight months, so that we have thirty periods in total. For each one, seventy divisions (clusters of funds) are created. Funds are allocated in those divisions according to the size of their AUM at the end of each period[19]. Then we define

[19] Out methodology follows closely previous work done by Marathe and Shawky (1999) and Nguyen-Thi-Thanh (2010).

divjumpassetUSD$_t$ as the percentage of funds that changed division from the beginning to the end of period t. We are careful to only compare funds that were active during all the duration of each period. Also, we only consider the US universe of funds to avoid introducing noise created from fluctuations in the exchange rates.



On the y-axis we have our constructed measure of changes between divisions *divjumpassetUSD*$_t$, and on the x-axis the corresponding total return of the market portfolio during the same period. As it can been seen from the graph there appears to be a negative relationship between the two, which is also statistically significant. Note that this is just an indication of the relationship between the rank of funds and the market conditions, under a simple linear regression, and thus it does not capture any second order effects (or a hump-shaped relationship). Hence, this is only weak evidence supporting our prediction in Remark 1, but we believe that there is much more to explore in the future in this direction.

3.5 Extension: Unobservable Investment Decision

In this section, we want to extend our model, and investigate the equilibrium where the investment decision of the fund managers cannot be observed by the investors. In this case, investors use the return of the fund managers' to both update their beliefs about managers' skill but also to understand whether or not they invested in their own project. In reality, it is indeed the case that investors do not know exactly the exposure of a fund manager to the systematic risk. They use instead a history of data of the fund return's comovement with the market return to infer the fund's statistical *beta*. Since the model we are examining here is static, the assumption in this section is that this inference is only made based on the proximity of the market return to fund's return.

The model considers only one period and it remains same as before, apart from a few changes outlined below. Firstly, an additional error ϵ has been introduced in order to make the manager's choice of investment unobservable by the investors- note, that without this tracking error, investors could perfectly observe the decision of managers based on whether or not $r = m$. Hence our model becomes:

$$\begin{aligned} r &= (1-\beta)\,a + \beta\,(m+\epsilon) \\ a &\sim \mathcal{N}(\alpha, \sigma^2\,) \\ m &\sim \mathcal{N}(\mu\,, \sigma_m^2) \\ \epsilon &\sim \mathcal{N}(0\,, \sigma_\epsilon^2) \end{aligned} \tag{3.1}$$

The manager's performance r is a weight average of the return of her idiosyncratic strategy a and that of the market m, and as before we study only the simple binary case where $\beta \in 0, 1$. The rest of the notation and ideas remain unchanged.

The posterior distribution of r, conditional on (β, s, s^m) is given by

$$\begin{aligned} r\,|\,\alpha, \beta, s, s^m &\sim \mathcal{N}\Big(\bar{r}(\alpha, \beta, s, s^m),\, \bar{\sigma}^2(\beta)\Big) \\ \bar{r}(\alpha, \beta, s, s^m) &\equiv (1-\beta)[(1-\psi)\alpha + \psi s] \\ &\qquad + \beta[(1-\psi_m)\mu + \psi_m s^m] \\ \bar{\sigma}^2(\beta) &\equiv (1-\beta)^2\psi\nu^2 + \beta^2(\psi_m\nu_m^2 + \sigma_\epsilon^2) \end{aligned} \tag{3.2}$$

Our goal is to study whether a monotonic cutoff equilibrium (introduced in the previous sections) exists under this alternative assumption. We believe that only such an equilibrium would be interesting and realistic to serve for further study. We move on to find a closed-form expression for the ex-post reputation φ, which is given by the following lemma.

Lemma 3.1. *The manager's posterior reputation is given by*

$$\varphi(r, m, s^m) = \left(1 + \frac{1-\pi}{\pi}\frac{\rho\big(r, L, l(s^m)\big)}{\rho\big(r, H, h(s^m)\big)}\right)^{-1}, \tag{3.3}$$

where

$$l(s^m) - h(s^m) = \frac{1-\psi}{\psi}(H - L), \tag{3.4}$$

and

$$\rho(r,\alpha,c) \quad = \quad \Phi\left(\frac{r-c(1+\psi)+\alpha\psi}{\nu\sqrt{1+\psi}}\right)\times\frac{\phi\left(\frac{r-\alpha}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}+\Phi\left(\frac{c-\alpha}{\nu}\right)\frac{\phi\left(\frac{r-m}{\sigma_\epsilon}\right)}{\sigma_\epsilon}, \quad (3.5)$$

Proof. In Appendix C.3. □

Using the above lemma, we can now eventually see whether this model can provide us with an equilibrium where the reputation $\varphi(r, m, s^m)$ is increasing in r. In fact, we get the following proposition:

Proposition 3.1. *A monotonic equilibrium under unobservable beta does not exist.*

Proof. In Appendix C.3. □

What this proposition shows is that the reputation $\varphi(r, m, s^m)$ cannot always be increasing in r under the assumption that investors do not observe the investment choices. That is to say that the assumption of unobservable investment choice under a static setting can lead us to counterintuitive equilibrium properties. We believe that in future research it could be interesting to study this realistic case under a dynamic setting where the inference of beta will be indeed based on the comovement of the market return with the fund's return.

3.6 Conclusions

The role of financial intermediaries and their characteristics have been greatly explored in the recent empirical literature. In this article, we have developed a theoretical model that describes how the strategic investment decisions of fund managers is influenced by their career concerns. In sum our argument is that those will tend to over-invest in market neutral strategies as a way to signal their ability. Moreover, we have described how managers' reputation depends on the market conditions; in particular, we find that the sensitivity of flows to performance is higher in bear markets than in bull markets and we discuss the competition between funds, measured by the changes in their rankings, as a function of the market conditions. Our model entails predictions about some directly observable fund characteristics such as their size and fees, as well as some indirectly observable quantities such as their reputation or their investment behavior depending on their signals. In our empirical section, we have managed to find support for many of the assumptions as well as predictions of our model. Moreover, we have extended our model to include the case when the manager's investment decision is not observable from the investors.

Finally, there are many ways forward with this research. The results of this model do not depend on the specific factor which funds use when they are tracking an index; one, may try to apply the same logic in funds that use factors other than the market return and test the corresponding empirical predictions. Also, using a slightly different interpretation of the investor's decision between allocating funds to a manager or to the market, one could think of an investor choosing between an active and a passive fund and use the closed form solution for fund's size, to see how the relative (total) size of the passive and active funds, depends on the market conditions.

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Chapter 4

Contracting on the Managerial Aftermarket: Market Sophistication and Termination

Our goal is to demonstrate that the relative job security that CEOs enjoy can be partly attributed to the high sophistication of the managerial labour market. To do this we build a theoretical model in which a representative investor proposes a contract to a manager, which also specifies the conditions of his termination. Production is a function of the manager's effort and ability, both of which are his private information. The former is a choice variable, whereas the latter follows a Geometric Brownian motion. The manager's post-termination payoff is generated by an exogenous managerial labour market, and it is equal to his expected ability. The market learns his ability with some given probability, which we interpret as its sophistication. Otherwise, it forms its posterior based on his termination time. Our main result is that the more sophisticated the market is, the more lenient the manager's contract becomes, which results in a longer tenure. To prove this we demonstrate that the investor's revenue maximisation problem encompasses a stopping problem. Its solution is to fire the manager when he reports that his ability is below a cutoff. We present a contract that implements the optimal stopping time, this uses a golden parachute to induce the manager to admit his incompetence.

4.1 Introduction

Should the relative job security of even under-performing CEOs be interpreted as a sign of management entrenchment? We definitely acknowledge that this is a significant contributing factor. Nevertheless, some of the instruments that protect an incompetent CEO, such as his severance package (golden parachute), are often negotiated before the beginning of his employment. Here we present the sophistication of the managerial labour market as another factor that should be considered. We define this sophistication to be the degree to which the managerial labour market can independently figure out the CEO's ability. We demonstrate that a highly sophisticated market will push a firm to offer a lenient contract to a CEO, even when it has all the bargaining power. This will protect an under-performing manager, and will lead to a longer expected tenure.

Our argument is the following. The post-termination career of a manager (he), whose ability is his private information, is also important for the representative investor (she) that employs him. This is because the more reputation the manager retains after being terminated, the higher his post-termination payoff. Hence, it is cheaper for the investor to initially attract him. However, the manager's post-termination payoff is not only a function of his reputation, since a sophisticated market will also be able to make some independent inference on his ability. But then the more sophisticated the market is, the less the investor knows about the manager's value in it. This is because the manager's ability is his private information, while his reputation is not. Therefore, an increase in the market's sophistication makes it harder for the investor to utilise the manager's post-termination career in order to reduce his current compensation. This decreases the value of terminating the manager for the investor, which results in the retainment of even a relatively under-performing manager.

It follows then that these lenient termination rules will result on only highly incompetent managers being fired and longer tenures. Interestingly, our intuition implies that the adoption of better corporate governance practises and an improvement on the ability of boards to screen their potential hires might have an adverse effect on the quality of available experienced managers.

More generally, our analysis investigates the effect of a CEO's termination on his reputation. The latter is interwoven with the value of his company, since it affects the behaviour of consumers and investors alike[1] . Henceforth, the conditions under which a CEO is fired and their effect on his reputation are an important consideration when negotiating his contract[2] In particular, our analysis focuses on deriving the reputational loss that a fired CEO incurs and showing how this depends on the length of his tenure.

To achieve the above we consider a model in which a representative investor makes a take-it-or-leave-it contractual offer to a manager. The manager's ability is his private information.

[1]For example see the following report from www.webershandwick.com.

[2]There have even been legal cases in which former CEOs have requested to be compensated for a loss in reputation. A nice article that covers both this and golden parachutes is that Howard Levitt's in the Financial Post.

Its initial value is generated by some fixed distribution and subsequently it evolves according to a geometric Brownian motion. The offered contract also specifies a cutoff on the reported ability, below which it is terminated. When fired the manager joins a managerial labour market, in which his payoff equals his expected ability. This market can independently figure out the manager's ability with some given probability, which we interpret as its degree of sophistication. However, when his ability remains private the market has to update its prior on his expected ability based on the realised termination time. We interpret this posterior expectation as the manager's reputation. Hence, the manager's post-termination expected payoff is a linear combination between his ability and reputation.

To derive our main results we calculate the revenue maximising cutoff in closed-form. We show that this is decreasing on the market's degree of sophistication[3]. Hence, a more sophisticated market will push the investor towards offering a more lenient contract to the manager. In addition, because the manager is fired on a lower ability level, this will result on a decrease on his competency when entering the labour market. Finally, we use the revenue maximising cutoff to demonstrate that an increase on the market's sophistication will also lead to a longer expected tenure for the manager.

Note that even though all three of the above results are phrased as a comparative static for the same market, they can also be interpreted as a comparison between different markets. In particular, as we explained above our aim is to make the argument that a highly sophisticated labour market, such as the managerial one, will tend to be dominated by more lenient contracts. Nevertheless, we are aware that this market also differs in many other aspects, however here we focus only on the affect of the market's sophistication on job security.

Another aspect of our analysis that is novel within this context is the informational value that termination times acquire. This is because the market updates its belief on the manager's ability based on the decision to fire him. It is unavoidable then that terminating a manager has a negative impact on his reputation. This also has an adverse effect on the investor's profits, since she captures part of the manager's post-termination payoff. On the other hand, continuing the employment of an incompetent manager is also inefficient. Hence the revenue maximising termination time has to balance those effects. We demonstrate that

[3]A more technical explanation of our main result is the following. The manager's post-termination payoff can be interpreted as an output of his partnership with the representative investor. Hence it is subject to the same inefficiencies that information asymmetry creates on production. What's more those inefficiencies are increasing in the dependency of the post-termination payoff on the manager's type, because this is his private information. Therefore, the higher the relative importance of the manager's actual ability, versus his reputation, the less valuable termination becomes for the investor. For example, if this payoff was purely based on the manager's reputation, then there would be no information asymmetry between the two with regards to the manager's post-termination payoff, and this would be the case where the value of termination would be the highest for the investor.

the optimal termination policy is a cutoff rule, that is the manager is fired when his ability falls below a certain cutoff, which we provide in closed-form.

A final contribution of our analysis is to demonstrate that Golden Parachutes fulfil an essential part of implementing the optimal termination time, which is to incentivise the manager to admit his incompetence. This is because the manager has superior knowledge on his ability to deliver profits, whereas the investor can only indirectly monitor how this ability evolves. Therefore, for the former to give up the possibility of future wages and take the associated penalty on his reputation it has to be that he is compensated by some other financial reward.

Our work is related to the vast literature of career concerns, due to the agent's post-termination payoff. What separates our analysis from this literature is that it mainly examines situations in which the agent does not have private information at the beginning of his employment. For example, the seminal contribution of Holmström (1999) assumes that the agent's ability is unknown by both him and the principal, and that the former attempts to influence the principal's learning process with his private action. A paper more relevant to our discussion is that of Gibbons and Murphy (1992). They consider a setup similar to Holmström's, however they allow the principal to write one-period incentive contracts. Their main result is that the closer to the end of his career the agent is, the more performance based his compensation becomes. This is because the agent's career concerns decrease, therefore the principal has to rely on direct incentives to induce performance. More recently, Prat and Jovanovic (2014) considered a similar setup but they allowed for long-term contracts. In contrast to the above two papers they find that effort is lower at the beginning of the agent's employment, because the initial uncertainty over his ability leaves a lot of room for him to manipulate the principal's beliefs.

A branch of this literature that is directly related to our paper allows for contract termination. Guriev and Kvasov (2005) consider an environment without information asymmetry, in which they demonstrate that efficient investment in the surplus of a contractual relationship can be achieved even with incomplete contracts. Milbourn (2003) considers a CEO who is replaced when the shareholder's posterior belief on her ability is below some exogenously set threshold. He shows that the higher the CEO's prior reputation is, the more performance based the optimal linear contract becomes. Another related paper is that of Demarzo and Sannikov (2016) in which they characterise the optimal stopping time of a principal-agent relationship in which both counter-parties gradually learn about the future profitability of their project. Finally, Madsen (2016) considers a model in which the agent possesses superior private information on when a project, which is always initially profitable, should be abandoned. He shows that the firm's optimal contract entails a deadline and a time depended golden parachute for the agent.

Due to its topic our paper is related to the vast literature that examines CEO compensation (for example see Edmans et al. (2012) and He et al. (2017)) and especially to those papers that focus on managerial turnover (Eisfeldt and Kuhnen (2013), Gayle et al. (2015), Hakenes and Katolnik (2017), Vasama (2016), Jenter and Lewellen (2017), Taylor (2010)).

Our work is also related to the literature of Dynamic Mechanism Design. This commonly considers an agent who has ex ante private information on his type, which evolves stochastically over time. The interested reader can find an excellent review of this literature on Pavan et al. (2014), which also unifies and solves a significant part of the earliest attempts. They provide methods to identify the contract that solves the above problem and results in the type of contracts that can be implemented. Eső and Szentes (2017) do the same but for a setting which includes both adverse selection and moral hazard. Bergemann and Strack (2015) analyse a continuous time specification of the former paper. Williams (2009, 2011) also considers a continuous time principal-agent model with both hidden actions and hidden states, but focuses on a less general model.

Our analysis is especially related to the branch of the aforementioned literature that also allows for stopping times. The most closely related paper is that of Garrett and Pavan (2012). Similar to us they consider a principal-agent relationship where only the agent observers his evolving ability, however they assume that both the principal's and the agent's post-termination payoffs are constants. They show that the principal's optimal contract entails cutoffs on the reported ability, below which the agent is fired. In addition, under common assumptions those cutoffs will be non-increasing in the history of reported abilities, and non-increasing over time. In other words, higher past ability or longer tenure result into more lenient current termination policies. This is because the inefficiency that information asymmetry creates in production is decreasing in both the agent's past history and his tenure. Another relevant paper is that of Kruse and Strack (2015). They also consider a model in which the agent's employment can be terminated and demonstrate how such a stopping rule can be implemented.

The rest of the paper is organized as follows. Section 4.2 introduces our model. The main analysis is undertaken in section 4.3. Finally, section 4.4 discusses two interesting extensions, the analysis of which can be found in the appendix, and section 4.5 concludes.

4.2 Model

This is a continuous time model $t \in [0, \infty)$. A representative investor, also referred to as the principal (she), makes a take-it-or-leave-it contractual offer to a manager, to whom we will refer to as the agent (he). The offered contract also specifies the conditions under which it will be terminated. At the point of its termination the principals switches to her outside option, which generates lump sum payoff $\omega_p \geq 0$. The agent also receives a lump sum payoff from a labour market (it), but only if he accepted the principal's offer[4]. Otherwise, the agent gets his outside option which gives lump sum payoff $\omega_a \geq 0$. Both the principal and the agent are risk neutral and discount the future with rate $r > 0$.

While being employed by the principal the agent produces for her flow payoff

$$y_t \;=\; \sqrt{a \cdot \theta_t} \cdot e_t$$

where $e_t \in [\,0,\, \sqrt{\kappa\,\theta_t}\,]$ is the flow level of effort chosen privately by the agent with flow cost $(e_t)^2/2$. We assume $\kappa > 1$, so that the surplus maximising level of effort is feasible. The agent's ability θ_t is also his private information and it evolves according to the Geometric Brownian motion

$$\mathrm{d}\theta_t \;=\; \theta_t \cdot \mu\,\mathrm{d}t + \theta_t \cdot \sigma\,\mathrm{d}B_t$$

where $\mu < r$ and B_t denotes the standard Brownian motion. The initial type θ_0 is drawn from the interval $[\,\underline{\theta},\, \overline{\theta}\,]$ and $\underline{\theta} > 0$. In addition, it is distributed according to smooth CDF $F(\theta_0)$ with density $f(\theta_0) > 0$. Following most of the literature, we assume that the inverse hazard rate $[1 - F(\theta_0)]/f(\theta_0)$ is non-increasing in θ_0.

To introduce career concerns suppose that the agent's payoff from this market equals some parameter $\Lambda > 0$ plus the market's inference on his current ability, that is his ability at the point of his termination. In particular, the market learns the agent's current ability with probability $\lambda \in [0, 1]$. We interpret λ as a measure of the labour market's sophistication. The only other information that the market acquires is the termination time of the agent's contract and the menu that was offered by the principal[5]. Hence it observers neither the agent's reports in this menu, nor the agent's production and wages.

What makes the model interesting is that even when the market does not learn the agent's current ability directly, it still updates its prior based on the time of termination. To be more specific, let h^t denote the principal's private history up to and including t, and $c_t(h^t) \in \{0, 1\}$ her decision to continue her partnership with the agent, with $c_t(h^t) = 1$ standing for continuation. This decision is a deterministic mapping of the stochastic history

[4]This is mainly for simplicity. As long as the firm is sufficiently productive most of our analysis holds even if the agent was able to join the labour market immediately.

[5]We could equivalently assume that the principal can disclose this mechanism. In addition, we could even allow her to disclose information on the agent's reports, however this would not alter our results, because the agent's payoff will turn out to be a linear function of the market's posterior on his ability.

h^t, which results in the following stochastic termination time.

$$\tau(c) \;=\; \inf\left\{\, t \geq 0 \,:\, c_t(h^t) = 0 \,\right\} \qquad \text{where} \qquad c = \{c_t(\cdot)\}_{t\geq 0}$$

For simplicity we assume that the agent has no limited liability. However, it will become obvious that for a wide set of parameters this will not be a binding constrain. The principal can fully commit to the contract she is offering, however the agent will not be able to commit in advance to staying in this contract.

The revelation principle holds, hence it is without loss to focus on direct and incentive compatible mechanisms. Let $\hat{\theta}_t$ denote the agent's report at t and $\hat{\theta}^t$ the whole reported path up to t. Therefore, the principal offers to the agent contract

$$\left\{\, w_t(\hat{\theta}^t),\, W_t(\hat{\theta}^t),\, e_t(\hat{\theta}^t),\, c_t(\hat{\theta}^t) \,\right\}_{t\geq 0}$$

which for each point t specifies the flow and lump sum wages, the recommended level of effort, and the termination policy as a function of the reported path $\hat{\theta}^t$.

To sum up the timing of our model is as follows. At the beginning of time the principal makes a take-it-or-leave-it offer to the agent. If the offer is rejected, then they both switch to their outside options. If it is accepted, then the agent starts reporting the path of his ability θ^t, which the contract translates into a corresponding level of production. Whenever the reported path implies a termination, that is $c_t(\hat{\theta}^t) = 0$, the agent's post-termination payoff is generated from the market, whereas the principal's continues to be equal to her outside option.

4.3 Analysis

Our analysis proceeds in the following way. In section 4.3.1 we derive an expression for the agent's post-termination payoff and solve the principal's first best problem. In section 4.3.2 we derive a representation of the principal's expected discounted revenue that does not depend on wages $w_t(\cdot)$ and $W_t(\cdot)$. We achieve this by imposing a restriction on the agent's action space, which makes our model equivalent to one where only his initial type is his private information. In section 4.3.3 we use this representation to characterise the optimal termination time of the agent's employment. This will also give us the principal's revenue under the imposed restriction. Section 4.3.4 presents wages that will implement the derived termination time and revenue, even without using the aforementioned restriction.

4.3.1 First Best

In the first best the principal can directly observe the agent's type, but for the model to be intersting we will continue to assume that the market only observes it with probability λ. Hence, with probability $1-\lambda$ it updates based only on the realised termination time. For the latter case let the market's posterior expectation on the ability of an agent that was terminated at time t be $m_t(c) = \mathbb{E}_{\theta_t}\big[\,\theta_t \,\big|\, \tau(c)=t\big]$, which we interpret as his reputation. Hereafter we will suppress the dependence of $\tau(c)$ and $m_t(c)$ on c in order to maintain a compact notation. It follows then that the agent's post-termination payoff can be written as

$$M(m_t\,,\,\theta_t) \;=\; \Lambda \,+\, (1-\lambda)\cdot m_t \,+\, \lambda\cdot\theta_t \tag{4.1}$$

Hence the more sophisticated the market is, the less the agent's post-termination payoff depends on his reputation[6].

To state the principal's revenue maximisation problem in the first best note that we have endowed her with all the bargaining power. Hence in the first best she captures all the expected value of the agent's production. Moreover, the principal also captures the agent's post-termination payoff, because he only gets access to the managerial market through her. Therefore, she solves

$$\max_{e,c,\theta^*} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_\theta\left[\int_0^\tau e^{-rt}\cdot\left(\sqrt{a\,\theta_t}\cdot e_t(\theta^t) - \frac{e_t(\theta^t)^2}{2}\right)\mathrm{d}t + e^{-r\tau}\cdot\Big(\omega_p + M(m_\tau\,,\,\theta_\tau)\Big)\,\middle|\,\theta_0\right]\mathrm{d}F(\theta_0)$$
$$-\;[1-F(\theta^*)]\cdot\omega_a \,+\, F(\theta^*)\cdot\omega_p \quad (\mathcal{P}_f)$$

where θ^* is the lowest initial type with which the principal contracts. This will be ignored in the following analysis since it is not of particular interest.

Note that the above expression first integrates over the initial type with measure $F(\theta_0)$, and then takes a conditional expectation over its evolution. It will become apparent in the next section that it is very useful to separate those two components of the agent's ability. Hence to achieve this let $z_t = \theta_t/\theta_0$ denote the proportional change of the agent's type from time zero to t. Therefore, z_t also follows a geometric Brownian motion, however its initial value is normalised to one.

The point-wise maximisation of production gives that the optimal level of effort is $e^f_t(\theta_t) = \sqrt{a\,\theta_t}$. Substitute this into ($\mathcal{P}_f$), and ignore the choice of θ^* to obtain that the

[6] An implicit assumption we have made is that the agent's reputation is only based on his termination time. An alternative model could also allow for some noise signal of the agent's performance while employed to be available to the market.

principal solves

$$\max_{c} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_z \left[\frac{a\,\theta_0}{2} \cdot \int_0^{\tau} e^{-rt}\, z_t \,\mathrm{d}t + e^{-r\tau} \cdot \Big(\omega_p + \Lambda + (1-\lambda)\, m_\tau + \lambda\, z_\tau\, \theta_0 \Big) \right] \mathrm{d}F(\theta_0) \qquad (\mathcal{T}_f)$$

If it wasn't for m_τ, the value of which is affected by the choice of the whole correspondence of the stopping rule $c(\cdot)$, this would be a canonical stopping problem. Nevertheless, the following lemma demonstrates this ($\mathcal{T}_f$) can be reformulated as such a stopping problem.

Lemma 4.1. *($\mathcal{T}_f$) is equivalent to*

$$\max_{\tau} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_z \left[\frac{a\,\theta_0}{2} \cdot \int_0^{\tau} e^{-rt}\, z_t \,\mathrm{d}t + e^{-r\tau} \cdot \Big(\omega_p + \Lambda + z_\tau \cdot \theta_0 \Big) \right] \mathrm{d}F(\theta_0) \qquad (\mathcal{T}_f')$$

Proof. In Appendix D.1. □

Interestingly, this is not a function of the market's sophistication, a property which will also pass to the solution of this stopping problem. Let

$$c \;=\; \left(\frac{\mu}{\sigma^2} - \frac{1}{2} \right) \,+\, \sqrt{ \left(\frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 \,+\, 2\,\frac{r}{\sigma^2} } \tag{4.2}$$

where $c > 0$ is only a function of parameters.

Proposition 4.1 (First Best)**.** *The optimal termination time is*

$$\tau^f(\theta_0) \;=\; \inf \big\{ t \geq 0 \,:\, z_t \,\leq\, q^f(\theta_0) \big\} \tag{4.3}$$

where

$$q^f(\theta_0) \;=\; \frac{c}{1+c} \cdot \frac{\omega_p + \Lambda}{\frac{a/2}{r-\mu} - 1} \cdot \frac{1}{\theta_0} \tag{4.4}$$

if $a > 2(r-\mu)$. Otherwise, $\tau^f(\theta_0) = 0$. In addition, $q^f(\theta_0)$ is decreasing in θ_0 and it is not a function of λ.

Proof. In Appendix D.1. □

The most important observation that we can make is that in the absence of information asymmetry, between the principal and the agent, the market's sophistication does not affect the termination decision. As we explained in the introduction, this is because this dependency is due to the inefficiencies that the information asymmetry creates during the contracting process.

A second interesting observation is that if the production process is sufficiently productive, i.e. $a > 2(r-\mu)$, then all initial types θ_0 are terminated on the same ability level. To see this let $Q^f(\theta_0) = \theta_0 \cdot q^f(\theta_0)$ be the terminal ability of initial type θ_0, that is his ability on $\tau^f(\theta_0)$. Then (4.4) gives that

$$Q^f(\theta_0) \;=\; \frac{c}{1+c} \cdot \frac{\omega_p + \Lambda}{\frac{a/2}{r-\mu} - 1}$$

which is not a function of θ_0. This is another property that will not hold in the second best, because the principal will use termination as a tool to decrease the agent's information rents. This is similar to the effect that information assymtery commonly has on the revenue maximising level of effort.

4.3.2 Revenue equivalence

For the analysis of this section it suffices that $M(m_t, \theta_t)$ is a twice continuously differentiable function, increasing in both the agent's ability and reputation at the point of his termination. For technical reasons we will also require that its partial derivative with respect to its second input is bounded above by some positive constant. In section 4.3.3 we will switch back to the linear version of $M(m_t, \theta_t)$ in order to derive our main results.

We start our analysis by showing that the principal's revenue can be written as a function of only production and termination relevant policies. Those are the recommended level of effort $e_t(\cdot)$ and the continuation decision $c_t(\cdot)$.

First, we impose a restriction on the agent's action space. Second, we use this restriction to derive a necessary representation of the agent's payoff that will only be a function of policies $e_t(\cdot)$ and $c_t(\cdot)$. Third, we equate this representation with one that is a function of wages. Fourth, we use this equality to calculate $w_t(\cdot)$ and $W_t(\cdot)$ on the principal's revenue.

The aforementioned restriction is that the agent can only use *consistent* deviations.

Definition. We call a deviation *consistent* if the agent is able to only misreport his initial type θ_0 and in addition is restricted to

- truthfully report the proportional change of his ability $z_t \;=\; \theta_t/\theta_0$
- and to mask any such misreport $\hat{\theta}_0 \in [\underline{\theta}, \overline{\theta}\,]$ by exerting flow effort

$$\hat{e}_t(\hat{\theta}_0, \theta^t) \;=\; e_t(\hat{\theta}^t) \cdot \sqrt{\hat{\theta}_0/\theta_0}$$

We say that the effort $\hat{e}_t(\hat{\theta}_0, \theta^t)$ "masks" the corresponding consistent deviation because it makes the agent's flow output $\sqrt{a\theta_0} \cdot \hat{e}_t(\hat{\theta}_0, \theta^t)$ equal to that of an agent whose initial types was indeed $\hat{\theta}_0$ and had the same path realisation of z_t. An equivalent interpretation of the above restriction is that the proportional change z_t is directly observable by the principal, whereas θ_0 is the agent's private information. This is similar to the two period example we used to introduce our model.

Let $\widehat{\tau}$ denote the termination time that is implied by $c_t(\hat{\theta}^t)$ under a consistent deviation. Similarly, we will write $z_{\widehat{\tau}}$ to denote the implied proportional change on which the contract is terminated. When the termination time is implied by truthful reporting we will instead use τ and z_τ, respectively. In this section, we will assume that the agent can commit on leaving the contract only in accordance with $\hat{\tau}$. The next section will demonstrate that this restriction is also not important for our results. Hence under a consistent deviation the agent's type when exiting the contract and entering the market is given by $\theta_0\, z_{\widehat{\tau}}$.

Then the expected discounted payoff of an agent that reported $\hat{\theta}_0$, while his true initial type was θ_0 is given by

$$\widehat{V}(\hat{\theta}_0, \theta_0) \;=\; \mathbb{E}_z\left[\int_0^{\widehat{\tau}} e^{-rt} \cdot \left(w_t(\hat{\theta}^t) - \frac{e_t(\hat{\theta}^t)^2}{2} \cdot \frac{\hat{\theta}_0}{\theta_0}\right) \mathrm{d}t \;+\; e^{-r\widehat{\tau}} \cdot M(m_{\widehat{\tau}}\,,\, \theta_0\, z_{\widehat{\tau}})\right] \tag{4.5}$$

Hence the agent's payoff has two parts. The first is a flow payoff up to the stochastic termination time $\widehat{\tau}$. The second is his post-termination payoff, which is generated by the market. It is useful to point out that the true initial type θ_0 appears only on the ratio that multiplies the cost of effort and in the market payoff. In particular, the actual polices and wages depend only on the reported type $\hat{\theta}_0$.

Next, we want to argue that there is a unique cutoff $\theta^* \in [\,\underline{\theta}\,,\, \overline{\theta}\,]$ above which all initial types accept the principal's offer. To see why this has to be true consider $\tilde{\theta}_0 > \theta_0$ and suppose that θ_0 accepts the principal's offer. Then $\tilde{\theta}_0$ can also accept the offer and deviate to reporting θ_0 as his initial type. Then he will obtain exactly the same expected wages as type θ_0, but with smaller cost of effort and higher market payoff. Hence he will also accept the offer. Then we conclude that the principal is restricted to contracting with initial types above a certain enodgenous cutoff θ^*.

Hence, the principal's revenue maximisation problem under consistent deviations, is

$$\max_{w,e,c,\theta^*} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_z\left[\int_0^{\tau} e^{-rt} \cdot \left(\sqrt{a\,\theta_0\, z_t} \cdot e_t(\theta^t) - w_t(\theta^t)\right) \mathrm{d}t + e^{-r\,\tau} \cdot \omega_p\right] \mathrm{d}\theta_0 + [1 - F(\theta^*)] \cdot \omega_p \tag{$\mathcal{P}$}$$

subject to

$$\theta_0 \;=\; \underset{\hat{\theta}_0}{\operatorname{argmax}}\; \widehat{V}(\hat{\theta}_0, \theta_0)\,, \quad \forall\; \theta_0 \in [\,\underline{\theta},\, \overline{\theta}\,] \tag{IC}$$

We want the derive a representation of the objective function of ($\mathcal{P}$) without the payments $w_t(\cdot)$. To achieve this let $V(\theta_0) = \widehat{V}(\theta_0, \theta_0)$ denote the agent's payoff on path. Also, let $M_2(\cdot)$ denote its partial derivative with respect to its second input. Then (IC) together with a generalised version of the envelop theorem implies the following.

Lemma 4.2 (Envelop Theorem)**.** *The agent's on path payoff is absolutely continuous and has the weak derivative*

$$V'(\theta_0) \;=\; \mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot\frac{e_t(\theta^t)^2}{2\,\theta_0}\,\mathrm{d}t \,+\, e^{-r\tau}\cdot z_\tau\cdot M_2(m_\tau\,,\,\theta_\tau)\Big] \tag{4.6}$$

Proof. In Appendix D.1. □

Therefore, we are able to write $V(\theta_0)$ in two possible ways. On the one hand, we can use (4.6), which provides an expression that does not depend on wages $w_t(\cdot)$. On the other hand, (IC) implies that we can take an expression of $V(\theta_0)$ by substituting $\widehat{\theta}_0 = \theta_0$ directly in (4.5). Hence contrary to the first representation, the second will be a function of $w_t(\cdot)$. Equating those two gives

$$\widehat{V}(\theta_0, \theta_0) \;=\; V(\theta^*) \,+\, \int_{\theta^*}^{\theta_0} V'(x)\,\mathrm{d}x \tag{4.7}$$

which we can further simplify by setting the payoff of the lowest participating type to his outside option ω_a. Next, we use (4.7) to solve for the discounted value of the expected wages of each type θ_0. Then we substitute this in the principal's revenue ($\mathcal{P}$) and use Fubini's Theorem to obtain the following representation.

Proposition 4.2 (Revenue Equivalence)**.** *The principal's revenue maximisation problem* ($\mathcal{P}$) *can equivalently be written as*

$$\max_{e,c}\;\int_{\theta^*}^{\overline{\theta}}\mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot\Big(\sqrt{a\,\theta_0\,z_t}\cdot e_t(\theta^t) - \big[1+\eta(\theta_0)/\theta_0\big]\cdot\frac{e_t(\theta^t)^2}{2}\Big)\mathrm{d}t$$
$$+\,e^{-r\tau}\cdot\Big(\omega_p + M(m_\tau\,,\,\theta_\tau) - \eta(\theta_0)\,z_\tau\,M_2(m_\tau,\theta_\tau)\Big)\Big]\,\mathrm{d}F(\theta_0) \,-\, [1-F(\theta^*)]\cdot\omega_a \,+\, F(\theta^*)\cdot\omega_p \tag{$\mathcal{P}'$}$$

where $\eta(\theta_0) = [1-F(\theta_0)]/f(\theta_0)$ denotes the inverse hazard rate of $F(\cdot)$.

Proof. In Appendix D.1. □

The first line of the above representation corresponds to the revenue that is generated during the agent's employment by the principal. The latter cannot capture all the surplus of this production because of the agent's rents, which are due to the asymmetry of information. Those rents appear in the constant $\eta(\theta_0)/\theta_0$ which increases the cost of production.

But note that the agent's type also affects his post-termination value, which is on the second line of the above representation. This dependence also results in information rents, but this time those are generated by the market payoff. Therefore, even in this case the principal will not manage to capture the full benefit that this market creates for the agent. The only case in which she will extract all this post-termination value is when $M_2(\cdot) = 0$, that is when the market is purely based on reputation and ignores actual ability.

4.3.3 Optimal production and termination

In this section we turn our attention to deriving the optimal production and termination policy τ. The point-wise optimal level of effort is

$$e_t^*(\theta^t) \;=\; \frac{\sqrt{a\,\theta_0}}{1+\eta(\theta_0)/\theta_0}\cdot\sqrt{z_t} \tag{4.8}$$

Substitute the point-wise optimal level of effort (4.8) in the necessary representation ($\mathcal{P}'$). This gives that the principal's flow payoff from employing an agent (θ_0, z_t) is

$$k(\theta_0)\cdot z_t \qquad \text{where} \qquad k(\theta_0) \;=\; \frac{a\,\theta_0/2}{[1+\eta(\theta_0)/\theta_0]^2} \tag{4.9}$$

We have assumed that $\eta(\theta_0)$ is non-increasing, hence $k(\theta_0)$ is increasing in θ_0. An interesting implication of (4.9) is that a higher initial type agent will be more valuable later in the contract than a lower initial type, even if they both have the same contemporaneous ability θ_t. This is because the inefficiencies that information asymmetry creates in production result on the lower initial types getting low incentive contracts.

For the rest of the analysis we switch back to the linear form of the agent's post-termination payoff $M(m_t\,,\,\theta_t)$, as given in (4.1). Substitute this form of $M(m_t\,,\theta_t)$ and the derived optimal effort in ($\mathcal{P}'$) to obtain that the principal solves

$$\max_{\tau}\int_{\theta^*}^{\overline{\theta}}\mathbb{E}_z\left[k(\theta_0)\cdot\int_0^{\tau}e^{-rt}\,z_t\,\mathrm{d}t+e^{-r\tau}\cdot\Big(\omega_p+\Lambda+(1-\lambda)\,m_\tau+\lambda\,z_\tau\,[\theta_0-\eta(\theta_0)]\Big)\right]\mathrm{d}F(\theta_0) \tag{$\mathcal{T}$}$$

where the choice over the cutoff θ^* has momentarily being ignored. ($\mathcal{T}$) is not a regular stopping problem, because the value of m_t is endogenous. Despite that, it can still be rephrased as one.

Lemma 4.3. $(\mathcal{T})$ *is equivalent to*

$$\max_{\tau} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_z \Big[k(\theta_0) \cdot \int_0^{\tau} e^{-rt} \, z_t \, \mathrm{d}t + e^{-r\tau} \cdot \Big(\omega_p + \Lambda + z_\tau \cdot [\theta_0 - \lambda \, \eta(\theta_0)] \Big) \Big] \, \mathrm{d}F(\theta_0) \qquad (\mathcal{T}')$$

Proof. In Appendix D.1. □

The important difference between $(\mathcal{T})$ and $(\mathcal{T}')$ is that the former depends on the posterior reputation m_τ, whereas the latter only on the stopping values z_τ. Therefore, in the second representation for each given θ_0 the principal faces a classic optimal stopping problem. As we mentioned in the introduction the value of stopping is decreasing in λ. This is because the agent's post-termination rents that are generated from the market are increasing in the market's sophistication.

This will affect the solution of this stopping problem. In particular, we demonstrate in Appendix D.2 that for each initial type θ_0 the principal solves

$$\max_{\tau} \; \mathbb{E}_z \Big[e^{-r\tau} \cdot \Big(K(\theta_0) - z_\tau \Big) \Big] \qquad (\mathcal{T}'_{\theta_0})$$

where $K(\theta_0) = \omega_p \, / \, [\frac{k(\theta_0)}{r-\mu} + \lambda \, \eta(\theta_0) - \theta_0]$. There we also argue that if $K(\theta_0)$ is negative, then stopping immediately is optimal. On the other hand, if $K(\theta)$ is positive, the principal solves a stopping problem that is similar to the optimal exercise time of the perpetual American put option. Our analysis in the appendix implies the following proposition.

Proposition 4.3 (Optimal Termination)**.** *The solution of* $(\mathcal{T}'_{\theta_0})$ *is*

$$\tau^*(\theta_0) \;=\; \inf \big\{ t \geq 0 \, : \, z_t \, \leq \, q(\theta_0) \big\} \qquad (4.10)$$

where

$$q^*(\theta_0) \;=\; \frac{c}{1+c} \cdot (\omega_p + \Lambda) \cdot \left[\frac{k(\theta_0)}{r - \mu} + \lambda \, \eta(\theta_0) - \theta_0 \right]^{-1} \qquad (4.11)$$

if the expression in the brackets is positive. Otherwise, $\tau^*(\theta_0) = 0$*.*

Proof. In Appendix D.1. □

Therefore, the solution of $(\mathcal{T}'_{\theta_0})$ is to terminate the employment of an agent whose initial type is θ_0 when his ability θ_t falls bellow the barrier $Q^*(\theta_0) = \theta_0 \cdot q^*(\theta_0)$. This barrier is also the terminal ability of initial type θ_0, that is his ability on $\tau^*(\theta_0)$. Interestingly, $Q^*(\theta_0)$ is not necessarily monotonic. This is because a higher initial type θ_0 is not only more productive while employed by the principal, but also generates a higher post-termination payoff for her.

Hence as θ_0 increases it is not obvious which effect will dominate. Despite that, we are still able to state the following results.

Corollary 4.1 (Terminal Abilities)**.** *Let $Q^*(\theta_0) = \theta_0 \cdot q^*(\theta_0)$ denote the level of ability on which the initial type θ_0 will be terminated. Then whenever $0 < Q^*(\theta_0) < \theta_0$ (equivalently $\tau^*(\theta_0) > 0$):*

- *a sufficient condition for $Q^*(\theta_0)$ to be decreasing in θ_0 is that the firm is relatively productive:*

$$a \;\geq\; \lambda \cdot (r - \mu) \cdot \left[1 + \eta(\underline{\theta})/\underline{\theta}\right]^3 \tag{$\mathcal{A}$}$$

- *$Q^*(\theta_0)$ is decreasing on the degree of the market's sophistication λ, for every $\theta_0 \in [\underline{\theta}, \overline{\theta}]$*

The same results holds for the cutoff $q^*(\theta_0)$.

Proof. In Appendix D.1. □

Our first comparative static is with respect to θ_0. Under condition ($\mathcal{A}$) this states that the higher the agent's initial ability θ_0 is, the lower his terminal one θ_τ becomes. This is because under this condition current production is relatively more important compared to the manager's post-termination payoff. We believe that such a restriction is especially relevant for big and productive firms, that is our model describes well the situation that a super-star CEO is facing.

The second statement of the corollary gives our main comparative static. This says that in a sophisticated labour market, where the agent's ability is an important factor of his post-termination payoff, his terminal ability will be relatively lower. As we explained before the higher the λ is, the higher the information asymmetry between the principal and the agent with respect to his post-termination payoff. This increases the agent's rents generated from his termination, which results in a decrease on the value that termination has for the principal.

The above comparative statics are on the agent's terminal ability. Next, we use (4.11) to derive the corresponding ones on the agent's tenure. A peculiarity of hitting times such as (4.10) is that their expected value is infinite. Hence, we cannot use the expected tenure $\mathbb{E}[\tau^*(\theta_0)]$ for our comparative statics. Instead, we rank the distributions of those stopping times using first order stochastic dominance. Let $G(\cdot \,|\, \theta_0, \lambda)$ denote the CDF of $\tau^*(\theta_0)$ for given initial ability θ_0 and market sophistication λ.

Corollary 4.2 (Tenures)**.** *Consider $\theta_0' > \theta_0$ and $\lambda' > \lambda$. Then:*

- *If ($\mathcal{A}$) is satisfied, $G(\cdot \,|\, \theta_0', \lambda)$ first order stochastically dominates $G(\cdot \,|\, \theta_0, \lambda)$.*

- $G(\cdot \,|\, \theta_0, \lambda')$ *first order stochastically dominates* $G(\cdot \,|\, \theta_0, \lambda)$.

Proof. In Appendix D.1. □

The proof is a simple application of the results we already have on the cutoff $q^*(\theta_0)$. This represents the relative change on the agent's ability below which he will be fired. Non-surprisingly, the probability to have been terminated before t is decreasing on the value of the cutoff. Hence, the above is an equivalent way to state Corollary 4.1.

We interpret the cutoff $q^*(\theta_0)$ as a measure of how strict the firing rule of θ_0 is. That is the smaller $q^*(\theta_0)$ the more lenient its contract. Naturally, the properties of $q^*(\theta_0)$ pass on to the implied distribution of the agent's tenure. Hence, under condition ($\mathcal{A}$) a higher initial type gets a more lenient contract. In addition, the more sophisticated the market is, the more lenient the agent's contract.

Finally, because it is not very relevant to our discussion we have not provided a solution for the θ^* the level of initial ability above which the manager is hired. This is obtained by equating the point-wise value (the value for each given θ_0) that ($\mathcal{T}'$) takes under the optimal termination time (4.10) with the sum of the outside options $\omega_p + \omega_a$. However, for Λ sufficiently high the principal would contract with all manager types, even if she was planning to fire them immediately after. In this case, her role would be closer to that of an agency that connects that agent with the managerial labour market.

The results of this section where derived under the assumption that the agent can only use consistent deviations. However, in the next one we will show that as long as ($\mathcal{A}$) holds we will be able to find a contract that implements them.

4.3.4 Implementation

In this section we demonstrate that under ($\mathcal{A}$) there exists a contract that implements both the policies and the payoffs that we identified in the previous section. This contract will not rely on the restrictions that we imposed on the agent's strategic space. In other words, the agent can use any deviation he wants.

Here we provide a contract that generates the policies and payoffs that we identified in the previous section. This will require from the agent to make a choice from a menu only at time zero. An equivalent interpretation is that the agent will only need to report his initial type. The menu that the principal offers to the agent is

$$\Big\{ w(\hat\theta_0, y_t),\; [W_t(\hat\theta_0)]_{t>0},\; W_0(\hat\theta_0) \Big\} \tag{$\mathcal{W}$}$$

In addition, the principal allows the agent to decide on his own when it is the optimal time to leave. The first component of this payoff is the linear flow wage

$$w_t(\hat{\theta}_0, y_t) \;=\; w(\hat{\theta}_0)\cdot y_t \qquad \text{where} \qquad w(\hat{\theta}_0) \;=\; \frac{\sqrt{a}}{1+\eta(\hat{\theta}_0)/\hat{\theta}_0} \tag{4.12}$$

To write the second component let

$$u(\hat{\theta}_0) \;=\; \frac{a\,\hat{\theta}_0/2}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2}$$

which we will shortly show that it is a constant that multiplies the manager's flow payoff under $w_t(\hat{\theta}_0, y_t)$. Then the contract's second instrument is the golden parachutes

$$W_t(\hat{\theta}_0) \;=\; \overline{W}(\hat{\theta}_0) - (1-\lambda)\cdot m_t \qquad \text{where} \qquad \overline{W}(\hat{\theta}_0) \;=\; \omega_p \cdot \frac{\frac{u(\hat{\theta}_0)}{r-\mu} - \lambda\,\hat{\theta}_0}{\frac{k(\hat{\theta}_0)}{r-\mu} + \lambda\,\eta(\hat{\theta}_0) - \hat{\theta}_0} \tag{4.13}$$

This is paid to the agent at any point $t > 0$ that he reports that his employment should be terminated. Therefore, in the context of our analysis this golden parachute is a tool that the principal uses to incentivise the agent to admit his inadequacy.

The third and final component of the menu that the principal offers to the agent is the signing bonus $W_0(\hat{\theta}_0)$, then functional form of which is given by

$$W_0(\hat{\theta}_0) \;=\; \omega_a + \int_{\theta^*}^{\theta} \widehat{U}_2(x,x)\,\mathrm{d}x - \widehat{U}(\theta_0,\theta_0) \tag{4.14}$$

where $\widehat{U}(\hat{\theta}_0, \theta_0)$ denotes the continuation payoff (excluding the signing bonus of time zero) of a manager whose initial ability is θ_0, but his choice from the above menu was that corresponding to $\hat{\theta}_0$. The functional form of this function is given below.

The rest of this section proves our implementation result. It shows that if ($\mathcal{A}$) holds, then the menu of contracts ($\mathcal{W}$) implements the optimal effort and termination time derived in the previous section. As we mentioned before, this part of the analysis will not impose any restrictions on the agent's action space. This will prove that those optimal policies are indeed the revenue maximising ones for the principal.

To begin the proof, for each $t > 0$ consider an agent that reported $\hat{\theta}_0$ while his initial ability was θ_0. His optimal level of effort is given by

$$\max_{e_t} \; w(\hat{\theta}_0)\cdot\sqrt{\theta_0\, z_t}\cdot e_t \;-\; \frac{(e_t)^2}{2}$$

the solution of which is $\widehat{e}^*(\hat{\theta}_0, \theta_0, z_t) = w(\hat{\theta}_0)\cdot\sqrt{\theta_0\, z_t}$. Hence it follows from the functional form of $w(\hat{\theta}_0)$ that if the initial ability has been truthfully reported, then the implemented level of effort is the revenue maximising one, as given in (4.8).

Next we demonstrate that if the initial ability θ_0 was truthfully reported, then the golden parachute $W_t(\hat{\theta}_0)$ implements the revenue maximising stopping time, as given in (4.10). The contract delegates to the agent the termination decision. Hence he also solves an optimal stopping problem. The agent's flow payoff while the contract continues is

$$\widehat{u}(\hat{\theta}_0, \theta_0) \cdot z_t \qquad \text{where} \qquad \widehat{u}(\hat{\theta}_0, \theta_0) \;=\; \frac{a\,\theta_0/2}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2}$$

Also, let $u(\theta_0) = \widehat{u}(\theta_0, \theta_0)$ denote the corresponding constant that multiplies z_t under truthful reporting. Therefore, it follows from the functional form of $W_t(\theta_0)$ that the agent's optimal stopping problem is

$$\max_{\tau} \;\; \mathbb{E}_z\Big[u(\hat{\theta}_0, \theta_0) \cdot \int_0^{\tau} e^{-rt}\, z_t \,\mathrm{d}t + e^{-r\tau} \cdot \Big(\lambda\, z_\tau\, \theta_0 + \overline{W}(\hat{\theta}_0)\Big)\Big] \tag{$\mathcal{T}_a$}$$

To solve this we use again our generic analysis in Appendix D.1.

Lemma 4.4 (Golden Parachutes)**.** *The solution of* ($\mathcal{T}_a$) *is*

$$\tau_a \;=\; \inf\big\{t \geq 0 \,:\, z_t \,\leq\, \widehat{q}(\hat{\theta}_0, \theta_0)\big\} \tag{4.15}$$

where

$$\widehat{q}(\hat{\theta}_0, \theta_0) \;=\; \frac{c \cdot \overline{W}(\hat{\theta}_0)}{1+c} \cdot \left[\frac{u(\hat{\theta}_0, \theta_0)}{r-\mu} - \lambda\,\theta_0\right]^{-1} \tag{4.16}$$

In addition, $\widehat{q}(\theta_0, \theta_0) = q(\theta_0)$ *and* $\widehat{q}(\hat{\theta}_0, \theta_0) = q(\hat{\theta}_0) \cdot \frac{\hat{\theta}_0}{\theta_0}$.

Proof. The stopping time τ_a and the associated barrier $\widehat{q}(\hat{\theta}_0, \theta_0)$ follow immediately from the solution of the optimal stopping problem of Appendix D.1. In addition, we obtain that $\widehat{q}(\theta_0, \theta_0) = q(\theta_0)$ by substituting $\overline{W}(\theta_0)$ in the left hand side of this inequality, which can also be used to obtain that $\widehat{q}(\hat{\theta}_0, \theta_0) = \widehat{q}(\hat{\theta}_0, \hat{\theta}_0) \cdot \hat{\theta}_0/\theta_0$. □

Hence we have shown that if the initial type was truthfully reported, then the given flow wage and golden parachute implement the revenue maximising stopping time.

It remains to implement the truthful reporting of the initial type θ_0, which is achieved with the signing bonus $W_0(\hat{\theta}_0)$. Our analysis in Appendix D.1 gives that the agent's payoff at time zero, net of $W_0(\hat{\theta}_0)$, is given by

$$\widehat{U}(\hat{\theta}_0, \theta_0) \;=\begin{cases} \frac{u(\hat{\theta}_0,\theta_0)}{r-\mu} \;+\; \left(\frac{u(\hat{\theta}_0,\theta_0)}{r-\mu} - \lambda\,\theta_0\right) \cdot \frac{\widehat{q}(\hat{\theta}_0,\theta_0)^{1+c}}{c} & \text{, if} \quad \widehat{q}(\hat{\theta}_0, \theta_0) \leq 1 \\ \lambda \cdot \theta_0 \;+\; \overline{W}(\hat{\theta}_0) & \text{, if} \quad \widehat{q}(\hat{\theta}_0, \theta_0) > 1 \end{cases} \tag{4.17}$$

From which the following result follows.

Lemma 4.5. *The signing bonus $W_0(\hat{\theta}_0)$ implements the truthful reporting of θ_0. In addition,*

$$\widehat{U}(\theta_0, \theta_0) \;+\; W_0(\theta_0) \;=\; \omega_a + \int_{\theta^*}^{\theta_0} V'(x)\,\mathrm{d}x \tag{4.18}$$

where $V'(x)$ is as given in (4.6), but calculated under the optimal effort and termination policies.

Proof. In Appendix D.1. □

Therefore, the proposed contract implements the revenue maximising effort level and termination time. In addition, (4.18) gives that the agent's payoff is the same with that we calculated in the previous section. Therefore, the principal's revenue has to also be the same, which completes the proof of Proposition 4.4.

Proposition 4.4 (Implementation)**.** *Suppose that ($\mathcal{A}$) holds. Then the menu of contracts ($\mathcal{W}$) implements the optimal policies $e^*(\theta^t)$ and $\tau^*(\theta_0)$ as identified in (4.8) and (4.10), respectively. In addition, the implied payoff for the principal and the agent are also the same with those of the previous section.*

Proof. Follows from the above discussion. □

4.4 Discussion

In order to obtain closed-form solutions, our analysis has imposed an important simplifying restriction on the way the market learns the agent's ability. This is that either it directly observes it, or it learns nothing. Hence, the agent's post-termination payoff becomes a linear combination between his reputation and actual ability. Nevertheless, we believe that the underlining intuition of our result, which is that the less the principal knows about the agent's post-termination value the smaller the value of termination becomes, is more general than our setting. In particular, it should hold even if the market obtains a private signal on the agent's ability.

Moreover, in an alternative model we could also assume that the agent's post-termination payoff was generated by a new contractual offer from a second principal. Generically, this would imply that this payoff is a function of the whole posterior CDF of the agent's ability, instead of only the corresponding expected value. This is because his continuation payoff would be generated by the rents he captures from the second contract, which commonly depend on the whole functional form of this posterior CDF.

In such a setting, it might even be that the first principal would find it optimal to disclose some of the agent's reports on his ability. This is a topic that has been studied much more extensively in Chapter 1. We believe that extending the results of the aforementioned chapter in the setting we discuss here is very interesting future research.

In Appendix D.3 we offer some results in a model that allows for the post-termination payoff to be generated by the contractual offer of a second principal, however we assume that the interaction between the first principal and the agent is public. Hence in this setting information provision is irrelevant, since the second principal acquires exogenously all the information that the first principal has.

We demonstrate that the first principal faces an optimal stopping problem that is very similar to that of our main analysis. Its solution is again a cutoff on the reported ability that only depends on the agent's initial type. In addition, if the production process of the first principal is sufficiently more productive than that of the second, then this cutoff is decreasing on the agent's initial ability. Therefore, we obtain again that agents with higher initial ability get more lenient contracts, and the intuition is the same as in our main analysis. Finally, the contract that implements this cutoff also uses a golden parachute in order to incentivise the agent to admit his incompetence.

4.5 Conclusions

We have considered the problem of a representative investor that hires a manager who can potentially be fired. After his termination the manager's payoff is generated by an exogenous labour market that learns his ability with some positive probability. In the state of the world where this ability remains hidden the market updates its prior on the manager's expected ability only based on the time of his firing. We interpreted this expectation as the manager's reputation and showed that in our assumed specification his post-termination payoff is a linear combination between his reputation and actual ability. Next, we demonstrated that the investor's revenue maximisation problem encompasses an optimal stopping problem. The solution of the latter is a cutoff on the manager's reported ability, below which his employment is terminated. We have shown that if the production process is sufficiently efficient, this cutoff is decreasing on the manager's initial ability. Therefore, the more qualified the manger is when negotiating his contract, the more lenient this contract will be. In addition, this cutoff is always decreasing on the probability by which the market learns the agent's ability, that is on the relative importance of the manager's ability versus his reputation. As a result, the more sophisticated the labour market is, the less competent the manager is when entering this market, and the longer the expected duration of his employment. Finally we presented a contract that implements the above stopping time. This uses a golden parachute to induce the manager to truthfully report his incompetence.

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Appendix A

Appendixes of Chapter 1

A.1 Proofs of section 1.4

Proof of Lemma 1.1. The revelation principle applies, hence it is without loss to focus on direct and incentive compatible mechanisms. To make the notation more compact, write the reported type as a subscript. S_2's revenue maximisation problem is the following one

$$\max_{p_L, p_H, q_L, q_H}$$

$$mu_2^s \, p_H + (1 - \mu_2^s)\, p_L$$

$$\begin{aligned} \text{s.t. } (\mathrm{IR}_L) \quad & \theta_L \, q_L - p_L \geq 0 \\ (\mathrm{IR}_H) \quad & \theta_H q_H - p_H \geq 0 \\ (\mathrm{IC}_L) \quad & \theta_L \, q_L - p_L \geq \theta_L \, q_H - p_H \\ (\mathrm{IC}_H) \quad & \theta_H q_H - p_H \geq \theta_H q_L - p_L \end{aligned}$$

Assuming that (IR_L) does not bind leads to a contradiction. Subsequently, this can be used to show that (IC_H) has to bind. Hence the above simplifies to the unconstrained maximisation problem

$$\max_{q_L, q_H} \ \mu_2^s \, \theta_H \, q_H + (\theta_L - \mu_2^s \theta_H)\, q_L$$

As a result the unique solution is to set $q_H^* = 1$, while the optimal probability of supplying the low type is

$$q_L^* = \begin{cases} 1 & \text{, if } \ \mu_2^s \leq \theta_L / \theta_H \\ 0 & \text{, if } \ \mu_2^s \geq \theta_L / \theta_H \end{cases}$$

This is implementable, because substituting the above solutions in (IC_L) gives

$$\begin{aligned} \theta_L q_L - p_L \,\geq\, \theta_L q_H - p_H \;\Leftrightarrow\; 0 \,\geq\, \theta_L q_H - p_H \\ \Leftrightarrow\; \theta_H(q_H - q_L) + \theta_L q_L \,\geq\, \theta_L q_H \;\Leftrightarrow\; q_H \,\geq\, q_L, \end{aligned}$$

which is satisfied. Because the (IR_L) binds the low type's payoff is zero. The high type's payoff can be obtained using the (IR_L) and (IC_H) constraints, which give that

$$\theta_H q_H - p_H \;\overset{\text{IC}_H}{=}\; \theta_H q_L - p_L \;\overset{\text{IR}_L}{=}\; (\theta_H - \theta_L) q_L$$

□

Proof of Lemma 1.2. For any countable S, and $s \in S$ let the ex ante probability of s to be realised be denoted by

$$g(s) \;=\; \mu_0\, g(s \,|\, \theta_H) + (1 - \mu_0)\, g(s \,|\, \theta_L)\,.$$

Then for all $s \in S$ such that $g(s) \neq 0$ it follows that

$$g(s \,|\, \theta_H) \;=\; \frac{\mu_1^s}{\mu_0}\, g(s) \qquad \text{and} \qquad g(s \,|\, \theta_L) \;=\; \frac{1 - \mu_1^s}{1 - \mu_0}\, g(s)$$

$$\begin{aligned} &\mu_0 \rho_H \,\mathbb{E}_s\left[Q(\mu_1^s) \,|\, \theta_H\right] + (1 - \mu_0)\rho_L \,\mathbb{E}_s\left[Q(\mu_1^s) \,|\, \theta_L\right] \\ &\qquad = \sum_{s \in S} Q(\mu_1^s) \Big\{ \mu_0 \rho_H g(s \,|\, \theta_H) + (1 - \mu_0)\rho_L g(s \,|\, \theta_L) \Big\} \\ &\qquad\qquad = \sum_{s \in S} Q(\mu_1^s) \Big\{ \rho_H \mu_1^s + \rho_L (1 - \mu_1^s) \Big\}\, g(s) \;=\; \mathbb{E}_g\big[J_f(\mu_1^s)\big]. \end{aligned}$$

□

Proof of Proposition 1.1. It follows from the discussion on the main part of the paper that information provision has an impact iff $\rho_L \leq \theta_L/\theta_H < \rho_H$, in which case the concave closure of $J_f(\mu)$ is given by

$$\mathcal{J}_f(\mu) \;=\; \begin{cases} J_f(\mu) & ,\text{ for } \;\mu \,\leq\, \mu^* \\ J_f(\mu^*)\frac{\mu - \mu^*}{1 - \mu^*} & ,\text{ for } \;\mu \,\geq\, \mu^* \end{cases} \qquad \text{where} \quad \mu^* \;=\; \frac{\theta_L/\theta_H - \rho_L}{\rho_H - \rho_L}$$

If $\mu_0 \,\leq\, \mu^*$, then $J_f(\mu_0) \,=\, \mathcal{J}_f(\mu_0)$ and as a result setting $\Pr(\mu \,=\, \mu_0) \,=\, 1$ is optimal, which is achieved when no information is provided to S_2. If $\mu_0 > \mu^*$, then the optimal $\tilde{g}(\mu)$ randomises between posteriors μ^* and 1. This is the solution of ($\mathcal{G}_f'$), where the choice variable is a distribution over posteriors $\tilde{g}(\mu)$. Hence a distribution over signals $g(s \,|\, \theta_1)$ solves ($\mathcal{G}_f$) if and only if it results in a randomisation between posteriors μ^* and 1.

It is without loss to focus on a binary signal $s \in \{\underline{s}, \overline{s}\}$. Let $\overline{s}$ be the signal realisation that gives $\mu_1^s = 1$. Then it has to be that $g_f\,(\overline{s} \,|\, \theta_L) = 0$. Therefore, $g_f\,(\underline{s} \,|\, \theta_L) = 1$. Finally, to find $g_f\,(\underline{s} \,|\, \theta_H)$ note that this has to satisfy

$$\mu^* \;=\; \frac{\mu_0\, g_f\,(\underline{s} \,|\, \theta_H)}{\mu_0\, g_f\,(\underline{s} \,|\, \theta_H) + 1 - \mu_0}$$

This signal is a solution of ($\mathcal{G}_f$), hence it is also part of the solution of ($\mathcal{P}_f$).

In addition, $\mu_0 > \mu^*$ equivalently implies that $\rho_H \mu_0 + (1-\mu_0)\rho_L > \theta_L/\theta_H$. But since $\rho_H \geq \rho_L$ this last inequality implies $\rho_H > \theta_L/\theta_H$. Hence $\rho_H > \theta_L/\theta_H$ can be ignored, as it is implied by (1.4). Finally, if $\rho_L = \mu_0$ then the informative signal collapses to no information provision, which is way the left hand side of (1.4) has a strict inequality. □

Proof of Lemma 1.3. To achieve more compact expressions we adopt the notation

$$\begin{aligned} q_L &= q_1(\theta_L) \\ q_H &= q_1(\theta_H) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathbb{E}_L &= \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_L\right] \\ \mathbb{E}_H &= \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_H\right] \end{aligned}$$

For convenience we copy below ($\mathcal{P}$) and all four of its constrains, using the shorter notation.

$$\begin{aligned} \max_{p_L, p_H, q_L, q_H, g} \quad & \mu_0 p_H + (1-\mu_0)\, p_L \\ \text{s.t. } (\text{IR}_L) \quad & \theta_L\, q_L + \rho_L\, \mathbb{E}_L - p_L \;\geq\; 0 \\ (\text{IR}_H) \quad & \theta_H q_H + \rho_H \mathbb{E}_H - p_H \;\geq\; 0 \\ (\text{IC}_L) \quad & \theta_L\, q_L + \rho_L\, \mathbb{E}_L - p_L \;\geq\; \theta_L q_H + \rho_L \mathbb{E}_H - p_H \\ (\text{IC}_H) \quad & \theta_H q_H + \rho_H \mathbb{E}_H - p_H \;\geq\; \theta_H q_L + \rho_H \mathbb{E}_L - p_L \end{aligned} \tag{$\mathcal{P}$}$$

Using the (IC_H) we infer that whenever the (IR_L) is satisfied the same is true for the (IR_H), since we have assumed for now that $\rho_H \geq \rho_L$. Suppose that (IR_L) was not binding on the maximum of ($\mathcal{P}$). Then S_1 could increase both p_L and p_H by the same constant $\varepsilon > 0$. This would increase her payoff and for ε small enough still satisfy all the constrains, which leads to a contradiction. Thus (IR_L) has to bind on the maximum of ($\mathcal{P}$), which gives

$$p_L \;=\; \theta_L\, q_L \;+\; \rho_L\, \mathbb{E}_L$$

Next, suppose that (IC_H) was not binding on the maximum of ($\mathcal{P}$). Then S_1 could increase p_H by a small enough constant $\varepsilon > 0$, which would increase her payoff and still satisfy all the constrains. Hence we obtain again a contradiction and (IC_H) has to bind, which gives

$$p_H = \theta_H(q_H - q_L) + \rho_H(\mathbb{E}_H - \mathbb{E}_L) + p_L$$

Substitute p_H and p_L in the objective function of ($\mathcal{P}$) to obtain

$$\begin{aligned}\mu_0 p_H + (1-\mu_0)\, p_L &= \mu_0\big[\theta_H(q_H - q_L) + \rho_H(\mathbb{E}_H - \mathbb{E}_L)\big] + p_L \\ &= \mu_0\, \theta_H q_H + (\theta_L - \mu_0\theta_H)\, q_L \\ &\quad + \mu_0\rho_H\, \mathbb{E}_H + (1-\mu_0)\rho_L\mathbb{E}_L - \mu_0(\rho_H - \rho_L)\mathbb{E}_L\end{aligned}$$

which is the objective function given on the statement of this Lemma. So far we have ignored the (IC_L). This can be rewritten as

$$\begin{aligned}p_H - p_L &\geq \theta_L(q_H - q_L) + \rho_L(\mathbb{E}_H - \mathbb{E}_L) \\ \Leftrightarrow \theta_H(q_H - q_L) + \rho_H(\mathbb{E}_H - \mathbb{E}_L) &\geq \theta_L(q_H - q_L) + \rho_L(\mathbb{E}_H - \mathbb{E}_L) \\ \Leftrightarrow (\theta_H - \theta_L)(q_H - q_L) &\geq (\rho_H - \rho_L)(\mathbb{E}_L - \mathbb{E}_H)\end{aligned}$$

which gives the ($\mathcal{P}_c$) constrain. □

Proof of Lemma 1.4. For any countable S, it has already been shown in Lemma 1.2 that

$$\mu_0\rho_H\, \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_H\big] + (1-\mu_0)\rho_L\mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_L\big] = \sum_{s\in S} Q(\mu_1^s)\left\{\rho_H\mu_1^s + \rho_L(1-\mu_1^s)\right\} g(s)$$

In addition, deleting irrelevant signals from S, i.e. those that occur with zero probability on path, and substituting $g(s\,|\,\theta_L) = \frac{1-\mu_1^s}{1-\mu_0}\, g(s)$ gives

$$\begin{aligned}\mu_0(\rho_H - \rho_L)\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_L\right] &= \sum_{s\in S} Q(\mu_1^s)\, \mu_0\, (\rho_H - \rho_L)\, g(s\,|\,\theta_L) \\ &= \sum_{s\in S} Q(\mu_1^s)\, \mu_0\, (\rho_H - \rho_L)\, \frac{1-\mu_1^s}{1-\mu_0}\, g(s)\end{aligned}$$

Thus, subtracting the second part above from the first gives

$$
\begin{aligned}
&\sum_{s\in S} Q(\mu_1^s)\left\{\rho_H\mu_1^s + \rho_L(1-\mu_1^s) - \mu_0\left(\rho_H-\rho_L\right)\frac{1-\mu_1^s}{1-\mu_0}\right\} g(s) \\
=\;& \sum_{s\in S}\frac{Q(\mu_1^s)}{1-\mu_0}\left\{(\rho_H-\rho_L)\mu_1^s(1-\mu_0) + \rho_L(1-\mu_0) - \mu_0\left(\rho_H-\rho_L\right)\left(1-\mu_1^s\right)\right\} g(s) \\
=\;& \sum_{s\in S}\frac{Q(\mu_1^s)}{1-\mu_0}\left\{(\rho_H-\rho_L)(\mu_1^s-\mu_0) + \rho_L(1-\mu_0)\right\} g(s) \\
=\;& \frac{\rho_H-\rho_L}{1-\mu_0}\sum_{s\in S} Q(\mu_1^s)\left\{\mu_1^s - \mu_0 + \frac{\rho_L(1-\mu_0)}{\rho_H-\rho_L}\right\} g(s) \\
=\;& \frac{\rho_H-\rho_L}{1-\mu_0}\sum_{s\in S} Q(\mu_1^s)\left\{\mu_1^s - \frac{\mu_0\rho_H-\rho_L}{\rho_H-\rho_L}\right\} g(s)
\end{aligned}
$$

□

Proof of Proposition 1.2. Proving that the proposed signal solves ($\mathcal{G}'$) follows exactly the same argumentation as in the first best, hence it is omitted. To show that it also solves ($\mathcal{P}'$) it remains to consider how it interacts with the constrain ($\mathcal{P}_c$), which can equivalently be written as

$$
q_H - q_L \;\geq\; (\rho_H-\rho_L)\Big(\mathbb{E}_g\left[\mathbb{1}\left\{\mu_1^s \leq \mu^*\right\} \mid \theta_L\right] - \mathbb{E}_g\left[\mathbb{1}\left\{\mu_1^s \leq \mu^*\right\} \mid \theta_H\right]\Big) \qquad (\mathcal{P}_c')
$$

It is always optimal to set $q_H^* = 1$. First suppose that $\theta_L/\theta_H \leq \mu_0$, which implies that the point-wise optimal supply for the low type is $q_L = 0$. But in this case ($\mathcal{P}_c'$) trivially holds, since its left hand side equals one, while its right hand side is always less than one. This proves that in this case the point-wise optimal signal also solves ($\mathcal{P}'$).

Hereafter, the proof only considers the diametrically opposite case $\theta_L/\theta_H > \mu_0$. Rewrite the right hand side of ($\mathcal{P}_c'$) as follows

$$
\begin{aligned}
&(\rho_H-\rho_L)\Big(\mathbb{E}_g\left[\mathbb{1}\left\{\mu_1^s \leq \mu^*\right\} \mid \theta_L\right] - \mathbb{E}_g\left[\mathbb{1}\left\{\mu_1^s \leq \mu^*\right\} \mid \theta_H\right]\Big) \\
&\qquad = \frac{\rho_H-\rho_L}{\theta_H-\theta_L}\sum_s Q(\mu_1^s)\Big(g(s\mid\theta_L) - g(s\mid\theta_H)\Big) \\
&\qquad = \frac{\rho_H-\rho_L}{\theta_H-\theta_L}\sum_s Q(\mu_1^s)\left(\frac{1-\mu_1^s}{1-\mu_0} - \frac{\mu_1^s}{\mu_0}\right) g(s) \\
&\qquad = \frac{\rho_H-\rho_L}{\theta_H-\theta_L}\sum_s Q(\mu_1^s)\frac{\mu_0-\mu_1^s}{(1-\mu_0)\mu_0}\, g(s)
\end{aligned}
$$

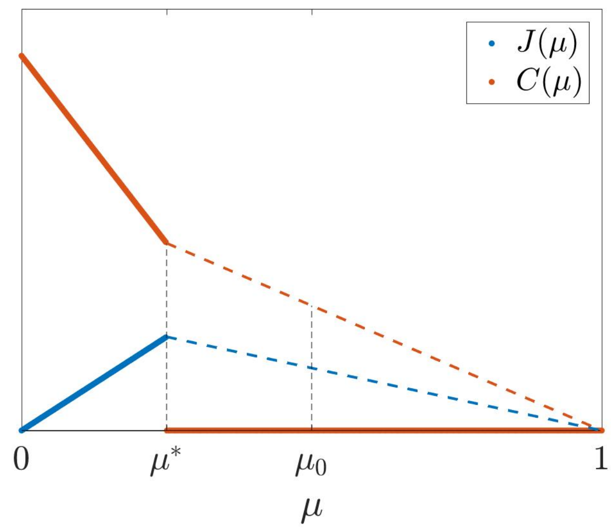


Figure A.1: The point-wise post contractual payoff of S_1 $J(\mu)$, and the point-wise value of the right hand side of $(\mathcal{P}_c'')$.

As a result, the constrain becomes

$$q_H - q_L \;\geq\; \mathbb{E}_{\tilde{g}}\big[C(\mu)\big] \quad \text{where} \quad C(\mu) \;=\; \frac{\rho_H - \rho_L}{\theta_H - \theta_L}\, Q(\mu)\, \frac{\mu_0 - \mu}{(1-\mu_0)\mu_0} \qquad (\mathcal{P}_c'')$$

Under no information provision the right hand side is equal to $C(\mu_0) = 0$, therefore if no information provision solves $(\mathcal{G}')$, then it also solves $(\mathcal{P}')$.

Instead suppose that information provision solves $(\mathcal{G}')$. Similarly to the first best, this implies that $\mu_0 > \mu^*$. Therefore the point-wise optimal supply, when $\theta_L/\theta_H > \mu_0$, is $q_H = q_L = 1$, while under the point-wise optimal signal

$$\mathbb{E}_{\tilde{g}}\big[C(\mu)\big] \;=\; \tilde{g}(\mu^*)C(\mu^*) \;>\; 0$$

Hence under the point-wise optimal solutions the left hand side of $(\mathcal{P}_c'')$ would be zero, whereas the right hand side would be positive.

Note that both $J(\mu)$ and $C(\mu)$ are piecewise linear below and above μ^*. Hence it is without loss to consider a distribution that induces only two posteriors $\mu^- \leq \mu^* < \mu^+$. For this part of the proof it will potentially be useful to refer to Figure (A.1). Suppose that $\mu^- < \mu^*$, which implies

$$J(\mu^-) < J(\mu^*) \quad \text{and} \quad C(\mu^-) > C(\mu^*).$$

Since the expectations $\mathbb{E}_{\tilde{g}}[J(\mu)]$ and $\mathbb{E}_{\tilde{g}}[C(\mu)]$ are linear combinations of the value of each function at μ^- and zero, it follows that switching to $\mu^- = \mu^*$ is strictly better. This is because it increases $\mathbb{E}_{\tilde{g}}[J(\mu)]$ and it decreases $\mathbb{E}_{\tilde{g}}[C(\mu)]$. Then S_1 is always better off by leaving the supply of the high type and the low realisation of the buyer's posterior on their point-wise optimal values. Hence $(\mathcal{P}')$ can be written as

$$\begin{aligned} \max_{q_L, \tilde{g}^-} \quad & (\theta_L - \mu_0 \theta_H)\, q_L + \tilde{g}^-\, J_1(\mu^*) \\ \text{s.t.} \quad & 1 - q_L \;\geq\; \tilde{g}^-\, C(\mu^*) \end{aligned}$$

where $\tilde{g}^- = \tilde{g}(\mu^*)$. When $\mu^- = \mu^*$, $\tilde{g}^-$ is a bijection of μ^+, hence it will be convenient to use the former as a choice variable. The objective function and the constrain are linear in both choice variables. Hence S_1 prefers to decrease q_L instead of $\tilde{g}^-$ when

$$\begin{aligned} \theta_L - \mu_0\theta_H \;\leq\; \frac{J_1(\mu^*)}{C(\mu^*)} \quad &\Leftrightarrow \quad \theta_L - \mu_0\theta_H \;\leq\; \frac{\frac{\rho_H - \rho_L}{1-\mu_0} Q(\mu^*)\left(\mu^* - \frac{\mu_0\rho_H - \rho_L}{\rho_H - \rho_L}\right)}{\frac{\rho_H - \rho_L}{\theta_H - \theta_L}\, \frac{Q(\mu^*)}{(1-\mu_0)\mu_0}\left(\mu_0 - \mu^*\right)} \\ &\Leftrightarrow \quad \frac{\frac{\theta_L}{\theta_H} - \mu_0}{1 - \frac{\theta_L}{\theta_H}}\left(1 - \frac{\mu^*}{\mu_0}\right) \;\leq\; \mu^* - \frac{\mu_0\rho_H - \rho_L}{\rho_H - \rho_L} \\ \Leftrightarrow \quad \frac{\theta_L}{\theta_H} - \mu_0 \;&\leq\; \left[\left(\frac{\theta_L}{\theta_H}\frac{1}{\mu_0} - 1\right) + \left(1 - \frac{\theta_L}{\theta_H}\right)\right]\mu^* - \frac{\mu_0\rho_H - \rho_L}{\rho_H - \rho_L}\left(1 - \frac{\theta_L}{\theta_H}\right) \\ \Leftrightarrow \quad \frac{\theta_L}{\theta_H} - \mu_0 \;&\leq\; \frac{\theta_L}{\theta_H}\,\frac{1-\mu_0}{\mu_0}\,\mu^* - \frac{\mu_0\rho_H - \rho_L}{\rho_H - \rho_L}\left(1 - \frac{\theta_L}{\theta_H}\right) \\ \Leftrightarrow \quad 1 - \mu_0 \;&\leq\; \frac{\theta_L}{\theta_H}\,\frac{1-\mu_0}{\mu_0}\,\mu^* + \frac{\rho_H(1-\mu_0)}{\rho_H - \rho_L}\left(1 - \frac{\theta_L}{\theta_H}\right) \end{aligned}$$

But remember that $\mu^* = \frac{\theta_L/\theta_H - \rho_L}{\rho_H - \rho_L}$. Hence the above equivalently becomes

$$1 \;\leq\; \frac{\theta_L}{\theta_H}\,\frac{1}{\mu_0}\,\frac{\theta_L/\theta_H - \rho_L}{\rho_H - \rho_L} + \frac{\rho_H}{\rho_H - \rho_L}\left(1 - \frac{\theta_L}{\theta_H}\right)$$

In addition, the above trade off is only relevant if $\frac{\theta_L}{\theta_H}\frac{1}{\mu_0} > 1$. Hence it suffices that

$$\rho_H - \rho_L \;\leq\; \frac{\theta_L}{\theta_H} - \rho_L + \rho_H - \rho_H\frac{\theta_L}{\theta_H} \quad \Leftrightarrow \quad 0 \;\leq\; \frac{\theta_L}{\theta_H}\,(1 - \rho_H)$$

which holds. As a result the point-wise optimal signal solves $(\mathcal{P}')$, while the probability of supplying the good to the low type becomes

$$q_L^* \;=\; \begin{cases} 1 - (\rho_H - \rho_L)[1 - g^*(\underline{s}\,|\,\theta_H)] & \text{, if } \mu_0 < \theta_L/\theta_H \\ 0 & \text{, if } \mu_0 \geq \theta_L/\theta_H \end{cases}$$

Finally, any solution of ($\mathcal{P}'$) is also a solution of ($\mathcal{P}$), which completes the proof. $\square$

Proof of Corollary 1.1. The payoff a low type is always zero, since the (IR_L) binds. For the high type, when information provision is possible we can use the binding (IC_H) and (IR_L) to obtain that his payoff is

$$(\theta_H - \theta_L) \cdot q_L^* + (\rho_H - \rho_L) \cdot \mathbb{E}_{g^*}[Q(\mu_1^s) \,|\, \theta_L] \tag{A.1}$$

where q_L^* and g^* are as given in Proposition 1.2. First, suppose that $\mu_0 \geq \theta_L/\theta_H$. If information provision was not possible, then the high type's payoff would be zero. However, when it is possible $q_L^* = 0$ and (A.1) becomes

$$(\rho_H - \rho_L) \cdot \mathbb{E}_g[Q(\mu_1^s) \,|\, \theta_L] > 0$$

hence he is strictly better off. Second, suppose that $\mu_0 < \theta_L/\theta_H$. If information provision was not possible, then the high type's payoff would be $\theta_H - \theta_L$. However, when it is possible

$$q_L^* \;=\; 1 - (\rho_H - \rho_L) \cdot [1 - g^*(\underline{s} \,|\, \theta_H)]$$

Hence (A.1) becomes

$$\begin{aligned}(\theta_H - \theta_L) \cdot \Big[1 - (\rho_H - \rho_L)[1 - g^*(\underline{s} \,|\, \theta_H)] + (\rho_H - \rho_L) \cdot g^*(\underline{s} \,|\, \theta_L)\Big] \\ = \; (\theta_H - \theta_L) \cdot \Big[1 + (\rho_H - \rho_L) \cdot g^*(\underline{s} \,|\, \theta_H)\Big]\end{aligned}$$

which is bigger than $\theta_H - \theta_L$, hence again he is better off. $\square$

Proof of Lemma 1.5. For the convenience of the reader ($\mathcal{P}$) together with all four of its constrains is copied below

$$\begin{aligned}
&\max_{p_L, p_H, q_L, q_H, g} \; \mu_0 p_H + (1 - \mu_0)\, p_L \\
&\text{s.t. } (\text{IR}_L) \quad \theta_L\, q_L + \rho_L\, \mathbb{E}_L - p_L \;\geq\; 0 \\
&\quad (\text{IR}_H) \quad \theta_H q_H + \rho_H \mathbb{E}_H - p_H \;\geq\; 0 \\
&\quad (\text{IC}_L) \quad \theta_L\, q_L + \rho_L\, \mathbb{E}_L - p_L \;\geq\; \theta_L q_H + \rho_L \mathbb{E}_H - p_H \\
&\quad (\text{IC}_H) \quad \theta_H q_H + \rho_H \mathbb{E}_H - p_H \;\geq\; \theta_H q_L + \rho_H \mathbb{E}_L - p_L
\end{aligned} \tag{$\mathcal{P}$}$$

where the compact notation introduced in the proof of Lemma 1.3 is used. First, we want to demonstrate that under negative correlation ($\mathcal{P}$) has two possible families of solutions.

First, suppose that

$$(\theta_H - \theta_L)q_L \;\geq\; (\rho_L - \rho_H)\mathbb{E}_L \tag{A.2}$$

which equivalently implies

$$\theta_H\, q_L + \rho_H \mathbb{E}_L - p_L \;\;\geq\;\; \theta_L\, q_L + \rho_L \mathbb{E}_L - p_L \;,$$

This together with the (IC_H) give

$$\theta_H\, q_H + \rho_H \mathbb{E}_H - p_H \;\;\geq\;\; \theta_L\, q_L + \rho_L \mathbb{E}_L - p_L$$

Next it is shown that a necessary implication of (A.2) is that (IR_L) and (IC_H) need to bind. Suppose (IR_L) does not bind, then prices $\tilde{p}_L = p_L + \varepsilon$ and $\tilde{p}_H = p_H + \varepsilon$, for $\varepsilon > 0$ small enough, increase S_1's payoff and satisfy all of the above constrains, which leads to a contradiction. Suppose (IC_H) does not bind, then $\tilde{p}_H = p_H + \varepsilon$, again for $\varepsilon > 0$ small enough, increases S_1's payoff and satisfies all of the above constrains, which leads to a contradiction. Hence rewriting the binding (IC_H) and the (IC_L) gives

$$\theta_H(q_H - q_L) + \rho_H(\mathbb{E}_H - \mathbb{E}_L) \;=\; p_H - p_L \;\geq\; \theta_L(q_H - q_L) + \rho_L(\mathbb{E}_H - \mathbb{E}_L)$$

and combining those two together gives

$$(\theta_H - \theta_L)(q_H - q_L) \;\geq\; (\rho_L - \rho_H)(\mathbb{E}_H - \mathbb{E}_L) \tag{A.3}$$

which is an equivalent expression of ($\mathcal{P}_c$). Finally, adding up the initial assumption (A.2) and (A.3) gives

$$(\theta_H - \theta_L)q_H \;\geq\; (\rho_L - \rho_H)\mathbb{E}_H \tag{A.4}$$

which will be used at the end of the proof.

Next the diametrically opposite case of solutions is considered, but it will be more convenient to start by assuming that it is (A.4) that holds with the opposite direction. That is suppose that the solution of ($\mathcal{P}$) satisfies

$$(\theta_H - \theta_L)q_H \;\leq\; (\rho_L - \rho_H)\mathbb{E}_H \tag{A.5}$$

which equivalently implies

$$\theta_L q_H + \rho_L \mathbb{E}_H - p_H \;\geq\; \theta_H q_H + \rho_H \mathbb{E}_H - p_H$$

This together with the (IC_L) give

$$\theta_L q_L + \rho_L \mathbb{E}_L - p_L \;\geq\; \theta_H q_H + \rho_H \mathbb{E}_H - p_H$$

Then following an argumentation similar to that of the previous case we can show that (IR_H) and (IC_L) need to bind. Hence rewriting the binding (IC_L) and the (IC_H) gives

$$\theta_H(q_H - q_L) + \rho_H(\mathbb{E}_H - \mathbb{E}_L) \;\geq\; p_H - p_L \;=\; \theta_L(q_H - q_L) + \rho_L(\mathbb{E}_H - \mathbb{E}_L)$$

and combining those two together gives (A.3) again. Combining the supposition (A.5) with (A.3) gives

$$(\theta_H - \theta_L)q_L \;\leq\; (\rho_L - \rho_H)\mathbb{E}_L\,,$$

which is the opposite of the first supposition considered (A.2). But this implies that the solution of ($\mathcal{P}$) satisfies

$$(\theta_H - \theta_L)q_L \;\geq\; (\rho_L - \rho_H)\mathbb{E}_L \quad\Leftrightarrow\quad (\theta_H - \theta_L)q_H \;\geq\; (\rho_L - \rho_H)\mathbb{E}_H$$

Hence, there are only two possible families of solutions for ($\mathcal{P}$). Either (A.2) holds, in which case (IR_L) and (IC_H) bind, or it holds with the reverse direction in which case (IR_H) and (IC_L) bind.

Next we want to show that (A.2) will always hold on the maximum. The proof is by contradiction. Suppose that (A.2) does not hold, then it has to be that it holds with the opposite direction. Therefore, the (IR_L) and (IC_H) bind. Those two give p_L and p_H, which can be substituted in the objective function of ($\mathcal{P}$) and (IC_L) to obtain

$$\max_{q_L, q_H, g} \begin{Bmatrix} [\theta_H - (1-\mu_0)\theta_L]q_H \;+ (1-\mu_0)\theta_L\, q_L \\ +\, [\rho_H - (1-\mu_0)\rho_L]\mathbb{E}_H + (1-\mu_0)\rho_L \mathbb{E}_L \end{Bmatrix} \tag{$\mathcal{P}'_L$}$$

$$\text{s.t.} \quad (\theta_H - \theta_L)\,(q_H - q_L) \;\geq\; (\rho_L - \rho_H)(\mathbb{E}_H - \mathbb{E}_L) \tag{A.3}$$

$$(\theta_H - \theta_L)q_H \;\leq\; (\rho_L - \rho_H)\mathbb{E}_H \tag{A.5}$$

where (A.5) is used instead of the inverse of (A.2), since those two are equivalent.

First, note that setting $q_H = \frac{\rho_L - \rho_H}{\theta_H - \theta_L}\,\mathbb{E}_H$ is always optimal. This is because increasing q_H relaxes (A.5) and increases the objective function of ($\mathcal{P}'_L$). Hence (A.5) has to bind. Second, note that increasing q_L increases the objective function of ($\mathcal{P}_L$). Hence ($\mathcal{P}_c$) has to bind. But this implies that (A.2) also binds.

However, this means that the maximum under (A.5) is also available under the initial supposition that (A.2) holds. Therefore, it is without loss to assume that it is the (IR_L) and (IC_H) that bind. But in this case the representation of Lemma 1.3 is still relevant. However, it is not necessarily true that (IR_H) will be satisfied, hence this has to be added to the contains. But using the binding (IR_L) and (IC_H) we can see that (IR_H) equivalently becomes (A.2). □

Proof of Proposition 1.3. First, we find the point-wise optimal signal under negative correlation. S_1's information provision problem is still the same as in ($\mathcal{G}$), however its reformulated version will be different. Following exactly the same approach as in the previous subsection we can show that ($\mathcal{G}$) equivalently becomes

$$\max_{\tilde{g}} \ \mathbb{E}_{\tilde{g}}\big[J^-(\mu)\big] \quad \text{s.t.} \ \ \mathbb{E}_{\tilde{g}}[\,\mu\,] = \mu_0 \tag{$\mathcal{G}^-$}$$

where

$$J^-(\mu) \ = \ \frac{\rho_H - \rho_L}{1-\mu_0}\, Q^-(\mu) \left(\mu - \frac{\mu_0\rho_H - \rho_L}{\rho_H - \rho_L}\right)$$

This is almost identical to the functional form of $J(\mu)$ under positive correlation, however $Q(\mu) = \mathbb{1}\{\mu \leq \mu^*\}(\theta_H - \theta_L)$ has to be replaced with

$$Q^-(\mu) \ = \ \mathbb{1}\{\mu \geq \mu^*\}(\theta_H - \theta_L)$$

The reason for this alteration is that when the correlation is negative S_2 offers the discount for high realisations of μ_1^s, instead of low ones. Therefore the graph of $J^-(\mu)$ under negative correlation can be obtained by flipping that of $J(\mu)$ around a vertical axis passing from μ^*. Hence $J^-(\mu)$ is flat at zero for $\mu < \mu^*$, then jumps to $J^-(\mu^*)$ and is subsequently decreasing. Thus a necessary condition for information provision to be strictly optimal is that

$$J^-(\mu^*) \ > \ 0 \quad \Leftrightarrow \quad \frac{\theta_L}{\theta_H} \ > \ \mu_0\rho_H$$

Moreover, following the same argumentation as in the case of positive correlation we get than information provision is strictly point-wise optimal if and only if

$$\max\{\rho_H, \mu_0\rho_H\} \ < \ \frac{\theta_L}{\theta_H} \ < \ \mu_0\rho_H + (1-\mu_0)\rho_L$$

But this is implied by

$$\rho_H \ < \ \frac{\theta_L}{\theta_H} \ < \ \mu_0\rho_H + (1-\mu_0)\rho_L$$

which is also a necessary and sufficient condition for information provision to be optimal in the first best under negative correlation. Hence under negative correlation the point-wise optimal signal in the first and second best is the same. The second inequality ensures that the discount is not achieved in the absence of information provision, while the first that a signal that achieves this discount can be constructed. This is a randomisation between posterior 0 and μ^*, and it is easy to check that the signal distribution provided in the statement of the proposition achieves that.

However, note that same way as in the case of positive correlation the constrain ($\mathcal{P}_c$) will bind under the point-wise optimal solutions when $\mu_0 < \theta_L/\theta_H$. Despite that, the argumentation used in the proof of Proposition 1.2 is still relevant and it shows that S_1 would always be better off by decreasing the probability of supplying the low type, instead of alternating the point-wise optimal signal.

On the other hand, when $\mu_0 > \theta_L/\theta_H$ and the point-wise optimal signal entails information provision, it is the ($\mathcal{P}_h$) that binds. Hence we have that

$$(\theta_H - \theta_L)q_L \;=\; (\rho_L - \rho_H)\mathbb{E}_L$$

where we use the compact notation introduced in the proof of Lemma 1.3. In this case, it is always optimal to set $q_H = 1$ and ($\mathcal{P}_c$) holds. Hence substitute the above equality in the objective function of ($\mathcal{P}'$) to obtain the unconstrained information provision problem

$$\max_{g} \;\left\{(\theta_L - \mu_0\theta_H)\cdot\frac{\rho_L - \rho_H}{\theta_H - \theta_L}\,\mathbb{E}_L + \mathbb{E}_g\big[J^-(\mu_1^s)\big]\right\} \qquad (\widetilde{\mathcal{G}})$$

where the second term is the standard part of the information provision problem of S_1, while the first is the one introduced by the binding ($\mathcal{P}_h$). We can further simplify this by noting that

$$\mathbb{E}_L \;=\; \mathbb{E}_g\left[Q^-(\mu_1^s)\cdot\frac{1-\mu_1^s}{1-\mu_0}\right]$$

Hence same way as before we can reformulate the above as a choice over unconditional posterior distributions $\tilde{g}(\mu)$.

$$\max_{\tilde{g}} \;\mathbb{E}_{\tilde{g}}\left[\widetilde{J}^-(\mu)\right] \quad \text{s.t.} \quad \mathbb{E}_{\tilde{g}}[\,\mu\,] = \mu_0 \qquad (\widetilde{\mathcal{G}}')$$

where

$$\begin{aligned}
\widetilde{J}^{-}(\mu) &= Q^{-}(\mu)\cdot(\theta_L-\mu_0\theta_H)\cdot\frac{\rho_L-\rho_H}{\theta_H-\theta_L}\cdot\frac{1-\mu}{1-\mu_0}+J^{-}(\mu)\\
&= Q^{-}(\mu)\cdot\frac{\rho_L-\rho_H}{1-\mu_0}\cdot\left\{\frac{\rho_L-\mu_0\rho_H}{\rho_L-\rho_H}-\mu+\frac{\mu_0\theta_H-\theta_L}{\theta_H-\theta_L}\cdot\big(\mu-1\big)\right\}\\
&= Q^{-}(\mu)\cdot\frac{\rho_L-\rho_H}{1-\mu_0}\cdot\left\{\frac{\rho_L-\mu_0\rho_H}{\rho_L-\rho_H}-\frac{\mu_0\theta_H-\theta_L}{\theta_H-\theta_L}-\mu\cdot\frac{(1-\mu_0)\cdot\theta_H}{\theta_H-\theta_L}\right\}
\end{aligned}$$

Hence, $\widetilde{J}^{-}(\mu)$ is equal to zero in $[0,\mu^*)$ and decreasing in $[\mu^*,1]$. In addition, we can show that $\widetilde{J}^{-}(\mu^*)>0$. It suffices to prove this for $\mu^*=1$, in which case the inequality becomes

$$\frac{\rho_L-\mu_0\rho_H}{\rho_L-\rho_H} \;>\; \frac{\mu_0\theta_H-\theta_L}{\theta_H-\theta_L}+1\cdot\frac{(1-\mu_0)\cdot\theta_H}{\theta_H-\theta_L} \;=\; 1$$

which holds as $\mu_0\in(0,1)$ and $\rho_L>\rho_H$. Therefore, $\widetilde{J}^{-}$ has the same shape with J^{-}, as a result they share the same optimal unconditional distribution over posteriors. Finally, in this case q_L is found be using the binding ($\mathcal{P}_h$) instead of ($\mathcal{P}_c$). □

Proof of Corollary 1.2. The payoff a low type is always zero, since his individual rationality constrain binds. For the high type, first suppose that $\mu_0\geq\theta_L/\theta_H$. Thus if information provision was not possible, then his payoff would be zero. But even when information provision is possible we have argued in the proof of Proposition 1.3 that his individual rationality constrains binds, hence his payoff is again zero. Finally, suppose that $\mu_0<\theta_L/\theta_H$. If information provision was not possible, then the high type's payoff would be $\theta_H-\theta_L$. However, when it is possible we can use the binding (IC_H) and (IR_L) to obtain that his payoff is

$$(\theta_H-\theta_L)\cdot q_L^{-}+(\rho_H-\rho_L)\cdot\mathbb{E}_{g^-}[Q(\mu_1^s)\,|\,\theta_L] \tag{A.6}$$

where q_L^{-} and g^{-} are as given in Proposition 1.2. In particular,

$$q_L^{-} \;=\; 1-(\rho_L-\rho_H)\cdot[1-g^{-}(\underline{s}\,|\,\theta_L)]$$

Hence (A.6) becomes

$$\begin{aligned}
(\theta_H-\theta_L)\cdot\Big[1-(\rho_L-\rho_H)\cdot[1-g^{-}(\underline{s}\,|\,\theta_L)]+(\rho_H-\rho_L)\cdot g^{-}(\underline{s}\,|\,\theta_L)\Big]&\\
= (\theta_H-\theta_L)\cdot\big[1-(\rho_L-\rho_H)\big]&
\end{aligned}$$

which is smaller than $\theta_H-\theta_L$. Hence in this case the high type buyer is worse off under information provision. □

A.2 Proofs of Section 1.5

A.2.1 Selling information

Our aim in this subsection is to prove Proposition 1.4. Therefore, in the subsequent analysis we allow S_1 to profit directly from selling information to S_2. Following Calzolari and Pavan (2006) we model this with an exogenous parameter $\gamma \in [0,1]$ that denotes the part of the ex ante benefit of information provision that S_1 captures from S_2. To be more specific, S_2's payoff for given posterior μ_2^s is

$$U_2^s \;=\; \mu_2^s\,\theta_H + \max\{0\,,\,\theta_L - \mu_2^s\theta_H\}$$

In addition, let U_2^{ND} denote S_2's payoff under no disclosure. Then S_1 captures

$$\begin{aligned}\gamma \cdot \left(\mathbb{E}_g[U^s] - U^{ND}\right) \;=\; &\gamma \cdot \mathbb{E}_g\left[\mathbb{1}\left(\mu_2^s \leq \frac{\theta_L}{\theta_H}\right)(\theta_L - \mu_2^s\theta_H)\right] \\ &- \gamma \cdot \max\left\{0\,,\,\theta_L - \left[\mu_0\theta_H + (1-\mu_0)\theta_L\right]\theta_H\right\}\end{aligned}$$

We impose no restrictions on the sign of the correlation, hence to facilitate the exposition let $\underline{\rho} = \min\{\rho_L, \rho_H\}$ and $\overline{\rho} = \max\{\rho_L, \rho_H\}$. Furthermore, we adopt the compact notation introduced in the proof of Lemma 1.3, and define $\mathbb{1}_2(\mu_2^s) \;=\; \mathbb{1}\left(\mu_2^s \leq \frac{\theta_L}{\theta_H}\right)$. Therefore, S_1 solves

$$\begin{aligned}\max_{p_L, p_H, q_L, q_H, g} \quad & \mu_0 p_H + (1-\mu_0)\,p_L + \gamma \cdot \mathbb{E}_g\left[\mathbb{1}_2(\mu_2^s)(\theta_L - \mu_2^s\theta_H)\right] \\ \text{s.t. } (\mathrm{IR}_L) \quad & \theta_L\,q_L + \rho_L\,\mathbb{E}_L - p_L \;\geq\; 0 \\ (\mathrm{IR}_H) \quad & \theta_H q_H + \rho_H \mathbb{E}_H - p_H \;\geq\; 0 \\ (\mathrm{IC}_L) \quad & \theta_L\,q_L + \rho_L\,\mathbb{E}_L - p_L \;\geq\; \theta_L q_H + \rho_L \mathbb{E}_H - p_H \\ (\mathrm{IC}_H) \quad & \theta_H q_H + \rho_H \mathbb{E}_H - p_H \;\geq\; \theta_H q_L + \rho_H \mathbb{E}_L - p_L\end{aligned} \tag{Γ}$$

where the constant part of $\gamma \cdot \left(\mathbb{E}_g[U^s] - U^{ND}\right)$ is suppressed.

Lemma A.2.1. *(Γ) equivalently becomes*

$$\max_{q_L,q_H,g} \begin{Bmatrix} \mu_0\,\theta_H q_H \; + (\theta_L - \mu_0\theta_H)\, q_L \\ + \mu_0\rho_H \left(\mathbb{E}_H - \mathbb{E}_L\right) + \rho_L\,\mathbb{E}_L + \gamma\cdot\mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s)(\theta_L - \mu_2^s\theta_H)\big] \end{Bmatrix} \tag{Γ'}$$

$$s.t. \quad (\theta_H - \theta_L)\,(q_H - q_L) \;\geq\; (\rho_H - \rho_L)\big(\mathbb{E}_L - \mathbb{E}_H\big) \tag{Γ_c}$$

The proof is identical to that of Lemma 1.5, so it is omitted. The only difference between (Γ') and the corresponding ($\mathcal{P}'$) is the addition of $\gamma\cdot\mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s)(\theta_L - \mu_2^s\theta_H)\big]$ on its second line. As in the main text, we start by ignoring the constrain (Γ') and focus on finding the point-wise optimal signal, which solves

$$\max_g \; \Big\{\mu_0\rho_H\left(\mathbb{E}_H - \mathbb{E}_L\right) + \rho_L\,\mathbb{E}_L + \gamma\cdot\mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s)(\theta_L - \mu_2^s\theta_H)\Big\} \tag{$\mathcal{G}_\gamma$}$$

Lemma A.2.2. *S_1's information provision problem equivalently becomes*

$$\max_g \;\; \mathbb{E}_g[J_\gamma(\mu_2^s)] \tag{$\mathcal{G}_\gamma$}$$

where its point-wise value $J_\gamma(\mu_2^s)$ is

$$J_\gamma(\mu_2^s) \;=\; \mathbb{1}_2(\mu_2^s)\cdot\left\{\mu_2^s\theta_H\cdot\left(\frac{1-\theta_L/\theta_H}{1-\mu_0} - \gamma\right) - \theta_L\cdot\left(\frac{1-\theta_L/\theta_H}{1-\mu_0}\cdot\frac{\mu_0\rho_H}{\theta_L/\theta_H} - \gamma\right)\right\} \tag{A.2.1}$$

Proof. We have already shown in the proof of Lemma 1.4 that

$$\mu_0\rho_H\cdot\big(\mathbb{E}_H - \mathbb{E}_L\big) + \rho_L\,\mathbb{E}_L \;=\; \frac{\rho_H-\rho_L}{1-\mu_0}\cdot(\theta_H-\theta_L)\cdot\mathbb{E}_g\left[\mathbb{1}_2(\mu_2^s)\cdot\left(\mu_1^s - \frac{\mu_0\rho_H - \rho_L}{\rho_H-\rho_L}\right)\right]$$

the right hand side of which can be further manipulated to obtain

$$\mu_0\rho_H\cdot\big(\mathbb{E}_H - \mathbb{E}_L\big) + \rho_L\,\mathbb{E}_L \;=\; \frac{\theta_H-\theta_L}{1-\mu_0}\cdot\mathbb{E}_g\Big[\mathbb{1}_2(\mu_2^s)\cdot(mu_2^s - \mu_0\rho_H)\Big]$$

Therefore, adding on the above the payoff obtained from selling information gives

$$\mathbb{E}_g\left[\mathbb{1}_2(\mu_2^s)\cdot\left\{\frac{\theta_H-\theta_L}{1-\mu_0}\cdot\Big(\mu_2^s - \mu_0\rho_H\Big) + \gamma\cdot\big(\theta_L - \mu_2^s\theta_H\big)\right\}\right] \;=$$
$$\mathbb{E}_g\left[\mathbb{1}_2(\mu_2^s)\cdot\left\{\mu_2^s\theta_H\cdot\left(\frac{1-\theta_L/\theta_H}{1-\mu_0} - \gamma\right) - \theta_L\cdot\left(\frac{1-\theta_L/\theta_H}{1-\mu_0}\cdot\frac{\mu_0\rho_H}{\theta_L/\theta_H} - \gamma\right)\right\}\right]$$

□

Notice that we conditioned the point-wise payoff J_γ on S_2's posterior on θ_2 (μ_2^s) instead of that on θ_1 (μ_1^s). This will make the expressions shorter and valid for both positive and negative correlation. Next, we express ($\mathcal{G}_\gamma$) as a choice over posteriors $\tilde{g}(\mu)$ on θ_2:

$$\max_{\tilde{g}} \; \mathbb{E}_{\tilde{g}}\big[J_\gamma(\mu)\big] \quad \text{s.t.} \quad \begin{cases} \mathbb{E}_{\tilde{g}}[\,\mu\,] = \mu_0\rho_H + (1-\mu_0)\rho_L \\ \mu \in \big[\underline{\rho}\,,\,\overline{\rho}\big] \end{cases} \tag{$\mathcal{G}'_\gamma$}$$

where note that the posterior μ is bounded by the transitioning probabilities ρ_L and ρ_H, as it is on θ_1 instead of θ_2. For the same reason its expected value has to be equal to $\mu_0\rho_H + (1-\mu_0)\rho_L$, instead of μ_0. To solve this we invoke the optimality condition that $\mathbb{E}_{\tilde{g}}\big[J_\gamma(\mu)\big] = \mathcal{J}_\gamma\big(\mu_0\rho_H + (1-\mu_0)\rho_L\big)$, where

$$\mathcal{J}_\gamma \;=\; \sup\big\{z \,|\, (\mu, z) \in \mathrm{co}(J_\gamma)\big\},$$

denotes the concave closure of J_γ.

Next, we want to understand how the graph of J_γ looks like. It will be convenient to consider different cases on the underline parameters, each one on a corresponding lemma. Throughout, we maintain the assumption that

$$\underline{\rho} \;\leq\; \frac{\theta_L}{\theta_H} \;<\; \overline{\rho} \tag{A.2.2}$$

so that information provision can have an impact on prices.

Lemma A.2.3. *Suppose that $\rho_H\mu_0 \geq \frac{\theta_L}{\theta_H}$, then $J_2(\mu) \leq 0$ for all $\mu \in \big[\underline{\rho}\,,\,\overline{\rho}\big]$.*

Therefore, information provision can never be strictly optimal under the above parametric restriction, since no information provision gives at least zero. This is identical to our conclusion in the baseline model.

Proof. First note that $\rho_H\mu_0 \geq \frac{\theta_L}{\theta_H}$ implies $\rho_H > \frac{\theta_L}{\theta_H}$, which together with (A.2.2) gives that $\rho_L \leq \frac{\theta_L}{\theta_H}$. Therefore, we only need to consider the case of positive correlation. As in the baseline model

$$J_\gamma(\mu) \;=\; 0\,, \quad \text{for all} \quad \mu \in \left(\frac{\theta_L}{\theta_H}\,,\,\rho_H\right]$$

Hence, it remain to prove that $J_\gamma(\mu) \leq 0$ for all $\mu \in \left[\rho_L, \frac{\theta_L}{\theta_H}\right]$. As J_γ is linear on this subset it will suffice to show that $J_\gamma\left(\frac{\theta_L}{\theta_H}\right)$ and $J_\gamma(\rho_L)$ are non-positive.

$$J_\gamma\left(\frac{\theta_L}{\theta_H}\right) \;=\; \frac{\theta_H - \theta_L}{1-\mu_0} \cdot \left(\frac{\theta_L}{\theta_H} - \mu_0\rho_H\right)$$

As a result,

$$J_\gamma\left(\frac{\theta_L}{\theta_H}\right) \;>\; 0 \quad\Leftrightarrow\quad \frac{\theta_L}{\theta_H} \;>\; \rho_H\mu_0 \tag{A.2.3}$$

Therefore, $J_\gamma\left(\frac{\theta_L}{\theta_H}\right) \leq 0$. Finally,

$$\begin{aligned} J_\gamma(\rho_L) \;&=\; \frac{\theta_H-\theta_L}{1-\mu_0}\cdot\Big(\rho_L-\mu_0\rho_H\Big)+\gamma\cdot\big(\theta_L-\rho_L\theta_H\big) \;\leq\; 0 \\ &\Leftrightarrow\quad \gamma \;\leq\; \frac{1-\theta_L/\theta_H}{1-\mu_0}\cdot\frac{\mu_0\rho_H-\rho_L}{\theta_L/\theta_H-\rho_L} \end{aligned}$$

But $\rho_H\mu_0 \geq \frac{\theta_L}{\theta_H}$ implies $\mu_0 \geq \frac{\theta_L}{\theta_H}$, as a result the right hand side of the above inequality is always greater or equal than one, from which it follows that it holds for any $\gamma \in [0,1]$. □

Next we want to consider the case where $\rho_H\mu_0 < \frac{\theta_L}{\theta_H}$, which we will further break down to two subcases.

Lemma A.2.4. *Suppose that* $\rho_H\mu_0 < \frac{\theta_L}{\theta_H}$ *and* $\gamma \;<\; \frac{1-\theta_L/\theta_H}{1-\mu_0}$, *then*

- J_γ *is linear and increasing in* $\left[\underline{\rho}\,,\, \frac{\theta_L}{\theta_H}\right]$ *and equals zero in* $\left(\frac{\theta_L}{\theta_H}\,,\,\overline{\rho}\right]$
- *becomes strictly positive on* $\frac{\theta_L}{\theta_H}$, *that is* $J_\gamma\left(\frac{\theta_L}{\theta_H}\right) \;>\; 0$

Proof. The second bullet point follows immediately from (A.2.3). To prove the first bullet point note that (A.2.1) gives

$$J_\gamma(\mu) \;=\; \begin{cases} \mu\cdot\theta_H\cdot\left(\frac{1-\theta_L/\theta_H}{1-\mu_0}-\gamma\right)-\theta_L\cdot\left(\frac{1-\theta_L/\theta_H}{1-\mu_0}\cdot\frac{\mu_0\rho_H}{\theta_L/\theta_H}-\gamma\right) & ,\text{ if } \;\mu\in\left[\underline{\rho}\,,\,\frac{\theta_L}{\theta_H}\right] \\ 0 & ,\text{ if } \;\mu\in\left(\frac{\theta_L}{\theta_H}\,,\,\overline{\rho}\right] \end{cases} \tag{A.2.4}$$

□

It will convenient to discuss the implications of the above lemma, after providing the one that characterises J_γ in the remaining parametric restriction.

Lemma A.2.5. *Suppose that* $\rho_H\mu_0 < \frac{\theta_L}{\theta_H}$ *and* $\gamma \;\geq\; \frac{1-\theta_L/\theta_H}{1-\mu_0}$, *then*

- J_γ *is linear, non-increasing, and strictly positive in* $\left[\underline{\rho}\,,\,\frac{\theta_L}{\theta_H}\right]$, *and equals zero in* $\left(\frac{\theta_L}{\theta_H}\,,\,\overline{\rho}\right]$
- *Extending the line that is* J_γ *in* $\left[\underline{\rho}\,,\,\frac{\theta_L}{\theta_H}\right]$ *to* $\overline{\rho}$ *would give a positive value, in other words*

$$J_\gamma\left(\frac{\theta_L}{\theta_H}\right)+J'_\gamma\left(\frac{\theta_L}{\theta_H}\right)\cdot\left(\overline{\rho}-\frac{\theta_L}{\theta_H}\right) \;\geq\; 0$$

Proof. The first bullet point follows from (A.2.3) and (A.2.4). For the second bullet point, note that the positive part of J_γ calculated at $\overline{\rho}$ becomes

$$\frac{\theta_H-\theta_L}{1-\mu_0}\cdot\Big(\overline{\rho}-\mu_0\rho_H\Big)+\gamma\cdot\big(\theta_L-\overline{\rho}\,\theta_H\big)\;\geq\;0$$

$$\Leftrightarrow\quad\frac{1-\theta_L/\theta_H}{\overline{\rho}-\theta_L/\theta_H}\cdot\frac{\overline{\rho}-\mu_0\rho_H}{1-\mu_0}\;\geq\;\gamma\quad\Leftrightarrow\quad\frac{\overline{\rho}-\overline{\rho}\cdot\frac{\theta_L}{\theta_H}}{\overline{\rho}-\frac{\theta_L}{\theta_H}}\cdot\frac{1-\mu_0\cdot\frac{\rho_H}{\overline{\rho}}}{1-\mu_0}\;\geq\;\gamma$$

where both of the two fractions on the left hand side of the last inequality are greater than one, hence it has to hold. □

Therefore, in the case covered in Lemma A.2.4 the shape of J_γ is similar to that obtained for J in the baseline model. As a result, its concave closure will be

$$\mathcal{J}_\gamma(\mu)\;=\;\begin{cases}J_\gamma(\mu) & \text{, if}\quad\mu\leq\frac{\theta_L}{\theta_H}\\ J_\gamma(\mu^*)\cdot\frac{\mu-\frac{\theta_L}{\theta_H}}{\overline{\rho}-\frac{\theta_L}{\theta_H}} & \text{, if}\quad\mu\geq\frac{\theta_L}{\theta_H}\end{cases}$$

Interestingly, the above concave closure will also be relevant under the parametric restriction of Lemma A.2.5, which follows from its second bullet point.

Therefore, arguing in the same way as in the main text we get that information provision is strictly optimal if and only if

$$\max\big\{\underline{\rho}\,,\;\rho_H\mu_0\big\}\;<\;\frac{\theta_L}{\theta_H}\;<\;\rho_H\mu_0+\rho_L(1-\mu_0)$$

in which case the optimal $\tilde{g}$ randomises between $\frac{\theta_L}{\theta_H}$ and $\overline{\rho}$.

The above discussion ignores the (Γ_c) constrain, however this is equivalent to that of the baseline case ($\mathcal{P}_c$). In addition, the incentives to maintain the point-wise optimal signal are even stronger for $\gamma>0$. Therefore, S_1 will again opt to decrease the probability of providing the good to the low type, when necessary, instead of altering the point-wise optimal signal. Hence, the solution of this extension is identical to that of the baseline model.

A.2.2 Static Type

Our aim in this subsection is to prove Proposition 1.5. Hence we assume that the buyer knows both his types when interacting with the first seller, which effectively implies a four element space $(\theta_1,\theta_2)\in\big\{(\theta_L,\theta_L),(\theta_H,\theta_L),(\theta_L,\theta_H),(\theta_H,\theta_H)\big\}$. As in the main text, to maintain a compact notation we will write $\{p_{LL}, p_{HL}, q_{LH}, q_{HH}\}$ instead of

$$\big\{p_1(\widehat{\theta}_L, \widehat{\theta}_L), p_1(\widehat{\theta}_H, \widehat{\theta}_L), q_1(\widehat{\theta}_L, \widehat{\theta}_H), q_1(\widehat{\theta}_H, \widehat{\theta}_H)\big\}$$

Also, when it is not causing confusion we will use the non-hated types even when referring to the reports.

A trivial case in which information provision can be optimal when (1.17) does not hold is that of perfect negative correlation, as we have shown in Remark 3. Hence in the remaining of this subsection we will assume that (1.17) holds and prove that in this case no information provision is always optimal.

Lemma A.2.6. *Let* $\mathbb{1}_2(\mu_2^s) = \mathbb{1}\left(\mu_2^s \leq \frac{\theta_L}{\theta_H}\right)$*, then for any choice of* $g(s \,|\, \theta_1, \theta_2)$ *:*

$$\mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s) \,|\, \theta_1, \theta_H\big] \;\leq\; \mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s) \,|\, \theta_1, \theta_L\big] \quad \textit{for} \quad \theta_1 \in \{\theta_L, \theta_H\} \tag{A.2.5}$$

and the inequality is strict if $g(s \,|\, \theta_1, \theta_2)$ *generates an impact on* S_2*'s price.*

Proof. Using the towering property we can equivalently rewrite (A.2.5) as

$$\mathbb{E}_g\Big[\mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s) \,|\, \theta_2 = \theta_H\big] - \mathbb{E}_g\big[\mathbb{1}_2(\mu_2^s) \,|\, \theta_2 = \theta_L\big] \,\Big|\, \theta_1\Big] \;\leq\; 0 \quad \text{for} \quad \theta_1 \in \{\theta_L, \theta_H\}$$

Hence it suffices to prove the statement of the lemma for the expression within the first expectation. This follows from a proof identical with that of Remark 1. □

Next, we want to consider the implication of the above lemma on the information structures that S_1 can use.

Lemma A.2.7. *It is without loss to focus on information structures that transmit transmits information about* θ_1*, but not about* θ_2*.*

Proof. The revelation principle holds. Hence it has to be that a (θ_H, θ_L) buyer prefers to truthfully report his type instead of (θ_H, θ_H) and visa versa

$$q_{HL}\,\theta_H - p_{HL} \;\geq\; q_{HH}\,\theta_H - p_{HH} \tag{A.2.6}$$

$$q_{HH}\,\theta_H + \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_H, \theta_H\right] - p_{HH} \;\geq\; q_{HL}\,\theta_H + \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_H, \theta_L\right] - p_{HL} \tag{A.2.7}$$

Adding those two up we get $\mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_H, \theta_H\right] \geq \mathbb{E}_g\left[Q(\mu_1^s) \,|\, \theta_H, \theta_L\right]$. But whenever an information structure creates an impact on prices we get that the above contradicts Lemma A.2.6. The same contradictions can be obtained for (θ_L, θ_L) and (θ_L, θ_H).

Therefore, any informative signal that transmits information on the second period type, in a way that impacts prices, cannot be implemented. Henceforth, it is without loss to ignore such information structures. □

Thus we further simplify our notation by writing $\mathbb{E}_L$ and $\mathbb{E}_H$ to denote $\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_1=\theta_L\right]$ and $\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_1=\theta_H\right]$, respectively.

We start by solving S_1's problem under the assumption that she creates a contract in which all four types will participate. To facilitate the exposition we provide here the four individual rationality constrains.

$$
\begin{aligned}
&(\mathrm{IR}_{LL}\ )\ \ q_{LL}\theta_L - p_{LL} \ \geq\ 0\\
&(\mathrm{IR}_{LH}\ )\ \ q_{LH}\theta_L + \mathbb{E}_H - p_{LH} \ \geq\ 0\\
&(\mathrm{IR}_{HL}\ )\ \ q_{HL}\theta_H - p_{HL} \ \geq\ 0\\
&(\mathrm{IR}_{HH})\ \ q_{HH}\theta_H + \mathbb{E}_H - p_{HH} \ \geq\ 0
\end{aligned}
$$

To maintain a compact notation will denote the incentive compatibility constrain that ensures that (θ_H,θ_L) does not want to report (θ_L,θ_H) as IC(HL,LH). We will use the same notation to refer to the rest of the IC constrains.

Note that Lemma A.2.7 together with inequalities (A.2.6) and (A.2.7) give (A.2.8) below, while (A.2.9) follows from the corresponding incentive compatibility constrains of the period 1 low types.

$$q_{HL}\,\theta_H - p_{HL} \ =\ q_{HH}\,\theta_H - p_{HH} \tag{A.2.8}$$

$$q_{LL}\ \theta_L - p_{LL} \ =\ q_{LH}\ \theta_L\ - p_{LH} \tag{A.2.9}$$

Lemma A.2.8. *Suppose that S_1 offers an contract in which all four types participate. Then her payoff maximisation problem equivalently becomes*

$$
\begin{aligned}
&\max_{p,g}\ \Big\{\mu_0\rho_H p_{HH} + \mu_0(1-\rho_H)p_{HL} + (1-\mu_0)\rho_L p_{LH} + (1-\mu_0)(1-\rho_L)p_{LL}\Big\}\\
&s.t\quad p_{LL}\ =\ q_{LL}\theta_L\,,\quad p_{LH}\ =\ q_{LH}\theta_L\,,\\
&\qquad\ p_{HL}\ =\ (q_{HL}-q_{HH})\theta_H + p_{HH}\\
&\qquad\ (q_{HH}-q_{LL}\,)\theta_H + q_{LL}\theta_L - (\mathbb{E}_L - \mathbb{E}_H\,)\ \geq\ p_{HH}\ \geq\ q_{HH}\theta_H\\
&\qquad\ (q_{HH}-q_{LH})\theta_H + q_{LH}\theta_L - (\mathbb{E}_L - \mathbb{E}_H)\ \geq\ p_{HH}\ \geq\ q_{HH}\theta_H
\end{aligned}
\tag{$\mathcal{P}_4$}
$$

Proof. We first want to show that the payoff of (θ_L,θ_L) is the lowest of the four types, which will imply that (i) IR_{LL} has to bind and (ii) we can ignore the other three individual rationality constrains. This follows from the derivations below

$$
\begin{aligned}
&\text{(A.2.9)} \quad q_{LH}\,\theta_L + \mathbb{E}_L - p_{LH} \;\geq\; q_{LL}\,\theta_L - p_{LL}\\
&\text{IC(HH,LL)} \quad q_{HH}\,\theta_H + \mathbb{E}_H - p_{HH} \;\geq\; q_{LL}\,\theta_H + \mathbb{E}_L - p_{LL}\\
&\qquad\qquad\qquad\qquad \Rightarrow \quad q_{HH}\,\theta_H + \mathbb{E}_H - p_{HH} \;\geq\; q_{LL}\,\theta_L - p_{LL}\\
&\text{IC(HL,LL)} \quad q_{HL}\,\theta_H - p_{HL} \;\geq\; q_{LL}\,\theta_H - p_{LL}\\
&\qquad\qquad\qquad\qquad \Rightarrow q_{HL}\,\theta_H - p_{HL} \;\geq\; q_{LL}\,\theta_L - p_{LL}
\end{aligned}
$$

Hence the binding IR_{LL} gives p_{LL}. To obtain p_{LH} substitute $q_{LL}\theta_L - p_{LL} = 0$ on the left hand side of (A.2.9).

We next want to further simplify the problem by eliminating redundant IC constrains. First, note that (A.2.8) ensures that (θ_H, θ_H) will not deviate to (θ_H, θ_L) and visa versa. Equation (A.2.9) implies the same for (θ_L, θ_H) and (θ_L, θ_L).

Next consider the incentives of the two period 1 high types (θ_H, θ_L) and (θ_H, θ_H) to deviate to either of the two period 1 low types (θ_L, θ_L) and (θ_L, θ_H). The derivations below show that if (θ_H, θ_H) prefers not to do any of those two potential deviations, then the same is true for (θ_H, θ_L).

$$
\begin{aligned}
&\text{IC(HH,LL)} \quad q_{HH}\theta_H + \mathbb{E}_H - p_{HH} \;\geq\; q_{LL}\theta_H + \mathbb{E}_L - p_{LL}\\
&\xrightarrow{\rho_H \geq \rho_L} \quad q_{HH}\theta_H - p_{HH} \;\geq\; q_{LL}\theta_H - p_{LL} \quad \xrightarrow{\text{(A.2.8)}} \quad q_{HL}\theta_H - p_{HL} \;\geq\; q_{LL}\theta_H - p_{LL}\\
&\text{IC(HH,LH)} \quad q_{HH}\theta_H + \mathbb{E}_H - p_{HH} \;\geq\; q_{LH}\theta_H + \mathbb{E}_L - p_{LH}\\
&\xrightarrow{\rho_H \geq \rho_L} \quad q_{HH}\theta_H - p_{HH} \;\geq\; q_{LH}\theta_H - p_{LH} \quad \xrightarrow{\text{(A.2.8)}} \quad q_{HL}\theta_H - p_{HL} \;\geq\; q_{LH}\theta_H - p_{LH}
\end{aligned}
$$

Hence IC(HH,LL) and IC(HH,LH) imply IC(HL,LL) and IC(HL,LH), respectively. The first part of the first inequality given in the statement of the Lemma follows immediately from IC(HH,LL), while the first pat of the second inequality follows after substituting in IC(HH,LH) the price p_{LH} using (A.2.9).

Finally consider the incentives of the two period 1 low types (θ_L, θ_L) and (θ_L, θ_H) to deviate to either of the two period 1 low types (θ_H, θ_L) and (θ_H, θ_H). The derivations below show that if (θ_L, θ_L) prefers not to do any of those two potential deviations, then the same is true for (θ_L, θ_H).

$$\text{IC(LL,HL)} \; q_{LL}\theta_L - p_{LL} \; \geq \; q_{HL}\theta_L - p_{HL}$$
$$\xRightarrow{(A.2.9)} \; q_{LH}\theta_L - p_{LH} \; \geq \; q_{HL}\theta_L - p_{HL} \; \xRightarrow{\rho_H \geq \rho_L} \; q_{LH}\theta_L + \mathbb{E}_L - p_{LH} \; \geq \; q_{HL}\theta_L + \mathbb{E}_H - p_{HL}$$
$$\text{IC(LL,HH)} \; q_{LL}\theta_L - p_{LL} \; \geq \; q_{HH}\theta_L - p_{HH}$$
$$\xRightarrow{(A.2.9)} \; q_{LH}\theta_L - p_{LH} \; \geq \; q_{HH}\theta_L - p_{HH} \; \xRightarrow{\rho_H \geq \rho_L} \; q_{LH}\theta_L + \mathbb{E}_L - p_{LH} \; \geq \; q_{HH}\theta_L + \mathbb{E}_H - p_{HH}$$

Substituting zero, which is the payoff of (θ_L, θ_L) on the left hand side of IC(LL,HH) gives that $p_{HH} \geq q_{HH}\theta_H$, whereas the same substitution on IC(LL,HL) gives

$$p_{HL} \; \geq \; q_{HL}\theta_H \; \xRightarrow{(A.2.8)} \; p_{HH} \geq q_{HH}\theta_H$$

which completes the proof. □

But note that the only effect of sending an informative signal in $(\mathcal{P}_4)$ is that it decreases the upper bound of p_{HH} and as a result of p_{HL}. Hence an informative signal is never strictly optimal. This result is due to the inclusion of (θ_L, θ_L), which makes it impossible for S_1 to charge (θ_L, θ_H) for the possibility of obtaining a discount.

We continue by considering the possibility of excluding (θ_L, θ_L) from S_1's contract.

Lemma A.2.9. *Suppose that S_1 uses a contract that excludes (θ_L, θ_L). Then S_2 charges price $\overline{p} = \theta_H$ irrespectively of the choice of $g(s \mid \theta_1)$.*

Proof. When (θ_L, θ_L) is excluded the lowest possible posterior on θ_2, with a signal that only depends on θ_1, is achieved when S_1 reveals the buyer as period 1 high type. In this case

$$\mu_2^s \; = \; \rho_H \; \geq \; \frac{\theta_L}{\theta_H}$$

from which the statement of the lemma follows. □

Therefore, again in this case no information provision is optimal, and since excluding (θ_L, θ_L) achieves nothing S_1 is no better off compared to the four type contract.

Another possibility that is potentially incentive compatible is to exclude both (θ_L, θ_L) and (θ_H, θ_L), but in this case it is trivial to argue that again S_2 will charge price $\overline{p} = \theta_H$ irrespectively of the choice of $g(s \mid \theta_1)$.

Hence we have considered all cases and we have shown that no information provision will always be optimal.

A.3 Isoelastic Cost

In this section we expand the baseline model by allowing q_t to take any positive value, but we introduce an isoelastic cost function. Hence the payoff of each seller is

$$\pi_t \;=\; p_t - c(q_t)\,, \quad \text{where} \quad c(q) \;=\; \frac{q^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \quad \text{for} \;\; \epsilon \;>\; 0\,,$$

and we interpret p_t as the total price charged by S_t for all of the traded quantity q_t. The buyer's payoff from each trade is

$$\theta_1\, q_1 - p_1 \qquad \text{and} \qquad b\,\theta_2\, q_2 - p_2\,,$$

when served by S_1 and S_2, respectively. The constant $b > 0$ is introduced to allow for the possibility that the buyer's valuation of each trade can vary in some deterministic way. To maintain the analysis as close to the baseline model as possible we assume that

$$\max\left\{\rho_L\,,\,\frac{\theta_L}{\theta_H}\right\} \;\leq\; \rho_H \tag{A.3.1}$$

which ensures (i) that the agent's type is positively correlated across sellers, and (ii) that if S_1 reveals the buyer as a high period 1 type, then S_2 will only supply a positive quantity to the period 2 high type. Finally, to demonstrate that controlling the flow of information can be payoff equivalent for S_1 to controlling access to S_2, we assume instead that S_1 simple gets to interact always first with the buyer and can commit on a single distribution even in the absence of a contract. Hence even if the buyer opts not to trade with her.

The analysis follows closely that of the baseline model. We first solve S_2's payoff maximisation problem and derive the buyer's information rents from his second contract. Subsequently, this is used to build and solve S_1's information provision problem in the first and second best, in subsections A.3.2 and A.3.3, respectively.

A.3.1 The buyer's post contractual payoff and outside option

We start by solving S_2's payoff maximisation problem for any given realisation of her posterior on the period 1 type μ_1^s, and period 2 type μ_2^s. As before, those two satisfy

$$\mu_2^s \;=\; \mu_1^s\,\rho_H + (1-\mu_1^s)\,\rho_L$$

S_2's problem is similar to that of the baseline model and its treatment can be found in the appendix. In the following lemma we provide the buyer's payoff, which is the only result needed to proceed with S_1's problem.

Lemma A.3.1. *The payoff of a low buyer type under S_2 is equal to zero, while that of the high one equals*

$$Q(\mu_1^s) \;=\; \begin{cases} b^{1+\epsilon}\cdot(\theta_H-\theta_L)\cdot\left(\frac{\theta_L-\mu_2^s\theta_H}{1-\mu_2^s}\right)^{\epsilon} & ,\ \textit{if}\quad \mu_1^s \le \mu^* \\ 0 & ,\ \textit{if}\quad \mu_1^s \ge \mu^* \end{cases} \tag{A.3.2}$$

Also, on the subset of posteriors $[0,\mu^*)$ it is decreasing, and strictly concave (convex) for

$$\mu_1^s \quad < (>) \quad \mu^* + \frac{1-\epsilon}{2\cdot(\rho_H-\rho_L)}\cdot\left(1-\frac{\theta_L}{\theta_H}\right)$$

Proof. Follows from the corresponding lemma of the section of multi-period contracts in Chapter 2 once you substitute $Q(\mu_1^s)$ with $B(\mu_2^s)$. □

Therefore, only a high period 2 type obtains a positive payoff under S_2, as in the baseline model. But contrary to it, his payoff is a continuous function of S_2's posterior. Interestingly, for $\epsilon \to 0$ the isoelastic model converges to the baseline one.

Next, we want to demonstrate how S_1 can enforce the buyer's participation in her contract by using the event $\{\emptyset\}$. This denotes a rejection of S_1's offer from the buyer, and we will show that it will not occur on the equilibrium path. We have assumed that S_1 can commit on the signal distribution g even if the buyer does not participate in her contract[1] . Hence, let $g(s\,|\,\emptyset)$ be the corresponding conditional distribution. Therefore, the outside options of a period 1 high and low type are

$$\rho_H\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\emptyset\right] \quad \text{and} \quad \rho_L\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\emptyset\right]$$

respectively. Thus, S_1's objective is to reduce $\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\emptyset\right]$ as much as possible. This can be made zero by introducing signal realisation $s_\emptyset \in S$ such that

$$\left.\begin{aligned} g(s_\emptyset\,|\,\emptyset\,) &= 1 \\ g(s_\emptyset\,|\,\theta_L\,) &= 0 \\ g(s_\emptyset\,|\,\theta_H) &= \varepsilon \end{aligned}\right\} \quad \Rightarrow \quad \Pr(\theta_1=\theta_H\,|\,s_\emptyset) \;=\; \frac{\mu_0\,\varepsilon}{\mu_0\,\varepsilon+\Pr(\emptyset)}$$

[1] The discussion here demonstrates how S_1 can reduce the buyer's outside option to zero. In the baseline model, even if S_1 was not able to commit without a contract on g, the solution of her information provision problem would be the same. To see this note that the argumentation on Section 1.2 does not rely on the architect controlling access to the designer.

Suppose S_1 was using a mechanism such that Pr($\emptyset$)=0, that is the buyer was always participating on path. Then $\Pr(\theta_1 = \theta_H \,|\, s_\emptyset) = 1$ and this would be true even if ε was an infinitesimal positive real number. Effectively, the buyer's outside option under such a mechanism would be zero as

$$\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\emptyset\right] \;=\; Q(1) \;=\; 0$$

In addition, by setting $\varepsilon \to 0$ the cost of including $s_\emptyset$ on the set of possible realisations S tends to zero, because the same is true for the probability of using it on path.

Hence there exists a mechanism that achieves the smallest possible outside option for the buyer, and the cost of doing so is zero since the effect of $s_\emptyset$ on S_1's information provision problem is infinitesimal. Moreover, the signal $s_\emptyset$ can be used in both the first and second best. Interestingly, this make the model equivalent to one in which S_1 controls not only the information provided to S_2, but also the buyer's access to her. Hence to save in space, and because $s_\emptyset$ does not occur on path, we will hereafter ignore it on both the proofs and statements of our results, and we will set the buyer's outside option immediately to zero.

A.3.2 The first best contract of Seller 1

We solve S_1's revenue maximisation problem under the assumption that if the buyer opts to participate in her mechanism, then his type is automatically reveal to her, but not to S_2. Hence she solves

$$\begin{aligned} \max_{p_1,q_1,g} \quad & \mu_0\Big(p_1(\theta_H) - c\big[q_1(\theta_H)\big]\Big) + (1-\mu_0)\Big(p_1(\theta_L) - c\big[q_1(\theta_L)\big]\Big) \\ \text{s.t. } (\text{IR}_L) \quad & \theta_L\, q_1(\theta_L) - p_1(\theta_L) + \rho_L\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_L\right] \;\geq\; 0 \\ (\text{IR}_H) \quad & \theta_H q_1(\theta_H) - p_1(\theta_H) + \rho_H \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_H\right] \;\geq\; 0 \end{aligned} \qquad (\mathcal{P}_f)$$

Both of the individual rationality constraints need to bind. Hence solve for the prices, and substitute those in S_1's objective function to obtain the unconstrained problem

$$\max_{q_1,g} \quad \left\{ \begin{aligned} &\mu_0\cdot\Big(\theta_H q_1(\theta_H) - c\big[q_1(\theta_H)\big]\Big) + (1-\mu_0)\cdot\Big(\theta_L q_1(\theta_L) - c\big[q_1(\theta_L)\big]\Big) \\ &\qquad + \mu_0\rho_H\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_H\right] + (1-\mu_0)\rho_L \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_L\right] \end{aligned} \right\} \qquad (\mathcal{P}_f')$$

The first line represents the surplus generated from the trade of period 1, while the second the buyer's ex ante post contractual payoff, which S_1 captures through the individual rationality constrains. We can use first order conditions to obtain that the optimal supply schedule in the first best is

$$q_f(\theta_1) \;=\; (\theta_1)^{\epsilon}$$

In the remaining of this subsection we focus on S_1's information provision problem

$$\max_{g} \quad \Big\{ \mu_0 \rho_H \, \mathbb{E}_g \left[Q(\mu_1^s) \,|\, \theta_H \right] + (1-\mu_0)\rho_L \mathbb{E}_g \left[Q(\mu_1^s) \,|\, \theta_L \right] \Big\} \tag{$\mathcal{G}_f$}$$

Lemma A.3.2. *In the first best, S_1's information provision problem equivalently becomes*

$$\max_{g} \;\; \mathbb{E}_g[J_f(\mu_1^s)] \tag{$\mathcal{G}_f$}$$

where its point-wise value $J_f(\mu_1^s)$ is

$$J_f(\mu_1^s) \;=\; Q(\mu_1^s) \cdot \left[\, \mu_1^s \cdot (\rho_H - \rho_L) + \rho_L \, \right] \tag{A.3.3}$$

Proof. Identical to that of Lemma 1.2. □

Then following the same argumentation as in the main text we get that S_1's information provision problem becomes

$$\max_{\tilde{g}} \; \mathbb{E}_{\tilde{g}}\big[J_f(\mu) \big] \quad \text{s.t.} \;\; \mathbb{E}_{\tilde{g}}[\, \mu \,] \;=\; \mu_0 \tag{$\mathcal{G}_f'$}$$

To solve this it is important to characterise the graph of $J_f(\mu)$. Given our existing assumption (A.3.1), the only case in which information provision can have an impact on the mechanism used by S_2 is when

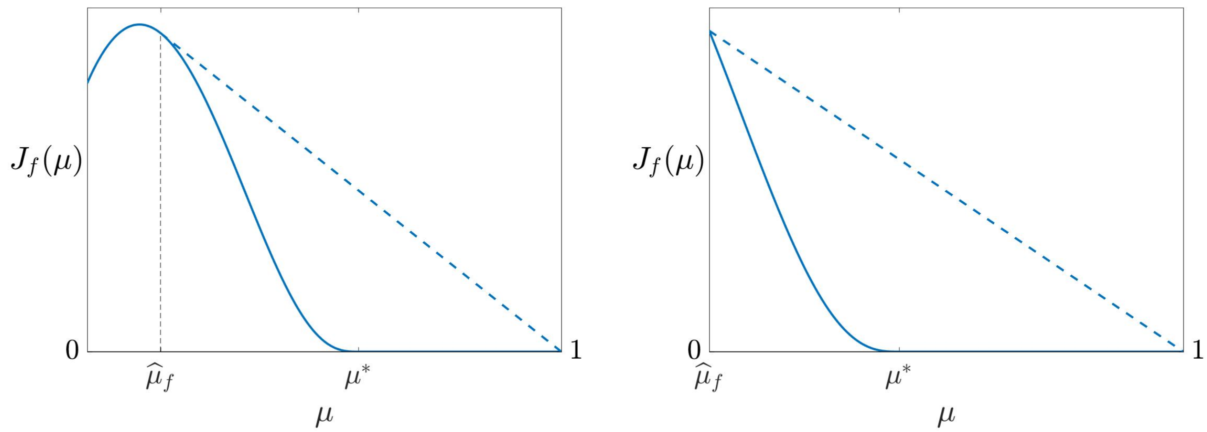
$$\rho_L \;<\; \frac{\theta_L}{\theta_H} \;<\; \rho_H \tag{A.3.4}$$

in which case $\mu^* \in (0,1)$. Also, define

$$\mu_f^{**} \;=\; \max \left\{ 0 \, , \, \frac{\beta_f^{**} - \rho_L}{\rho_H - \rho_L} \right\} \quad \text{where} \quad \beta_f^{**} \;=\; \frac{2\, \theta_L / \theta_H}{2 + (\epsilon - 1)\big(1 - \frac{\theta_L}{\theta_H}\big)}$$

Lemma A.3.3. *Suppose that (A.3.4) holds. Then $J_f(\mu)$ is positive on $[0, \mu^*)$ and equals zero on $[\mu^*, 1]$. Moreover,*

- *It changes monotonicity at most once*
- *If $\epsilon \leq 1$, then it is strictly concave on $[0, \mu^*]$.*
- *If $\epsilon > 1$, then it is strictly concave in $[0, \mu_f^{**}]$, and strictly convex in $[\mu_f^{**}, \mu^*]$.*



(a) small ρ_L (b) big ρ_L

Figure A.2: A representative graph of J_f. The dashed line denotes its concave closure, when this is above J_f.

Proof. Follows as a subcase of Lemma B.1.2 of Chapter 2. In particular, the functional form of $J_f(\mu)$ is the same with that of $J_t(\mu, \lambda)$ on the $(\lambda = 0)$ boundary. □

Plots (A.2a) and (A.2b) demonstrate the two possible cases of J_f under (A.3.4). In the former plot it is initially increasing and subsequently decreasing, whereas in the latter it is only decreasing. An intuitive way to understand the shape of J_f is to consider its value on the prior μ_0, where the underline trade off between $Q(\mu_0)$ and $\mu_0\rho_H + (1 - \mu_0)\rho_L$ becomes apparent. That is between the rents captured by a period 2 high type and the probability of the buyer to be one. The higher the μ_0 is, the smaller the $Q(\mu_0)$, which has a negative impact on S_1's post contractual payoff. However, only a period 2 high type captures positive rents, which creates an effect opposite from the above.

As in the baseline model, we solve S_1's information provision problem ($\mathcal{G}'_f$) by invoking the optimality condition $\mathbb{E}_{\tilde{g}}[J_f(\mu)] = \mathcal{J}_f(\mu_0)$, where

$$\mathcal{J}_f(\mu) \;=\; \sup\left\{z \,|\, (\mu, z) \in \mathrm{co}(J_f)\right\},$$

denotes the concave closure of J_f. In Figure (A.2) whenever $\mathcal{J}_f(\mu) > J_f(\mu)$ this is represented by the dashed line. If $\mu < \hat{\mu}_f$, then there is not a linear combination of points of J_f that achieves something above $J_f(\mu)$. Henceforth, on this set of points $J_f(\mu) = \mathcal{J}_f(\mu)$ and there is no benefit from information provision. On the other hand, for $\mu > \hat{\mu}_f$ finding such a linear combination is possible. In fact, in this case $\mathcal{J}_f(\mu)$ is the line that connects $J_f(1)$ to the tangency point $\hat{\mu}_f$, which is the unique solution of

$$J_f(\hat{\mu}_f) + J_f'(\hat{\mu}_f)(1-\hat{\mu}_f) \;=\; 0 \tag{A.3.5}$$

Interestingly, the tangency point is not necessarily in $[0,1]$, as demonstrated in Plot (A.2b), in which case we instead use the corner solution $\hat{\mu}_f = 0$.

Proposition A.3.1. *In the first best, an informative signal strictly solves S_1's payoff maximisation problem ($\mathcal{P}_f$) iff (A.3.4) holds and $\mu_0 > \hat{\mu}_f$. If those two conditions hold, then an optimal signal is $s \in \{\underline{s}, \overline{s}\}$ with distribution*

$$g_f\left(\underline{s}\,|\,\theta_L\right) \;=\; 1\,, \qquad \textit{and} \qquad g_f\left(\underline{s}\,|\,\theta_H\right) \;=\; \frac{1-\mu_0}{\mu_0}\frac{\hat{\mu}_f}{1-\hat{\mu}_f}. \tag{A.3.6}$$

In addition, the optimal supply schedule is $q_f(\theta_1) \;=\; (\theta_1)^{\epsilon}$.

Proof. The above discussion implies that the concave closure of $J_f(\mu)$ is given by

$$\mathcal{J}_f(\mu) \;=\; \begin{cases} J_f(\mu) & \text{, for } \mu \,\le\, \hat{\mu}_f \\ J_f(\hat{\mu}_f) + J_f'(\hat{\mu}_f)(\mu-\hat{\mu}_f) & \text{, for } \mu \,\ge\, \hat{\mu}_f \end{cases} \tag{A.3.7}$$

The functional form of $\hat{\mu}_f$ follows as a subcase of Proposition 2.1.2 when $\psi_i = 0$. This is

$$\hat{\mu}_f \;=\; \max\left\{0\,,\, \frac{\hat{\beta}_f-\rho_L}{\rho_H-\rho_L}\right\} \quad \text{where} \quad \hat{\beta}_i \;=\; \frac{\omega_1-\sqrt{(\omega_1)^2-4\omega_0\omega_2}}{2\omega_2}$$

and

$$\omega_0 \;=\; \frac{\theta_L}{\theta_H}\,, \quad \omega_1 \;=\; 1+\frac{\theta_L}{\theta_H}+\epsilon\cdot\left(1-\frac{\theta_L}{\theta_H}\right) \quad \omega_2 \;=\; 1+\frac{\epsilon}{\rho_H}\cdot\left(1-\frac{\theta_L}{\theta_H}\right) \tag{A.3.8}$$

If $\mu_0 \le \hat{\mu}_f$, then $J_f(\mu_0) = \mathcal{J}_f(\mu_0)$ and as a result setting $\Pr(\mu = \mu_0) = 1$ is optimal, which is achieved when no information is provided to S_2. If $\mu_0 > \hat{\mu}_f$, then the optimal signal randomises between posteriors $\hat{\mu}_f$ and 1. This is the solution of ($\mathcal{G}_f'$). Hence if $s \in \{\underline{s}, \overline{s}\}$ solves ($\mathcal{G}_f$), then it has to be that $g_f\left(\overline{s}\,|\,\theta_L\right) = 0$, so that the posterior that $\overline{s}$ implies is one. Therefore, $g_f\left(\underline{s}\,|\,\theta_L\right) = 1$. Finally, to find $g_f\left(\underline{s}\,|\,\theta_H\right)$ note that this has to satisfy

$$\hat{\mu}_f \;=\; \frac{\mu_0\, g_f\left(\underline{s}\,|\,\theta_H\right)}{\mu_0\, g_f\left(\underline{s}\,|\,\theta_H\right)+1-\mu_0} \tag{A.3.9}$$

But if this signal solves ($\mathcal{G}_f$), then it also solves ($\mathcal{P}_f'$), and as a result ($\mathcal{P}_f$). □

A.3.3 The second best contract of Seller 1

Next we analyse the second best, where θ_1 is the buyer's private information. S_1 solves

$$\begin{aligned}
&\max_{p_1(\widehat{\theta}_1),\, q_1(\widehat{\theta}_1),\, g(s\,|\,\widehat{\theta}_1)} \left\{ \mu_0\Big(p_1(\widehat{\theta}_H) - c\big[q_1(\widehat{\theta}_H)\big]\Big) + (1-\mu_0)\,\Big(p_1(\widehat{\theta}_L) - c\big[q_1(\widehat{\theta}_L)\big]\Big) \right\} \\
&\text{s.t. } (\text{IR}_L\,),\ \ (\text{IR}_H\,), \\
&(\text{IC}_L\,)\ \ \theta_L\, q_1(\widehat{\theta}_L) \ + \rho_L\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_L\right] - p_1(\widehat{\theta}_L\,) \\
&\qquad\qquad\qquad \geq\ \theta_L\, q_1(\widehat{\theta}_H)\ + \rho_L\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_H\right] - p_1(\widehat{\theta}_H) \qquad\qquad (\mathcal{P}) \\
&(\text{IC}_H)\ \ \theta_H q_1(\widehat{\theta}_H) + \rho_H \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_H\right] - p_1(\widehat{\theta}_H) \\
&\qquad\qquad\qquad \geq\ \theta_H\, q_1(\widehat{\theta}_L)\ \ + \rho_H \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\widehat{\theta}_L\,\right] - p_1(\widehat{\theta}_L)
\end{aligned}$$

where the individual rationality constraints, (IR_L) and (IR_H), are as in the previous subsection. Hereafter, we will use $\{p_L, p_H, q_L, q_H\}$ instead of $\{p_1(\widehat{\theta}_L), p_1(\widehat{\theta}_H), q_1(\widehat{\theta}_L), q_1(\widehat{\theta}_H)\}$, in order to maintain a compact notation. Similarly to the first best, it is convenient to reduce the number of constrains by substituting the transfers p_L and p_H.

Lemma A.3.4. *In the second best, S_1's payoff maximisation problem equivalently becomes*

$$\begin{aligned}
\max_{q_1, g}\ &\left\{ \mu_0 \cdot \Big(\theta_H q_H - c(q_H)\Big) + (1-\mu_0)\cdot\left(\frac{\theta_L - \mu_0\theta_H}{1-\mu_0}\cdot q_L - c(q_L)\right)\right. \\
&\left. +\ \mu_0\rho_H\, \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_H\right] + (1-\mu_0)\rho_L \mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_L\right] -\ \mu_0(\rho_H - \rho_L)\mathbb{E}_g\left[Q(\mu_1^s)\,|\,\theta_L\right] \right\} \qquad (\mathcal{P}')
\end{aligned}$$

$$s.t\quad (\theta_H - \theta_L)\,(q_H - q_L)\ \geq\ (\rho_H - \rho_L)\,\Big(\mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_L\big] - \mathbb{E}_g\big[Q(\mu_1^s)\,|\,\theta_H\big]\Big) \qquad (\mathcal{P}_c)$$

Proof. Identical to that of Lemma 1.3. □

The first line represents the surplus of the first period trade that S_1 is able to capture. Similarly, the second line is the part of the buyer's ex ante payoff that S_1 captures. Both lines are smaller than the corresponding ones of $(\mathcal{P}'_f)$, because of the period 1 high type's rents. The point-wise optimal production level is

$$q_1^*(\theta_1)\ =\ \begin{cases} (\theta_H)^\epsilon & ,\text{ if }\ \theta_1 = \theta_H \\ (\xi)^\epsilon & ,\text{ if }\ \theta_1 = \theta_L \end{cases}\ ,\ \ \text{where}\ \ \xi\ =\ \max\left\{0\,,\ \theta_L - (\theta_H - \theta_L)\cdot\frac{\mu_0}{1-\mu_0}\right\} \qquad \text{(A.3.10)}$$

However, because of $(\mathcal{P}_c)$ we will not always be able to find a contract that implements this supply schedule. We will say that a signal is *impactful* if $\mathbb{E}_g\big[Q(\beta_0^s)\,|\,\theta_L\big] > \mathbb{E}_g\big[Q(\beta_0^s)\,|\,\theta_H\big]$.

As we have argued in Remark 1 whenever a signal influences S_2's contract, this inequality has to hold.

Lemma A.3.5 (Implementation). *For $\rho_H > \rho_L$:*

- *For any $q_1(\theta_1)$ combined with an impactful signal, there exists a constant b high enough for those to not be implementable.*
- *Conversely, the point-wise optimal $q_1^*(\theta_1)$ combined with an uninformative signal is implementable.*
- *A sufficient condition for $q_1^*(\theta_1)$ combined with any signal to be implementable is that*

$$\left(\frac{\theta_H}{\theta_L}\right)^{\epsilon} \;\geq\; 1 + (\rho_H - \rho_L) \cdot b^{1+\epsilon} \tag{A.3.11}$$

Proof. To obtain the first statement note that ($\mathcal{P}_c$) can equivalently be re-written as

$$\begin{aligned} q_1(\theta_H) - q_1(\theta_L) \;\geq\; & (\rho_H - \rho_L)\, b^{1+\epsilon} \\ & \times \left(\mathbb{E}_g \left[\left(\frac{\theta_L - \theta_H \mu_2^s}{1-\mu_2^s} \right)^{\epsilon} \middle| \, \theta_L \right] - \mathbb{E}_g \left[\left(\frac{\theta_L - \theta_H \mu_2^s}{1-\mu_2^s} \right)^{\epsilon} \middle| \, \theta_H \right] \right) \end{aligned}$$

Hence as long as the right hand side is positive, it can be made infinitely large by increasing b. The second statement follows trivially from the above expression, as in the case of an uninformative signal the two expectations on the second line of its right hand side are equal to each other. Thus this becomes zero. For the third statement note that the left hand side of the above expression is bigger than $(\theta_H)^\epsilon - (\theta_L)^\epsilon$. Also, the second line of its right hand side is smaller than $\mathbb{E}_s\big[Q(\beta_0^s)\,|\,\theta_L\big] \leq Q(0) = (\theta_L)^\epsilon$. Therefore ($\mathcal{P}_c$) is implied by

$$(\theta_H)^\epsilon - (\theta_L)^\epsilon \;\geq\; b^{1+\epsilon}\,(\rho_H - \rho_L)\,(\theta_L)^\epsilon\,,$$

which after rearranging gives (A.3.11). □

The lemma sheds light on the limitation of point-wise maximisation, for both production and information provision, in this setup. To be more precise, when the buyer's report has post contractual value, in the sense that an informative signal is optimal, S_1 may not always be able to provide the right incentives for the buyer to share this information. In particular, the first statement above shows that when the interaction of period 2 is much more important than that of period 1, that is b is relatively big or equivalently θ_L and θ_H are relatively small, S_1 may have to significantly constrain the amount of information she shares with S_2. In fact if $b \to \infty$, then on this limiting case S_1 cannot provide any information at all. Dworczak (2016*a*,*b*) has demonstrated the same limitation by showing in a setting closer to an action that the only type of mechanism that is always implementable, under an aftermarket for the bidder that acquires the object, is one that does not share information on his reported type. On the other hand, when no information provision is optimal, as in Calzolari and Pavan (2006), this is always implementable.

The rest of the analysis will focus on deriving the point-wise optimal signal. First, because for the right choice of b, or θ_H/θ_L, condition (A.3.11) will always be satisfied. Second, because for this case the graphical approach used in the previous subsection still applies, which makes the solution of S_1's information provision problem much more tractable and ease to demonstrate[2]. S_1's information provision problem, ignoring the implementation constrain (A.3.11), becomes

$$\max_g \; \Big\{ \mu_0 \rho_H \, \mathbb{E}_g \left[Q(\mu_1^s) \,|\, \theta_H \right] + (1-\mu_0)\rho_L \mathbb{E}_g \left[Q(\mu_1^s) \,|\, \theta_L \right] \; - \; \mu_0(\rho_H - \rho_L)\mathbb{E}_g \left[Q(\mu_1^s) \,|\, \theta_L \right] \Big\} \quad (\mathcal{G})$$

Lemma A.3.6. *In the second best, S_1's information provision problem equivalently becomes*

$$\max_g \;\; \mathbb{E}_g[J(\mu_1^s)] \qquad (\mathcal{G})$$

where its point-wise value $J(\mu_1^s)$ is

$$J(\mu_1^s) \;=\; \frac{\rho_H - \rho_L}{1-\mu_0} \cdot Q(\mu_1^s) \cdot \left(\mu_1^s - \frac{\mu_0 \rho_H - \rho_L}{\rho_H - \rho_L} \right) \qquad \text{(A.3.12)}$$

Proof. Identical to that of Lemma 1.4. □

Similarly to the baseline model ($\mathcal{G}$) is transformed to a choice of distributions over posteriors $\tilde{g}(\mu)$, that is it equivalently becomes

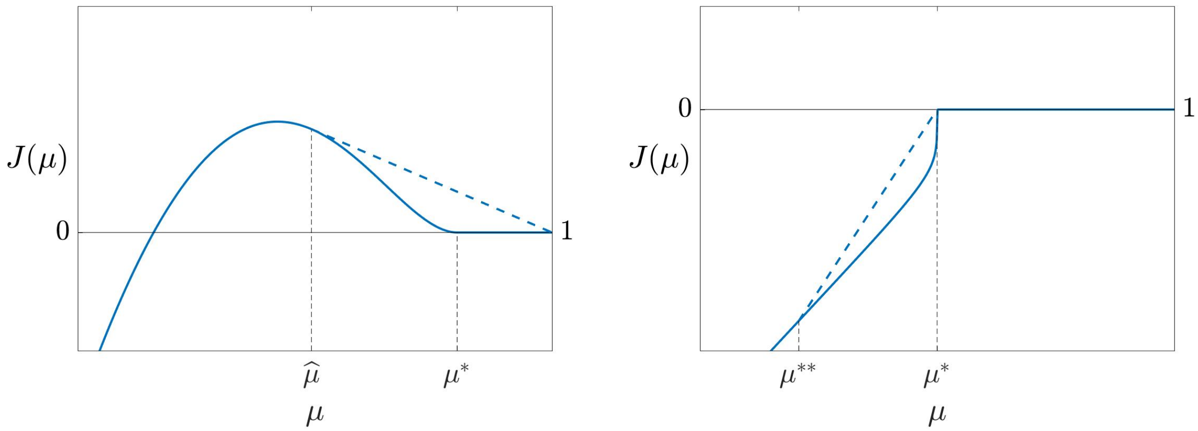
$$\max_{\tilde{g}} \; \mathbb{E}_{\tilde{g}}\big[J(\mu)\big] \quad \text{s.t.} \;\; \mathbb{E}_{\tilde{g}}[\,\mu\,] \;=\; \mu_0 \qquad (\mathcal{G}')$$

This is solved by invoking the optimality condition $\mathbb{E}_{\tilde{g}}\big[J(\mu)\big] = \mathcal{J}(\mu_0)$, where

$$\mathcal{J}(\mu) \;=\; \sup\big\{ z \,|\, (\mu, z) \in \text{co}(J_0) \big\}$$

denotes the concave closure of J. To find $\mathcal{J}$ the shape of J needs to be characterised, which

[2]Despite that, the implications of a binding implementability constrain pose a very interesting question, worthy of further research.



(a) $\mu_0\rho_H < \frac{\theta_L}{\theta_H}$ (b) $\mu_0\rho_H \geq \frac{\theta_L}{\theta_H}$

Figure A.3: A representative graph of J. The dashed line denotes its concave closure, when this is above J.

is undertaken in the following Lemma. To shorten its statement define

$$\mu^{**} = \max\left\{0\ ,\ \frac{\beta^{**} - \rho_L}{\rho_H - \rho_L}\right\}\ , \quad \text{and} \quad \beta^{**} \equiv \frac{\theta_L}{\theta_H}\frac{2\big(1-\mu_0\rho_H\big) + (\epsilon-1)\big(1-\frac{\theta_L}{\theta_H}\big)\frac{\mu_0\rho_H}{\theta_L/\theta_H}}{2(1-\mu_0\rho_H) + (\epsilon-1)\big(1-\frac{\theta_L}{\theta_H}\big)}\ .$$

As in the first best, we only need to consider the shape of J when (A.3.4) holds, since otherwise information provision has no impact on S_2's contract.

Lemma A.3.7. *Assume throughout that* (A.3.4) *holds. Suppose* $\mu_0\rho_H < \frac{\theta_L}{\theta_H}$*, then* $J(\mu)$*:*

- *changes monotonicity at most once, and if* $\mu^* > 0$ *it falls to it from above.*
- *If* $\epsilon \leq 1$*, then it is strictly concave on* $[0, \mu^*]$*.*
- *If* $\epsilon > 1$*, then it is strictly concave in* $[0, \mu^{**}]$*, and strictly convex in* $[\mu^{**}, \mu^*]$*.*

If instead $\mu_0\rho_H \geq \frac{\theta_L}{\theta_H}$*, then* $J(\mu)$ *is non-positive.*

Proof. Follows as a subcase of Lemma B.1.2 in Chapter 2. □

Figure (A.3) shows the two possible cases of $J(\mu)$. That first one, shown in Plot (A.3a), is qualitatively similar to the graph of J_f from the previous subsection. The second, presented in Plot (A.3b), is substantially different, and in particular in non-positive. This is because if $\mu_0\rho_H$ is relatively high, then convincing the period 1 high type to report truthfully, under an informative signal, becomes too expensive. Similarly, to the baseline model.

Hence if $\rho_H \mu_0 < \theta_L/\theta_H$, then the analysis is identical to that of the first best, that is the dashed line on Plot (A.3a) represents $\mathcal{J}(\mu)$ for $\mu > \hat{\mu}$, and $\hat{\mu}$ is found by solving the tangency condition

$$J(\hat{\mu}) + J'(\hat{\mu})(1-\hat{\mu}) \;=\; 0 \tag{A.3.13}$$

On the other hand, if $\rho_H \mu_0 \geq \theta_L/\theta_H$, then no information provision has to be optimal since it always achieves at least zero.

Proposition A.3.2. *Suppose throughout that the implementation condition (A.3.11) holds. Then the point-wise optimal supply schedule, as given in (A.3.10), solves S_1's payoff maximisation problem ($\mathcal{P}$). In addition, an informative signal strictly solves ($\mathcal{P}$) iff*

$$\max\{\rho_L,\, \rho_H\mu_0\} \;<\; \frac{\theta_L}{\theta_H} \;<\; \rho_H\,, \qquad \textit{and} \qquad \mu_0 > \hat{\mu} \tag{A.3.14}$$

If those two conditions hold, then an optimal signal is $s \in \{\underline{s}, \overline{s}\}$ with distribution

$$g^*\left(\underline{s}\,|\,\theta_L\right) \;=\; 1\,, \qquad \textit{and} \qquad g^*\left(\underline{s}\,|\,\theta_H\right) \;=\; \frac{1-\mu_0}{\mu_0}\frac{\hat{\mu}}{1-\hat{\mu}} \tag{A.3.15}$$

***Proof*.** Whenever (A.3.11) holds, the point-wise optimal supply schedule is implementable under any signal. Hence the solution of ($\mathcal{G}$) will also solve ($\mathcal{P}'$), and as a result ($\mathcal{P}$).

Then we can focus on finding the $\tilde{g}$ that solves ($\mathcal{G}'$). It follows from the above discussion that whenever $\mu_0\rho_H \geq \theta_L/\theta_H$ no information provision is optimal. If instead $\mu_0\rho_H < \theta_L/\theta_H$, but (A.3.4) does not hold, then information provision has no impact on S_2's contract, in which case no information provision is again optimal.

Hence it remains to consider $\mu_0\rho_H < \theta_L/\theta_H$ when (A.3.4) holds. As we argued before in this case the concave closure of J is

$$\mathcal{J}(\mu) \;=\; \begin{cases} J(\mu) & \text{, for } \;\mu \;\leq\; \hat{\mu} \\ J(\hat{\mu}) + J'(\hat{\mu})(\mu-\hat{\mu}) & \text{, for } \;\mu \;\geq\; \hat{\mu} \end{cases} \tag{A.3.16}$$

where the functional form of the tangency point $\hat{\mu}$ follows as a subcase of Proposition 2.1.2 in Chapter 2 when $\psi_i = \mu_0\rho_H$.

$$\hat{\mu} \;=\; \max\left\{0\,,\, \frac{\hat{\beta}-\rho_L}{\rho_H-\rho_L}\right\} \quad \textit{where} \quad \hat{\beta} \;=\; \frac{\omega_1 - \sqrt{(\omega_1)^2 - 4\omega_0\omega_2}}{2\omega_2} \tag{A.3.17}$$

and

$$\omega_0 \;=\; \frac{\theta_L}{\theta_H} + \frac{\mu_0\rho_H k}{1-\mu_0}\,,\;\; \omega_1 \;=\; 1 + \frac{\theta_L}{\theta_H} + k\frac{1+\mu_0}{1-\mu_0}\,,\;\; \omega_2 \;=\; 1 + \frac{k}{\rho_H(1-\mu_0)}\,, \tag{A.3.18}$$

and $k = \epsilon\left(1 - \frac{\theta_L}{\theta_H}\right)$ is introduced to maintain a compact notation. Note that if the tangency point is negative, then we use the corner solution instead, which is zero. Hence we can conclude that information provision is strictly optimal iff $\mu_0 > \hat{\mu}$, in which case the solution of ($\mathcal{G}'$) randomises between $\hat{\mu}$ and one. Let $\overline{s}$ be the signal that results in $\mu = 1$. Then it has to be that $g^*(\overline{s}\,|\,\theta_L) = 0$. Therefore, the binary signal $s \in \{\underline{s}, \overline{s}\}$ solves ($\mathcal{G}$) if $g^*(\underline{s}\,|\,\theta_L) = 1$ and $g^*(\underline{s}\,|\,\theta_H)$ satisfies

$$\hat{\mu} \;=\; \frac{\mu_0\, g^*(\underline{s}\,|\,\theta_H)}{\mu_0\, g^*(\underline{s}\,|\,\theta_H) + 1 - \mu_0} \tag{A.3.19}$$

□

Compared to the baseline model, it is harder to infer the shape of the subset of transitioning probabilities for which information provision is strictly optimal. This is because instead of the linear constrain (1.12) that used to define the diagonal of the triangle that was this set for the baseline model, we now have the non-linear constrain $\mu \geq \hat{\mu}$. Despite that, we can still make the following claim. Define

$$(\hat{\rho}_L, \hat{\rho}_H) \;=\; \begin{cases} (\tilde{\rho}_L, 1) & \text{, if } \mu_0 < \theta_L/\theta_H \\ \left(0, \frac{\theta_L/\theta_H}{\mu_0}\right) & \text{, if } \mu_0 \geq \theta_L/\theta_H \end{cases}$$

where $\tilde{\rho}_L$ is the ρ_L that solves $\mu_0 + (1-\mu_0)\rho_L = \hat{\beta}$ when $\rho_H = 1$ and $\mu_0 < \theta_L/\theta_H$.

Corollary A.1. *The set of points for which (A.3.14) is satisfied is a convex subset of*

$$\left\{(\rho_L, \rho_H) \;:\; \rho_L \;\in\; \left[\hat{\rho}_L\,,\, \frac{\theta_L}{\theta_H}\right] \;\; \text{and} \;\; \rho_H \;\in\; \left[\frac{\theta_L}{\theta_H}\,,\, \hat{\rho}_H\right]\right\} \tag{A.3.20}$$

In addition, if for given point (ρ_L, ρ_H) information provision is strictly optimal in the second best, then the same is true for the first best.

For a graphical illustration of the above result also check Plot (1.6b). The comparison between the first and second best is of special interest. As shown the set of transitioning probabilities ρ_L and ρ_H for which information provision is optimal is larger in the former setup. This is due to the information rents captured by the period 1 high type, which create an additional cost that S_1 has to incur if she opts for an informative signal.

Proof . The functional form of $\hat{\mu}$ has being given in (A.3.17) and (A.3.18). It follows from the proof of Proposition 2.1.2 in Chapter 2 that $\hat{\beta}$ is the smaller of the two solutions of

$$\omega_2\beta^2 - \omega_1\beta + \omega_0 \;=\; 0 \tag{A.3.21}$$

with respect to β. Hence the left hand side of the above is zero when calculated on $\hat{\beta}$. If we calculate it on $\beta = \rho_H$ we get

$$\omega_2\rho_H^2 - \omega_1\rho_H + \omega_0 \;=\; \rho_H^2 - \rho_H\left(\frac{\theta_L}{\theta_H}+1\right)\frac{\theta_L}{\theta_H} \;=\; (1-\rho_H)\left(\frac{\theta_L}{\theta_H}-\rho_H\right) \;\leq\; 0$$

Therefore, the left hand side of (A.3.21) is non-positive for $\beta \in [\hat{\beta}, \rho_H]$.

$$\mu_0 \;\geq\; \frac{\hat{\beta}-\rho_L}{\rho_H-\rho_L} \quad\Leftrightarrow\quad \underbrace{\mu_0\rho_H + (1-\mu_0)\rho_L}_{=\beta_0} \;\geq\; \hat{\beta}$$

But the biggest possible value of β_0 is ρ_H, hence we infer that

$$\mu_0 \;\geq\; \frac{\hat{\beta}-\rho_L}{\rho_H-\rho_L} \quad\Leftrightarrow\quad Q(\rho_H,\rho_L) \;\leq\; 0\,, \qquad \text{where} \quad Q(\rho_H,\rho_L) \;=\; \omega_2\beta_0^2 - \omega_1\beta_0 + \omega_0$$

Next, we want to demonstrate that $Q(\rho_H, \rho_L)$ is convex in (ρ_H, ρ_L). Calculate

$$\frac{\partial Q}{\partial \rho_H} \;=\; \mu_0\cdot(2\omega_2\beta_0-\omega_1) - \left(\frac{\beta_0}{\rho_H}\right)^2\frac{k}{1-\mu_0} + \frac{k\,\mu_0}{1-\mu_0} \quad\Rightarrow$$

$$\frac{\partial^2 Q}{\partial \rho_H^2} \;=\; 2\omega_2\mu_0^2 - 2\frac{k\,\mu_0}{1-\mu_0}\frac{\beta_0}{\rho_H^2} + 2\frac{k\beta_0\rho_L}{\rho_H^3}$$

Substitute β_0 and ω_2 above to obtain

$$\frac{\partial^2 Q}{\partial \rho_H^2} \;=\; 2\omega_2\mu_0^2 - 2\frac{k\,\mu_0^2}{1-\mu_0}\frac{1}{\rho_H} + 2k(1-\mu_0)\frac{\rho_L^2}{\rho_H^3} \;=\; 2\mu_0^2 + 2k(1-\mu_0)\frac{\rho_L^2}{\rho_H^3}\,.$$

Similarly, calculate the partial derivative with respect to ρ_L and subsequently substitute ω_2 to obtain

$$\frac{\partial Q}{\partial \rho_L} \;=\; (1-\mu_0)(2\omega_2\beta_0-\omega_1) \;=\; -(1-\mu_0)\omega_1 + 2(1-\mu_0)\beta_0 + 2k\left(\mu_0 + (1-\mu_0)\frac{\rho_L}{\rho_H}\right)\,.$$

Differentiate the first expression above with respect to ρ_L, which appears only in β_0, to obtain the second order partial derivative below. Also, to obtain the cross-derivative below differentiate the second equivalent expression above with respect to ρ_H.

$$\frac{\partial^2 Q}{\partial \rho_L^2} \;=\; (1-\mu_0)^2 2\omega_2\,, \qquad \text{and} \qquad \frac{\partial^2 Q}{\partial \rho_L\partial\rho_H} \;=\; 2(1-\mu_0)\left(\mu_0 - \frac{k\rho_L}{\rho_H^2}\right)\,.$$

As a result, both of the second order partial derivatives are positive. Hence for $Q(\rho_H, \rho_L)$ to be convex it suffices that

$$\frac{\partial^2 Q}{\partial \rho_H^2}\frac{\partial^2 Q}{\partial \rho_L^2} \geq \left(\frac{\partial^2 Q}{\partial \rho_L \partial \rho_H}\right)^2 \Leftrightarrow$$

$$4(1-\mu_0)^2\omega_2\left(\mu_0^2 + k(1-\mu_0)\frac{\rho_L^2}{\rho_H^3}\right) \geq 4(1-\mu_0^2)^2\left(\mu_0^2 - 2\mu_0\frac{k\rho_L}{\rho_H^2} + \frac{k^2\rho_L^2}{\rho_H^4}\right)$$

Cancel out the $4(1-\mu_0)^2$ terms and substitute ω_2 to equivalently obtain

$$\mu_0^2 + k(1-\mu_0)\frac{\rho_L^2}{\rho_H^3} + \frac{k\mu_0^2}{1-\mu_0}\frac{1}{\rho_H} + \frac{k^2\rho_L^2}{\rho_H^4} \geq \mu_0^2 - 2\mu_0\frac{k\rho_L}{\rho_H^2} + k^2\frac{\rho_L^2}{\rho_H^4} \Leftrightarrow$$

$$(1-\mu_0)\frac{\rho_L^2}{\rho_H^2} + \frac{\mu_0^2}{1-\mu_0} + 2\mu_0\frac{\rho_L}{\rho_H} \geq 0\,,$$

which holds. Hence we have demonstrated that $Q(\rho_H, \rho_L)$ is convex, for all (ρ_H, ρ_L) such that $\rho_L < \theta_L/\theta_H < \rho_H$, which implies that the set of (ρ_H, ρ_L) for which $Q(\rho_H, \rho_L) < 0$ is convex. Therefore, the set of (ρ_H, ρ_L) for which $\mu_0 > \frac{\hat{\beta}-\rho_L}{\rho_H-\rho_L}$ is convex.

Next, consider the linear constrain $\rho_L < \theta_L/\theta_H$. In particular, note that

$$\rho_L = \frac{\theta_L}{\theta_H} \quad \Rightarrow \quad \beta_0 \geq \frac{\theta_L}{\theta_H}$$

But the tangency point $\hat{\mu}$ is always in $[0, \mu^*]$, which implies that $\hat{\beta} \leq \theta_L/\theta_H$. Then $\beta_0 \geq \hat{\beta}$, which in turn implies that

$$Q\left(\rho_H, \frac{\theta_L}{\theta_H}\right) \leq 0$$

As a result, $Q(\rho_H, \theta_L/\theta_H) \leq 0$ for all $\rho_H \leq \hat{\rho}_H = \min\left\{1\,,\ \frac{\theta_L/\theta_H}{\mu_0}\right\}$, which is our second linear constrain. This describes the boundary of the set on the vertical axis that keeps ρ_L constant.

Next, suppose that $\mu_0 \geq \theta_L/\theta_H$, then

$$\mu_0\hat{\rho}_H + (1-\mu_0)\rho_L \geq \mu_0\hat{\rho}_H\,\frac{\theta_L}{\theta_H} \geq \hat{\beta}_0\,,$$

as a result $Q(\hat{\rho}_H, \rho_L) \leq 0$ for all $\rho_L \leq \theta_L/\theta_H$. Suppose instead that $\mu_0 < \theta_L/\theta_H$, and let $\tilde{\rho}_L$ denote the unique solution of ρ_L for $Q(1, \rho_L) = 0$. Then again $Q(1, \tilde{\rho}_L) = 0$. To gather the above result let $\hat{\rho}_L = 0$ in the first case, and $\hat{\rho}_L = \tilde{\rho}_L$ in the second, so that

$$(\hat{\rho}_L, \hat{\rho}_H) = \begin{cases} (\tilde{\rho}_L, 1) & \text{, if } \mu_0 < \theta_L/\theta_H \\ \left(0, \frac{\theta_L/\theta_H}{\mu_0}\right) & \text{, if } \mu_0 > \theta_L/\theta_H \end{cases}$$

Then the horizontal line that connects $(\hat{\rho}_L, \hat{\rho}_H)$ with $(\theta_L/\theta_H, \hat{\rho}_H)$ is the northern boundary of the points for which information provision is optimal, and the vertical line that connects $(\theta_L/\theta_H, \theta_L/\theta_H)$ with $(\theta_L/\theta_H, 1)$ is the eastward. Finally, the set is convex as $Q(\rho_H, \rho_L)$ is convex, and its bounded on the left by $\hat{\rho}_L$ and below by θ_L/θ_H.

To result on the comparison between the first and second best follows from noting that Proposition 2.1.2, in Chapter 2, implies that $\hat{\beta}_i$ is increasing in ψ_i, the relevant value of which for the first best in zero and for the second best $\mu_0\rho_H$. □

As in the baseline model, a case of special interest is when the buyer's type is the same under both sellers.

Corollary A.2. *Suppose that the buyer's type is perfectly correlated across sellers, that is $\rho_L = 0$ and $\rho_H = 1$, then no information provision is optimal.*

Proof. First, suppose that $\rho_H\mu_0 \geq \theta_L/\theta_H$, which for $\rho_H = 1$ becomes $\mu_0 \geq \theta_L/\theta_H$. Then we have already argued that no information provision is optimal. Suppose instead that $\rho_H\mu_0 < \theta_L/\theta_H$, which for $\rho_H = 1$ becomes $\mu_0 < \theta_L/\theta_H$. Then $J(\mu_0) = 0$ has at most two solutions since it changes monotonicity at most once. One of the solutions is μ^*, and it is ease to see that the other is μ_0. But then it has to be that $\mu_0 < \mu^*$, since J always falls to the point $(\mu^*, 0)$ from above. Hence, μ_0 is below the maximum of J, which implies that $\mu_0 < \hat{\mu}$. Thus, $J(\mu_0) = \mathcal{J}(\mu_0)$, which implies that no information provision is optimal. □

Appendix B

Appendixes of Chapter 2

B.1 Proofs for multi-period contracts

Proof of Lemma 2.1.1. The dependence on t and s is dropped. The revelation principle applies. To make the notation more compact, write the reported type as a subscript. S_2's revenue maximisation problem is the following one

$$\begin{aligned}
\max_{p,q} \quad & \beta \cdot \Big(p_H - c(q_H)\Big) + (1-\beta) \cdot \Big(p_L - c(q_L)\Big) \\
\text{s.t. } (\mathrm{IR}_L) \quad & b\,\theta_L\,q_L - p_L \geq 0 \\
(\mathrm{IR}_H) \quad & b\,\theta_H q_H - p_H \geq 0 \\
(\mathrm{IC}_L) \quad & b\,\theta_L\,q_L - p_L \geq b\,\theta_L\,q_H - p_H \\
(\mathrm{IC}_H) \quad & b\,\theta_H q_H - p_H \geq b\,\theta_H q_L - p_L
\end{aligned}$$

where both the constrains and the objective function are written in terms of per period payoffs. Assuming that (IR_L) does not bind leads to a contradiction. Subsequently, this can be used to show that (IC_H) has to bind. Hence the above simplifies to the unconstrained maximisation problem

$$\max_{q} \; \beta \cdot \left(b\,\theta_H\,q_H - c(q_H)\right) + (1-\beta) \cdot \left(b\,\frac{\theta_L - \beta\theta_H}{1-\beta} q_L - c(q_L)\right)$$

For $\beta < \theta_L/\theta_H$ the objective function is concave, hence the unique solution is given by the first order conditions

$$c'(q_H) = b \cdot \theta_H \quad \text{and} \quad c'(q_L) = b \cdot \frac{\theta_L - \beta\theta_H}{1-\beta}$$

This is implementable, because substituting the above solutions in (IC$_L$) gives

$$b\,\theta_L q_L - p_L \,\geq\, b\,\theta_L q_H - p_H \;\Leftrightarrow\; 0 \,\geq\, b\,\theta_L q_H - p_H$$
$$\Leftrightarrow\; b\,\theta_H(q_H - q_L) + b\,\theta_L q_L \,\geq\, b\,\theta_L q_H \;\Leftrightarrow\; q_H \,\geq\, q_L,$$

which is satisfied. Because the (IR$_L$) binds the low type's period payoff is zero. The high type's period payoff can be obtained using the (IR$_L$) and (IC$_H$) constrains, which give that

$$b\,\theta_H q_H - p_H \;=\; b\,\theta_H q_L - p_L \;=\; b\,(\theta_H - \theta_L) q_L \;=\; b\,(\theta_H - \theta_L)\left(b\,\frac{\theta_L - \beta\theta_H}{1-\beta}\right)^{\epsilon}$$

Hence a constant stream of the above payoff up to infinity gives $B(\beta)$. To obtain the results on its derivatives note that $\frac{\theta_L - \beta\theta_H}{1-\beta} = \theta_H - \frac{\theta_H - \theta_L}{1-\beta}$. Hence, on its non-flat part

$$B'(\beta) \;=\; -b^{1+\epsilon}\,(\theta_H - \theta_L)\left(\frac{\theta_L - \beta\theta_H}{1-\beta}\right)^{\epsilon-1}\,\epsilon\,\frac{\theta_H - \theta_L}{(1-\beta)^2}$$

Then the first expression below is obtained by gathering terms, while the second from differentiating again.

$$B'(\beta) \;=\; -\frac{B(\beta)}{1-\beta}\,\epsilon\,\frac{\theta_H - \theta_L}{\theta_L - \beta\theta_H} \quad \text{and} \quad B''(\beta) \;=\; \frac{B'(\beta)}{1-\beta}\left(2 + (1-\epsilon)\frac{\theta_H - \theta_L}{\theta_L - \beta\theta_H}\right), \qquad \text{(B.1.1)}$$

Thus, the statements on the monotonicity and concavity of $B(\beta)$ on its non-flat part follow immediately from the above. □

Lemma B.1.1. *It is without loss of generality to only consider one-shot deviations. In those a θ^t buyer type reports truthfully θ^{t-1}, potentially misreports θ_t as $\hat{\theta}_t$, and subsequently switches back to truthful reporting.*

Proof of Lemma B.1.1. Necessity is trivial. For sufficiency suppose that type θ^t has a profitable deviation to report $\{\hat{\theta}_t, .., \hat{\theta}_{t'}\}$ up to t' and then switch to truthfulness. But on t' a realised type $\{.., \hat{\theta}_t, .., \hat{\theta}_{t'-1}, \theta_{t'}\}$ that was truthful faces the same payoff on t' as a misreported one that ends with type $\theta_{t'}$. Hence, the one-shot deviation constrains implies that $\{\hat{\theta}_t, .., \hat{\theta}_{t'-1}, \theta_{t'}\}$, that is a deviation that misreports only up to $t'-1$ is better. Applying the same argument shows that any deviation with finite horizon t' is no better than truth-telling. Infinitely long deviations can be arbitrarily well approximated by finite ones, hence as long as the one-shot constrains bind up to a constant difference $\varepsilon > 0$ the same contradiction is obtained. To maintain notation light this ε is ignored on the main text. □

Proof of Proposition 2.1.1. Lemma B.1.1 shows that it is without loss to only consider one shot deviations. Hence hereafter IC(θ^t) will refer exclusivity to the incentive compatibility constrains obtained under one-shot deviations. To maintain a compact notation, let $\hat{\theta}^t = \{\theta^{t-1}, \hat{\theta}_t\}$ denote a history of truthful reports up to $t-1$ followed by a potential misreport $\hat{\theta}_t$. In addition, denote a generic history θ^{t-1} followed by $\theta_t = \theta_H$ as θ^t_H, and similarly define θ^t_L. Then the payoff of a θ^t buyer type under a one shot deviation is

$$\begin{aligned}\widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1}) \;&=\; \theta_t q_t(\hat{\theta}^t) - p_t(\hat{\theta}^t) + \gamma\delta\, \mathbb{E}_\theta\big[\widehat{U}_{t+1}(\theta_{t+1}, \theta_{t+1}, \hat{\theta}^t)\,|\,\theta_t\big] \\ &\quad + (1-\gamma)\,\delta\, \Pr(\theta_{t+1} = \theta_H\,|\,\tau = t, \theta_t)\,\mathbb{E}_g\big[B(\beta^s_t)\,|\,\hat{\theta}^t\big]\end{aligned}$$

In period t the buyer obtains the quantity and price corresponding to type $\hat{\theta}_t$, however his actual valuation corresponds to the realised type θ_t. The distribution of the signal s is conditioned on the reported type $\hat{\theta}^t$, but the probability of the buyer to be a high type in S_2's contract is only a function of the actual type θ_t. Let the on path payoff of a θ^t buyer type be given by $U_t(\theta^t) = \widehat{U}_t(\theta_t, \theta_t, \theta^{t-1})$, then the corresponding individual rationality and incentive compatibility constrains become

$$\begin{aligned}&\text{IR}(\theta^t) \quad U_t(\theta^t) \;\geq\; 0 \\ &\text{IC}(\theta^t) \quad U_t(\theta^t) \;\geq\; \widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1})\end{aligned} \tag{B.1.2}$$

where the buyer's outside option is set to zero, since we have assumed that if the buyer rejects S_1's offer, then S_2 does not trade with him. Next, consider the following problem

$$\begin{aligned}(\mathcal{P}^H) \quad \max_{q,g}\; \mathbb{E}_\theta\left[\sum_{t=0}^{\infty} \delta^t \Big(p_t(\theta^t) - c\big[q_t(\theta^t)\big]\Big)\right] \\ \text{subject to IR}(\theta^t) \text{ and IC}(\theta^t),\end{aligned} \tag{B.1.3}$$

which is similar to ($\mathcal{P}$), but it ignores the individual rationality constrains of all high types IR(θ^t_H) and the incentive compatibility constrains of all low types IC(θ^t_L). Ignoring the former set of constrains is without loss of generality as $U_t(\theta^t_H) \geq U_t(\theta^t_L)$, however this is not true for the IC(θ^t_L) constrains. Despite that, if the solution of ($\mathcal{P}^H$) happens to also satisfy this set of constrains, then it is a solution of ($\mathcal{P}$).

The rest of the proof shows that in ($\mathcal{P}^H$), for every (p, q, g) there exists a p' such that the objective function under (p', q, g) is no less than under (p, q, g), and that both the IR(θ^t_L) and IC(θ^t_H) constrains bind. Solving for p' from the constrains and substituting in the objective function will give ($\mathcal{P}'$). Hence if (q, g) is a solution of ($\mathcal{P}'$), then there exists p' such that (p', q, g) is also a solution of ($\mathcal{P}^H$). Finally, p' will be substituted in the previously ignored

IC(θ_L^t) so that a sufficient condition is obtained for (p', q, g) to be a solution of ($\mathcal{P}$), which only depends on policies (q, g).

The argument is recursive. Suppose that IR(θ_L^t) and IC(θ_H^t) bind for all periods up to and including t', but not for $t'+1$. For simplicity, denote $p_t(\theta_H^{t'})$ by p_H, $p_t(\theta_L^{t'})$ by p_L, and similarly use $\{p_{LL}, p_{HL}, p_{LH}, p_{HH}\}$ for the possible combinations up to $t'+1$. Moreover, adopt the same notational change for the IR and IC constrains. Suppose that IR$_{HL}$ does not bind, then let

$$\big(\tilde{p}_H, \tilde{p}_{HH}, \tilde{p}_{HL}\big) \;=\; \big(p_H - \delta\varepsilon,\, p_{HH} + \varepsilon,\, p_{HL} + \varepsilon\big)$$

and increase ε until it does. Under this transformation IR$_L$ and IC$_H$ continue to bind and S$_1$ is indifferent between the two contracts. The same argument works if IR$_{LL}$ does not bind. Suppose instead that IC$_{LH}$ does not bind, then let

$$\big(\tilde{p}_L,\, \tilde{p}_H,\, \tilde{p}_{LH}\big) \;=\; \big(p_L - \delta\varphi_L\varepsilon,\, p_H + \delta(\varphi_H - \varphi_L)\varepsilon,\, p_{LH} + \varepsilon\big),$$

and increase ε until it does. Under this transformation IR$_L$ and IC$_H$ continue to bind and S$_1$ is actually better off. Finally, suppose that IC$_{HH}$ does not bind, then let

$$\big(\tilde{p}_H,\, \tilde{p}_{HH}\big) \;=\; \big(p_H - \delta\varphi_H\varepsilon,\, p_{HH} + \varepsilon\big),$$

and increase ε until it does. Under this transformation IR$_L$ and IC$_H$ continue to bind and S$_1$ is indifferent between the two contracts. Hence, if both IR(θ_L^t) and IC(θ_H^t) bind for all periods up to and including t', and IC(θ_L^t) is ignored, then there exists an alternative contract that implements the same policies, is not worse for S$_1$, and has all constrains binding up to $t'+1$. In addition, the regular one period argumentation shows that IR(θ_L^0) and IC(θ_H^0) have to bind, from which the recursive argument follows.

Hence, it is without loss to assume that IR(θ_L^t) and IC(θ_H^t) bind. The former gives

$$p_t(\theta_L^t) \;=\; \theta_L q_t(\theta_L^t) + \gamma\delta\varphi_L \widehat{U}_{t+1}(\theta_H, \theta_H, \theta_L^t) + (1-\gamma)\delta\rho_L \mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]\,,$$

and the latter

$$U_t(\theta_H^t) \;=\; \theta_H q_t(\theta_L^t) - p_t(\theta_L^t) + \gamma\delta\varphi_H \widehat{U}_{t+1}(\theta_H, \theta_H, \theta_L^t) \\ + [1-\gamma]\delta\rho_H \mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]\,.$$

In both equations above it has been used that $\widehat{U}_{t+1}(\theta_L, \theta_L, \theta_L^t) = U_{t+1}(\{\theta_L, \theta_L^t\})$, which in turn is equal to zero because the IR($\{\theta_L, \theta_L^t\}$) constrain binds. Substitute the derived expression for $p_t(\theta_L^t)$ in that for $U_t(\theta_H^t)$ to obtain

$$\begin{aligned} U_t(\theta_H^t) \;=\; & (\theta_H - \theta_L) q_t(\theta_L^t) + \gamma\delta(\varphi_H - \varphi_L)\widehat{U}_{t+1}(\theta_H, \theta_H, \theta_L^t) \\ & + (1-\gamma)\delta(\rho_H - \rho_L)\mathbb{E}_g[B(\beta_t^s)|\theta_L^t]\,. \end{aligned} \tag{B.1.4}$$

In addition, because IC($\{\theta_L^t, \theta_H\}$) binds, I get that $\widehat{U}_{t+1}(\theta_H, \theta_H, \theta_L^t) = U_{t+1}(\{\theta_L^t, \theta_H\})$, the expression of which follows the same pattern with the above equation. Using the same argument repeatedly and substituting forward gives the functional form provided for $U_t^H(\theta^{t-1})$ in the main text. In particular, for a period 1 high types that becomes

$$\begin{aligned} U_0(\theta_H) \;=\; \sum_{t=0}^{\infty} \gamma^t \delta^t (\varphi_H - \varphi_L)^t \Big[& (\theta_H - \theta_L) q_t(L^t) \\ & + \delta(1-\gamma)(\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_t^s) \,|\, L^t\big]\Big)\Big]\,. \end{aligned} \tag{B.1.5}$$

It follows by the definition of $U_0(\theta_0)$ that the on path expected discounted payments on period 1 satisfies

$$\mathbb{E}_\theta\left[\sum_{t=0}^{\infty} \gamma^t \delta^t \theta_t q_t(\theta^t) \,\middle|\, \theta_0\right] - U_0(\theta_0) \;=\; \mathbb{E}_\theta\left[\sum_{t=0}^{\infty} \gamma^t \delta^t p_t(\theta^t) \,\middle|\, \theta_0\right]$$

It has been shown that IR(θ_0) binds, hence for a low type substitute $U_0(\theta_L) = 0$, whereas for a high type the expression derived in (B.1.5). Finally, substitute the expected discounted transfers on the objective function of ($\mathcal{P}$) to obtain ($\mathcal{P}'$).

To complete the proof note that by definition the transfers offered to a high type make him indifferent between deviating and not, after every history θ^{t-1}. Hence for the derived solution to be implementable it suffices that IC(θ_L^t) is satisfied. This is

$$\begin{aligned} U_t(\theta_L^t) \;\geq\; & \theta_L q_t(\theta_H^t) - p_t(\theta_H^t) + \gamma\,\delta\,\varphi_L \widehat{U}_{t+1}(\theta_H, \theta_H, \theta_H^t) \\ & + \delta\,(1-\gamma)\,\rho_L \mathbb{E}_g\big[B(\beta_t^s) \,|\, \theta_H^t\big]\,, \end{aligned}$$

where $\widehat{U}_{t+1}(\theta_L, \theta_L, \theta_H^t) = U_{t+1}(\{\theta_H^t, \theta_L\}) = 0$ has already been used on the continuation value on the right hand side. Substitute $U_t(\theta_L^t) = 0$ on the left hand side, $\widehat{U}_{t+1}(\theta_H, \theta_H, \theta_H^t) =$

$U_{t+1}(\{\theta_H^t, \theta_H\})$ on the right one, rearrange and add the rest of the parts of $U_t(\theta_H^t)$ to obtain

$$(\theta_H - \theta_L)q_t(\theta_H^t) + \gamma\,\delta\,(\varphi_H - \varphi_L)U_{t+1}(\{\theta_H^t, \theta_H\}) \\ + \;\delta\,(1-\gamma)\,(\rho_H - \rho_L)\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_H^t\big] \;\geq\; U_t(\theta_H^t)$$

Finally, substitute the recursive expression of $U_t(\theta_H^t)$ as it appears on (B.1.4), and note that by definition $U_{t+1}(\{\theta_H^t, \theta_H\}) = U_{t+1}^H(\theta_H^t)$ and $U_{t+1}(\{\theta_L^t, \theta_H\}) = U_{t+1}^H(\theta_L^t)$ to obtain $(\mathcal{P}_c)$. □

Proof of Corollary 2.1.1. The third line of $(\mathcal{P}')$ represents the information rents, which only affect the production of the L^t histories. Hence, point-wise maximisation on any other history simple optimises its first line, which represents the surplus from production, and gives the first-best level of effort. In contrast, for every t the production relevant payoff that corresponds to the L^t history is

$$\Pr(L^t)\gamma^t\delta^t\Bigg\{\theta_L q_t(L^t) - \frac{q_t(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} - \frac{\mu_0(\varphi_H-\varphi_L)^t}{\Pr(L^t)}(\theta_H-\theta_L)q_t(L^t)\Bigg\} \\ = \;\Pr(L^t)\gamma^t\delta^t\Bigg\{\xi_t\, q_t(L^t) - \frac{q_t(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\Bigg\},$$

the point-wise maximisation of which gives $(\xi_t)^\epsilon$. Substitute the derived point-wise optimal quantities in $(\mathcal{P}_c)$ to obtain

$$(\theta_H-\theta_L)\Bigg[(\theta_H)^\epsilon - (\xi_t)^\epsilon \;+\; \sum_{t'=t+1}^{\infty}\big[(\varphi_H-\varphi_L)\gamma\delta\big]^{t'-t}\Big((\theta_L)^\epsilon - (\xi_t)^\epsilon\Big)\Bigg] \;\geq\; \delta(\rho_H-\rho_L) \\ \times\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Pr(\tau=t'\,|\,\tau>t-1)\Bigg\{\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_L^t L_{t+1}^{t'}\big] - \mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_H^t L_{t+1}^{t'}\big]\Bigg\}, \tag{B.1.6}$$

for all L^t histories, and for the remaining ones substitute ξ_t with θ_L. Note that $(\theta_L)^\epsilon \geq (\xi_t)^\epsilon$. Hence the left hand side above is bigger than $(\theta_H-\theta_L)[(\theta_H)^\epsilon - (\theta_L)^\epsilon]$. In addition, the right hand side of (B.1.6) is smaller than

$$\delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Pr(\tau=t'\,|\,\tau>t-1)\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_L^t L_{t+1}^{t'}\big] \\ \leq\; \delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\big[(\varphi_H-\varphi_L)\delta\big]^{t'-t}\Pr(\tau=t'\,|\,\tau>t-1)B(0) \\ \leq\; \delta(\rho_H-\rho_L)B(0) \;=\; \delta(\rho_H-\rho_L)b^{1+\epsilon}(\theta_H-\theta_L)\,(\theta_L)^\epsilon,$$

where the first inequality follows from noting that $B(\cdot)$ is decreasing, and the second because getting $B(0)$ in period $t+1$ for sure is better than any other realisation of τ. Hence combining the two equations together gives (2.1.4). □

Proof of Lemma 2.1.2. For brevity denote the event $(\tau = t)$ as (t) in the probabilities below. In ($\mathcal{G}_t$) change the order of the summations and multiply the probabilities to obtain

$$\sum_s B(\beta_t^s)\Big\{\sum_{\theta^t}\big[\Pr(s,t\,|\,\theta^t)\Pr(\theta^t)\Pr(\theta_t=\theta_H|\,\theta_t)\big] \\ -\mu_0(\varphi_H-\varphi_L)^t(\rho_H-\rho_L)\Pr(s,t\,|\,L^t)\Big\}. \tag{B.1.7}$$

Moreover, note that

$$\sum_{\theta^t}\Pr(s,t\,|\,\theta^t)\Pr(\theta^t)\Pr(\theta_t=\theta_H|\theta_t) \\ = \varphi_H\sum_{\theta_H^t}\Pr(s,t\,|\,\theta_H^t)\Pr(\theta_H^t)+\varphi_L\sum_{\theta_L^t}\Pr(s,t\,|\,\theta_L^t)\Pr(\theta_L^t)$$

To transform the above equation note that for the period t high types

$$\Pr(\theta_t=\theta_H\,|\,s,t) = \frac{\sum_{\theta_H^t}\Pr(s,t\,|\,\theta_H^t)\Pr(\theta_H^t)}{\Pr(s,t)} \Leftrightarrow \mu_t^s\Pr(s,t) = \sum_{\theta_H^t}\Pr(s,t\,|\,\theta_H^t)\Pr(\theta_H^t)$$

Similarly, for the period t low types

$$(1-\mu_t^s)\Pr(s,t) = \sum_{\theta_L^t}\Pr(s,t\,|\,\theta_L^t)\Pr(\theta_L^t)$$

Finally, for the history L^t

$$\Pr(s,t\,|\,L^t) = \frac{\Pr(s,t,L^t)}{\Pr(L^t)} = \lambda_t^s\frac{\Pr(s,t)}{\Pr(L^t)} = \frac{\lambda_t^s}{(1-\varphi_L)^t}\frac{\Pr(s,t)}{1-\mu_0}$$

Hence, substitute the above in (B.1.7) to obtain

$$\sum_s B(\beta_t^s)\Big\{\rho_H\mu_t^s \Pr(s,t) + \rho_L(1-\mu_t^s)\Pr(s,t) - \psi_t\lambda_t^s \Pr(s,t)\Big\}$$
$$= \sum_s B(\beta_t^s)\Big\{\rho_H\mu_t^s + \rho_L(1-\mu_t^s) - \psi_t\lambda_t^s\Big\}\Pr(s,t) \;=\; \sum_s B(\beta_t^s)\big(\beta_t^s - \psi_t\lambda_t^s\big)g_t(s)\Pr(t)\,. \tag{B.1.8}$$

□

To shorten the statement of the subsequent lemma, let D denote the domain of $J_t(\mu,\lambda)$, and define its following subsets

$$\overline{D} \equiv \big\{(\mu,\lambda)\in D : \mu = 0, \text{ or } \lambda = 0, \text{ or } \mu+\lambda = 1\big\}\,,$$
$$D_0 \equiv \big\{(\mu,\lambda)\in D : \mu \geq \mu^*\big\} \quad \text{and} \quad D_I \equiv \overline{D}^c \cap D_o\,.$$

Thus $\overline{D}$ denotes the boundary of the domain, D_0 the subset of posteriors for which $J_t(\mu,\lambda)$ is flat, i.e. equal to zero, and D_I that of interior points on which $J_t(\mu,\lambda)$ is not flat.

Lemma B.1.2. *$J_t(\mu,\lambda)$ is flat for all $\mu \in D_0$. Also, it is neither concave, nor convex for all $(\mu,\lambda)\in D_I$, and for an immediate termination $D_I = \{\emptyset\}$.*

- *On the boundary ($\mu = 0$): it is linear and decreasing on λ.*
- *On the boundary ($\lambda = 0$), and when $\Psi_t < \frac{\theta_L}{\theta_H}$ also on ($\mu+\lambda = 1$):*
  - *It changes monotonicity at most once, and if $\mu^* > 0$ it falls to μ^* from above.*
  - *If $\epsilon \leq 1$, then it is strictly concave on $[0,\mu^*]$.*
  - *If $\epsilon > 1$, then it is strictly concave on $[0,\mu_i^{**}]$, and strictly convex on $[\mu_i^{**},\mu^*]$, where*

$$\mu_i^{**} = \max\left\{0\,,\, \frac{\beta_i^{**}-\rho_L}{\rho_H-\rho_L}\right\} \quad \textit{and} \quad \beta_i^{**} \equiv \frac{\theta_L}{\theta_H}\,\frac{2\big(1-\Psi_i\big)+(\epsilon-1)\big(1-\frac{\theta_L}{\theta_H}\big)\frac{\Psi_i}{\theta_L/\theta_H}}{2(1-\Psi_i)+(\epsilon-1)\big(1-\frac{\theta_L}{\theta_H}\big)}\,.$$

- *On the boundary ($\mu+\lambda = 1$), when $\Psi_t \geq \frac{\theta_L}{\theta_H}$: It is negative and strictly increasing for all $\mu < \mu^*$ Otherwise, it is equal to zero. In addition,*
  - *If $\epsilon \geq 1$, then it is strictly concave on $[0,\mu^*]$.*
  - *If $\epsilon < 1$ and $(1-\epsilon)(1-\frac{\theta_L}{\theta_H}) \geq 2(1-\Psi_t)$, then its strictly convex on $[0,\mu^*]$.*

– *If* $\epsilon < 1$ *and* $(1-\epsilon)(1-\frac{\theta_L}{\theta_H}) < 2(1-\Psi_t)$ *it is strictly concave on* $[0, \mu_i^{**}]$*, and strictly convex on* $[\mu_i^{**}, \mu^*]$.

Proof of Lemma B.1.2. The time subscripts are suppressed. It is copied here from (B.1.1) that for every $\beta \leq \theta_L/\theta_H$:

$$B'(\beta) \;=\; -\frac{B(\beta)}{1-\beta}\,\epsilon\,\frac{\theta_H-\theta_L}{\theta_L-\beta\theta_H} \quad \text{and} \quad B''(\beta) \;=\; \frac{B'(\beta)}{1-\beta}\left(2+(1-\epsilon)\frac{\theta_H-\theta_L}{\theta_L-\beta\theta_H}\right),$$

Hence, for $\mu \leq \mu^*$ differentiating and re-arranging gives that

$$\frac{\partial J(\mu,\lambda)/\partial\,\mu}{\rho_H-\rho_L} \;=\; B'(\beta)(\beta-\psi\,\lambda)+B(\beta) \;\Rightarrow\; \frac{\partial^2 J(\mu,\lambda)/\partial\,\mu^2}{(\rho_H-\rho_L)^2} \;=\; B''(\beta)(\beta-\psi\,\lambda)+2B'(\beta)$$
$$=\; \frac{B'(\beta)}{1-\beta}\left[2(1-\psi\lambda)+(1-\epsilon)(b-\psi\lambda)\frac{\theta_H-\theta_L}{\theta_L-\beta\,\theta_H}\right].$$

Likewise,

$$\frac{\partial J(\mu,\lambda)}{\partial\,\lambda} \;=\; -\psi B(\beta), \qquad \frac{\partial^2 J(\mu,\lambda)}{\partial\lambda\,\partial\mu} \;=\; -\psi(\rho_H-\rho_L)B'(\beta), \quad \text{and} \quad \frac{\partial^2 J(\mu,\lambda)}{\partial\lambda^2} \;=\; 0\,.$$

To prove that $J(\mu,\lambda)$ is neither concave nor convex on any of its interior points, that is not on the flat side of its domain, it suffices to show that its Hessian matrix is indefinite. This is given by

$$D^2 J(\mu,\lambda) \;=\; \begin{pmatrix} \frac{\partial^2 J}{\partial\mu^2} & \frac{\partial^2 J}{\partial\mu\partial\lambda} \\ \cdot & \frac{\partial^2 J}{\partial\lambda^2} \end{pmatrix}$$

Hence, its determinant is $|D^2 J| = -\left(\frac{\partial^2 J}{\partial\mu\partial\lambda}\right)^2 < 0$, from which it follows that it is indefinite.

To prove the statements for the boundaries $(\mu+\lambda=1)$ and $(\lambda=0)$, define the following linear combination of (μ',λ') and (μ'',λ''), for $w \in [0,1]$ and $\mu'' \neq \mu'$,

$$\begin{pmatrix} \bar\mu \\ \bar\lambda \end{pmatrix} \;=\; (1-w)\begin{pmatrix} \mu' \\ \lambda' \end{pmatrix} + w\begin{pmatrix} \mu'' \\ \lambda'' \end{pmatrix}.$$

This implies that

$$w = \frac{\bar\mu-\mu'}{\mu''-\mu'} \;\Rightarrow \bar\lambda \;=\; \frac{\lambda''-\lambda'}{\mu''-\mu'}(\bar\mu-\mu')+\lambda' \;=\; \frac{\lambda''-\lambda'}{\mu''-\mu'}\left(\frac{\bar b-\rho_L}{\rho_H-\rho_L}-\mu'\right)+\lambda'$$
$$\Rightarrow\; (\bar b-\psi\,\bar\lambda) \;=\; \bar b\underbrace{\left(1-\frac{\psi}{\rho_H-\rho_L}\frac{\lambda''-\lambda'}{\mu''-\mu'}\right)}_{\equiv\zeta}-\psi\underbrace{\left[\lambda'-\frac{\lambda''-\lambda'}{\mu''-\mu'}\left(\frac{\rho_L}{\rho_H-\rho_L}+\mu'\right)\right]}_{\equiv\zeta'}$$

Let $\Psi \equiv \zeta'/\zeta$ and note that for the subsets $(\lambda = 0)$ and $(\mu + \lambda = 1)$

$$\begin{pmatrix} \mu' & \mu'' \\ \lambda' & \lambda'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} \zeta \\ \zeta' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 + \frac{\psi}{\rho_H - \rho_L} \\ 1 + \frac{\rho_L}{\rho_H - \rho_L} \end{pmatrix}$$
$$\Psi = 0, \quad \frac{\psi \rho_H}{\rho_H - \rho_L + \psi},$$

respectively. In addition, substituting $\psi = \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^t (\rho_H - \rho_L)$ I get that

$$\frac{\psi \rho_H}{\rho_H - \rho_L + \psi} = \frac{\mu_0 \rho_H}{(1-\mu_0)\left(\frac{\varphi_H - \varphi_L}{1-\varphi_L}\right)^{-t} + \mu_0} \leq \mu_0 \rho_H < 1.$$

Now the characterisation of $J(\mu, \lambda)$ on its boundaries can be obtained. Note that by moving $w \in [0, 1]$ I essentially move $J(\mu, \lambda)$ on the two specified sides. Moreover, because μ is a linear transformation of w its monotonicity and concavity changes at the same values of μ, and as a result of β. Hence, define

$$\bar{J}(w) \equiv J\big[\bar{\mu}(w), \bar{\lambda}(w)\big] = \zeta\, B(\bar{\beta})(\bar{\beta} - \Psi), \quad \textit{where} \quad \bar{\beta} = \bar{\mu}(\rho_H - \rho_L) + \rho_L$$
$$\text{and} \quad \bar{\mu} = w(\mu'' - \mu') + \mu'.$$

Then algebra similar to that used to simplify the partial derivatives of $J(\mu, \lambda)$ implies that

$$\bar{J}''(w) = (\rho_H - \rho_L)^2(\mu'' - \mu')^2 \zeta \frac{B'(\bar{\beta})}{1 - \bar{\beta}} \left[2(1-\Psi) + (1-\epsilon)(\bar{\beta} - \Psi)\frac{\theta_H - \theta_L}{\theta_L - \bar{\beta}\,\theta_H}\right].$$

Hence, to find the set of $\bar{\beta}$'s for which $\bar{J}(w)$ is convex solve

$$\begin{aligned} \bar{J}''(w) \geq 0 \;&\Leftrightarrow\; 2(1-\Psi) + (1-\epsilon)(\bar{\beta} - \Psi)\frac{\theta_H - \theta_L}{\theta_L - \bar{\beta}\,\theta_H} \leq 0 \\ \Leftrightarrow\; \beta\left[2(1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\right] &\geq 2(1-\Psi)\frac{\theta_L}{\theta_H} - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\Psi. \end{aligned} \tag{B.1.9}$$

Next, four cases are considered. First suppose that $\epsilon \leq 1$ and $\theta_L/\theta_H \geq \Psi$, which implies

$$\frac{\theta_L}{\theta_H} \geq \Psi \;\Rightarrow\; \begin{cases} (1-\Psi) \geq \left(1 - \frac{\theta_L}{\theta_H}\right) & \Rightarrow \quad (1-\Psi) - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right) \geq 0 \\ (1-\Psi)\frac{\theta_L}{\theta_H} \geq \left(1 - \frac{\theta_L}{\theta_H}\right)\Psi & \Rightarrow \quad (1-\Psi)\frac{\theta_L}{\theta_H} - (1-\epsilon)\left(1 - \frac{\theta_L}{\theta_H}\right)\Psi \geq 0 \end{cases}$$

Hence, (B.1.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H}\frac{2(1-\Psi)-(1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi)-(1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)} \;\geq\; \frac{\theta_L}{\theta_H},$$

which is never satisfied on $[0,\theta_L/\theta_H)$. Second, suppose that $\epsilon > 1$ and $\theta_L/\theta_H \geq \Psi$, then (B.1.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H}\frac{2(1-\Psi)+(\epsilon-1)\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi)+(\epsilon-1)\left(1-\frac{\theta_L}{\theta_H}\right)}$$

where the right hand side is equal to θ_L/θ_H for $\theta_L/\theta_H = \Psi$ and strictly less than it and positive for $\theta_L/\theta_H > \Psi$. Hence, in the former subcase it is always concave in $[0,\theta_L/\theta_H]$, while in the latter there is a point in this interval above which it becomes strictly convex. Third, suppose that $\epsilon \geq 1$ and $\theta_L/\theta_H < \Psi$. Then the second line of (B.1.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H}\frac{2(1-\Psi)+(\epsilon-1)\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi)+(\epsilon-1)\left(1-\frac{\theta_L}{\theta_H}\right)} \;\geq\; \frac{\theta_L}{\theta_H},$$

Hence, similar to the first case $\bar{J}(w)$ is always concave on its non-flat part. Forth, suppose that $\epsilon < 1$ and $\theta_L/\theta_H < \Psi$. Consider the subcase where $2(1-\Psi) \leq (1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)$, which implies that $2(1-\Psi)\frac{\theta_L}{\theta_H} \leq (1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)\Psi$. If the first inequality holds with equality, then (B.1.9) is trivially satisfied. Otherwise, it becomes

$$\beta \;\leq\; \frac{\theta_L}{\theta_H}\frac{(1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}-2(1-\Psi)}{(1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)-2(1-\Psi)},$$

the right hand side of which is strictly bigger than θ_L/θ_H. Hence, in this subcase the function is always convex in $[0,\theta_L/\theta_H]$. Finally, consider the subcase where $2(1-\Psi) > (1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)$, for which (B.1.9) becomes

$$\beta \;\geq\; \frac{\theta_L}{\theta_H}\frac{2(1-\Psi)-(1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\Psi}{\theta_L/\theta_H}}{2(1-\Psi)-(1-\epsilon)\left(1-\frac{\theta_L}{\theta_H}\right)},$$

the right hand side of which is strictly smaller than θ_L/θ_H. Hence, in this subcase there exists a point in $[0,\theta_L/\theta_H)$ above which $\bar{J}(w)$ is strictly convex and below strictly concave. □

Proof of Proposition 2.1.2. The statement for the interior of the domain of $J_t(\mu,\lambda)$, for the first, and for third bullet point follow immediately from Lemma B.1.2. To obtain the second let

$$\bar{J}_i(\mu) \;\equiv\; \widetilde{J}_i\Big(\mu(\rho_H-\rho_L)+\rho_L\Big)\,, \quad \text{where} \quad \widetilde{J}_i(\beta) \;=\; \zeta_i B(\beta)(\beta-\Psi_i)$$

and instead of (2.1.6) solve

$$\widetilde{J}_i(\hat{\mu}_i)+\widetilde{J}_i'(\hat{\beta}_i)(\rho_H-\hat{\beta}_i) \;=\; 0\,, \tag{B.1.10}$$

in $[\Psi_i, \theta_L/\theta_H]$. It is ease to so that

$$\hat{\mu}_i \;\equiv\; \max\left\{\frac{\hat{\beta}_i-\rho_L}{\rho_H-\rho_L}\,,\,0\right\}.$$

To solve (B.1.10) note that

$$\begin{aligned}\widetilde{J}_i(\beta) \;=\; \zeta_i B(\beta)(\beta-\Psi_i) \;\Rightarrow\; \widetilde{J}_i(\beta) \;&=\; \zeta_i B(\beta)+\zeta_i B'(\beta)(\beta-\Psi_i)\\ &=\; \zeta_i\,\frac{B(\beta)}{1-\beta}\left[1-\beta-\epsilon(\theta_H-\theta_L)\frac{\beta-\Psi_i}{\theta_L-\beta\theta_H}\right]\end{aligned}$$

Hence, (B.1.10) equivalently becomes

$$\begin{aligned}&(\hat{\beta}_i-\Psi_i)(1-\hat{\beta}_i)+\left[1-\hat{\beta}_i-\epsilon(\theta_H-\theta_L)\frac{\hat{\beta}_i-\Psi_i}{\theta_L-\hat{\beta}_i\theta_H}\right](\rho_H-\hat{\beta}_i) \;=\; 0\\ &\Leftrightarrow\quad (1-\hat{\beta}_i)(\rho_H-\Psi_i) \;=\; \epsilon\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\hat{\beta}_i-\Psi_i}{\frac{\theta_L}{\theta_H}-\hat{\beta}_i}(\rho_H-\hat{\beta}_i) \\ &\Leftrightarrow\quad \frac{\theta_L}{\theta_H}-\hat{\beta}_i\left(1+\frac{\theta_L}{\theta_H}\right)+\hat{\beta}_i^2 \;=\; \frac{\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)}{\rho_H-\Psi_i}\left[\hat{\beta}_i(\rho_H+\Psi_i)-\hat{\beta}_i^2-\Psi_i\rho_H\right]\end{aligned} \tag{B.1.11}$$

which after cancelling out terms and rearranging becomes

$$\begin{aligned}&\omega_2\hat{\beta}_i^2-\omega_1\hat{\beta}_i+\omega_0 \;=\; 0\,, \quad \text{where} \quad \omega_0 \;\equiv\; \frac{\theta_L}{\theta_H}+\Psi_i\rho_H\frac{\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)}{\rho_H-\Psi_i}\,,\\ &\omega_1 \;\equiv\; 1+\frac{\theta_L}{\theta_H}+\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)\frac{\rho_H+\Psi_i}{\rho_H-\Psi_i}\,, \quad \text{and} \quad \omega_2 \;\equiv\; 1+\frac{\epsilon\left(1-\frac{\theta_L}{\theta_H}\right)}{\rho_H-\Psi_i}\,.\end{aligned} \tag{B.1.12}$$

In addition, it has already been shown that in the relevant parametric case $\bar{J}(\beta)$ changes its monotonicity and concavity at most once for $\beta < \theta_L/\theta_H$. Hence, because the function is initially concave, it is the smaller root that is the relevant one, while the bigger exists because of the shape that $B(\beta)$ would have for $\beta > \theta_L/\theta_H$ if it wasn't becoming flat there. Hence, the solution is given by

$$\hat{\beta}_i \equiv \frac{\omega_1 - \sqrt{(\omega_1)^2 - 4\omega_0\omega_2}}{2\omega_2} \tag{B.1.13}$$

To prove the statement on its monotonicity with respect to Ψ_i equivalently rewrite the second line of (B.1.11) as

$$\hat{\beta}_i - 1 + \epsilon\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\hat{\beta}_i - \Psi_i}{\rho_H - \Psi_i}\,\frac{\rho_H - \hat{\beta}_i}{\frac{\theta_L}{\theta_H} - \hat{\beta}_i} \;=\; 0\,, \tag{B.1.14}$$

where it follows from the above discussion that for the relevant solution given in (B.1.13) it has to be that $\Psi_i < \hat{\beta}_i < \frac{\theta_L}{\theta_H} \le \rho_H$. Hence the partial derivative of the left hand side of (B.1.14) with respect to $\hat{\beta}_i$ is

$$\frac{\partial LHS(\text{B.1.14})}{\partial\hat{\beta}_i} \;=\; 1 + \epsilon\left(1 - \frac{\theta_L}{\theta_H}\right)\left[\frac{1}{\rho_H - \Psi_i}\,\frac{\rho_H - \hat{\beta}_i}{\frac{\theta_L}{\theta_H} - \hat{\beta}_i} + \frac{\hat{\beta}_i - \Psi}{\rho_H - \Psi_i}\,\frac{\rho_H - \frac{\theta_L}{\theta_H}}{(\frac{\theta_L}{\theta_H} - \hat{\beta}_i)^2}\right] \;>\; 0\,,$$

whereas that with respect to Ψ_i is

$$\frac{\partial LHS(\text{B.1.14})}{\partial\Psi_i} \;=\; \epsilon\left(1 - \frac{\theta_L}{\theta_H}\right)\frac{\rho_H - \hat{\beta}_i}{\frac{\theta_L}{\theta_H} - \hat{\beta}_i}\,\frac{\hat{\beta}_i - \rho_H}{(\rho_H - \Psi_i)^2} \;<\; 0\,.$$

Then the implicit function theorem gives that $\partial\hat{\beta}_i/\partial\Psi_i > 0$. □

Proof of Corollary 2.1.2 . The restriction $\varphi_H = 1 - \varphi_L = 1$ implies that $\mu_t^s + \lambda_t^s = 1$, as a history where a high type changes to a low one never occurs. Hence it follows from Lemma B.1.2 that

$$J_t(\mu,\lambda) \;=\; \bar{J}_t(\mu) \;=\; \left(1 + \frac{\psi_t}{\rho_H - \rho_L}\right)B(\beta)\,(\beta - \Psi_t)\;,$$

where to make notation more compact the shorthand $\beta = (\rho_H - \rho_L)\mu + \rho_L$ is used. Substitute $\varphi_H = 1 - \varphi_L = 1$ to get that $\Psi_t = \frac{\mu_0}{1-\mu_0}$ and $\psi_t = \mu_0$, and $\rho_H = 1 - \rho_L = 1$, which gives that $\beta = \mu$, to obtain

$$\bar{J}_t(\mu) \;=\; \bar{J}_0(\mu) \;=\; \frac{1}{1-\mu_0}B(\mu)(\mu - \mu_0)$$

Note that $\mu_t = \mu_0$. Then an argumentation identical to that used in the proof of Corollary A.2 shows that no information provision is optimal. □

Proof of Proposition 2.1.3. First, it is shown that for all $(\mu, \lambda) \in D$ there exist points $(\mu', 0) \in (\lambda = 0)$ and $(\mu'', 1-\mu'') \in (\mu + \lambda = 1)$ such that

$$\mathcal{J}_t(\mu, \lambda) \;=\; \frac{\mu - \mu''}{\mu' - \mu''}\,\bar{\mathcal{J}}_f(\mu') \;+\; \frac{\mu' - \mu}{\mu' - \mu''}\,\bar{\mathcal{J}}_t(\mu'')\,. \tag{B.1.15}$$

To prove this first take point $(\mu_i, \lambda_i) \in D$ and let weight $\bar{\omega}_i$ be the one that gives this as a linear combination of $(\mu_i, 0)$ and $(\mu_i, 1-\mu_i)$. Then this solves

$$\bar{\omega}_i(1-\mu_i) + (1-\bar{\omega}_i)0 \;=\; \lambda_i$$

where $\bar{\omega}_i \in [0,1]$ since $\mu_i + \lambda_i \leq 1$. Therefore,

$$\begin{aligned} &\bar{\omega}_i J_t(\mu_i, 1-\mu_i) + (1-\bar{\omega}_i)J_t(\mu_i, 0) \\ &= B(\beta_i)\beta_i - [\bar{\omega}_i(1-\mu_i) + (1-\bar{\omega}_i)0]\psi_t \;=\; J_t(\mu_i, \lambda_i)\,, \end{aligned}$$

which implies that $J_t(\mu_i, \lambda_i)$ can always be obtained as a linear combination of the value of J_t on two corresponding points on the boundaries $(\lambda = 0)$ and $(\mu + \lambda = 1)$.

Next, note that the concave closure $\mathcal{J}_t(\mu, \lambda)$ on every $(\mu, \lambda) \in D$ is a linear combination of J_t over a subset of D, call it $D(\mu, \lambda)$. Take any point $(\mu_i, \lambda_i) \in D(\mu, \lambda)$ that is also interior, and substitute $J_t(\mu_i, \lambda_i)$ with an additional linear combination between $(\mu_i, 0)$ and $(\mu_i, 1-\mu_i)$, while keeping the weight that multiplies $J_t(\mu_i, \lambda_i)$ constant. This leaves the value of $\mathcal{J}_t(\mu, \lambda)$ unchanged, as its follows from the above discussion that

$$\begin{aligned} J_t(\mu, \lambda) \;&=\; \sum_{(\mu_j, \lambda_j) \in D(\mu, \lambda)} \omega_j\, J_t(\mu_j, \lambda_j) \;=\; ... + \omega_i J_t(\mu_i, \lambda_i) \\ &=\; ... + \omega_i \bar{\omega}_i J_t(\mu_i, 1-\mu_i) + \omega_i(1-\bar{\omega}_i)J_t(\mu_i, 0)\,. \end{aligned}$$

Repeating the same process for all interior points of $D(\mu, \lambda)$ gives that $\mathcal{J}_t(\mu, \lambda)$ can be written as a linear combination of J_t over points belonging to $(\lambda = 0)$ and $(\mu + \lambda = 1)$ exclusively.

To prove (B.1.15) suppose that it does not hold for a point $(\mu, \lambda) \in D$. Then by maintaining the same probabilities of the posteriors to be on each of the two boundaries, $\Pr(\lambda_t^s = 0)$ and $\Pr(\mu_t^s + \lambda_t^s = 1)$, and changing the conditional probabilities so that the two corresponding conditional expectations are equal to the concave closures $\bar{\mathcal{J}}_f(\mu)$ and $\bar{\mathcal{J}}_t(\mu)$, respectively, the whole expectation increases, which leads to a contradiction.

Therefore, $\mathcal{J}_t(\mu, \lambda)$ can always be expressed as a linear combination of its value on the two boundaries. To find the correct weight for each set of points (μ, λ), $(\mu', 0)$, and $(\mu'', 1-\mu'')$

solve

$$\begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \omega \begin{pmatrix} \mu' \\ 0 \end{pmatrix} + (1-\omega) \begin{pmatrix} \mu'' \\ 1-\mu'' \end{pmatrix} \Rightarrow \omega = \frac{\mu - \mu''}{\mu' - \mu''} .$$

Hence to fully characterise $\mathcal{J}_t(\mu, \lambda)$ it remains to pin down μ' and μ''. To do this let x be the slope of the line that connects $(\mu', 0)$, (μ, λ), and $(\mu'', 1-\mu'')$. Then for $\mu \neq \mu'$, which also implies $\mu'' \neq \mu'$, this is equal to:

$$x = \frac{\lambda - 0}{\mu - \mu'} = \frac{1-\mu''-0}{\mu''-\mu'} \Rightarrow \begin{cases} \lambda & = x(\mu - \mu') \\ 1-\mu'' & = x(\mu''-\mu') \end{cases} \Rightarrow \begin{cases} \mu' & = \mu - \frac{\lambda}{x} \\ \frac{1-\mu'}{1-\mu''} & = 1 + \frac{1}{x} \end{cases} .$$

This in turn implies

$$\frac{\mu' - \mu}{\mu' - \mu''} = \frac{\lambda}{1-\mu''} \quad \text{and}$$

$$\frac{\mu - \mu''}{\mu' - \mu''} = 1 - \frac{\mu'-\mu}{\mu'-\mu''} = 1 - \frac{\lambda}{1-\mu'} \frac{1-\mu'}{1-\mu''} = 1 - \frac{\lambda}{1-\mu'} \left(1 + \frac{1}{x}\right) ,$$

which gives that

$$\begin{aligned} (1-\mu') \frac{\mu-\mu''}{\mu'-\mu''} &= 1 - \mu' - \lambda \left(1 + \frac{1}{x}\right) \\ = 1 - \mu + \frac{\lambda}{x} - \lambda \left(1 + \frac{1}{x}\right) &= 1 - \mu - \lambda . \end{aligned}$$

Then define

$$\begin{aligned} \widehat{\mathcal{J}}_t(x; \mu, \lambda) \;&\equiv\; \frac{\mu-\mu''}{\mu'-\mu''} \bar{\mathcal{J}}_f(\mu') + \frac{\mu'-\mu}{\mu'-\mu''} \bar{\mathcal{J}}_t(\mu'') \\ &= (1-\mu') \frac{\mu-\mu''}{\mu'-\mu''} \frac{\bar{\mathcal{J}}_f(\mu')}{1-\mu'} + (1-\mu'') \frac{\mu'-\mu}{\mu'-\mu''} \frac{\bar{\mathcal{J}}_t(\mu'')}{1-\mu''} \\ &= (1-\mu-\lambda) \frac{\bar{\mathcal{J}}_f(\mu')}{1-\mu'} + \lambda \frac{\bar{\mathcal{J}}_t(\mu'')}{1-\mu''} , \end{aligned}$$

and note that it follows from the above argumentation that

$$\mathcal{J}_t(\mu, \lambda) = \max_x \widehat{\mathcal{J}}_t(x; \mu, \lambda) \quad \text{s.t} \quad \mu', \mu'' \in [0,1] \tag{B.1.16}$$

To write the constrain in terms of x solve for μ'', which is given by

$$x = \frac{1-\mu''-\lambda}{\mu''-\mu} \Leftrightarrow \mu'' = \frac{1-\lambda+x\mu}{1+x} .$$

As a result, for μ' and μ'' to be in the interval $[0,1]$ it has to be that

$$0 \le \mu - \frac{\lambda}{x} \le 1 \;\Rightarrow\; x \in \left(-\infty\,,\, -\frac{\lambda}{1-\mu}\right] \cup \left[\frac{\lambda}{\mu}\,,\, +\infty\right)$$
$$0 \le \frac{1-\lambda+x\mu}{1+x} \le 1 \;\Rightarrow\; x \in \left(-\infty\,,\, -\frac{1-\lambda}{\mu}\right] \cup \left[-\frac{\lambda}{1-\mu}\,,\, +\infty\right)\,,$$

respectively. Then using that $1 > \mu + \lambda$ gives that the intersection of the above two sets is

$$x \in \left(-\infty\,,\, -\frac{1-\lambda}{\mu}\right] \cup \left[\frac{\lambda}{\mu}\,,\, +\infty\right)\,,$$

which is the desired form for the constrain of (B.1.16). □

B.2 Proofs for continuous types

To avoid repetition, the subsequent lemma provides a sufficient and necessary condition for implementation that will be applied to the contracts offered by both S_1 and S_2 .

Lemma B.2.1 (Implementation)**.** *For given price $p : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ suppose that the payoff of a θ type buyer, when reporting $\hat{\theta}$, is*

$$\widehat{V}(\hat{\theta},\theta) \equiv v(\hat{\theta},\theta) - p(\hat{\theta}) \tag{B.2.1}$$

where $v : [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is absolute continuous in the second variable with weak derivative $v_2 : [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$. Then truthful reporting is implementable only if v_2 is increasing in the first variable. In addition, if this holds then the price

$$p(\theta) = v(\theta,\theta) - \int_{\underline{\theta}}^{\theta} v_2(x,x)\,\mathrm{d}x \tag{B.2.2}$$

ensures that truthful reporting in optimal.

Proof of Lemma D.1.1. First, necessity is proven. Suppose that truthful reporting is optimal and let $V(\theta) = \widehat{V}(\theta,\theta)$, then for any $\theta_1, \theta_2 \in [\underline{\theta}, \overline{\theta}]$ such that $\theta_1 < \theta_2$:

$$V(\theta_2) \ge \widehat{V}(\theta_1,\theta_2) = V(\theta_1) + \int_{\theta_1}^{\theta_2} \widehat{V}_2(\theta_1,\theta)\,\mathrm{d}\theta$$
$$V(\theta_1) \ge \widehat{V}(\theta_2,\theta_1) = V(\theta_2) - \int_{\theta_1}^{\theta_2} \widehat{V}_2(\theta_2,\theta)\mathrm{d}\theta$$

where the subscript 2 indicates the partial derivative with respect to the second entry. Rearranging the two inequalities and combining them gives

$$\int_{\theta_1}^{\theta_2} \widehat{V}_2(\theta_2, \theta) \, \mathrm{d}\theta \;\geq\; \int_{\theta_1}^{\theta_2} \widehat{V}_2^s(\theta_1, \theta) \, \mathrm{d}\theta.$$

As this has to hold for any choice of θ_1 and θ_2, as defined above, it follows that $v_2(\hat{\theta}, \theta) = \widehat{V}_2(\hat{\theta}, \theta)$ has to be non-decreasing on $\hat{\theta}$. Second, sufficiency is proven. Suppose $\hat{\theta} < \theta$, then

$$\begin{aligned} v(\hat{\theta}, \theta) - p(\hat{\theta}) \;&=\; v(\hat{\theta}, \theta) - v(\hat{\theta}, \hat{\theta}) + \int_{\underline{\theta}}^{\widehat{\theta}} v_2(x, x) \, \mathrm{d}x \\ &=\; \int_{\widehat{\theta}}^{\theta} v_2(\hat{\theta}, x) \, \mathrm{d}x + \int_{\underline{\theta}}^{\widehat{\theta}} v_2(x, x) \, \mathrm{d}x \\ =\; \int_{\widehat{\theta}}^{\theta} \Big\{ v_2(\hat{\theta}, x) - v_2(x, x) \Big\} \, \mathrm{d}x + \int_{\underline{\theta}}^{\theta} v_2(x, x) \, \mathrm{d}x \;&\leq\; \int_{\underline{\theta}}^{\theta} v_2(x, x) \, \mathrm{d}x \;=\; v(\theta, \theta) - p(\theta) \end{aligned}$$

As a result reporting $\hat{\theta} = \theta$ is no worse than any $\hat{\theta} < \theta$. The proof for $\hat{\theta} > \theta$ is similar hence it is omitted. □

Proof of Lemma 2.2.1. S_2's revenue for given choice of policies $p_2(\theta_2)$ and $q_2(\theta_2)$ is

$$\int_{\underline{\theta}_2}^{\overline{\theta}_2} \Big\{ p_2(\theta_2) - c[q_2(\theta_2)] \Big\} \, \mathrm{d}F_2^s(\theta_2), \tag{B.2.3}$$

The buyer's payoff, when reporting $\hat{\theta}_2$ instead of his actual type θ_2, is

$$\widehat{V}_2^s(\hat{\theta}_2, \theta_2) \;=\; b\,\theta_2 q_2(\hat{\theta}_2) - p_2(\hat{\theta}_2)\,.$$

It is ease to argue that Theorem 2 of Milgrom and Segal (2002) applies in this setting. Hence their envelop theorem gives that

$$\frac{\mathrm{d}V_2^s(\theta_2)}{\mathrm{d}\theta_2} \;=\; b\,q_2(\theta_2)\,.$$

As a result, integrating and equating with $V_2^s(\theta_2) = \widehat{V}_2^s(\theta_2, \theta_2)$ gives

$$b \int_{\underline{\theta}_2}^{\theta_2} q_2(x) \, \mathrm{d}x + V_2^s(\underline{\theta}_2) \;=\; b\,\theta_2 q_2(\theta_2) - p_2(\hat{\theta}_2)\,.$$

Then the objective function of (2.2.2) follows from substituting in (B.2.3) the price $p_2(\theta_2)$, as given from the above equality, applying Fubini's Theorem, and setting $V_2^s(\underline{\theta}_2) = 0$.

Finally, note that it follows from Lemma D.1.1 that a sufficient and necessary condition for the choice of $q_2(\theta_2)$ to be implementable is that it is non-decreasing. $\square$

Proof of Proposition 2.2.1. Under static types $\bar{V}_2^s(\theta_1) = V_2^s(\theta_1)$. Hence S_1's ex ante payoff from the buyer's contract with S_2 can equivalently be rewritten as follows

$$\begin{aligned}
\int_{\underline{\theta}_1}^{\bar{\theta}_1} \mathbb{E}_g\big[\bar{V}_2^s(\theta_1)\,|\,\theta_1\big]\mathrm{d}F_1(\theta_1) \;&=\; \int_{\underline{\theta}_1}^{\bar{\theta}_1} \sum_s \big\{\bar{V}_2^s(\theta_1)g(s\,|\,\theta_1)\big\}\,\mathrm{d}F_1(\theta_1) \\
=\; \sum_s \int_{\underline{\theta}_1}^{\bar{\theta}_1} \bar{V}_2^s(\theta_1)f_1^s(\theta_1)\mathrm{d}\theta_1\, g(s) \;&=\; \sum_s \int_{\underline{\theta}_1}^{\bar{\theta}_1} [1 - F_1^s(\theta_1)]\,\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\mathrm{d}\theta_1\, g(s) \\
=\; \int_{\underline{\theta}_1}^{\bar{\theta}_1} \sum_s \mu_1^s(\theta_1)\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}g(s\,|\,\theta_1)\,\mathrm{d}F_1(\theta_1) \;&=\; b\int_{\underline{\theta}_0}^{\bar{\theta}_0} \sum_s \big\{\mu_1^s(\theta_1)q_1^s(\theta_1)g(s\,|\,\theta_1)\,\big\}\mathrm{d}F_1(\theta_1)
\end{aligned}$$

where the last equality is due to the fact that $\theta_1 = \theta_2$ under static types. It follows from (B.2.3) that under no information provision the quantity implemented by S_2 is

$$b^\epsilon \max\big\{0\,,\,\theta_1 - \mu_1(\theta_1)\big\}^\epsilon,$$

but since $\mu_1(\theta_1)$ is non-increasing by assumption, then whenever $\mu_1(\underline{\theta}_1) > \underline{\theta}_1$ there exists a set $[\underline{\theta}_1, \theta_1^+]$ of positive measure for which the supplied quantity is zero. Consider a binary signal $s \in \{s^-, s^+\}$ such that

$$\begin{aligned}
g(s^-\,|\,\theta_1) \;&=\; 1\,, \quad \text{if} \quad \theta_1 \le \theta^+ \\
g(s^+\,|\,\theta_1) \;&=\; 1\,, \quad \text{if} \quad \theta_1 > \theta^+
\end{aligned}$$

which essentially reveals the corresponding set in which the buyer's type belongs. It is ease to show that the corresponding inverse hazard rates are

$$\mu_1^-(\theta_1) \;=\; \begin{cases} \frac{F_1(\theta^+)-F_1(\theta_1)}{f_1(\theta_1)} & ,\text{ if } \;\theta_1 \le \theta^+ \\ \text{not defined} & ,\text{ if } \;\theta_1 > \theta^+ \end{cases} \quad \text{and} \quad \mu_1^+(\theta_1) \;=\; \begin{cases} \text{not defined} & ,\text{ if } \;\theta_1 \le \theta^+ \\ \mu_1(\theta_1) & ,\text{ if } \;\theta_1 > \theta^+ \end{cases}$$

Note that both of them are non-increasing in θ_1, hence after the realisation of each of them S_2 implements the corresponding point-wise optimal quantity. As a result, S_1's ex ante revenue under this binary signal becomes

$$b^{1+\epsilon}\int_{\underline{\theta}_1}^{\theta^+}\mu_1^-(\theta_1)\max\left\{0\,,\,\theta_1-\mu_1^-(\theta_1)\right\}^\epsilon \mathrm{d}F_1(\theta_1)$$
$$+\;b^{1+\epsilon}\int_{\theta^+}^{\bar{\theta}_1}\mu_1(\theta_1)\max\left\{0\,,\,\theta_1-\mu_1(\theta_1)\right\}^\epsilon \mathrm{d}F_1(\theta_1)$$

The second line is identical to S_1's ex ante payoff under no information provision, but the first would be zero instead. On the other hand, under the constructed binary signal there will be at least a few types in $[\underline{\theta}_1,\theta^+]$ that will be supplied a positive quantity, as $\mu_1^-(\theta^+)=0$. Hence, for a subset of types close to θ^+ the supplied quantity will be strictly positive. □

Proof of Lemma 2.2.2. It is ease to argue that Theorem 2 of Milgrom and Segal (2002) applies in this setting. Hence their envelop theorem gives that

$$\frac{\mathrm{d}V_1(\theta_1)}{\mathrm{d}\theta_1}\;=\;q_1(\theta_1)+\mathbb{E}_g\left[\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\theta_1\right].$$

As a result, integrating and equating with $V_1(\theta_1)=\widehat{V}_1(\theta_1,\theta_1)$ gives

$$\int_{\underline{\theta}_1}^{\theta_1}\frac{\mathrm{d}V_1(x)}{\mathrm{d}x}\,\mathrm{d}x\;=\;\theta_1q_1(\theta_1)-p_1(\theta_1)+\mathbb{E}_g\left[\bar{V}_2^s(\theta_1)\,|\,\theta_1\right].$$

Hence substitute in the objective function of (2.2.3) the price $p_1(\theta_1)$, as given from the above equality, apply Fubini's Theorem, and set $V_1(\underline{\theta}_1)=0$ to obtain

$$\int_{\underline{\theta}_1}^{\bar{\theta}_1}\Bigg\{q_1(\theta_1)\left[\theta_1-\mu_1(\theta_1)\right]-c[q_1(\theta_1)]$$
$$+\,\mathbb{E}_g\left[\bar{V}_2^s(\theta_1)-\frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\theta_1\right]\Bigg\}\mathrm{d}F_1(\theta_1) \tag{B.2.4}$$

Then the functional form of the point-wise optimal production follows from the first order condition of the above. Lemma D.1.1 implies that for the pair (q_1^*,g) to be implementable it has to be that

$$\frac{\partial\,\widehat{V}_1(\hat{\theta}_1,\theta_1)}{\partial\,\theta_1}\;=\;q_1^*(\hat{\theta}_1)+\mathbb{E}_g\left[\frac{\mathrm{d}\bar{V}_1(\theta_1)}{\mathrm{d}\theta_1}\,\middle|\,\hat{\theta}_1\right]$$

is non-decreasing in $\hat{\theta}_1$, from which condition (2.2.5) is derived. Note that $q_1^*(\hat{\theta}_1)$ is increasing in $\hat{\theta}_1$, while under no information provision the above expectation is constant with respect to it. Hence no information provision is implementable.

Finally, to derive the objective function of (2.2.4) rewrite the second line of (B.2.4) as follows

$$\int_{\underline{\theta}_1}^{\overline{\theta}_1} \mathbb{E}_g\left[\bar{V}_2^s(\theta_1) - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\,\bigg|\,\theta_1\right]\Big\}\mathrm{d}F_1(\theta_1)$$
$$= \int_{\underline{\theta}_1}^{\overline{\theta}_1}\sum_s\left\{\bar{V}_2^s(\theta_1) - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\right\}g(s|\theta_1)f_1(\theta_1)\,\mathrm{d}\theta_1$$
$$= \sum_s\int_{\underline{\theta}_1}^{\overline{\theta}_1}\left\{\bar{V}_2^s(\theta_1) - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}\right\}f_1^s(\theta_1)\,\mathrm{d}\theta_1\,g(s)$$
$$= \sum_s\int_{\underline{\theta}_1}^{\overline{\theta}_1}\left\{\frac{1-F_1^s(\theta_1)}{f_1^s(\theta_1)} - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}\right\}\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1}f_1^s(\theta_1)\mathrm{d}\theta_1 g(s)\,.$$

□

Proof of Proposition 2.2.2. First, it is shown that $\frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1} \geq 0$. This function has been defined as

$$\bar{V}_2^s(\theta_1) \;=\; \mathbb{E}_{\theta_2}\big[V_2^s(\theta_2)\,|\,\theta_1, s\big]\,,$$

where it follows from (2.2.1) that $V_2^s(\theta_2)$ is non-decreasing, because $\frac{\mathrm{d}V_2^s(\theta_2)}{\mathrm{d}\theta_2} = b\,q_2^s(\theta_2) \geq 0$. But then since higher values of θ_1 induce a conditional CDF $F_2(\cdot\,|\,\theta_1)$ that FOSD lower values, it follows that $\bar{V}_2^s(\theta_1)$ is non-decreasing in θ_1.

Next, it is shown that under any deterministic signal it has to be that $\mu_1^s(\theta_1) \leq \mu_1(\theta_1)$. To do this fix a partition $\{\Theta_1^s\}_{s\in S}$ of $[\underline{\theta}_1, \overline{\theta}_1]$ and note that the probability of each signal s to be realised is

$$g(s) \;=\; \int_{\theta_1\in\Theta_1^s} f_1(\theta_1)\,\mathrm{d}\theta_1\,.$$

As a result, the posterior density and CDF are

$$f_1^s(\theta_1) \;=\; \begin{cases} f_1(\theta_1)/g(s) & \text{, if } \;\theta_1\in\Theta_1^s \\ 0 & \text{, if } \;\theta_1\notin\Theta_1^s\end{cases} \qquad \text{and} \qquad F_1^s(\theta_1) \;=\; \int_{x\leq\theta_1\,\cap\, x\in\Theta_1^s} f_1(x)\,\mathrm{d}x\;\frac{1}{g(s)}\,.$$

respectively. Hence whenever $\mu_1^s(\theta_1)$ is defined, that is for $\theta_1\in\Theta_1^s$, this is given by

$$\mu_1^s(\theta_1) \;=\; \frac{1-F_1^s(\theta_1)}{f_1^s(\theta_1)} \;=\; \int_{x>\theta_1\,\cap\, x\in\Theta_1^s} f_1(x)\,\mathrm{d}x\;\frac{1}{f_1(\theta_1)}$$
$$\leq \int_{x>\theta_1} f_1(x)\,\mathrm{d}x\;\frac{1}{f_1(\theta_1)} \;=\; \mu_1(\theta_1)$$

To finish the proof note that the objective function of (2.2.4) under the deterministic signal that uses the partition $\{\Theta_1^s\}_{s\in S}$ becomes

$$\sum_s \int_{\theta_1 \in \Theta_1^s} \Big\{\mu_1^s(\theta_1) - \mu_1(\theta_1)\Big\} \frac{\mathrm{d}\bar{V}_2^s(\theta_1)}{\mathrm{d}\theta_1} \,\mathrm{d}F_1(\theta_1) \;\le\; 0$$

But under no information provision the same objective function is equal to zero, and this is implementable, hence this signalling structure is optimal among deterministic signals. □

***Proof of Proposition** 2.2.3.* The proof is an extension of the treatment of continuous types undertaken by Calzolari and Pavan (2006), which can be found in the Appendix of their paper. A discussion similar to theirs can demonstrated that the incentives to provide information to S_2 are the strongest when S_1 captures all the expected benefit from this information provision. Moreover, in this case S_1's payoff is equivalent, up to a constant transformation, to as if she was integrated with S_2. Hence, if it can be shown that no information provision is optimal in this case, then the same is true when S_1 captures none of this expected benefit from information provision.

To do this set $\varphi = 1$ and suppose momentarily that S_1 and S_2 were integrated and solve the corresponding dynamic mechanism design problem. Denote the realised history $\{\theta_1, \theta_2\}$ by θ^2, and let $\theta^1 = \theta_1$. Then S_1 solves:

$$\mathbb{E}_{\theta^t}\left[\max_{p_t, q_t} \sum_{t=1}^{2} \Big\{p_t(\theta^t) - c\big[q_t(\theta^t)\big]\Big\}\right], \tag{B.2.5}$$

subject to the individual rationality constrains of period 1, and the incentive compatibility constrains of period 1 and period 2. First, the above problem is solved under the following restriction on the set of deviations used by the buyer.

- In period 1: he freely chooses the report $\hat{\theta}_1$
- In period 2: he is restricted to truthfully report if his type was redrawn, or not.

Let $\widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1)$ denote the payoff of an buyer that reported $\hat{\theta}_1$ in period 1, his type was redrawn, and he subsequently reported $\hat{\theta}_2$, while his actual type was θ_2.

$$\widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1) \;=\; b\,\theta_2\, q_2(\hat{\theta}_1, \hat{\theta}_2) - p_2(\hat{\theta}_1, \hat{\theta}_2)$$

In addition, let $U_2(\theta_2; \hat{\theta}_1) \equiv \widehat{U}_2(\hat{\theta}_2, \theta_2; \hat{\theta}_1)$ be the corresponding value under truthful reporting of θ_2. Note that the actual value of θ_1 is irrelevant in terms of his incentives to report $\hat{\theta}_2$.

Hence, incentive compatibility implies that

$$U_2(\theta_2;\hat{\theta}_1) \;=\; \max_{\hat{\theta}_2\in[0,1]} \widehat{U}_2(\hat{\theta}_2,\theta_2;\hat{\theta}_1)\,, \quad \text{for all} \quad \hat{\theta}_1 \in [0,1]$$

In all of the subsequent discussion Theorem 2 of Milgrom and Segal (2002) applies. Hereafter, it will simple be invoked as *envelop theorem*. Then the envelop theorem gives that

$$\frac{\mathrm{d}U_2(\theta_2;\hat{\theta}_1)}{\mathrm{d}\theta_2} \;=\; b\,q_2(\hat{\theta}_1,\theta_2)$$

Set $U_2(0\,;\hat{\theta}_1) = 0$ and note that

$$U_2(\theta_2;\hat{\theta}_1) \;=\; \int_0^{\theta_2} \frac{\mathrm{d}U_2(x;\hat{\theta}_1)}{\mathrm{d}x}\,\mathrm{d}x \;\Leftrightarrow$$
$$b\,\theta_2\,q_2(\hat{\theta}_1,\theta_2) - p_2(\hat{\theta}_1,\theta_2) \;=\; \int_0^{\theta_2} b\,q_2(\hat{\theta}_1,\theta_2)\,\mathrm{d}x$$

Since this hold for any $\hat{\theta}_1$, it also holds for $\hat{\theta}_1 = \theta_1$. Hence the part of (B.2.5) that follows a redraw of θ_2, after a realisation of θ_1, can be rewritten as

$$\begin{aligned}
&\mathbb{E}_{\theta_2}\Big[p_2(\theta_1,\theta_2) - c\big[q_2(\theta_1,\theta_2)\big]\Big] \\
&\quad = \int_0^1 \left\{ b\,\theta_2\,q_2(\theta_1,\theta_2) - c\big[q_2(\theta_1,\theta_2)\big] - \int_0^{\theta_2} b\,q_2(\theta_1,\theta_2)\,\mathrm{d}x \right\} f_1(\theta_2)\,\mathrm{d}\theta_2 \\
&\quad = \int_0^1 \left\{ b\,\big[\theta_2 - \mu_1(\theta_2)\big]\,q_2(\theta_1,\theta_2) - c\big[q_2(\theta_1,\theta_2)\big] \right\} f_1(\theta_2)\,\mathrm{d}\theta_2 \qquad \text{(B.2.6)}
\end{aligned}$$

Next, denote the expected payoff of the buyer in period 1 under such a deviation by

$$\begin{aligned}
\widehat{U}_1(\hat{\theta}_1,\theta_1) \;=\; &\theta_1 q_1(\hat{\theta}_1) - p_1(\hat{\theta}_1) + \rho\Big(\,b\,\theta_1 q_2(\hat{\theta}_1,\hat{\theta}_1) - p_2(\hat{\theta}_1,\hat{\theta}_1)\,\Big) \\
&+ (1-\rho)\mathbb{E}_{\theta_2}\Big[\,\widehat{U}_2(\hat{\theta}_2,\theta_2;\hat{\theta}_1)\,\Big]\,,
\end{aligned}$$

and let $U_1(\theta_1) = \widehat{U}_1(\theta_1,\theta_1)$. Then under truthful reporting the envelop theorem implies that

$$\frac{\mathrm{d}U_1(\theta_1)}{\mathrm{d}\theta_1} \;=\; q_1(\theta_1) + \rho b\,q_2(\theta_1,\theta_1)$$

As a result, using the same manipulations as above I obtain that the part of the (B.2.5) that corresponds to the realisation of θ_1 and of it not being redrawn can be written as

$$\big[\theta_1 - \mu_1(\theta_1)\big]\, q_1(\theta_1) - c\big[q_1(\theta_1)\big] + \rho\Big\{ b\,\big[\theta_1 - \mu_1(\theta_1)\big]\, q_2(\theta_1, \theta_1) - c\big[q_2(\theta_1, \theta_1)\big]\Big\} \tag{B.2.7}$$

Then point-wise maximisation of (B.2.6) and (B.2.7) gives that

$$q_t^*(\theta_t) \;=\; \max\Big\{0\,,\, b^t\big[\theta_t - \mu_1(\theta_t)\big]\Big\}^{\epsilon} \tag{B.2.8}$$

This solves the optimisation of (B.2.5) under a restriction on the buyer's action space. But it is ease to show using Lemma D.1.1 that this supply schedule is also implementable without the imposed restriction. This in turn gives that S_1 can achieve the same payoff in the problem where the buyer has access to his full action space, as in the restricted one. Hence, the supply schedule (B.2.8) is a solution to her optimisation problem (B.2.5).

Switching back to the non-integrated S_1 and S_2 case, note that (B.2.8) is what the latter would chose under no information provision. Hence, it follows from the discussion in the beginning of this proof that no information provision is optimal for her. □

B.3 Proofs for Moral Hazard

Proof of Lemma 2.3.1. The dependence on (τ, s) is dropped. The revelation principle applies and since the agent's type is static, it is without loss to focus on contracts that pay period wage $w^b(\theta, y_t)$ in $t \in \{\tau, .., \infty\}$. Moreover, observe that using the reported type, a perfect estimate of the effort can be deducted by P_b. Hence, any misalignment between this estimate and the recommended effort can be punished strongly enough for the agent to mask it. As a result a report $\hat{\theta}$ implies choice of effort

$$\hat{e}^b(\hat{\theta}, \theta) \;=\; e^b(\hat{\theta}) \cdot \frac{\hat{\theta}}{\theta}.$$

For simplicity drop the dependence on y_t, as this will occur only off path and write the reported type as a subscript for $w^b(\theta)$ and $e^b(\theta)$. P_b's payoff maximisation problem is essentially a static one.

$$\begin{aligned}
\max_{e,w} \quad & \beta\left(b\,\theta_H e_H^b - w_H^b\right) + (1-\beta)\big(b\,\theta_L e_L^b - w_L^b\big) \\
\text{s.t. } (\text{IR}_L) \quad & w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; 0 \\
(\text{IR}_H) \quad & w_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; 0 \\
(\text{IC}_L) \quad & w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} \\
(\text{IC}_H) \quad & w_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}},
\end{aligned}$$

where both the constrains and the objective function are written in per period payoff. Assuming that (IR_L) does not bind leads to a contradiction. Subsequently, this can be used to show that (IC_H) has to also bind. Hence the above simplifies to the unconstrained maximisation problem

$$\max_{e}\; \beta\left(b\,\theta_H\, e_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right) + (1-\beta)\left(b\,\theta_L\, e_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\,\frac{1-\beta\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}}{1-\beta}\right)$$

The objective function is concave, hence the first order conditions give that

$$e^b(\theta_H) \;=\; (b\,\theta_H)^{\epsilon} \quad \text{and} \quad e^b(\theta_L) \;=\; b^{\epsilon}\cdot\left(\frac{(1-\beta)\,\theta_L}{1-\beta\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}}\right)^{\epsilon}.$$

This is implementable as (IC_L) equivalently becomes

$$\begin{aligned}
w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left[\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} - 1\right] \Leftrightarrow \\
w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} \;\geq\; w_L^b - \frac{(e_L^b)^2}{2}\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}} - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left[\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}} - 1\right] \Leftrightarrow \\
\theta_H e_H^b \;\geq\; \theta_L e_L^b
\end{aligned}$$

which is satisfied for the derived effort choices. Because the IR_L binds the low type's period payoff is zero. The high type's period payoff can be obtained using the (IR_L) and (IC_H)

constrains, which give that

$$w_H^b - \frac{(e_H^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} = w_L^b - \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}} = \frac{(e_L^b)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left[1-\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}\right]$$

$$= \frac{(b\,\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left[1-\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}\right]\left(\frac{1-\beta}{1-\beta\cdot\left(\frac{\theta_L}{\theta_H}\right)^{1+\frac{1}{\epsilon}}}\right)^{1+\epsilon}.$$

Hence a constant stream of the above payoff up to infinity gives $B(\beta)$. Moreover,

$$B'(\beta) = -(1+\epsilon)K\left(\frac{1-\beta}{1-\beta\kappa}\right)^{\epsilon}\frac{1-\kappa}{(1-\beta\kappa)^2} = -B(\beta)\frac{(1+\epsilon)(1-\kappa)}{(1-\beta)(1-\beta\kappa)} < 0$$

for all $\beta \in [0,1)$. In addition,

$$B''(\beta) = -(1+\epsilon)K(1-\kappa)\frac{-\epsilon(1-\beta)^{\epsilon-1}(1-\beta\kappa)^{\epsilon+2}+\kappa(\epsilon+2)(1-\beta)^{\epsilon}(1-\beta\kappa)^{\epsilon+1}}{(1-\beta\kappa)^{2(\epsilon+2)}}$$

$$= B'(\beta)\left(\frac{\kappa(\epsilon+2)}{1-\beta\kappa}-\frac{\epsilon}{1-\beta}\right) > 0 \Leftrightarrow \frac{\kappa(\epsilon+2)}{1-\beta\kappa} < \frac{\epsilon}{1-\beta} \Leftrightarrow \beta > 1-\epsilon\frac{1-k}{k}.$$

□

Proof of Proposition 2.3.1. Lemma B.1.1 applies in this setting. Hence it is without loss to only consider one-shot deviations. In those a θ^t agent type reports truthfully θ^{t-1}, potentially misreports θ_t as $\hat{\theta}_t$, and subsequently switches back to truthful reporting. Hereafter, IC(θ^t) will refer exclusivity to the incentive compatibility constrains obtained under one-shot deviations. To maintain a compact notation, let $\hat{\theta}^t = \{\theta^{t-1}, \hat{\theta}_t\}$ denote a history of truthful reports up to $t-1$ followed by a potential misreport $\hat{\theta}_t$. In addition, denote a generic history θ^{t-1} followed by $\theta_t = \theta_H$ as θ_H^t, and similarly define $\hat{\theta}_L^t$. Then the payoff of a θ^t agent type under a one shot deviation is

$$\widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1}) = w_t^a(\hat{\theta}^t) - \frac{e_t^a(\hat{\theta}^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\hat{\theta}_t}{\theta_t}\right)^{1+\frac{1}{\epsilon}} + f_t(\hat{\theta}^t)\gamma\delta\,\mathbb{E}_\theta\big[\widehat{U}_{t+1}(\theta_{t+1}, \theta_{t+1}, \hat{\theta}^t)\,|\,\theta_t\big]$$
$$+ [1-f_t(\hat{\theta}^t)\gamma]\,\delta\,\Pr(\theta_{t+1}=\theta_H\,|\,\tau=t,\theta_t)\,\mathbb{E}_g\big[B(\beta_t^s)\,|\,\hat{\theta}^t\big]$$

In period t the agent obtains the wage of the reported type $\hat{\theta}_t$, but since his type is θ_t he actually has to mask this deviation which is why there is this adjustment on the cost of effort. The probability of continuation $\gamma f_t(\hat{\theta}^t)$ is only a function of $\hat{\theta}^t$, however under one shot deviations the agent's type is truthfully reported in $\widehat{U}_{t+1}$ and the expectation over it depends on the actual type θ_t. Similarly, the probability of the agent to be a high type in P_b's contract, $\Pr(\theta_{t+1} = \theta_H \,|\, \tau = t, \theta_t)$, is only a function of the realised type θ_t. In contrast, the distribution of the signal s, provided by P_a to P_b, is contingent on the reported type $\hat{\theta}^t$.

Let the on path payoff of a θ^t type agent be given by $U_t(\theta^t) = \widehat{U}_t(\theta_t, \theta_t, \theta^{t-1})$, then the corresponding individual rationality and incentive compatibility constrains become

$$\begin{aligned} \mathrm{IR}(\theta^t) \quad U_t(\theta^t) \;&\geq\; 0 \\ \mathrm{IC}(\theta^t) \quad U_t(\theta^t) \;&\geq\; \widehat{U}_t(\hat{\theta}_t, \theta_t, \theta^{t-1}) \end{aligned} \tag{B.3.1}$$

where the agent's outside option is zero because if he decides to terminate the contract in $t < \tau$, then P_a can ensure that P_b will not approach the agent. Doing so is always beneficial for P_a, because it lowers the agent's outside option. Next, consider the following problem

$$\begin{gathered} (\mathcal{P}^H) \qquad \max_{w,e,f,g} \mathbb{E}_\theta \left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1}) \gamma^t \delta^t \Big(\theta_t \, e_t^a(\theta^t) - w_t^a(\theta^t) \Big) \right] \\ \text{subject to } \mathrm{IR}(\theta^t) \text{ and } \mathrm{IC}(\theta^t), \end{gathered} \tag{B.3.2}$$

which is similar to ($\mathcal{P}$), but it ignores the individual rationality constrains of all high types $\mathrm{IR}(\theta_H^t)$ and the incentive compatibility constrains of all low types $\mathrm{IC}(\theta_L^t)$. Ignoring the former set of constrains is without loss of generality as $U_t(\theta_H^t) \geq U_t(\theta_L^t)$, however this is not true for the $\mathrm{IC}(\theta_L^t)$ constrains. Despite that, if the solution of ($\mathcal{P}^H$) happens to also satisfy this set of constrains, then it is a solution of ($\mathcal{P}$).

The rest of the proof shows that in ($\mathcal{P}^H$), for every (w, e, f, g) there exists a w' such that the objective function under (w', e, f, g) is no less than under (w, e, f, g), and that both the $\mathrm{IR}(\theta_L^t)$ and $\mathrm{IC}(\theta_H^t)$ constrains bind. Solving for w' from the constrains and substituting in the objective function will give ($\mathcal{P}'$). As a result, I will have shown that if (e, f, g) is a solution of ($\mathcal{P}'$), then there exists w' such that (w', e, f, g) is also a solution of ($\mathcal{P}^H$). Finally, w' will be substituted in the previously ignored $\mathrm{IC}(\theta_L^t)$ so that a sufficient condition is obtained for (w', e, f, g) be a solution of ($\mathcal{P}$), which is only in terms of policies (e, f, g).

The argument is recursive. Suppose that $\mathrm{IR}(\theta_L^t)$ and $\mathrm{IC}(\theta_H^t)$ bind for all periods up to and including t', but not for $t'+1$. For simplicity, denote $w_t^a(\theta_H^{t'})$ by w_H, $w_t^a(\theta_L^{t'})$ by w_L, and similarly use $\{w_{LL}, w_{HL}, w_{LH}, w_{HH}\}$ for the possible combinations up to $t'+1$. Moreover, adopt the same notational change for the IR and IC constrains. Suppose that IR_{HL} does not bind, then let

$$\big(\tilde{w}_H, \tilde{w}_{HH}, \tilde{w}_{HL}\big) \;=\; \big(w_H + \delta\varepsilon,\, w_{HH} - \varepsilon,\, w_{HL} - \varepsilon\big)$$

and increase ε until it does. Under this transformation IR_L and IC_H continue to bind and P_a is indifferent between the two contracts. The same argument works if IR_{LL} does not bind. Suppose instead that IC_{LH} does not bind, then let

$$\big(\tilde{w}_L,\, \tilde{w}_H,\, \tilde{w}_{LH}\big) \;=\; \big(w_L + \delta\varphi_L\varepsilon,\, w_H - \delta(\varphi_H - \varphi_L)\varepsilon,\, w_{LH} - \varepsilon\big),$$

and increase ε until it does. Under this transformation IR_L and IC_H continue to bind and P_a is actually better off. Finally, suppose that IC_{HH} does not bind, then let

$$\big(\tilde{w}_H,\, \tilde{w}_{HH}\big) \;=\; \big(w_H + \delta\varphi_H\varepsilon,\, w_{HH} - \varepsilon\big),$$

and increase ε until it does. Under this transformation IR_L and IC_H continue to bind and P_a is indifferent between the two contracts. Hence, if both $\text{IR}(\theta_L^t)$ and $\text{IC}(\theta_H^t)$ bind for all periods up to and including t', and $\text{IC}(\theta_L^t)$ is ignored, then there exists an alternative contract that implements the same policies, is not worse for P_a, and has all constrains binding up to $t'+1$. In addition, the regular one period argumentation shows that $\text{IR}(\theta_L^0)$ and $\text{IC}(\theta_H^0)$ have to bind, from which the recursive argument follows.

Hence, it is without loss to assume that $\text{IR}(\theta_L^t)$ and $\text{IC}(\theta_H^t)$ bind, which gives

$$-w_t^a(\theta_L^t) \;=\; -\frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} + f_t(\theta_L^t)\gamma\delta\varphi_L\widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + [1 - f_t(\theta_L^t)\gamma]\delta\rho_L\mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]$$

and

$$\begin{aligned} U_t(\theta_H^t) \;=\; & w_t^a(\theta_L^t) - \frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}} \\ & + f_t(\theta_L^t)\gamma\delta\varphi_H\widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + [1 - f_t(\theta_L^t)\gamma]\delta\rho_H\mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t], \end{aligned}$$

respectively. Note that in both equations above it has been used that $\widehat{U}_{t+1}(\theta_L,\theta_L,\theta_L^t) = U_{t+1}(\{\theta_L,\theta_L^t\}) = 0$, where the first equality follows from the ignored $\text{IC}(\{\theta_L,\theta_L^t\})$ and the second from the fact that $\text{IR}(\{\theta_L,\theta_L^t\})$ binds. Substitute the first line in the second to obtain that

$$\begin{aligned} U_t(\theta_H^t) \;=\; & \frac{e_t^a(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1 - \frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) \\ & + f_t(\theta_L^t)\gamma\delta(\varphi_H - \varphi_L)\widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t) + [1 - f_t(\theta_L^t)\gamma]\delta(\rho_H - \rho_L)\mathbb{E}_g[B(\beta_t^s)|\hat{\theta}^t]. \end{aligned}$$

But because the $IC(\{\theta_L^t,\theta_H\})$ holds, I get that $U_{t+1}(\{\theta_L^t,\theta_H\}) = \widehat{U}_{t+1}(\theta_H,\theta_H,\theta_L^t)$ and the same argument can be used repeatedly for any $t' > t+1$. Hence, substitute forward to obtain the functional form given for $U_t^H(\theta^{t-1})$. In particular, for period 0 that becomes

$$
\begin{aligned}
U_0(\theta_H) \;=\; & \sum_{t=0}^{\infty} f_0^{t-1}(L^{t-1})\gamma^t\delta^t(\varphi_H-\varphi_L)^t \\
& \times\left[\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right)+\delta[1-f_t(L^t)](\rho_H-\rho_L)\mathbb{E}_g\big[B(\beta_t^s)\,|\,L^t\big]\Big)\right]. \quad \text{(B.3.3)}
\end{aligned}
$$

It follows by the definition of $U_0(\theta_0)$ that the on path expected discounted payments In period 0 satisfies

$$
U_0(\theta_0)+\mathbb{E}_\theta\left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t\frac{e_t^a(\theta^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\,\middle|\,\theta_0\right] \;=\; \mathbb{E}_\theta\left[\sum_{t=0}^{\infty} f_0^{t-1}(\theta^{t-1})\gamma^t\delta^t w_t^a(\theta^t)\,\middle|\,\theta_0\right]
$$

It has being shown that IR(θ_0) binds, hence for a low type substitute above $U_0(\theta_L)=0$, while for a high type substitute the expression obtained in (B.1.5). Finally, substituting the expected discounted payments on the objective function gives ($\mathcal{P}'$).

To complete the proof note that by definition the wages given to a high type makes him indifferent between deviating or not, after every history θ^{t-1}. For a low type the IC(θ_L^t) becomes

$$
\begin{aligned}
U_t(\theta_L^t) \;\geq\; & w_t^a(\theta_H^t)-\frac{e_t^a(\theta_H^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H}{\theta_L}\right)^{1+\frac{1}{\epsilon}}+f_t(\theta_H^t)\gamma\,\delta\,\varphi_L U_{t+1}(\{\theta_H^t,\theta_H\}) \\
& +\delta\,[1-f_t(\theta_H^t)\gamma]\,\varphi_L\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_H^t\big]\,,
\end{aligned}
$$

which after substituting $U_t(\theta_L^t)=0$ equivalently becomes

$$
\begin{aligned}
U_t(\theta_H^t) \;\leq\; & \frac{e_t^a(\theta_H^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right)+\delta\;\Pr(\tau>t+1\,|\,\tau>t,\theta_H^t)\,(\varphi_H-\varphi_L)\,U_{t+1}(\theta_H^t,\theta_H) \\
& +\delta\;\Pr(\tau=t+1\,|\,\tau>t,\theta_H^t)(\rho_H-\rho_L)\mathbb{E}_g\big[B(\beta_t^s)\,|\,\theta_H^t\big].
\end{aligned}
$$

Finally, substituting the provided expression for $U_t^H(\theta^{t-1})$, and re-arranging gives ($\mathcal{P}_c$).

□

Proof of Corollary 2.3.1. The third line of ($\mathcal{P}'$) represents the information rents, which only affect the production of the L^t histories. Hence, point-wise maximisation on any other history simple optimises its first line, which represents the surplus from production, and gives the first-best level of effort. In contrast, for every t the production relevant payoff that corresponds to the L^t history is

$$\Pr(L^t) f_0^{t-1}(L^{t-1})\gamma^t\delta^t\left\{\theta_L e_t^a(L^t) - \frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}} - \frac{\mu_0(\varphi_H-\varphi_L)^t}{\Pr(L^t)}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right)\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right\} = \Pr(L^t) f_0^{t-1}(L^{t-1})\gamma^t\delta^t\left\{\theta_L e_t^a(L^t) - \xi_t\frac{e_t^a(L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right\}$$

the point-wise maximisation of which gives θ_L/ξ_t. Hence substitute the derived effort in the value functions of ($\mathcal{P}_c$) to obtain

$$\begin{aligned}
&\frac{(\theta_H)^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right) - \frac{e_t^*(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right) + \left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right)\\
&\quad\times\sum_{t'=t+1}^{\infty}\left[(\varphi_H-\varphi_L)\gamma\delta\right]^{t'-t}\left(f_t^{t'-1}(\theta_H^t L_{t+1}^{t'-1})\frac{(\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}} - f_t^{t'-1}(\theta_L^t L_{t+1}^{t'-1})\frac{e_{t'}^*(\theta_L^t L_{t+1}^{t'})^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\right)\\
&\quad\geq\ \delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\left[(\varphi_H-\varphi_L)\delta\right]^{t'-t}\Big\{\Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_L^t L_{t+1}^{t'})\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_L^t L_{t+1}^{t'}\big]\\
&\qquad\qquad - \Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_H^t L_{t+1}^{t'})\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_H^t L_{t+1}^{t'}\big]\Big\}
\end{aligned}\tag{B.3.4}$$

Note that $(\theta_L)^\epsilon \geq e_t^*(\theta_L^t)$, hence the second line of (B.3.4) is non-negative under a non-decreasing termination policy. Hence, its left hand side is bigger than

$$\frac{(\theta_H)^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right) - \frac{e_t^*(\theta_L^t)^{1+\frac{1}{\epsilon}}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right)\ \geq\ \frac{\theta_H^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(\frac{\theta_H^{1+\frac{1}{\epsilon}}}{\theta_L^{1+\frac{1}{\epsilon}}}-1\right) - \frac{\theta_L^{1+\epsilon}}{1+\frac{1}{\epsilon}}\left(1-\frac{\theta_L^{1+\frac{1}{\epsilon}}}{\theta_H^{1+\frac{1}{\epsilon}}}\right).$$

In addition, the right hand side of (B.3.4) is smaller than

$$\begin{aligned}
&\delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\left[(\varphi_H-\varphi_L)\delta\right]^{t'-t}\Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_L^t L_{t+1}^{t'})\mathbb{E}_g\big[B(\beta_{t'}^s)\,|\,\theta_L^t L_{t+1}^{t'}\big]\\
&\quad\geq\ \delta(\rho_H-\rho_L)\sum_{t'=t}^{\infty}\left[(\varphi_H-\varphi_L)\delta\right]^{t'-t}\Pr(\tau_a=t'\,|\,\tau_a>t-1,\theta_L^t L_{t+1}^{t'})B(0)\\
&\quad\geq\ \delta(\rho_H-\rho_L)B(0)\ =\ \delta(\rho_H-\rho_L)\frac{1-\kappa}{1-\delta}\frac{(b\theta_L)^{1+\epsilon}}{1+\frac{1}{\epsilon}},
\end{aligned}$$

where the first inequality follows from noting that $B(\cdot)$ is decreasing, and the second because getting $B(0)$ in period t+1 for sure is better than any other distribution for τ. Hence combining the two equations together gives the first sufficient condition. For $\epsilon = 1$, this becomes

$$\frac{(\theta_H^2 - \theta_L^2)^2}{\theta_H^2 \theta_L^2} \;\geq\; \frac{\delta\, b^s}{1-\delta}(\rho_H - \rho_L)(\theta_H^2 - \theta_L^2)\frac{\theta_L^2}{\theta_H^2},$$

which after substituting $\kappa = \theta_L^2/\theta_H^2$ gives the derived sufficient condition. □

Proof of Lemma 2.3.3. The time subscripts are suppressed. Differentiating and rearranging gives that

$$\begin{aligned} B'(\beta) \;&=\; K(1+\epsilon)\left(\frac{1-\beta}{1-\beta\kappa}\right)^{\epsilon} \frac{-(1-\beta\kappa)+\kappa(1-\beta)}{(1-\beta\kappa)^2} \\ &= -K(1+\epsilon)(1-\kappa)\frac{(1-\beta)^{\epsilon}}{(1-\beta\kappa)^{\epsilon+2}} = -B(\beta)\,\frac{(1+\epsilon)(1-\kappa)}{(1-\beta)(1-\beta\kappa)} < 0. \end{aligned}$$

Also,

$$\begin{aligned} B''(\beta) \;&=\; K(1+\epsilon)(1-\kappa)\frac{(1-\beta)^{\epsilon-1}}{(1-\beta\kappa)^{\epsilon+3}}\Big(\epsilon(1-\beta\kappa) - (\epsilon+2)\kappa(1-\beta)\Big) \\ &= -B'(\beta)\,\frac{\epsilon(1-\beta\kappa)-(\epsilon+2)\kappa(1-\beta)}{(1-\beta)(1-\beta\kappa)}. \end{aligned}$$

As a result,

$$\frac{\partial J_t(\eta,\lambda)/\partial\eta}{\rho_H - \rho_L} \;=\; B'(\beta)(\beta - \psi\,\lambda) + B(\beta) \;=\; B(\beta)\left[1 - (\beta - \psi\,\lambda)\frac{(1+\epsilon)(1-\kappa)}{(1-\beta)(1-\beta\kappa)}\right],$$

and

$$\begin{aligned} \frac{\partial^2 J_t(\eta,\lambda)/\partial\eta^2}{(\rho_H-\rho_L)^2} \;&=\; B''(\beta)(\beta-\psi\,\lambda) + 2B'(\beta) \\ &= -B'(\beta)\left[\frac{\epsilon(1-\beta\kappa)-(\epsilon+2)\kappa(1-\beta)}{(1-\beta)(1-\beta\kappa)}(\beta-\psi\,\lambda) - 2\right]. \end{aligned}$$

Finally,

$$\frac{\partial J_t(\eta,\lambda)}{\partial\lambda} \;=\; -\psi B(\beta), \qquad \frac{\partial^2 J_t(\eta,\lambda)}{\partial\lambda\partial\eta} \;=\; -\psi(\rho_H-\rho_L)B'(\beta), \qquad \frac{\partial^2 J_t(\eta,\lambda)}{\partial\lambda^2} \;=\; 0.$$

To prove that $J_t(\eta,\lambda)$ is neither concave nor convex on any of its interior points it suffices to show that its Hessian matrix is indefinite. This is given by

$$D^2 J_t(\eta,\lambda) \;=\; \begin{pmatrix} \frac{\partial^2 J}{\partial\eta^2} & \frac{\partial^2 J}{\partial\eta\partial\lambda} \\ \cdot & \frac{\partial^2 J}{\partial\lambda^2} \end{pmatrix}$$

Hence, its determinant is $|D^2J| = -\left(\frac{\partial^2 J}{\partial\eta\partial\lambda}\right)^2 < 0$, from which it follows that it is indefinite.

To prove the statements for the boundaries $(\eta + \lambda = 1)$ and $(\lambda = 0)$, define the following linear combination of (η', λ') and (η'', λ''), for $w \in [0,1]$ and $\eta'' \neq \eta'$,

$$\begin{pmatrix} \bar{\eta} \\ \bar{\lambda} \end{pmatrix} = (1-w)\begin{pmatrix} \eta' \\ \lambda' \end{pmatrix} + w\begin{pmatrix} \eta'' \\ \lambda'' \end{pmatrix}.$$

This implies that

$$\begin{aligned} w = \frac{\bar{\eta}-\eta'}{\eta''-\eta'} \Rightarrow \bar{\lambda} &= \frac{\lambda''-\lambda'}{\eta''-\eta'}(\bar{\eta}-\eta') + \lambda' = \frac{\lambda''-\lambda'}{\eta''-\eta'}\left(\frac{\bar{b}-\rho_L}{\rho_H-\rho_L} - \eta'\right) + \lambda' \\ \Rightarrow (\bar{\beta} - \psi\,\bar{\lambda}) &= \bar{\beta}\underbrace{\left(1 - \frac{\psi}{\rho_H-\rho_L}\frac{\lambda''-\lambda'}{\eta''-\eta'}\right)}_{=\zeta} - \psi\underbrace{\left[\lambda' - \frac{\lambda''-\lambda'}{\eta''-\eta'}\left(\frac{\rho_L}{\rho_H-\rho_L} + \eta'\right)\right]}_{=\zeta'} \end{aligned}$$

Let $\Psi = \zeta'/\zeta$ and note that for the subsets $(\lambda = 0)$ and $(\eta + \lambda = 1)$

$$\begin{aligned} \begin{pmatrix} \eta' & \eta'' \\ \lambda' & \lambda'' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} \zeta_0 \\ \zeta_1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 + \frac{\psi}{\rho_H-\rho_L} \\ 1 + \frac{\rho_L}{\rho_H-\rho_L} \end{pmatrix} \\ \Psi &= 0, \quad \frac{\psi\rho_H}{\rho_H-\rho_L+\psi}, \end{aligned}$$

respectively. In addition, substituting $\psi = \frac{\mu_0}{1-\mu_0}\left(\frac{\varphi_H-\varphi_L}{1-\varphi_L}\right)^t(\rho_H-\rho_L)$ I get that

$$\frac{\psi\rho_H}{\rho_H-\rho_L+\psi} = \frac{\mu_0\rho_H}{(1-\mu_0)\left(\frac{\varphi_H-\varphi_L}{1-\varphi_L}\right)^{-t} + \mu_0} \leq \mu_0\rho_H < 1.$$

Now the characterisation of $J_t(\eta, \lambda)$ on its boundaries can be obtained. Note that by moving $w \in [0,1]$ I essentially move $J_t(\eta, \lambda)$ on the two specified sides. Moreover, because η is a linear transformation of w its monotonicity and concavity changes at the same values of η, and as a result of β.

$$\begin{aligned} \bar{J}(w) = J\left[\bar{\eta}(w), \bar{\lambda}(w)\right] = \zeta_0 B(\bar{\beta})(\bar{\beta} - \Psi), \quad \textit{where} \quad \bar{\beta} &= \bar{\eta}(\rho_H-\rho_L) + \rho_L \\ \text{and} \quad \bar{\eta} &= w(\eta''-\eta') + \eta'. \end{aligned}$$

Then algebra similar to that used to simplify the partial derivatives of $J_t(\eta, \lambda)$ implies that

$$\begin{aligned}
\bar{J}'(w) &= (\rho_H - \rho_L)(\eta'' - \eta')\zeta_0 B(\bar{\beta}) \left(1 - (\bar{\beta} - \Psi)\,\frac{1+\epsilon}{(1-\bar{\beta})(1-\bar{\beta}\kappa)}\right) \\
\bar{J}''(w) &= (\rho_H - \rho_L)^2(\eta'' - \eta')^2\zeta_0[-B'(\bar{\beta})] \\
&\quad \times \left(\frac{\epsilon(1-\bar{\beta}\kappa) - (\epsilon+2)\kappa(1-\bar{\beta}))}{(1-\bar{\beta})(1-\bar{\beta}\kappa)}\,(\bar{\beta} - \Psi) - 2\right)
\end{aligned}$$

First, note that $\bar{J}'(w) \geq 0$ if and only if

$$(1-\bar{\beta})(1-\bar{\beta}\kappa) \;\geq\; (\bar{\beta}-\Psi)(1+\epsilon)(1-\kappa) \;\Leftrightarrow\; \big[1+\Psi(1+\epsilon)(1+\kappa)\big] - \big[2+\epsilon(1-\kappa)\big]\bar{\beta} + \kappa\bar{\beta}^2 \geq 0,$$

solving for the roots of the left hand side gives

$$\frac{2+\epsilon(1-\kappa) \pm \sqrt{[2+\epsilon(1-\kappa)]^2 - 4\kappa - 4\kappa(1-\kappa)(1+\epsilon)\Psi}}{2\kappa}.$$

The roots are not necessarily real numbers, but when they are the (+) one is always greater than one as $\frac{2+\epsilon(1-\kappa)}{2\kappa} > \frac{1}{\kappa} \geq 1$. Also, it is ease to show that the (−) one is positive. Simplifying we get that

$$\bar{J}'(w) \geq 0 \Leftrightarrow \begin{cases} \forall\ \bar{\beta} \in [0,1] & \text{, if } \Psi \geq \frac{1}{\kappa} + \frac{\epsilon^2(1-\kappa)}{4(1+\epsilon)\kappa} \\ \bar{\beta} \leq \beta^*(\Psi) & \text{, if } \Psi \leq \frac{1}{\kappa} + \frac{\epsilon^2(1-\kappa)}{4(1+\epsilon)\kappa} \end{cases}.$$

However for $\Psi \in [0,1)$, which is the case here, only the second line is relevant. Similarly, I get that $\bar{J}'(w) \geq 0$ if and only if

$$\begin{gathered}
2(1-\bar{\beta})(1-\bar{\beta}\kappa) \;\leq\; \left\{[\epsilon(1-\kappa) - 2\kappa] + 2\kappa\bar{\beta}\right\}(\bar{\beta} - \Psi) \Leftrightarrow \\
2 - 2(1+\kappa)\bar{\beta} + 2\kappa\bar{\beta}^2 \leq [\epsilon(1-\kappa) - 2\kappa](\bar{\beta} - \Psi) + 2\kappa\bar{\beta}^2 - 2\kappa\Psi\bar{\beta} \Leftrightarrow \\
2(1-\Psi\kappa) + (1-\kappa)\Psi\epsilon \;\leq\; \bar{\beta}\Big[2(1-\Psi\kappa) + (1-\kappa)\epsilon\Big]
\end{gathered}$$

where the expression in the brackets of the right hand side is positive for $\Psi \in [0,1)$, hence this equivalently becomes

$$\bar{\beta} \;\geq\; \frac{2(1-\Psi\kappa) + (1-\kappa)\Psi\epsilon}{2(1-\Psi\kappa) + (1-\kappa)\epsilon},$$

where again for $\Psi \in [0,1)$ the right hand side is positive and smaller than one. In addition, note that the because $\bar{J}(w)$ is initially increasing it has a maximum on $\beta^*(\Psi)$, but this implies that the function is concave there, hence $\beta^*(\Psi) \leq \beta^{**}(\Psi) < 1$. □

B.4 Proofs for Endogenous Termination

The main variables are

$$f_t \;=\; \mu_t x_t^H + (1-\mu_t)x_t^l\,, \qquad \eta_t \;=\; \frac{\mu_t(1-\gamma x_t^H)}{1-\gamma f_t}\,, \qquad \mu_{t+1} \;=\; \frac{\mu_t x_t^H + (1-\mu_t)x_t^l\varphi_L}{f_t}\,.$$

The following derivatives will be repeatedly used in the subsequent analysis. Derivatives of f_t:

$$\frac{\partial f_t}{\partial \mu_t} \;=\; x_t^H - x_t^l\,, \qquad \frac{\partial f_t}{\partial x_t^l} \;=\; 1-\mu_t\,, \qquad \frac{\partial f_t}{\partial x_t^H} \;=\; \mu_t\,.$$

Derivatives of η_t:

$$\frac{\partial \eta_t}{\partial \mu_t} \;=\; \frac{\eta_t^2}{\mu_t^2}\frac{1-\gamma x_t^l}{1-\gamma x_t^H} \;=\; \frac{\eta_t(1-\eta_t)}{\mu_t(1-\mu_t)}\,, \qquad \frac{\partial \eta_t}{\partial x_t^l} \;=\; \frac{\gamma(1-\mu_t)}{1-\gamma f_t}\eta_t\,, \qquad \frac{\partial \eta_t}{\partial x_t^H} \;=\; -\frac{\gamma\mu_t}{1-\gamma f_t}(1-\eta_t)\,.$$

Derivatives of μ_{t+1}:

$$\frac{\partial \mu_{t+1}}{\partial \mu_t} \;=\; (1-\varphi_L)\frac{x_t^H\,x_t^l}{f_t^2}\,, \qquad \frac{\partial \mu_{t+1}}{\partial x_t^l} \;=\; -(1-\varphi_L)\frac{(1-\mu_t)\mu_t}{f_t^2}x_t^H\,, \qquad \frac{\partial \mu_{t+1}}{\partial x_t^H} \;=\; \frac{\mu_t}{f_t}(1-\mu_{t+1})\,.$$

As argued in the main text, P_a's problem has the following recursive representation

$$\begin{aligned} V_t(\mu_t) \;=\; \max_{x_t^H,x_t^l}\; \Big\{ & \mu_t u_H + (1-\mu_t)u_l + \delta\gamma f_t V_{t+1}(\mu_{t+1}) + \delta(1-\gamma f_t)\mathcal{J}_0(\eta_t) \\ & + h_t^1(1-x_t^H) + h_t^0 x_t^H + l_t^1(1-x_t^l) + l_t^0 x_t^l \Big\}\,, \end{aligned} \qquad \text{(B.4.1)}$$

where h_t^1 and h_t^0 are the Lagrange multipliers for $0 \le x_t^H \le 1$, and l_t^1, l_t^0 are the corresponding multipliers for $0 \le x_t^l \le 1$. Let $v_t(x_t^H, x_t^l, \mu_t)$ denote the first line of the above objective function, that is the expression in the parenthesis, but without the constrains. The problem is solved under the following generic assumption.

Assumption. $\mathcal{J}_0$ is twice continuously differentiable and concave. Both u_H and u_l are positive. Also,

$$u_H \;>\; (1-\delta)\mathcal{J}_0(1) \qquad \text{and} \qquad \frac{u_H}{1-\delta\gamma} + \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1) \;>\; \mathcal{J}_0(0) + \mathcal{J}_0'(0). \qquad \text{(B.4.2)}$$

The proof proceeds as follows. First, it is shown that $x_t^H = 1$ is optimal under the supposition that V_{t+1} is twice continuously differentiable and concave. Second, it is shown that in this case V_t is also twice continuously differentiable and concave. Third, the contraction mapping theorem is used to show that $V_t = V_{t+1} = V$, and that V is indeed concave and twice continuously differentiable. Forth, some sufficient conditions are provided for each of the three possible solutions for x_t^t (interior, $x_t^l = 0$, and $x_t^l = 1$) to be relevant.

There is actually one value on which the functional form of V_t and V_{t+1} can easily be derived, which is when the agent's current reputation is one.

Lemma B.4.1. *For $\mu_t = 1$ it is strictly optimal to continue, $x_t^H(1) = 1$. Moreover,*

$$V(1) \;=\; \frac{u_H}{1-\delta\gamma} \;+\; \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1)\,. \tag{B.4.3}$$

Proof. For $\mu_t = 1$ the principal's dynamic problem simplifies to

$$V(1) \;=\; \max_{x_t^H} \;\; u_H + \delta\gamma x_t^H V(1) + \delta(1-\gamma x_t^H)\mathcal{J}_0(1).$$

The payoff from always continuing $x_t^H = 1$ is

$$\frac{u_H}{1-\delta\gamma} \;+\; \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1).$$

On the other hand, that of stopping $x_t^H = 0$ is $\mathcal{J}_0(1)$. Hence, no-stopping is proffered to stopping when

$$\frac{u_H}{1-\delta\gamma} \;+\; \frac{\delta(1-\gamma)}{1-\delta\gamma}\,\mathcal{J}_0(1) \;>\; \mathcal{J}_0(1) \quad\Leftrightarrow\quad u_H \;>\; (1-\delta)\mathcal{J}_0(1),$$

which has been assumed to hold. □

This allows the derivation of the following result.

Lemma B.4.2. *Suppose that $V_{t+1}(\mu_{t+1})$ is twice continuously differentiable and concave, then always continuing a high type, $x_t^H(\mu_t) = 1$, is strictly optimal for every $\mu_t \in (0,1]$.*

Proof. Differentiating gives

$$\begin{aligned}\frac{\partial v_t}{\partial x_t^H} \;&=\; \delta\gamma\mu_t V_{t+1}(\mu_{t+1}) + \delta\gamma\mu_t(1-\mu_{t+1})V'_{t+1}(\mu_{t+1}) - \delta\gamma\mu_t\mathcal{J}_0(\eta_t) - \delta\gamma\mu_t(1-\eta_t)\mathcal{J}'_0(\eta_t)\\ &=\; \delta\gamma\mu_t\Big[\,V_{t+1}(\mu_{t+1}) + (1-\mu_{t+1})V'_{t+1}(\mu_{t+1}) - \mathcal{J}_0(\eta_t) - (1-\eta_t)\mathcal{J}'_0(\eta_t)\,\Big]\end{aligned}$$

But note that

$$\frac{\partial}{\partial \mu_{t+1}}\Big[V_{t+1}(\mu_{t+1}) + (1-\mu_{t+1})V'_{t+1}(\mu_{t+1})\Big] \;=\; (1-\mu_{t+1})V''_{t+1}(\mu_{t+1}) \;\leq\; 0$$
$$\Rightarrow \quad V_{t+1}(\mu_{t+1}) + (1-\mu_{t+1})V'_{t+1}(\mu_{t+1}) \;\geq\; V_{t+1}(1)$$

Moreover,

$$\frac{\partial}{\partial \eta_t}\Big[-\mathcal{J}_0(\eta_t) - (1-\eta_t)\mathcal{J}'_0(\eta_t)\Big] \;=\; -(1-\eta_t)\mathcal{J}''_0(\eta_t) \;\geq\; 0$$
$$\Rightarrow \quad -\mathcal{J}_0(\eta_t) - (1-\eta_t)\mathcal{J}'_0(\eta_t) \;\geq\; -\mathcal{J}_0(0) - \mathcal{J}'_0(0).$$

As a result,

$$\frac{\partial v_t}{\partial x_t^H} \;\geq\; \delta\gamma\mu_t\Big(V_{t+1}(1) - \mathcal{J}_0(0) - \mathcal{J}'_0(0)\Big) \;>\; 0,$$

because $V_{t+1}(1) = V(1)$, as identified in the previous lemma, which implies that the derivative $\partial v_t/\partial x_t^H$ has to be strictly positive. This in turn implies that it is always strictly optimal to set $x_t^H = 1$. □

As a result (B.4.1) simplifies to

$$V_t(\mu_t) \;=\; \max_{x_t^l}\; \Big\{\mu_t u_H + (1-\mu_t)u_l + \delta\gamma f_t V_{t+1}(\mu_{t+1}) + \delta(1-\gamma f_t)\mathcal{J}_0(\eta_t) + l_t^1(1-x_t^l) + l_t^0 x_t^l\Big\} \quad \text{(B.4.4)}$$

where

$$f_t \;=\; \mu_t + (1-\mu_t)x_t^l\,, \quad \eta_t \;=\; \frac{\mu_t(1-\gamma)}{1-\gamma f_t}\,, \quad \mu_{t+1} \;=\; \frac{\mu_t + (1-\mu_t)x_t^l\varphi_L}{f_t}\,. \quad \text{(B.4.5)}$$

Differentiating with respect to x_t^l gives

$$\frac{\partial v_t}{\partial x_t^l} \;=\; \delta\gamma(1-\mu_t)\bigg(V_{t+1}(\mu_{t+1}) - (1-\varphi_L)\frac{\mu_t}{f_t}V'_{t+1}(\mu_{t+1}) - \mathcal{J}_0(\eta_t) + \eta_t\mathcal{J}'_0(\eta_t)\bigg) \;=\; l_t^1 - l_t^0\,,$$
$$\frac{\partial^2 v_t}{\partial (x_t^l)^2} \;=\; \delta\gamma(1-\mu_t)\bigg((1-\varphi_L)^2\frac{\mu_t^2(1-\mu_t)}{f_t^2}\,V''_{t+1}(\mu_{t+1}) + \frac{\gamma(1-\mu_t)}{1-\gamma f_t}\,\eta_t^2\mathcal{J}''_0(\eta_t)\bigg) \;\leq\; 0\,. \quad \text{(B.4.6)}$$

Moreover, it follows from the Envelop Theorem that

$$\begin{aligned} V_t'(\mu_t) \;=\;& u_H - u_l + \delta\gamma(1-x_t^l)V_{t+1}(\mu_{t+1}) + \delta\gamma(1-\varphi_L)\frac{x_t^l}{f_t}V_{t+1}'(\mu_{t+1}) \\ & - \delta\gamma(1-x_t^l)\mathcal{J}_0(\eta_t) + \delta(1-\gamma x_t^l)\frac{\eta_t}{\mu_t}\mathcal{J}_0'(\eta_t)\,. \end{aligned} \tag{B.4.7}$$

Substitute the first order condition from (B.4.6) in (B.4.7) to obtain

$$\begin{aligned} V_t'(\mu_t)\mu_t - V_t(\mu_t) \;=\;& -u_l - \delta\gamma x_t^l V_{t+1}(\mu_{t+1}) - \delta(1-x_t^l\gamma)\mathcal{J}_0(\eta_t) \\ & + \delta\gamma(1-\varphi_L)\frac{\mu_t x_t^l}{f_t}V_{t+1}'(\mu_{t+1}) + \delta(1-\gamma x_t^l)\eta_t\mathcal{J}_0'(\eta_t) \quad \Rightarrow \\ V_t'(\mu_t)\mu_t - V_t(\mu_t) \;=\;& -u_l + x_t^l\,\frac{l_t^0 - l_t^1}{1-\mu_t} - \delta\mathcal{J}_0(\eta_t) + \delta\eta_t\mathcal{J}_0'(\eta_t)\,. \end{aligned} \tag{B.4.8}$$

Lemma B.4.3. *Suppose that $V_{t+1}(\mu_{t+1})$ is twice continuously differentiable and concave, then the same is true for $V_t(\mu_t)$. Moreover,*

$$\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \geq 0 \quad \textit{and} \quad \frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \geq 0.$$

Proof. The first statement of the lemma follows trivially. The rest of the proof focuses on proving concavity and the two derivatives. First, suppose that the non-negative constrain binds so that

$$l_t^0 > 0 \Rightarrow \begin{cases} x_t^l = 0 \\ l_t^1 = 0 \end{cases},$$

then total differentiation, with respect to μ_t, on the last line of (B.4.8) gives

$$V_t''(\mu_t)\mu_t \;=\; -(u_H - u_l) + \delta\mathcal{J}_0''(\eta_t)\eta_t\frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \leq 0. \tag{B.4.9}$$

This is because for $x_t^l = 0$,

$$\eta_t \;=\; \frac{\mu_t(1-\gamma)}{1-\gamma\mu_t} \;\Rightarrow\; \frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \;=\; (1-\gamma)\frac{1+\gamma\mu_t}{(1-\gamma\mu_t)^2} \;>\; 0.$$

Second, suppose that $x_t^l \in (0,1)$. Total differentiation on (B.4.8) gives (B.4.9) again, but now $x_t^l(\mu_t)$ is not known so the derivative of $\eta_t(\mu_t)$ cannot be calculated. Instead use total differentiation, with respect to μ_t, on the foc of x_t^l on (B.4.6), which for an interior solution gives

$$(1-\varphi_L)\frac{\mu_t}{f_t}V_{t+1}''(\mu_{t+1})\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \;=\; \mathcal{J}_0''(\eta_t)\eta_t\frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t}, \tag{B.4.10}$$

hence for this case it suffices to show that $\mathrm{d}\mu_{t+1}/\mathrm{d}\mu_t \geq 0$. This will imply that the left hand side of (B.4.10) is negative, which in turn will give the same for the right hand side, from which it will also follow that $\mathrm{d}\eta_t/\mathrm{d}\mu_t \geq 0$. Simple differentiation on the function form of μ_{t+1} gives

$$\mu_{t+1} \;=\; \frac{\mu_t + (1-\mu_t)x_t^l\varphi_L}{\mu_t + (1-\mu_t)x_t^l} \quad\Rightarrow\quad \frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \;=\; \frac{1-\varphi_L}{f_t^2}\left(x_t^l - \mu_t(1-\mu_t)\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t}\right).$$

For an interior solution, the derivative $\mathrm{d}x_t^l/\mathrm{d}\mu_t$ can also be derived by using the implicit function theorem on the foc of (B.4.6). It follows immediately from this that if $V''_{t+1}(\mu_{t+1}) = 0$, then $\mathrm{d}x_t^l/\mathrm{d}\mu_t \leq 0$, which in turn implies $\mathrm{d}\mu_{t+1}/\mathrm{d}\mu_t \geq 0$. Instead suppose that $V''_{t+1}(\mu_{t+1}) < 0$, then the implicit function theorem gives that

$$-\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t} \;=\; \left[-(1-\varphi_L)^2\frac{\mu_t x_t^l}{f_t^3} + \frac{\eta_t^2(1-\eta_t)}{\mu_t(1-\mu_t)}\frac{\mathcal{J}_0''(\eta_t)}{\delta V''_{t+1}(\mu_{t+1})}\right] \Bigg/ \left[(1-\varphi_L)^2\frac{\mu_t^2(1-\mu_t)}{f_t^3} + \frac{\gamma(1-\mu_t)\eta_t^2}{1-\gamma f_t}\frac{\mathcal{J}_0''(\eta_t)}{\delta V''_{t+1}(\mu_{t+1})}\right].$$

If $-\mathrm{d}x_t^l/\mathrm{d}\mu_t \geq 0$, then $\mathrm{d}\mu_{t+1}/\mathrm{d}\mu_t \geq 0$ follows immediately. Instead suppose that it is negative, in which case the following lower bound can be derived

$$-\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t} \;<\; 0 \quad\Rightarrow\quad -\frac{\mathrm{d}x_t^l}{\mathrm{d}\mu_t} \;\geq\; \frac{-(1-\varphi_L)^2\mu_t x_t^l/f_t^3}{(1-\varphi_L)^2\mu_t^2(1-\mu_t)/f_t^3} \;=\; -\frac{x_t^l}{\mu_t(1-\mu_t)}.$$

Hence, for $V''_{t+1}(\mu_{t+1}) < 0$ and $-\mathrm{d}x_t^l/\mathrm{d}\mu_t < 0$ I have that

$$\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \;\geq\; \frac{1-\varphi_L}{f_t^2}\left(x_t^l - \mu_t(1-\mu_t)\frac{x_t^l}{\mu_t(1-\mu_t)}\right) \;=\; 0\,,$$

which proves the concavity of $V_{t+1}(\mu_t)$ for interior solutions and also gives that $\mathrm{d}\eta_t/\mathrm{d}\mu_t \geq 0$.

Third, suppose that the constrain of the upper bound binds, then

$$l_t^1 > 0 \Rightarrow \begin{cases} x_t^l = 1 \\ l_t^0 = 0 \end{cases} \Rightarrow \begin{cases} \mu_{t+1} = \mu_t + (1-\mu_t)\varphi_L \\ \eta_t = \mu_t \end{cases}.$$

Substituting the above in $V_t'(\mu_t)$, as it is given in (B.4.7), gives

$$\begin{aligned} V_t'(\mu_t) \;&=\; u_H - u_l + \delta\gamma(1-\varphi_L)V'_{t+1}(\mu_{t+1}) + \delta(1-\gamma)\mathcal{J}_0'(\mu_t) \\ \Rightarrow\; V''(\mu_t) \;&=\; \delta\gamma(1-\varphi_L)^2 V''_{t+1}(\mu_{t+1}) + \delta(1-\gamma)\mathcal{J}_0''(\mu_t) \;\leq\; 0, \end{aligned}$$

which proves that $V_t(\mu_t)$ is concave in all three possible solutions. $\square$

Proposition B.4.1. *The recursive representation* $V(\mu_t)$ *exists and it is unique. Moreover,* $V(\mu_t)$ *is twice continuously differentiable and concave. Finally,*

$$\frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \geq 0 \quad \textit{and} \quad \frac{\mathrm{d}\eta_t}{\mathrm{d}\mu_t} \geq 0.$$

Proof. Let $\mathcal{V}\big([0,1]\big)$ denote the set of bounded, twice continuously differentiable, and concave functions such that $V:[0,1]\to\mathbb{R}$, and consider the operator $T:\mathcal{V}\big([0,1]\big)\to\mathcal{V}\big([0,1]\big)$, given by

$$\begin{aligned} T(V) \;&=\; \Big\{\mu u_H + (1-\mu)u_l + \delta\gamma f V(\mu^*) + \delta(1-\gamma f^*)\mathcal{J}_0(\eta^*)\Big\} \\ \text{where} \quad f^* \;&=\; \mu + (1-\mu)x^*\,, \quad \eta^* \;=\; \frac{\mu(1-\gamma)}{1-\gamma f^*}\,, \quad \mu^* \;=\; \frac{\mu + (1-\mu)x^*\varphi_L}{f^*} \end{aligned}$$

where x^* is the same as the solution for x_t^l derived above when the continuation value was $V_{t+1}(\cdot)$. Then both of Blackwell's sufficient conditions are satisfied. Hence $T(V)$ is a contraction, which implies that it has a unique solution in $\mathcal{V}\big([0,1]\big)$, which is a solution to P_a problem. Finally, because the solution is in $\mathcal{V}\big([0,1]\big)$ all the previous results derived for concave $V_{t+1}(\cdot)$ hold. $\square$

Next, some results are provided for the optimal x_t^l. As a shorthand let $V'(1) = \lim\limits_{\mu_t \Rightarrow 1} V'(\mu_t)$.

Lemma B.4.4.

$$\begin{aligned} V(1) - (1-\varphi_L)V'(1) \;&=\; \frac{1}{1-\delta\gamma(1-\varphi_L)x_t^l(1)}\Bigg(\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l \\ &+ \delta\bigg[(1-\gamma)\frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma} + \gamma(1-\varphi_L)(1-x^l(1))\bigg]\mathcal{J}_0(1) - \delta(1-\varphi_L)[1-\gamma x_t^l(1)]\mathcal{J}_0'(1)\Bigg) \end{aligned} \tag{B.4.11}$$

Proof. Note that $x^l(\mu_t)$ has to be continuous, as the objective function is twice differentiable in its domain. Then substituting in (B.4.7) gives

$$\begin{aligned} V'(1) \;=\; u_H - u_l + \delta\gamma(1-x^l(1))V(1) + \;\delta\gamma(1-\varphi_L)x^l(1)V'(1) \\ - \delta\gamma(1-x^l(1))\mathcal{J}_0(1) + \delta(1-\gamma x^l(1))\mathcal{J}_0'(1)\,, \end{aligned}$$

which can be rearranged to get

$$V'(1) = \frac{1}{1-\delta\gamma(1-\varphi_L)x^l(1)}\Big(u_H - u_l + \delta\gamma(1-x^l(1))V(1) - \delta\gamma(1-x^l(1))\mathcal{J}_0(1) + \delta(1-\gamma x^l(1))\mathcal{J}_0'(1)\Big).$$

Hence substitute the above expression of $V'(1)$ in the right hand side of (B.4.11) to obtain

$$V(1) - (1-\varphi_L)V'(1) = \frac{-(1-\varphi_L)}{1-\delta\gamma(1-\varphi_L)x^l(1)} \times \Big(u_H - u_l - \delta\gamma(1-x^l(1))\mathcal{J}_0(1) + \delta(1-\gamma x^l(1))\mathcal{J}_0'(1)\Big) + \frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma(1-\varphi_L)x^l(1)}V(1).$$

Finally substitute the functional form of $V(1)$, provided in (B.4.3), and gather terms to obtain (B.4.11). □

Lemma B.4.5. *Continuing a low type is strictly optimal for $\mu_t \to 1$, if*

$$\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l - \frac{1-\delta}{1-\delta\gamma}\big[1-\delta\gamma(1-\varphi_L)\big]\mathcal{J}_0(1) + \big[1-\delta(1-\varphi_L)\big]\mathcal{J}_0'(1) \;\geq\; 0,. \quad \text{(B.4.12)}$$

If the above inequality is reversed, then stopping is optimal. Finally, if it holds with equality then any $x^l \in [0,1]$ is a solution.

Proof. It follows from (B.4.6) that

$$\lim_{\mu_t \Rightarrow 1} \frac{\partial v}{\partial x_t^l}\frac{1/(\delta\gamma)}{1-\mu_t} = V(1) - (1-\varphi_L)V'(1) - \mathcal{J}_0(1) + \mathcal{J}_0'(1)$$

Hence, substituting the result of the previous lemma gives that

$$\lim_{\mu_t \Rightarrow 1} \frac{\partial v}{\partial x_t^l}\frac{1/(\delta\gamma)}{1-\mu_t} = \frac{1}{1-\delta\gamma(1-\varphi_L)x^l(1)} \times \left(\frac{\varphi_L\, u_H}{1-\delta\gamma} + (1-\varphi_L)u_l - \frac{1-\delta}{1-\delta\gamma}\big[1-\delta\gamma(1-\varphi_L)\big]\mathcal{J}_0(1) + \big[1-\delta(1-\varphi_L)\big]\mathcal{J}_0'(1)\right).$$

The statement follows from noting that the above is the marginal benefit from increasing x^l when $\mu_t \to 1$, the sign of which does not depend on x^l itself. □

Proposition B.4.2. *Having $x^l(\mu_t) = 0$ for all $\mu_t \in [0,1]$ is suboptimal iff (B.4.12) holds.*

In contrast, if it holds in the reversed direction and

$$\frac{\varphi_L u_H}{1-\delta\gamma}+(1-\varphi_L)u_l+\delta\left[(1-\gamma)\frac{1-\delta\gamma(1-\varphi_L)}{1-\delta\gamma}+\gamma(1-\varphi_L)\right]\mathcal{J}_0(1)-\delta(1-\varphi_L)\mathcal{J}_0'(1) \ \leq \ J_0(0)\,, \tag{B.4.13}$$

then $x_t^l(\mu_t)=0$ is optimal for all $\mu_t \in [0,1]$. Otherwise, there exists $\tilde{\mu}$ such that $x_t^l(\mu_t)=0$ is optimal iff $\mu_t > \tilde{\mu}$.

Proof. Suppose the lower bound binds, that is the low type is stopped, then

$$l_t^0 > 0 \Rightarrow \begin{cases} x_t^l = 0 \\ l_t^1 = 0 \end{cases},$$

hence the foc of (B.4.6) becomes

$$\frac{-l_t^0}{\delta\gamma(1-\mu_t)} = V(1)-(1-\varphi_L)V'(1)-\mathcal{J}_0(\eta)+\eta\mathcal{J}_0'(\eta), \quad \text{for} \quad \eta = \frac{\mu_t(1-\gamma)}{1-\gamma\mu_t}.$$

For this to be a solution it has to be that $l_t^0 \geq 0$, which holds if and only if

$$V(1)-(1-\varphi_L)V'(1) \ \leq \ \mathcal{J}_0(\eta)-\eta\mathcal{J}_0'(\eta). \tag{B.4.14}$$

Total differentiation on the right hand side of this inequality gives $-\eta\,\mathcal{J}_0''(\eta)\,\frac{\mathrm{d}\eta}{\mathrm{d}\mu_t}$, which is positive. Hence if this inequality is satisfied for any $\mu_t \in [0,1]$, then this is for a convex set $[\tilde{\mu},1]$. Finally, note that for $\mu_t = 1$, the above η becomes one, and (B.4.14) turns into the opposite of (B.4.12). Hence, if the former is satisfied, then the latter can never hold, which implies that stopping the low type with probability one is never optimal.

In contrast, if (B.4.14) holds for $\mu_t = 0$

$$V(1)-(1-\varphi_L)V'(1) \ \leq \ \mathcal{J}_0(0),$$

then stopping the low type with probability one is optimal for any $\mu_t \in [0,1]$. The left hand side of the above condition, given in (B.4.11), depends on $x^l(1)$. Despite that, Lemma B.4.5 gives that whenever (B.4.12) is not satisfied, then $x^l(1)=0$ from which the last statement follows. □

Appendix C

Appendixes of Chapter 3

C.1 Omitted Proofs

Proof of Lemma 3.1. Using (3.3) it is ease to argue that both idiosyncratic and index tracking strategies have to be played with positive probability. This is because the effect of the reputation $\varphi_\beta(\cdot)$ on the manager's payoff is bounded, whereas that of current return r is not. But this implies that $\varphi_0(\cdot)$ is calculated using Bayesian updating, and as a result it cannot be a function of r, since in this case r provides no information on the manager's ability α.

Fix s^m, then the manager's expected payoff while investing in an index tracking strategy $\beta = 1$ is not a function of s. On the other hand, her payoff under the idiosyncratic strategy is a function of r. In particular, it follows from the definition of monotonic equilibria that this is increasing in s, which proves that the manager's equilibrium strategy is a cut-off one, as presented in (3.4).

In addition, the indifference condition that defines $h(s^m)$ is

$$\begin{aligned}\mathbb{E}_r\Big[a \,+\, \delta\cdot\lambda\cdot[\varphi_0(r,s^m)\cdot(u^H-u^L) \,+\, u^L\,]\;\Big|\; s=h(s^m),\,\alpha=H\Big]\\ =\;\mathbb{E}_r\Big[m \,+\, \delta\cdot\lambda\cdot[\varphi_0(s^m)\cdot(u^H-u^L) \,+\, u^L\,]\;\Big|\; s^m\Big]\end{aligned}$$

while the one that defines $l(s^m)$ is

$$\begin{aligned}\mathbb{E}_r\Big[a \,+\, \delta\cdot\lambda\cdot[\varphi_0(r,s^m)\cdot(u^H-u^L) \,+\, u^L\,]\;\Big|\; s=l(s^m),\,\alpha=L\Big]\\ =\;\mathbb{E}_r\Big[m \,+\, \delta\cdot\lambda\cdot[\varphi_0(s^m)\cdot(u^H-u^L) \,+\, u^L\,]\;\Big|\; s^m\Big]\end{aligned}$$

But the right hand sides of the above two equations are the same, hence the two expressions on the right hand sides are equal. Therefore, the two conditional normals that are used in the two right hand sides have to be the same, which implies that

$$(1-\psi)\cdot H \,+\, \psi\cdot h(s^m) \;=\; (1-\psi)\cdot L \,+\, \psi\cdot l(s^m)$$

from which (3.5) follows. □

Proof of Lemma 3.2. The time subscripts is suppressed, when no ambiguity is created. The same is true for the signal s^m in the cutoffs $h(s^m)$ and $l(s^m)$. To find the posterior $\varphi_0(r)$ calculate

$$\mathbb{P}\left(r,\beta=0 \,\middle|\, s^m,H\right) \;=\; \mathbb{P}\left(r \,\middle|\, \beta=0,s^m,H\right)\times\mathbb{P}\left(\beta=0 \,\middle|\, s^m,H\right),$$

where

$$\mathbb{P}\left(\beta=0 \,\middle|\, s^m,H\right)=\mathbb{P}(s\geq h\,|\,s^m,H)=\Phi\left(-\frac{h-H}{\nu}\right), \tag{C.1.1}$$

and

$$\mathbb{P}\left(r\,|\,\beta=0,s^m,H\right)=\int_h^\infty \phi\left(\frac{r-(1-\psi)H-\psi s}{\sqrt{\psi}\,\nu}\right)\times\frac{1}{\sqrt{\psi}\,\nu}\phi\left(\frac{s-H}{\nu}\right)\frac{1/\nu}{\Phi\left(-\frac{h-H}{\nu}\right)}\,\mathrm{d}s$$

Hence, substituting gives that

$$\mathbb{P}\left(r,\beta=0\,|\,s^m,H\right)\;=\;\int_h^\infty \phi\left(\frac{r-(1-\psi)H-\psi s^i}{\sqrt{\psi}\,\nu}\right)\frac{\phi\left(\frac{s-H}{\nu}\right)}{\sqrt{\psi}\,\nu^2}\,\mathrm{d}s. \tag{C.1.2}$$

Let $\tilde{s}=(s-H)/\nu$, then the above becomes

$$\begin{aligned}
&\int_{\frac{h-H}{\nu}}^\infty \phi\left(\frac{r-H}{\sqrt{\psi}\,\nu}-\sqrt{\psi}\,\tilde{s}\right)\frac{\phi(\tilde{s})}{\sqrt{\psi}\,\nu}\,\mathrm{d}\tilde{s}\\
&=\frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}\int_{\frac{h-H}{\nu}}^\infty \phi\left(\frac{\tilde{s}-\frac{r-H}{\nu(1+\psi)}}{1/\sqrt{1+\psi}}\right)\sqrt{1+\psi}\,\mathrm{d}\tilde{s}\\
&=\frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}\Phi\left(\frac{r-h(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right).
\end{aligned} \tag{C.1.3}$$

Repeat the same process to find $\mathbb{P}\left(r\,|\,\beta=0,s^m,L\right)$ and observe that it follows from Bayes' rule that

$$\varphi_0(r)=\left(1+\frac{1-\pi}{\pi}\frac{\mathbb{P}\left(r,\beta=0\,|\,s^m,L\right)}{\mathbb{P}\left(r,\beta=0\,|\,s^m,H\right)}\right)^{-1}, \tag{C.1.4}$$

from which the provided formula follows. To derive φ_1 use Bayes' rule to get that

$$\varphi_1 = \left(1 + \frac{1-\pi}{\pi}\frac{\mathbb{P}\left(\beta = 1 \,|\, s^m, L\right)}{\mathbb{P}\left(\beta = 1 \,|\, s^m, H\right)}\right)^{-1}, \tag{C.1.5}$$

where $\mathbb{P}\left(\beta = 1 \,|\, s^m, \alpha\right) = 1 - \mathbb{P}\left(\beta = 0 \,|\, s^m, \alpha\right)$, which has been derived above. □

To prove our existence theorem we need to following three lemmas.

Lemma C.1.1. *If $M(\cdot)$ is the normal hazard function, then for $a \geq b$ we have,*

$$M(a) - M(b) \leq a - b \tag{C.1.6}$$

Proof. Since the hazard function is a continuous function, we can use the Mean Value Theorem, which says that for any $a > b$ there exists a $\xi \in (a, b)$ such that $M(a) - M(b) = M'(\xi)(a - b)$. Therefore, it is sufficient to prove that $M'(\xi) < 1$ for any ξ. To prove that, note that $M(\cdot)$ is convex, and hence $M'(\cdot)$ is increasing, so it would be sufficient to prove that $\lim_{x\to\infty} M'(x) = 1$. Now we use the following inequality for the normal hazard function. We know that for $x > 0$,

$$x < M(x) < x + \frac{1}{x} \tag{C.1.7}$$

But this easily implies that $M(x)$ has x as its asymptote as $x \to \infty$ (that is $\lim_{x\to\infty} M(x) - x = 0$). Finally this implies that $\lim_{x\to\infty} M'(x) = 1$ and this completes the proof (note the limit exists because $M'(\cdot)$ is increasing and bounded, as $M'(x) = M(x)(M(x) - x) < 1 + \frac{1}{x^2} < 2$). □

Lemma C.1.2. *The time subscripted is suppressed. A sufficient condition for $\varphi_0(r, s^m)$ to be increasing in the manager's performance r is that*

$$(H - L) \cdot \frac{1-\psi}{\psi} \;\geq\; l(s_1^m) - h(s_1^m). \tag{C.1.8}$$

Proof. Suppress inputs (r, s^m), and super/sub-scripts. Differentiating gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = -\frac{\varphi(1-\varphi)}{\nu\sqrt{1+\psi}}\left[-\frac{H-L}{\nu\psi\sqrt{1+\psi}} + M\left(-\frac{r - l(1+\psi) + L\psi}{\nu\sqrt{1+\psi}}\right) - M\left(-\frac{r - h(1+\psi) + H\psi}{\nu\sqrt{1+\psi}}\right)\right] \tag{C.1.9}$$

Let

$$\begin{aligned} \delta^L &= l(1+\psi) - L\psi \\ \delta^H &= h(1+\psi) - H\psi, \end{aligned} \tag{C.1.10}$$

then the above is positive if and only if

$$\frac{H-L}{\nu\psi\sqrt{1+\psi}} \geq M\left(\frac{\delta^L - r}{\nu\sqrt{1+\psi}}\right) - M\left(\frac{\delta^H - r}{\nu\sqrt{1+\psi}}\right) \tag{C.1.11}$$

But using Lemma C.1.1 we see that the right hand side is bounded above by

$$\frac{\delta^L - \delta^H}{\nu\sqrt{1+\psi}} = \frac{(l-h)(1+\psi) + (H-L)\psi}{\nu\sqrt{1+\psi}}. \tag{C.1.12}$$

Hence, a sufficient condition for the inequality to hold is that

$$\frac{H-L}{\psi} \geq (l-h)(1+\psi) + (H-L)\psi \;\Leftrightarrow\; (H-L)\frac{1-\psi}{\psi} \geq l-h. \tag{C.1.13}$$

□

Lemma C.1.3. *For $c > 0$, let*

$$\mu(x) = \left(1 + c\frac{\Phi(a_0 + b\,x)}{\Phi(a_1 + b\,x)}\right)^{-1}. \tag{C.1.14}$$

Suppose $b > 0$, then $\mu'(x) > 0 \Leftrightarrow a_1 < a_0$, whereas $b < 0$ implies that $\mu'(x) > 0 \Leftrightarrow a_1 > a_0$.

Proof. Differentiating gives

$$\mu'(x) = -b\mu(x)[1-\mu(x)] \times [M(-a_0 - b\,x) - M(-a_1 - b\,x)]\,.$$

Then the statement follows simple from the fact that $M(\cdot)$ is increasing . □

Proof of Proposition 3.1. Suppress time subscript t. Also suppress the signal s^m in the cutoffs $h(s^m)$ and $l(s^m)$, and in the reputations $\varphi_0(\cdot)$ and $\varphi_1(\cdot)$.

We start by proving existence. As we have argued in Lemma 3.1, in any monotonic equilibrium the optimal strategy of a high and low type manager is to pick $\beta = 0$ whenever her signal s is above the cutoffs h and l, respectively. In addition, another necessary implication is that h and l satisfy (3.5).

But then Lemma C.1.2 together with (3.5) give that $\varphi_0(r)$ is indeed increasing in r. Hence, the manager's best response to the functional forms of $\varphi_0(\cdot)$ and φ_1 as given Lemma 3.2 is to indeed use the cutoff strategies that Lemma 3.1 describes.

All that remains to prove existence is to shown that those cutoffs always exists. To do this note that the manager's payoff maximisation problem when picking the first period's beta is as given in (3.3). Let her expected payoff when picking $\beta = 0$ be denoted by

$$v_0(s,\alpha) \;=\; (1-\psi)\cdot\alpha + \psi\cdot s \,+\, \delta\cdot\lambda\cdot\mathbb{E}_r\left[\log\left(\varphi_0(r)(u^H - u^L) + u^L\right)\,\Big|\, s,\alpha\right],$$

whereas for $\beta = 1$ this becomes

$$v_1 \;=\; (1-\psi_m)\cdot\mu \,+\, \psi_m\cdot s^m \,+\, \delta\cdot\lambda\cdot\log\left(\varphi_1(u^H - u^L) + u^L\right).$$

But then v_1 is bounded, while $v(s,\alpha)$ goes from minus to plus infinity. Hence the manager uses both the low and high beta strategy depending on s. Next, we provide the equation that defines those cutoffs. Rewrite l as a function of h according to

$$l(h) - L = h - H + \frac{H-L}{\psi},$$

and substitute this equality in $\varphi_0(r)$ and φ_1 to obtain the following two functions, in which only h appears out of the two equilibrium cutoffs. Substituting in $\varphi_0(r)$ gives

$$\tilde{\varphi}_0(r,h) \;=\; \left(1 + \frac{1-\pi}{\pi}\cdot\rho(r)\cdot\frac{\Phi\left(\frac{r-h\cdot(1+\psi)+H\psi-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\Phi\left(\frac{r-h\cdot(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right)}\right)^{-1}, \tag{C.1.15}$$

where h is introduced as an input of the function. Similarly, substituting in φ_1 gives

$$\tilde{\varphi}_1(h) \;=\; \left(1 + \frac{1-\pi}{\pi}\cdot\frac{\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}{\Phi\left(\frac{h-H}{\nu}\right)}\right)^{-1} \tag{C.1.16}$$

Then the cutoff h is given by the high types indifference condition $v_0(h,H) \;=\; v_1$, which using the above notation becomes

$$\begin{aligned}\delta\cdot\lambda\cdot\int\log\left[\tilde{\varphi}_0(r,h)(u^H-u^L)+u^L\right]\cdot\phi\left(\frac{r-(1-\psi)H-\psi h}{\sqrt{\psi}\nu}\right)\frac{1}{\sqrt{\psi}\nu}\,\mathrm{d}r \\ =\; \delta\cdot\lambda\cdot\log\left[\tilde{\varphi}_1(h)(u^H-u^L)+u^L\right] + (1-\psi_m)\cdot\mu + \psi_m\cdot s^m - (1-\psi)\cdot H - \psi\cdot h\end{aligned} \tag{C.1.17}$$

where $\phi(\cdot)$ is the density of the standard normal distribution. To prove existence we demonstrate that (C.1.17) equation has at least one solution. Let $LHS(h)$ denote the left hand side of (C.1.17), $RHS(h)$ its right hand side, and $\Delta(h) = LHS(h) - RHS(h)$ their difference. Observe that all the parts of the above equation apart from the last line are bounded. As a result,

$$\begin{aligned}\lim_{h\to-\infty} \Delta(h) &= -\infty \\ \lim_{h\to+\infty} \Delta(h) &= +\infty.\end{aligned} \tag{C.1.18}$$

Then it follows from the continuity of this function that there exists at least one point where $\Delta(h) = 0$. Hence we have proven existence.

Next we show that (3.8) is indeed a sufficient condition for uniqueness. In particular, we will argue that (3.8) implies that $\Delta(h)$ is increasing in h. First, note that $LHS(h)$ is increasing in h, because $\tilde{\varphi}_0(r, h)$ is increasing in both r and h. We have already argued why this is true for r. For h the claim is a direct implication of Lemma C.1.3.

Hence it suffices to identify a condition for $RHS(h)$ to be decreasing. Lemma C.1.3 implies that $\tilde{\varphi}_1(h)$ is increasing in h. This is the opposite monotonicity, however we can use the fact that the following expression has a relatively simple upper bound

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}h}\log\left[\tilde{\varphi}_1(h)(u^H - u^L) + u^L\right] &= \frac{\tilde{\varphi}_1(h)[1-\tilde{\varphi}_1(h)]/\nu}{\tilde{\varphi}_1(h) + \frac{u^L}{u^H - u^L}} \times \left[M\left(-\frac{h-H}{\nu}\right) - M\left(-\frac{l(h)-L}{\nu}\right)\right] \\ &\leq \frac{1}{\nu}\left[M\left(-\frac{h-H}{\nu}\right) - M\left(-\frac{l(h)-L}{\nu}\right)\right] = \frac{1}{v}\int_{\frac{L-(1-\psi)H}{\psi}}^{H} M'\left(\frac{x-h}{\nu}\right)\mathrm{d}x \leq \frac{H-L}{\psi\nu^2}\end{aligned} \tag{C.1.19}$$

Hence, a sufficient condition for the right hand side to be decreasing, which will imply uniqueness, is that

$$\delta\lambda^i \frac{H-L}{\psi\nu^2} \;\leq\; \psi\,,$$

which equivalently gives (3.8). □

Proof of Proposition 3.2. We know that $\varphi_0(r, s^m)$ is increasing in r. Hence, it suffices to prove the conjectured result for $r \to -\infty$. The dependence on s^m is suppressed. Let $k = -h(1+\psi) + H\psi$. To find the limit $\lim_{r\to-\infty} \varphi_0(r)$ we first need to calculate.

$$\lim_{r\to-\infty} \frac{\Phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)\exp\left(\frac{2(H-L)r-(H^2-L^2)}{2\nu^2\psi(1+\psi)}\right)}. \tag{C.1.20}$$

Because both the numerator and the denominator go to zero as r goes to minus infinity this limit becomes

$$\frac{e^{\frac{H^2-L^2}{2\nu^2\psi(1+\psi)}} \lim_{r\to-\infty} \frac{\phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\nu\sqrt{1+\psi}}}{e^{\frac{(H-L)r}{\nu^2\psi(1+\psi)}} \times \left[\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)\frac{H-L}{\nu^2\psi(1+\psi)} + \frac{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}{\nu\sqrt{1+\psi}}\right]}.$$

In addition, algebra implies the following simplification

$$\frac{\phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)} = \exp\left(\frac{2(r+k)-\frac{H-L}{\psi}}{2\nu^2(1+\psi)\psi/(H-L)}\right). \tag{C.1.21}$$

This in turn gives

$$e^{-\frac{(H-L)r}{\nu^2\psi(1+\psi)}}\frac{\phi\left(\frac{r+k-(H-L)/\psi}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)} = \exp\left(\frac{2k-\frac{H-L}{\psi}}{2\nu^2(1+\psi)\psi/(H-L)}\right).$$

Hence the limit becomes

$$\exp\left(\frac{2k+H+L-\frac{H-L}{\psi}}{2\nu^2(1+\psi)\psi/(H-L)}\right) \times \lim_{r\to-\infty}\left(\frac{\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}\frac{H-L}{\nu\psi\sqrt{1+\psi}}+1\right)^{-1},$$

where

$$\lim_{r\to-\infty}\frac{\Phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)}{\phi\left(\frac{r+k}{\nu\sqrt{1+\psi}}\right)} = \lim_{x\to\infty}\frac{1-\Phi(x)}{\phi(x)} = 0 \tag{C.1.22}$$

Hence, substituting k we obtain that

$$\lim_{r\to-\infty}\varphi_0(r) = \left(1+\frac{1-\pi}{\pi}\exp\left[\left(H-\frac{H-L}{2\psi}-h\right)\frac{H-L}{\psi\nu^2}\right]\right)^{-1}.$$

Next, we want to show that the above is greater than $\varphi_1(r)$ for every h. This holds if and only if

$$\exp\left[\left(H-\frac{H-L}{2\psi}-h\right)\frac{H-L}{\psi\nu^2}\right] < \frac{\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}{\Phi\left(\frac{h-H}{\nu}\right)} \tag{C.1.23}$$

which can equivalently be rewritten as

$$\left(H-\frac{H-L}{2\psi}-h\right)\frac{H-L}{\psi\nu^2} < \log\frac{\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}{\Phi\left(\frac{h-H}{\nu}\right)}. \tag{C.1.24}$$

Differentiating the left hand side minus the right hand side we get

$$-\frac{H-L}{\psi\nu^2}+\frac{1}{\nu}M\left(\frac{H-h}{\nu}\right)-\frac{1}{\nu}M\left(\frac{H-h}{\nu}-\frac{H-L}{\nu\psi}\right)\leq-\frac{H-L}{\psi\nu^2}+\frac{H-L}{\psi\nu^2}=0 \quad \text{(C.1.25)}$$

Hence it suffices to check that

$$\lim_{h\to-\infty}\frac{\Phi\left(\frac{h-H}{\nu}\right)}{\exp\left(\frac{(H-L)h}{\psi\nu^2}\right)\Phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}\leq\exp\left[\left(\frac{H-L}{2\psi}-H\right)\frac{H-L}{\psi\nu^2}\right].$$

Similar argumentation with above gives that the limit on the left hand side becomes

$$\begin{aligned}\lim_{h\to-\infty}&\frac{\phi\left(\frac{h-H}{\nu}\right)}{\exp\left(\frac{(H-L)h}{\psi\nu^2}\right)\phi\left(\frac{h-H+(H-L)/\psi}{\nu}\right)}\\ &=\lim_{h\to-\infty}\exp\left(\frac{2(h-H)+\frac{H-L}{\psi}}{2\nu^2}\frac{H-L}{\psi}-\frac{(H-L)h}{\psi\nu^2}\right)\\ &=\exp\left[\left(\frac{H-L}{2\psi}-H\right)\frac{H-L}{\nu^2\psi}\right]. \quad \text{(C.1.26)}\end{aligned}$$

Hence, the above inequality holds. □

Proof of Proposition 3.3. The Input s^m is suppressed. First, note that h is the solution of (C.1.17), that is the solution of $\Delta(h)=0$, where $\Delta(h)$ is defined under the equation as the difference of its left hand side from its right hand side. Second, the optimal cutoff under no career concerns for the high type $c(H)$ is the one that corresponds to the solution of this equation for $\delta=0$, as this corresponds to the case when the next period is irrelevant. Let $h(\delta)$ denote the solution of (C.1.17) as a function of δ. Then it follows from the implicit function theorem that

$$\frac{\mathrm{d}h(\delta)}{\mathrm{d}\delta}=-\left.\frac{\partial\Delta(h)/\partial\delta}{\partial\Delta(h)/\partial h}\right|_{h=h(\delta)}. \quad \text{(C.1.27)}$$

But it follows from the limits calculated in (C.1.18) that the unique monotonic equilibrium needs to have $\partial\Delta(h)/\partial h>0$. Moreover, calculating the derivative on the numerator for some generic h gives

$$\frac{\partial\Delta(h)}{\partial\,\delta}=\lambda\,\mathbb{E}_r\left[\log\left[\tilde{\varphi}_0(r,h)(u^H-u^L)+u^L\right]-\log\left[\tilde{\varphi}_1(h)(u^H-u^L)+u^L\right]\,\middle|\,s=h,H\right],$$

but it follows from Proposition 3.2 that this is positive, because the difference inside the expectation is positive for every h. As a result, for every $\delta\geq0$ we get that $\mathrm{d}h(\delta)/\mathrm{d}\delta<0$, which through (3.5) implies the same for the cutoff used by the low type.

Finally, note that λ and δ enter (C.1.17) in exactly the same way, hence the same result can be stated for λ. □

Lemma C.1.4. *In the unique monotonic equilibrium, for every prior reputation $\pi > 1/2$ there exists a lower bound $\bar{s}^m(\pi)$, defined as the solution of $\varphi_1(s^m) = 1/2$, such that for every $s^m > \bar{s}^m$ we have $\varphi_1(s^m) > 1/2$, and $\bar{s}^m(\pi^i)$ is increasing in π.*

In addition, for every $s^m \geq \bar{s}^m(\pi)$ the cutoffs $h(s^m)$ and $l(s^m)$ are increasing in π, and the same is true for the posterior reputations $\varphi_0^(r, s^m)$ and $\varphi_1(s^m)$.

Proof. In the proof of Proposition 3.1 is has been shown that in the unique monotonic equilibrium there exists $\tilde{\varphi}_1$ such that $\varphi_1(s^m) = \tilde{\varphi}_1[h(s^m)]$, and its functional form is given in (C.1.16). Moreover, it is an immediate implication of Lemma C.1.3 that this is increasing in h, and it is ease to verify that

$$\lim_{h \to +\infty} \tilde{\varphi}_1(h) \ = \ \pi. \tag{C.1.28}$$

In addition, it follows from (C.1.17), which defines $h(s^m)$, that

$$(1 + \psi)H + \psi h(s^m) + \delta\lambda \log\left(\frac{u^H}{u^L}\right) \geq \ (1 - \psi_m)\mu + \psi_m s^m.$$

This provides a lower bound for $h(s^m)$, which is in an increasing function of s^m, and shows that

$$\lim_{s^m \to +\infty} h(s^m) \ = \ +\infty, \tag{C.1.29}$$

from which the existence of the cutoffs follows. It monotonicity follows from using the implicit function theorem on the equation that defines it

$$\tilde{\varphi}_1\big[\pi, h\big(\bar{s}(\pi)\big)\big] \ = \ 1/2, \tag{C.1.30}$$

where note that $\tilde{\varphi}_1$ is increasing in both π and h, and it has been argued in Proposition 3.4 that $h(\cdot)$ is also an increasing function.

For the second statement, it follows from (3.5) that is suffices to prove it for $h(s^m)$. Using the implicit function theorem on (C.1.17) we get that

$$\frac{\mathrm{d}h}{\mathrm{d}\pi} \ = \ -\frac{\partial\Delta/\partial\pi}{\partial\Delta/\partial h}, \tag{C.1.31}$$

where direct differentiation gives $\partial\Delta/\partial h = \psi > 0$ and that

$$\frac{\partial \Delta}{\partial \pi} = \frac{\delta\,\lambda}{\pi(1-\pi)}\mathbb{E}_r\left[\frac{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}{\tilde{\varphi}_0+\frac{u^L}{u^H-u^L}} - \frac{\tilde{\varphi}_1(1-\tilde{\varphi}_1)}{\tilde{\varphi}_1+\frac{u^L}{u^H-u^L}}\,\middle|\, s=h, H\right], \tag{C.1.32}$$

where the inputs r and s^m have been suppressed. Some basic calculus shows that for every $\tilde{\varphi} \in [1/2, 1]$ the ratio

$$\frac{\tilde{\varphi}(1-\tilde{\varphi})}{\tilde{\varphi}+\frac{u^L}{u^H-u^L}} \tag{C.1.33}$$

is decreasing in $\tilde{\varphi}$. Moreover, we have from Proposition 3.2 that $\tilde{\varphi}_0(r,h) > \tilde{\varphi}_1(h)$ for every $r \in \mathbb{R}$. But we already showed that $\tilde{\varphi}_1(h) > 1/2$ for every $s^m \geq \bar{s}^m(\pi)$. Hence, we get that $\partial\Delta/\partial\pi < 0$, which implies the second statement.

Finally, the third statement follows trivially from noting that the direct derivative of both the posteriors with respect to π is positive, and the fact that both are increasing in $h(s^m)$, implied by Lemma C.1.3, for which it has already being argued that it is increasing in π. □

Proof of Proposition 3.5. First, consider the investment decision of a high type manager, for which the probability of choosing the low beta strategy, conditional on the market signal s^m, is

$$\mathbb{P}\big(\beta = 0 \,\big|\, s^m\big) = \mathbb{P}\big(s \geq h(s^m) \,\big|\, s^m\big) = \mathbb{P}\big(h^{-1}(s) \geq s^m \,\big|\, s^m\big), \tag{C.1.34}$$

since it was shown in Proposition 3.4 that $h(\cdot)$ is increasing. Moreover, for given s^m the distribution of m is normal and is given by

$$m \,|\, s^m \sim \mathcal{N}\Big((1-\psi_m)\mu + \psi_m s^m\,,\, \psi_m \nu_m^2\Big). \tag{C.1.35}$$

Let $\tilde{m} = [m - (1-\psi_m)\mu]/\psi^m$. Then

$$\tilde{m} \,|\, s^m \sim \mathcal{N}\Big(s^m\,,\, \nu_m^2/\psi_m\Big), \tag{C.1.36}$$

while the ex-ante distribution of s^m is

$$s^m \sim \mathcal{N}\big(\mu\,,\, \sigma_m^2 + \nu_m^2\big), \tag{C.1.37}$$

As a result using again the properties of Bayesian updating with normal distributions we get that

$$s^m \,|\, \tilde{m} \sim \mathcal{N}\left(\tilde{\psi}\mu + (1-\tilde{\psi})\tilde{m}\,,\; \frac{\tilde{\psi}\nu_m^2}{\psi_m}\right), \tag{C.1.38}$$

where $\tilde{\psi}_m = (\sigma_m^2 + \nu_m^2)/(\sigma_m^2 + \nu_m^2 + \upsilon_m^2/\psi_m)$. Hence for for every $\hat{m}$, m such that $\hat{m} > m$, the distribution of corresponding normal that generates s^m conditional on $\hat{m}$ first order stochastically dominates the one of m. This immediately implies that

$$\mathbb{P}\big(\beta = 0 \,\big|\, \hat{m}\big) \;<\; \mathbb{P}\big(\beta = 0 \,\big|\, m\big). \tag{C.1.39}$$

Hence under better observed market conditions the manager is less likely to have chosen to invest in her idiosyncratic strategy. The second statement of the proposition follows from noting that

$$\frac{\mathrm{d}\varphi_0(r, s^m)}{\mathrm{d}r} \;\geq\; 0 \;=\; \frac{\mathrm{d}\varphi_1(s^m)}{\mathrm{d}r}, \tag{C.1.40}$$

To calculate the left derivative it is more convenient to use the equivalent $\tilde{\varphi}_0$ function from the proof of proposition 3.1. The derivative of this can be calculated in a manner similar to that used in the proof of Lemma C.1.2 to be

$$\frac{\mathrm{d}\tilde{\varphi}_0(r,h)}{\mathrm{d}r} \;=\; \frac{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}{\nu\sqrt{1+\psi}}\left[\frac{H-L}{\nu\psi\sqrt{1+\psi}} - \int_{\underline{x}}^{\overline{x}} M'\left(x + h\sqrt{1+\psi}/\nu\right)\mathrm{d}x\right],$$

where $M(\cdot)$ is the hazard rate of the standard normal distribution,

$$\underline{x} \;=\; -\frac{r + H\psi}{\nu\sqrt{1+\psi}} \quad \text{and} \quad \overline{x} \;=\; \underline{x} + \frac{(H-L)/\psi}{\nu\sqrt{1+\psi}}. \tag{C.1.41}$$

Next we want to show that this derivative is decreasing in s^m. This appears in $\tilde{\varphi}_0$ only indirectly through the cutoff $h(s^m)$, which it has already being shown to be an increasing function. Hence calculate

$$\frac{\mathrm{d}^2\tilde{\varphi}_0(r,h)}{\mathrm{d}r\mathrm{d}h} \;=\; \frac{1-2\tilde{\varphi}_0}{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}\left(\frac{\mathrm{d}\tilde{\varphi}_0(r,h)}{\mathrm{d}r}\right)^2 - \frac{\tilde{\varphi}_0(1-\tilde{\varphi}_0)}{\nu^2}\int_{\underline{x}}^{\overline{x}} M''\left(x + h\sqrt{1+\psi}/\nu\right)\mathrm{d}x\,,$$

the second line of which is always negative, as $M(\cdot)$ is a convex function. The first line is negative as long as $\tilde{\varphi}_0(r,h) > 1/2$. But we have already argued in Proposition 3.2 that $\tilde{\varphi}_0(r,h) > \tilde{\varphi}_1(h)$, and in Lemma C.1.4 that there exists lower bound $\bar{s}^m(\pi)$ such that for all $s^m \geq \bar{s}^m(\pi)$ it has to be that $\tilde{\varphi}_1(h) > 1/2$. Moreover, the same Lemma gives that $\bar{s}^m(\pi)$ is an increasing function and it is ease to verify that for bounded m

$$\lim_{\pi \to 1} \mathbb{P}(\phi_1(s^m) < 1/2 \,|\, m) \; = \; 0. \tag{C.1.42}$$

Hence, indeed $\dfrac{\mathrm{d}\varphi_0(r, s^m)}{\mathrm{d}r}$ is decreasing in s^m, from which the second statement of the proposition also follows.

□

Proof of equation 3.11. We have:

$$\begin{aligned}\mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \; = \; & \mathbb{P}(\varphi_1^1 > \varphi_1^2 \,|\, s^m)\, \mathbb{P}(1, 1 \,|\, s^m) \\ & + \mathbb{P}(\varphi_1^1 > \varphi_0^2 \,|\, s^m)\, \mathbb{P}(1, 0 \,|\, s^m) + \mathbb{P}(\varphi_0^1 > \varphi_1^2 \,|\, s^m)\, \mathbb{P}(0, 1 \,|\, s^m) \\ & + \mathbb{P}(\varphi_0^1 > \varphi_0^2 \,|\, s^m)\, \mathbb{P}(0, 0 \,|\, s^m),\end{aligned} \tag{C.1.43}$$

It follows immediately from Lemma C.1.4 that $\varphi_1^2 > \varphi_1^1$. Moreover, Proposition 3.2 gives that $\varphi_0^2 > \varphi_1^2$, hence we also have that $\varphi_0^2 > \varphi_1^1$. As a result the above becomes

$$\mathbb{P}(\varphi^1 > \varphi^2 \,|\, s^m) \; = \; \mathbb{P}(\varphi_0^1 > \varphi_1^2 \,|\, s^m)\, \mathbb{P}(0, 1 \,|\, s^m) + \mathbb{P}(\varphi_0^1 > \varphi_0^2 \,|\, s^m)\, \mathbb{P}(0, 0 \,|\, s^m), \tag{C.1.44}$$

we can only be certain about the monotonicity of the probability of both managers invest in their idiosyncratic portfolio which is deceasing given large s^m. The rest of the terms can not be monotonic as we have observed through simulations.

□

C.2 Investment and AUM in the Second Period

Here, first we derive the optimal investment decision of a manager in the second period. Second, we use this to calculate her AUM as a function of her posterior reputation, which we later use in order to derive her continuation payoff from period 2. To avoid repetition we consider the extended model in which there are two fund managers. In this the investor's preferences are given by

$$v(i, z_t^{ij}) = \begin{cases} \exp(z_t^{i1} - \bar{z}) \cdot (1 - f_t^i) \cdot R_t^i & , i = 1, 2 \\ \exp(m_t) & , i = m \end{cases}$$

Hence, in this case there are two independent preference shocks , one for each fund. The results of the baseline more can be obtained by setting the fees of the second manager equal to one, which will ensure that no investor will invest in her fund.

We solve the second period backwards by first considering the manager's investment decision when the funds have already been allocated. The manager's expected payoff is

$$\mathbb{E}\left[\log\left(A_2^i f_2^i R_2^i\right) \,|\, s_2^i, s_2^m, \beta_2^i, \alpha\right] \;=\; \log\left(A_2^i f_2^i\right) \,+\, \mathbb{E}\left[r_2^i \,|\, s_2^i, s_2^m, \beta_2^i, \alpha\right]$$

As a result the manager's objective when choosing her investment strategy β_2^i in the second period is to simply maximise the expected return r_2^i. Thus, she invests in her alpha only if

$$\mathbb{E}\left[r_2^i \,|\, s_2^i, s_2^m, \beta_2^i = 0, \alpha\right] \;\geq\; \mathbb{E}\left[r_2^i \,|\, s_2^i, s_2^m, \beta_2^i = 1, \alpha\right] \tag{C.2.1}$$

It is known that the posterior distributions of a_2^i and m_2, after conditioning on s_2^i and s_2^m, are also normal distributions with known expected values. Let $\psi = \sigma^2/(\sigma^2 + \nu^2)$ and $\psi_m = \sigma_m^2/(\sigma_m^2 + \nu_m^2)$. Then (C.2.1) becomes

$$(1-\psi)\cdot\alpha \,+\, \psi\cdot s_2^i \;\geq\; (1-\psi_m)\cdot\mu \,+\, \psi_m\cdot s_2^m\,,$$

which allows us to derive the manager's optimal investment strategy in the second period. This is a cutoff rule such that she invests in her alpha only if $s_2^i \geq c(\alpha, s_2^m)$, where

$$c(\alpha, s_2^m) \;=\; \frac{\psi_m}{\psi}\cdot s_2^m \,+\, \frac{1-\psi_m}{\psi}\cdot\mu \,-\, \frac{1-\psi}{\psi}\cdot\alpha \tag{C.2.2}$$

Thus, for the same market conditions a high type manager invests relatively more frequently on her alpha in the second period, as $c(H, s_2^m) \;<\; c(L, s_2^m)$ implies

$$\mathbb{P}[s_2^i \geq c(H, s_2^m)] > \mathbb{P}[s_2^i \geq c(L, s_2^m)] \;\Rightarrow\; \mathbb{P}(\beta_2^i = 0 \,|\, m_2, \alpha = H) > \mathbb{P}(\beta_2^i = 1 \,|\, m_2, \alpha = L),$$

where the second line required to infer s_2^m from the realised m_2. We will frequently need to condition expectations with respect to m_t instead of s_t^m, because we do not have in our data some measure of the latter in our data.

An important point that needs to be made is that the cutoffs $c(\alpha, s_2^m)$ are not the optimal ones for the investors. This is because those are risk-neutral, while the managers are risk-averse. Following the same argumentation as above we can show that the optimal cutoff for the investors is

$$c^*(\alpha, s_2^m) \;=\; c(\alpha, s_2^m) + \frac{\psi_m\sigma_m^2 - \psi\sigma^2}{2\,\psi}. \tag{C.2.3}$$

Thus the investor's optimal cutoff is adjusted by a "risk-loving" factor. For example, suppose that $\psi_m\sigma_m^2 \,>\, \psi\sigma^2$, that is investing in the market is relatively more risky conditional on the information that the manager has at her disposal when making the decision. Then an investor would require a higher level of confidence on her alpha s_2^i in order to also agree that relying on it is preferable to 'gambling' with r_2^m.

Let u_2^α denote the equilibrium payoff of an investor in the second period, conditional on investing with a manager of type α, but net of his preference shock z_t^{ij} and fees f_2^i. Then this is given by

$$u_2^\alpha = \mathbb{P}[s_2^i \geq c_\alpha(s_2^m)]\,\mathbb{E}[R_2^i \,|\, s_2^i \geq c_\alpha(s_2^m)] + \mathbb{P}[s_2^i \leq c_\alpha(s_2^m)]\,\mathbb{E}[R_2^i \,|\, s_2^i \leq c_\alpha(s_2^m)], \qquad \text{(C.2.4)}$$

which has a closed form representation that can be derived using the formulas of the moment generating function of the truncated normal distribution. We avoid providing this here as it does not facilitate the understanding of the model in any meaningful way. However, it is important to point out that when the market's posterior variance $\psi_m\sigma_m^2$ is much bigger than that of the alpha-based strategy $\psi\sigma^2$ then the misalignment between the manager's and the investors' preferences could be so substantial that a low type manager would be preferable simply because she is more reluctant to use her alpha. We exclude that by assuming that $u_2^H > u_2^L$, because if the parameters of the model were such that investing in an index tracking strategy was so attractive, then there would be little need for professional investors.

Let φ^i denote the public posterior belief on manager i's ability α^i at the beginning of period two. Then the investor's expected payoff, net of fees and the preferences shock, from opting for fund i is

$$u_2^i \;=\; \varphi^i(u_2^H - u_2^L) + u_2^L,$$

and the corresponding actual payoff is $e^{z_t^{ij}}(1 - f_t^i)u_2^i$. In addition, each investor has an outside option, which is to ignore the financial intermediaries and instead invest directly on m_2, which gives expected payoff

$$u^m \;=\; \mathbb{E}[\exp(m_t)] \;=\; e^{\mu+\sigma_m^2/2}.$$

To avoid repetition note that in a manner similar to the one above we can define

$$u_1^i \;=\; \pi^i(u_1^H - u_1^L) + u_1^L,$$

as the expected net payoff of an investor active in the first period. However, in this case the functional form of u_1^α will be completely different, as the cutoffs used by the managers in the first period will be influenced by their career concerns. We will derive those under a market equilibrium in the next subsection.

To ensure that when the lowest preference shocks are realised the investor would rather invest directly in the market we will assume that

$$(1 - f_2^i) \cdot u_2^H \; < \; u^m \cdot e^{\bar{z}} \tag{C.2.5}$$

We are now ready to derive the AUM of fund i in the beginning of period t, as only a function of net expected payoffs and announced fees.

Lemma C.2.1. *In any market equilibrium the AUM of fund i, competing against fund k, in period t is*

$$\left(\frac{(1-f_t^i)u_t^i}{u^m}\right)^{\lambda^i}\left(1 - \frac{\lambda^i}{\lambda^i + \lambda^k}\left(\frac{(1-f_t^k)u_t^k}{u^m}\right)^{\lambda^k}\right). \tag{C.2.6}$$

***Proof*.** To simplify the algebra drop the investor superscript and time subscripts. Also let $\xi^i = \log(1-f^i)u^i$, $i = 1, 2$ and $\xi^m = \log u^m + \bar{z}$. For an investor to prefer fund 1 to both directly investing in the market and to fund 2, it has to be that

$$\exp(z^1 - \bar{z}) \cdot (1 - f^1) \cdot u^1 \; \geq \; u^m \; \Leftrightarrow \; z^1 \; \geq \; \xi^m - \xi^1$$

and

$$\exp(z^1)(1-f^1)u^1 \; \geq \; \exp(z^2)(1-f^2)u^2 \; \Leftrightarrow \; z^1 + \xi^1 - \xi^2 \; \geq \; z^2,$$

respectively. Hence the proportion of the market that fund 1 captures is

$$\begin{aligned}
&\mathbb{P}\left(z^1 \; \geq \; \xi^m - \xi^1 \; \cap \; z^1 + \xi^1 - \xi^2 \; \geq \; z_2\right) \\
&\qquad = \int_{\xi^m - \xi^1}^{\infty} \mathbb{P}\left(z^1 + \xi^1 - \xi^2 \; \geq \; z_2 \,\middle|\, z^1\right) \mathrm{d}\mathbb{P}(z^1) \\
&\qquad = \int_{\xi^m - \xi^1}^{\infty} \left(1 - e^{-\lambda^2(z^1 + \xi^1 - \xi^2)}\right) \lambda^1 e^{-\lambda^1 z^1} \mathrm{d}z^1 \\
&\qquad = e^{-\lambda_1(\xi^m - \xi^1)} - e^{-\lambda^2(\xi^1 - \xi^2)} \frac{\lambda^1}{\lambda^1 + \lambda^2} e^{-(\lambda^1 + \lambda^2)(\xi^m - \xi^1)} \\
&\qquad = \left(\frac{(1-f^1)u^1}{u^m \cdot e^{\bar{z}}}\right)^{\lambda^1} \cdot \left(1 - \frac{\lambda^1}{\lambda^1 + \lambda^2}\left(\frac{(1-f^2)u^2}{u^m \cdot e^{\bar{z}}}\right)^{\lambda^2}\right)
\end{aligned}$$

The proof for fund 2 is equivalent. □

The proof calculates (C.2.6) as the probability of the intersection of two events. The first is that investor j prefers fund i to fund k. The second is that fund i is preferred to direct investment in the market.

To obtain the assets for the case where there is only one manager set $f^2 = 1$ to get:

$$\left(\frac{(1-f_t^i)\cdot u_t^i}{u^m \cdot e^{\bar{z}}}\right)^{\lambda^i} \tag{C.2.7}$$

C.3 Unobservable Investment Decision

Lemma C.3.1. *For generic a and b:*

$$\phi(a-bx)\phi(x) = \phi\left(x\sqrt{1+b^2} - \frac{a\,b}{\sqrt{1+b^2}}\right)\phi\left(\frac{a}{\sqrt{1+b^2}}\right). \tag{C.3.1}$$

In addition, for generic $\underline{x}$:

$$\int_{\underline{x}}^{\infty}\phi(a-bx)\phi(x)\,\mathrm{d}x = \Phi\left(\frac{a\,b}{\sqrt{1+b^2}} - \underline{x}\sqrt{1+b^2}\right)\phi\left(\frac{a}{\sqrt{1+b^2}}\right)\frac{1}{\sqrt{1+b^2}}. \tag{C.3.2}$$

Proof. The first equation follow from

$$\begin{aligned}
\phi(a-bx)\phi(x)2\pi \;&=\; \exp\left(-\frac{a^2-2abx+b^2x}{2} - \frac{x^2}{2}\right)\\
=\; \exp&\left(-\frac{(1+b^2)x^2-2abx+\frac{a^2b^2}{1+b^2}}{2} - \frac{a^2-\frac{a^2b^2}{1+b^2}}{2}\right)\\
=\; \exp&\left(-\frac{1}{2}\left(\sqrt{1+b^2}\,x - \frac{a\,b}{\sqrt{1+b^2}}\right)^2 - \frac{1}{2}\frac{a^2}{1+b^2}\right).
\end{aligned} \tag{C.3.3}$$

The second equation follows trivially from the first. □

To make notation more compact write $\bar{r}_H(s)$ and $\bar{r}_L(s)$ instead of $\bar{r}(\alpha, \beta = 0, s, s^m)$ and $\bar{r}_1$ instead of $\bar{r}(\alpha, \beta = 1, s, s^m)$. Similarly, write $\bar{\sigma}_\beta^2$ instead of $\bar{\sigma}^2(\beta)$. Also, let

$$\xi^2 \;\equiv\; \sigma_\epsilon^2\frac{\beta_0^2}{(1-\beta_0)^2}. \tag{C.3.4}$$

Define the following function

$$\begin{aligned}
\rho(r,\alpha,c) \;=\; \Phi&\left(\frac{(\tilde{r}-\alpha)\psi\nu}{\xi\sqrt{\xi^2+\psi^2\nu^2}} - \sqrt{1+\frac{\psi^2\nu^2}{\xi^2}}\frac{c-\alpha}{\nu}\right)\\
&\times\frac{\phi\left(\frac{\tilde{r}-\alpha}{\sqrt{\xi^2+\psi^2\nu^2}}\right)}{\sqrt{\xi^2+\psi^2\nu^2}} + \Phi\left(\frac{c-\alpha}{\nu}\right)\frac{\phi\left(\frac{r-m}{\sigma_\epsilon}\right)}{\sigma_\epsilon},
\end{aligned} \tag{C.3.5}$$

which under the restriction that $\beta_0 = 0$ simplifies to

$$\rho(r,\alpha,c) \;=\; \Phi\left(\frac{r-c(1+\psi)+\alpha\psi}{\nu\sqrt{1+\psi}}\right)\times\frac{\phi\left(\frac{r-\alpha}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}+\Phi\left(\frac{c-\alpha}{\nu}\right)\frac{\phi\left(\frac{r-m}{\sigma_\epsilon}\right)}{\sigma_\epsilon}, \quad \text{(C.3.6)}$$

Proof of Lemma 3.1. Drop dependence on s^m both in the cutoffs and on the expectations. First, calculate the probability of r and β to be realised under the cutoff h. For $\beta = \beta_0$ define the new random variable

$$\tilde{r} \;\equiv\; \frac{r-\beta_0\, m}{1-\beta_0} \;=\; a+\frac{\beta_0}{1-\beta_0}\,\epsilon, \quad \text{(C.3.7)}$$

for which we have

$$\begin{aligned}\tilde{r}\,|\,s,m \;&\sim\; \mathcal{N}\Big((1-\psi)H+\psi s,\;\xi^2\Big)\\ \xi^2 \;&\equiv\; \sigma_\epsilon^2\frac{\beta_0^2}{(1-\beta_0)^2}\end{aligned} \quad \text{(C.3.8)}$$

as a result

$$\Pr\left(\tilde{r},\beta_0\,|\,H\right) = \int_h^\infty \phi\left(\frac{\tilde{r}-(1-\psi)H-\psi s}{\xi}\right)\frac{1}{\xi}\phi\left(\frac{s-H}{\nu}\right)\frac{1}{\nu}\,\mathrm{d}s \quad \text{(C.3.9)}$$

Below we switch the variable of integration to $\tilde{s} = (s-H)/\nu$ and use the above lemma

$$\Pr\left(\tilde{r},\beta_0\,|\,H\right) \;=\; \int_{\frac{h-H}{\nu}}^\infty \phi\left(\frac{\tilde{r}-H}{\xi}-\frac{\psi\nu}{\xi}\tilde{s}\right)\phi(\tilde{s})\frac{1}{\xi}\,\mathrm{d}\tilde{s}$$

$$= \Phi\left(\frac{\tilde{r}-H}{\xi}\frac{\psi\nu/\xi}{\sqrt{1+\psi^2\nu^2/\xi^2}}-\sqrt{1+\frac{\psi^2\nu^2}{\xi^2}}\frac{h-H}{\nu}\right)\times\phi\left(\frac{(\tilde{r}-H)/\xi}{\sqrt{1+\psi^2\nu^2/\xi^2}}\right)\frac{1/\xi}{\sqrt{1+\psi^2\nu^2/\xi^2}}, \quad \text{(C.3.10)}$$

which after some algebra gives that

$$\Pr\left(\tilde{r},\beta_0\,|\,H\right) \;=\; \Phi\left(\frac{(\tilde{r}-H)\psi\nu}{\xi\sqrt{\xi^2+\psi^2\nu^2}}-\sqrt{1+\frac{\psi^2\nu^2}{\xi^2}}\frac{h-H}{\nu}\right)\times\phi\left(\frac{\tilde{r}-H}{\sqrt{\xi^2+\psi^2\nu^2}}\right)\frac{1}{\sqrt{\xi^2+\psi^2\nu^2}} \quad \text{(C.3.11)}$$

For $\beta = 1$, we have that $r = m+\epsilon$, hence

$$\Pr\left(r,\beta_1\,|\,H\right) \;=\; \phi\left(\frac{r-m}{\sigma_\epsilon}\right)\frac{1}{\sigma_\epsilon}\Phi\left(\frac{h-H}{\nu}\right) \quad \text{(C.3.12)}$$

Hence, we have an expression for

$$\Pr(r \,|\, H) \;=\; \Pr\left(\tilde{r} = \frac{r - \beta_0\, m}{1-\beta_0}, \beta_0 \,\middle|\, H\right) + \Pr\left(r, \beta_1 \,|\, H\right) \tag{C.3.13}$$

$$\Pr\left(\tilde{r} = \frac{r - \beta_0\, m}{1-\beta_0}, \beta_0 \,\middle|\, H\right) = \Pr\left(r, \beta = 0 \,|\, H\right) = \frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}}\Phi\left(\frac{r - h(1+\psi) + H\psi}{\nu\sqrt{1+\psi}}\right). \tag{C.3.14}$$

The expressions for the low type are identical, hence form this we can derive the posterior reputation of the manager. □

Proof of Proposition 3.1. We want to investigate if $\phi(r, m, s^m)$ can be always increasing in r. From Lemma 3 (19), sufficient b see if ρ can always be decreasing:

$$\rho = \frac{\rho_L}{\rho_H} \tag{C.3.15}$$

$$\rho = \frac{\Phi\left(\frac{r-l(1+\psi)+L\psi}{\nu\sqrt{1+\psi}}\right)\frac{\phi\left(\frac{r-L}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}} + \Phi\left(\frac{l-L}{\nu}\right)\frac{\phi\left(\frac{r-\mu}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon}}{\Phi\left(\frac{r-h(1+\psi)+H\psi}{\nu\sqrt{1+\psi}}\right)\frac{\phi\left(\frac{r-H}{\nu\sqrt{\psi(1+\psi)}}\right)}{\nu\sqrt{\psi(1+\psi)}} + \Phi\left(\frac{h-H}{\nu}\right)\frac{\phi\left(\frac{r-\mu}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon}} \tag{C.3.16}$$

Set $\nu\sqrt{\psi(1+\psi)} = \sigma_\varepsilon$ else it is non going to be decreasing

After substitution

$$\rho = \frac{\varepsilon^{A_1 r - C_1}\Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) + d_1}{\varepsilon^{A_2 r - C_2}\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right) + d_2} \tag{C.3.17}$$

With

$$A_1 = \frac{L-m}{\sigma_\varepsilon^2}, C_1 = \frac{L^2-m^2}{2\sigma_\varepsilon^2}, C_1 = \frac{L^2-m^2}{2\sigma_\varepsilon^2}, b_1 = l(1+\psi) - L_\psi, d_1 = \Phi\left(\frac{l-L}{\nu}\right) \tag{C.3.18}$$

Note $A_1 < A_2$.

Then we take the derivative with respect to r, which is proportional to:

$$\rho' \propto e^{A_1 r - C_1} e^{A_2 r - C_2} \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)(A_1 - A_2)$$
$$+ e^{A_1 r - C_1} e^{A_2 r - C_2} \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) \Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right) \frac{1}{\nu\sqrt{1+\psi}} \left(M\left(\frac{b_1-r}{\nu\sqrt{1+\psi}}\right) M\left(\frac{b_2-r}{\nu\sqrt{1+\psi}}\right)\right)$$
$$+ d_2\left[\varepsilon^{A_1 r - C_1} A_1 \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) + \varepsilon^{A_1 r - C_1} \frac{1}{\nu\sqrt{1+\psi}} \phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)\right] - d_1[\epsilon^{A_2 r - C_2} A_2 \Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)$$
$$+ \epsilon^{A_2 r - C_2} \frac{1}{\nu\sqrt{1+\psi}} \phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)] \quad \text{(C.3.19)}$$

We want the derivative of ρ to be negative for every r, m, which is proportional with:

$$\frac{\rho'}{\epsilon^{A_1 r - C_1} \epsilon^{A_2 r - C_2}} \propto d_2\left[A_1 \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) + \frac{1}{\nu\sqrt{1+\psi}} \phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)\right]$$
$$+ d_1\left[A_2 \Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right) \frac{1}{\nu\sqrt{1+\psi}} \phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)\right]$$
$$+ d_2\left[A_1 \frac{\Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)}{e^{A_2 r - C_2}} + \frac{1}{\nu\sqrt{1+\psi}} \frac{\phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)}{e^{A_2 r - C_2}}\right]$$
$$- d_1\left[A_2 \frac{\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)}{e^{A_1 - C_1}} \frac{1}{\nu\sqrt{1+\psi}} \frac{\phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)}{e^{A_1 - C_1}}\right]$$

We take any m such that $A_1, A_2 < 0$. Then $d_2[A_1 \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right) + \frac{1}{\nu\sqrt{1+\psi}} \phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)] + d_1[A_2]\Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right) \frac{1}{\nu\sqrt{1+\psi}} \phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)]$ is finite because $\Phi(.) \in [0,1]$ and $M(a) - M(b) \leq a - b$ for $a > b$ (Lemma 3).

We will investigate as $r \to \infty$. Also $\frac{\phi(.)}{e^{A_2 r - C}} \to 0$ as r $\to +\infty$

In addition we already know:

$$\frac{d_2 A_1 \Phi\left(\frac{r-b_1}{\nu\sqrt{1+\psi}}\right)}{e^{A_2 r - C_2}} - \frac{d_1 A_2 \Phi\left(\frac{r-b_2}{\nu\sqrt{1+\psi}}\right)}{e^{A_1 r - C_1}} \quad \text{(C.3.20)}$$

as $r \to +\infty$

$$\frac{d_2 d_1 e^{C_2} e^{r(A_1 - A_2)} - d_1 A_2 e^{C_1}}{e^{r A_1}} \quad \text{(C.3.21)}$$

we know that $A_1 - A_2 < 0$ so $e^{r(A_1 - A_2) \to +\infty}$ so

$$\frac{0 - d_1 A_2 e^{C_1}}{e^r A_1} \tag{C.3.22}$$

also we know that $A_1 < 0$ so $e^r A_1 \to 0$
and therefore $-\frac{d_1 A_2 e^{c_1}}{0} \implies +\infty$ as $A_2 < 0$.
So we finally we conclude that ρ' cannot always be negative.

$\square$

Appendix D

Appendixes of Chapter 4

D.1 Omitted proofs

Proof of Lemma 4.1. We want to transform the part of ($\mathcal{T}_f$) that depends on the reputation m_τ. Apply the following transformation

$$\mathbb{E}_{\theta_0}\Big[\mathbb{E}_\tau\big[e^{-r\tau}\cdot m_\tau \,\big|\, \theta_0\big]\Big] \;=\; \mathbb{E}_\tau\Big[e^{-r\tau}\cdot \mathbb{E}_{\theta_0}\big[m_\tau \,\big|\, \tau\big]\Big]$$

but m_τ is not a function of θ_0, hence $\mathbb{E}_{\theta_0}[m_\tau|\tau] = m_\tau$. Therefore,

$$\begin{aligned}
\mathbb{E}_{\theta_0}\Big[\mathbb{E}_\tau\big[e^{-r\tau}\cdot m_\tau \,\big|\, \theta_0\big]\Big] \;&=\; \mathbb{E}_\tau\big[e^{-r\tau}\cdot m_\tau \,\big|\, \tau\big] \\
&=\; \mathbb{E}_\tau\Big[e^{-r\tau}\cdot \mathbb{E}_{z,\theta_0}\big[\theta_0\cdot z_\tau \,\big|\, \tau\big]\Big] \\
&=\; \mathbb{E}_{\theta_0}\Big[\mathbb{E}_z\big[e^{-r\tau}\cdot\theta_0\cdot z_\tau \,\big|\, \theta_0\big]\Big]
\end{aligned}$$

where the last line follows from noting that the realisation of the path z together with θ_0 implies both τ and z_τ. Finally, substitute the above back in ($\mathcal{T}_f$) to obtain ($\mathcal{T}_f'$). □

Proof of Proposition 4.1. Follows as a subcase of the general stopping problem solved at Appendix D.2. □

Proof of Lemma 4.2. (IC) implies that

$$V(\theta_0) \;=\; \max_{\hat\theta_0}\; \widehat{V}(\hat\theta_0, \theta_0) \tag{D.1.1}$$

In (4.5) θ_0 appears only on the ratio $\hat\theta_0/\theta_0$ that multiplies the effort and in the market payoff.

Hence taking the partial derivative of $\widehat{V}(\hat{\theta}_0, \theta_0)$ with respect to θ_0 gives

$$\frac{\partial \widehat{V}(\hat{\theta}_0, \theta_0)}{\partial \theta_0} \;=\; \mathbb{E}_z\Big[\int_0^{\widehat{\tau}} e^{-rt} \cdot \frac{e_t(\hat{\theta}^t)^2}{2} \frac{\hat{\theta}_0}{(\theta_0)^2}\, \mathrm{d}t \,+\, e^{-r\widehat{\tau}} \cdot z_{\widehat{\tau}} \cdot M_2(m_{\widehat{\tau}}\,,\, \theta_{\widehat{\tau}})\Big] \tag{D.1.2}$$

Suppose momentarily that we could use the Envelop Theorem. Then this would give

$$V'(\theta_0) \;=\; \frac{\partial \widehat{V}(\hat{\theta}_0, \theta_0)}{\partial \theta_0}\Big|_{\hat{\theta}_0 = \theta_0} \;=\; \mathbb{E}_z\Big[\int_0^{\tau} e^{-rt} \cdot \frac{e_t(\theta^t)^2}{2\,\theta_0}\, \mathrm{d}t \,+\, e^{-r\,\tau} \cdot z_\tau \cdot M_2(m_\tau\,,\, \theta_\tau)\Big]$$

However, our solutions are not necessarily given by first order conditions. Therefore, we have to rely on one of its generalisations given by Milgrom and Segal (2002), and to be more precise on Theorem 2 of their paper.

To use this theorem it suffices to demonstrate that the partial derivative given in (D.1.2) is bounded above by a function that is integrable with respect to θ_0 for all $\hat{\theta}_0 \in [\underline{\theta}, \overline{\theta}]$. First, note that we have assumed that $e_t \leq \sqrt{\kappa\, \theta_t}$. In addition, it is a known result that

$$\mathbb{E}[z_t] \;=\; e^{\mu\, t} \quad \Rightarrow \quad \mathbb{E}[\, e^{-r\, t} \cdot z_t\,] \;=\; e^{-(r-\mu)\, t} \;\leq\; 1$$

Hence $e^{-rt} \cdot z_t$ is a super-martingale. Let $\overline{M}_2 > 0$ denote the constant which we have assumed that bounds $M_2(\cdot)$ from above. Then

$$\mathbb{E}[\, e^{-r\,\widehat{\tau}} \cdot z_{\widehat{\tau}} \cdot M_2(m_{\widehat{\tau}}, \theta_{\widehat{\tau}})\,] \;\leq\; \mathbb{E}[\, e^{-r\,\widehat{\tau}} \cdot z_{\widehat{\tau}}\,] \cdot \overline{M}_2 \;\leq\; \overline{M}_2$$

where the second inequality follows from Doob's optional sampling theorem. Applying the above in (D.1.2) gives that

$$\begin{aligned}\frac{\partial \widehat{V}(\hat{\theta}_0, \theta_0)}{\partial \theta_0} \;&\leq\; \mathbb{E}_z\Big[\int_0^{\widehat{\tau}} e^{-rt} \cdot z_t \cdot \frac{\kappa \cdot (\hat{\theta}_0)^2}{2 \cdot (\theta_0)^2}\, \mathrm{d}t\Big] + \overline{M}_2 \\ &\leq\; \mathbb{E}_z\Big[\int_0^{\infty} e^{-rt} \cdot z_t \cdot \frac{\kappa \cdot (\hat{\theta}_0)^2}{2 \cdot (\theta_0)^2}\, \mathrm{d}t\Big] + \overline{M}_2 \;=\; \frac{1}{r-\mu}\, \frac{\kappa \cdot (\hat{\theta}_0)^2}{2 \cdot (\theta_0)^2} + \overline{M}_2\end{aligned}$$

which is integrable with respect to θ_0 for all $\hat{\theta}_0 \in [\underline{\theta}, \overline{\theta}]$. Thus, (4.6) follows from the aforementioned thoerem. $\square$

Proof of Proposition 4.2. We want to obtain ($\mathcal{P}'$). Substituting truthful reporting $\hat{\theta}_0 = \theta_0$ in (4.5) gives that

$$\widehat{V}(\theta_0, \theta_0) \;=\; \mathbb{E}_z\Big[\int_0^{\tau} e^{-rt} \cdot \Big(w_t(\theta^t) - \frac{e_t(\theta^t)^2}{2}\Big) \mathrm{d}t \,+\, e^{-r\tau} \cdot M(m_\tau\,,\, \theta_\tau)\Big] \tag{D.1.3}$$

Hence we can equivalently write (4.7) as

$$\mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot w_t(\theta^t)\,\mathrm{d}t\Big] \;=\; \mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot\frac{e_t(\theta^t)^2}{2}\,\mathrm{d}t \;-\; e^{-r\tau}\cdot M(m_\tau\,,\,\theta_\tau)\Big] \;+\; \int_{\theta^*}^{\theta_0} V'(x)\,\mathrm{d}x \;+\; \omega_a$$

Therefore, integrating both sides with measure $f(\theta_0)$ gives

$$\int_{\theta^*}^{\overline{\theta}}\mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot w_t(\theta^t)\,\mathrm{d}t\Big]\mathrm{d}F(\theta_0) \;=\; \int_{\theta^*}^{\overline{\theta}}\mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot\frac{e_t(\theta^t)^2}{2}\,\mathrm{d}t - e^{-r\tau}\cdot M(m_\tau\,,\,\theta_\tau)\Big]\mathrm{d}F(\theta_0) \;+\; \int_{\theta^*}^{\overline{\theta}}\int_{\theta^*}^{\theta_0} V'(x)\,\mathrm{d}x\,\mathrm{d}F(\theta_0) \;+\; [1-F(\theta^*)]\cdot\omega_a$$

Then we use Fubini's Theorem to obtain that

$$\int_{\theta^*}^{\overline{\theta}}\int_{\theta^*}^{\theta_0} V'(x)\,\mathrm{d}x\,\mathrm{d}F(\theta_0) \;=\; \int_{\theta^*}^{\overline{\theta}}\frac{1-F(\theta_0)}{f(\theta_0)}\cdot V'(\theta_0)\,\mathrm{d}F(\theta_0)$$

Hence substituting $V'(\theta_0)$, as given in (4.6), gives that

$$\int_{\theta^*}^{\overline{\theta}}\mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot w_t(\theta^t)\,\mathrm{d}t\Big]\mathrm{d}F(\theta_0) \;=\; \int_{\theta^*}^{\overline{\theta}}\mathbb{E}_z\Big[\int_0^\tau e^{-rt}\cdot\big[1+\eta(\theta_0)/\theta_0\big]\frac{e_t(\theta^t)^2}{2}\,\mathrm{d}t \;-\; e^{-r\tau}\cdot\Big(M(m_\tau\,,\,\theta_\tau) \;-\; \eta(\theta_0)\,z_\tau\,M_2(m_\tau,\theta_\tau)\Big)\Big]\mathrm{d}F(\theta_0) \;+\; [1-F(\theta^*)]\cdot\omega_a$$

Finally, we substitute the left hand side of the above in ($\mathcal{P}$) to obtain ($\mathcal{P}'$). □

Proof of Lemma 4.3. Identical to that of Lemma 4.1. □

Proof of Proposition 4.3. Follows as a subcase of the general stopping problem solved at Appendix D.2. □

Proof of Corollary 4.1. Substituting the functional form of $q(\theta_0)$ we obtain that

$$Q(\theta_0) \;\propto\; \left[\frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} + \lambda\cdot\frac{\eta(\theta_0)}{\theta_0} - 1\right]^{-1}$$

where we have ignored the positive constants that multiply it. Hence, differentiating we obtain that

$$Q'(\theta_0) \;\propto\; -\frac{\partial\,[\eta(\theta_0)/\theta_0]}{\partial\,\theta_0}\cdot\left[\frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} + \lambda\cdot\frac{\eta(\theta_0)}{\theta_0} - 1\right]^{-2}\cdot\left[-\frac{a/(r-\mu)}{[1+\eta(\theta_0)/\theta_0]^3} + \lambda\right]$$

But $\eta(\theta_0)$ is non-increasing by assumption. Hence the above derivative is negative if and only if

$$a \;\geq\; \lambda \cdot (r-\mu) \cdot \big[1 + \eta(\theta_0)/\theta_0\big]^3$$

which is implied by condition ($\mathcal{A}$). □

Proof of Corollary 4.2. Let $Z_0 = 1$ and

$$\mathrm{d}Z_t \;=\; Z_t \cdot \mu \,\mathrm{d}t + Z_t \cdot \sigma \,\mathrm{d}B_t$$

where B_t denotes the standard Brownian motion. Also, for given $q \in (0,1)$ define the hitting time

$$\tau(q) \;=\; \inf\big\{t \geq 0 \;:\; Z_t \;\leq\; q\big\}$$

It is trivial to argue that $\Pr(\tau(q) \leq t \,|\, q)$ is decreasing in q. Hence we can prove the two statements using the corresponding results we have on the cutoff $q(\theta_0)$. □

We copy below the a generic lemma on static implementation. It will be convenient to state it using an also generic notation, and then simple use its implications on the subsequent proof.

Lemma D.1.1 (Implementation)**.** *For given payment $w : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ suppose that the payoff of a θ type agent, when reporting $\hat{\theta}$, is*

$$\widehat{U}(\hat{\theta}, \theta) \;=\; w(\hat{\theta}) + u(\hat{\theta}, \theta) \tag{D.1.4}$$

where $u : [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ is absolute continuous in the second variable with weak derivative $u_2 : [\underline{\theta}, \overline{\theta}] \times [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$. Then truthful reporting is implementable only if u_2 is increasing in the first variable. In addition, if this holds then the transfer

$$w(\theta) \;=\; \int_{\underline{\theta}}^{\theta} u_2(x,x)\,\mathrm{d}x - u(\theta,\theta) \tag{D.1.5}$$

ensures that truthful reporting in optimal.

Proof of Lemma D.1.1. First, necessity is proven. Suppose that truthful reporting is optimal and let $U(\theta) = \widehat{U}(\theta, \theta)$, then for any $\theta_1, \theta_2 \in [\underline{\theta}, \overline{\theta}]$ such that $\theta_1 < \theta_2$:

$$U(\theta_2) \;\geq\; \widehat{U}(\theta_1, \theta_2) \;=\; U(\theta_1) + \int_{\theta_1}^{\theta_2} \widehat{U}_2(\theta_1, \theta)\,\mathrm{d}\theta$$

$$U(\theta_1) \;\geq\; \widehat{U}(\theta_2, \theta_1) \;=\; U(\theta_2) - \int_{\theta_1}^{\theta_2} \widehat{U}_2(\theta_2, \theta)\mathrm{d}\theta$$

where the subscript 2 indicates the partial derivative with respect to the second entry. Rearranging the two inequalities and combining them gives

$$\int_{\theta_1}^{\theta_2} \widehat{U}_2(\theta_2,\theta)\,\mathrm{d}\theta \;\geq\; \int_{\theta_1}^{\theta_2} \widehat{U}_2^s(\theta_1,\theta)\,\mathrm{d}\theta.$$

As this has to hold for any choice of θ_1 and θ_2, as defined above, it follows that $u_2(\hat{\theta},\theta) = \widehat{U}_2(\hat{\theta},\theta)$ has to be non-decreasing on $\hat{\theta}$. Second, sufficiency is proven. Suppose $\hat{\theta} < \theta$, then

$$\begin{aligned} w(\hat{\theta}) + u(\hat{\theta},\theta) \;&=\; u(\hat{\theta},\theta) - u(\hat{\theta},\hat{\theta}) + \int_{\underline{\theta}}^{\widehat{\theta}} u_2(x,x)\,\mathrm{d}x \\ &=\; \int_{\widehat{\theta}}^{\theta} u_2(\hat{\theta},x)\,\mathrm{d}x + \int_{\underline{\theta}}^{\widehat{\theta}} u_2(x,x)\,\mathrm{d}x \\ &=\; \int_{\widehat{\theta}}^{\theta} \Big\{u_2(\hat{\theta},x) - u_2(x,x)\Big\}\,\mathrm{d}x + \int_{\underline{\theta}}^{\theta} u_2(x,x)\,\mathrm{d}x \;\leq\; \int_{\underline{\theta}}^{\theta} u_2(x,x)\,\mathrm{d}x \;=\; w(\theta) + u(\theta,\theta) \end{aligned}$$

As a result reporting $\hat{\theta} = \theta$ is no worse than any $\hat{\theta} < \theta$. The proof for $\hat{\theta} > \theta$ is similar hence it is omitted. □

Proof of Lemma 4.5. The agent's total payoff at time zero is

$$W_0(\hat{\theta}_0) \;+\; \widehat{U}(\hat{\theta}_0,\theta_0)$$

$\widehat{U}(\hat{\theta}_0,\theta_0)$ has two functional forms depending on if $\widehat{q}(\hat{\theta}_0,\theta_0) \leq 1$ or not. However, the function is absolute continuous on θ_0. In particular, if $\widehat{q}(\hat{\theta}_0,\theta_0) > 1$, then

$$\widehat{U}(\hat{\theta}_0,\theta_0) \;=\; \lambda\cdot\theta_0 \;+\; \overline{W}(\hat{\theta}_0) \quad\Rightarrow\quad \widehat{U}_2(\hat{\theta}_0,\theta_0) \;=\; \lambda \tag{D.1.6}$$

which is constant, and as a result non-decreasing in $\hat{\theta}_0$. Next, we demonstrate the same result for the case where $\widehat{q}(\hat{\theta}_0,\theta_0) \leq 1$. Substituting $\widehat{q}(\hat{\theta}_0,\theta_0) = q(\hat{\theta}_0)\,\hat{\theta}_0/\theta_0$ and cancelling out some terms gives

$$\widehat{U}(\hat{\theta}_0,\theta_0) \;=\; \frac{a\,\theta_0/[2(r-\mu)]}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} + \frac{(\theta_0)^{-c}}{c}\cdot\left(\frac{a/[2(r-\mu)]}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \lambda\right)\cdot(\hat{\theta}_0)^{1+c}\cdot q(\hat{\theta}_0)^{1+c}$$

Therefore, differentiating the above with respect to θ_0 we obtain that

$$
\begin{aligned}
\widehat{U}_2(\hat{\theta}_0, \theta_0) &= \frac{a/[2(r-\mu)]}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \left(\frac{a/[2(r-\mu)]}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \lambda \right) \cdot \left[\frac{\hat{\theta}_0}{\theta_0} \cdot q(\hat{\theta}_0) \right]^{1+c} \\
&= \frac{a/[2(r-\mu)]}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \left(\frac{a/[2(r-\mu)]}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \lambda \right) \cdot \widehat{q}(\hat{\theta}_0, \theta_0)^{1+c}
\end{aligned} \tag{D.1.7}
$$

The fact that the above is increasing in $\hat{\theta}_0$ follows from two observations. First, we are currently focusing on the case where $\widehat{q}(\hat{\theta}_0, \theta_0) \leq 1$. Second, ($\mathcal{A}$) together with Corollary 4.1 imply that $\hat{\theta}_0 q(\hat{\theta}_0)$ is decreasing in $\hat{\theta}_0$.

In addition, we have already shown that $\widehat{q}(\hat{\theta}_0, \theta_0) = q(\hat{\theta}_0)\hat{\theta}_0/\theta_0$, where again we get from Corollary 4.1 that the denominator is decreasing in $\hat{\theta}_0$. Therefore, $\widehat{q}(\hat{\theta}_0, \theta_0)$ is decreasing in $\hat{\theta}_0$. This implies that as $\hat{\theta}_0$ increases $\widehat{U}_2(\hat{\theta}_0, \theta_0)$ switches from (D.1.6) to (D.1.7). Moreover, on this unique switch it has to be that $\widehat{q}(\hat{\theta}_0, \theta_0) = 1$. Hence, $\widehat{U}_2(\hat{\theta}_0, \theta_0)$ is continuous with respect to $\hat{\theta}_0$. Thus $\widehat{U}_2(\hat{\theta}_0, \theta_0)$ is increasing for all $\hat{\theta}_0$ and θ_0.

Therefore, it follows immediately from Lemma D.1.1 that the proposed contract implements the revenue maximising effort level and termination time. In addition, it follows from (4.14) that the agent's payoff at period zero is

$$
\widehat{U}(\theta_0, \theta_0) + W_0(\theta_0) = \omega_a + \int_{\theta^*}^{\theta} \widehat{U}_2(x, x)\, \mathrm{d}x
$$

where

$$
\widehat{U}_2(\theta_0, \theta_0) = \begin{cases} \frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} - \left(\frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} - \lambda \right) \cdot q(\theta_0)^{1+c} & , \text{if } q(\theta_0) \leq 1 \\ \lambda & , \text{if } q(\theta_0) \geq 1 \end{cases} \tag{D.1.8}
$$

But remember that the agent's payoff in Section 3.3 has been given from Lemma 4.2 to be

$$
\begin{aligned}
V'(\theta_0) &= \mathbb{E}_z \left[\int_0^{\tau} e^{-rt} \cdot \frac{e_t(\theta^t)^2}{2\,\theta_0} \mathrm{d}t + e^{-r\tau} \cdot z_\tau \cdot M_2(m_\tau, \theta_\tau) \right] \\
&= \mathbb{E}_z \left[\frac{a/2}{[1+\eta(\theta_0)/\theta_0]^2} \cdot \int_0^{\tau} e^{-rt} z_t \, \mathrm{d}t + \lambda \cdot e^{-r\tau} z_\tau \right]
\end{aligned}
$$

where the second line follows from substituting the optimal effort of (4.8) and the linear function form of $M(\cdot)$. Hence for $q(\theta_0) > 1$ we get that $\widehat{U}_2(\theta_0, \theta_0) = V'(\theta_0)$.

It remains to prove the same for the case where $q(\theta_0) < 1$. Since the optimal stopping time is a hitting time on a linear barrier we can use the results of Appendix D.2.2 to further simplify the above to the following expression.

$$\begin{aligned} V'(\theta_0) &= \frac{a/2}{[1+\eta(\theta_0)/\theta_0]^2} \cdot \frac{1-q(\theta_0)^{1+c}}{r-\mu} + \lambda \cdot q(\theta_0)^{1+c} \\ &= \frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} - \left(\frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} - \lambda \right) \cdot q(\theta)^{1+c} \end{aligned} \tag{D.1.9}$$

where again comparing expressions we get that $\widehat{U}_2(\theta_0, \theta_0) = V'(\theta_0)$. This proves (4.18).

□

D.2 Optimal stopping problems

D.2.1 The problem and its solution

We present here a more detailed treatment of the optimal stopping problems we encounter in the paper. Those have the following form. Consider probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ with an one dimensional $\mathcal{F}_t$-Brownian motion B_t. The process of interest will be described by the stochastic differential equation

$$\mathrm{d}Z_t = \mu \cdot Z_t \, \mathrm{d}t + \sigma \cdot Z_t \, \mathrm{d}B_t \tag{D.2.1}$$

where $Z_0 = 1$. Our aim is to maximise the functional

$$J[\tau] = \mathbb{E}_P \left[\int_0^\tau e^{-r\,t} \, a \cdot Z_t \, \mathrm{d}t + e^{-r\,\tau} \cdot \big(B \cdot Z_\tau + C \big) \right] \tag{D.2.2}$$

where $a, B, C, r, \sigma > 0$ and $r > \mu$.

First, assume that $a \leq (r-\mu)B$. Let $b = (r-\mu)B$ and $\bar{c} = (r-\mu)C$. Then

$$\mathbb{E}_P \left[e^{-r\,\tau} \cdot (B \cdot Z_\tau + C) \right] = \mathbb{E}_P \left[\int_\tau^\infty e^{-r\,t} \cdot (b \cdot Z_t + \bar{c}) \, \mathrm{d}t \right]$$

Hence $J[\tau]$ equivalently becomes

$$J[\tau] = \mathbb{E}_P \left[\int_0^\tau e^{-r\,t} \, a \cdot Z_t \, \mathrm{d}t + \int_\tau^\infty e^{-r\,t} \cdot (b \cdot Z_t + \bar{c}) \, \mathrm{d}t \right]$$

But then $\tau = 0$ is optimal since $bZ_t + \bar{c} > aZ_t$ for any positive Z_t.

Hereafter we consider the opposite case $a > B\cdot(r-\mu)$. To do this note that

$$\mathbb{E}_P\left[\int_\tau^\infty e^{-r\,t}\,a\cdot Z_t\,\mathrm{d}t\right] \;=\; \mathbb{E}_P\left[a\cdot\mathbb{E}_P\left[\int_\tau^\infty e^{-r\,t}Z_t\,\mathrm{d}t\,\middle|\,Z_\tau\right]\right] \;=\; \mathbb{E}_P\left[a\cdot\frac{e^{-r\tau}Z_\tau}{r-\mu}\right]$$

Therefore, add and subtract $\mathbb{E}_P\left[a\,e^{-r\tau}Z_\tau/(r-\mu)\right]$ in the left hand side below to obtain that

$$\begin{aligned}\mathbb{E}_P\left[\int_0^\tau e^{-r\,t}\,a\cdot Z_t\,\mathrm{d}t\right] \;&=\; \mathbb{E}_P\left[\int_0^\infty e^{-r\,t}\,a\cdot Z_t\,\mathrm{d}t\right]-\mathbb{E}_P\left[a\cdot\frac{e^{-r\tau}Z_\tau}{r-\mu}\right]\\ &=\; a\cdot\mathbb{E}_P\left[\int_0^\infty e^{-r\,t}\cdot e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma B_t}\,\mathrm{d}t\right]-\mathbb{E}_P\left[a\cdot\frac{e^{-r\tau}Z_\tau}{r-\mu}\right]\\ &=\; a\cdot\int_0^\infty e^{-(r-\mu)t}\mathrm{d}t-\mathbb{E}_P\left[a\cdot\frac{e^{-r\tau}Z_\tau}{r-\mu}\right]\\ &=\; \frac{a}{r-\mu}\;-\;\mathbb{E}_P\left[a\cdot\frac{e^{-r\tau}Z_\tau}{r-\mu}\right]\end{aligned}$$

Therefore $J[\tau]$ equivalently becomes

$$\begin{aligned}J[\tau]\;&=\;\frac{a}{r-\mu}+\mathbb{E}_P\left[e^{-r\,\tau}\cdot\left\{C-\left(\frac{a}{r-\mu}-B\right)\cdot Z_\tau\right\}\right]\\ &=\;\frac{a}{r-\mu}+\left(\frac{a}{r-\mu}-B\right)\cdot\mathbb{E}_P\left[e^{-r\,\tau}\cdot\left(\frac{C}{\frac{a}{r-\mu}-B}-Z_\tau\right)\right]\end{aligned}\tag{D.2.3}$$

Hence, we can focus on selecting the stopping time that maximises the expectation on the second line above. This is a problem that has already being considered in McDonald and Siegel (1986). In the aforementioned paper it is demonstrated that the optimal stopping time takes the following form.

$$\tau^*\;=\;\left\{t\geq 0\;:\;Z_t\leq q\right\}\quad\text{where}\quad q\;=\;\frac{c}{1+c}\cdot\frac{C}{\frac{a}{r-\mu}-B}\tag{D.2.4}$$

where c is as define in (4.2). Hence if $q>1$, then again $\tau=0$ and

$$J[\tau^*]\;=\;B+C$$

Otherwise, if $q\leq 1$ they demonstrate that the value of $J[\tau]$ on its optimum is

$$J[\tau^*]\;=\;\frac{a}{r-\mu}+\left(\frac{a}{r-\mu}-B\right)\cdot\frac{q^{1+c}}{c}$$

And those two functional forms are continuous on parameters, since $q=1$ implies that the two given functional forms are also equal.

D.2.2 Expected discounted payoffs

In this subsection we demonstrate how standard results in hitting times with linear barriers can be used in our setting to derive the expected discounted value of a process before and after its stopping.

Consider the following Brownian motion with drift

$$\mathrm{d}X_t \;=\; \mu \cdot \mathrm{d}t \,+\, \sigma \cdot \mathrm{d}B_t \tag{D.2.5}$$

with initial condition $X_0 = 0$ and an associate barrier $a > 0$, which implies the following hitting time

$$\tau_a \;=\; \inf\left\{t \geq 0 \,:\, X_t = a\right\} \tag{D.2.6}$$

We state without proving the following result.

Lemma D.2.1. *For every $r > 0$*

$$\mathbb{E}\left[e^{-r\tau_a}\right] \;=\; \exp\left[a \cdot \left(\frac{\mu}{\sigma^2} - \sqrt{\frac{2\,r}{\sigma^2} + \frac{\mu^2}{\sigma^4}}\right)\right] \tag{D.2.7}$$

Next, consider the Geometric Brownian Motion

$$\mathrm{d}Z_t \;=\; Z_t \cdot \mu \,\mathrm{d}t + Z_t \cdot \sigma \,\mathrm{d}B_t$$

with initial condition $Z_0 = 1$. We state without proving that

$$Z_t \;=\; e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}$$

Let $q \in (0, 1)$ be a constant barrier and τ_q the associated hitting time of Z_t, that is

$$\tau_q \;=\; \inf\left\{t \geq 0 \,:\, Z_t = q\right\}$$

To related τ_q to τ_a we solve

$$Z_t \;=\; q \;\Leftrightarrow\; \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t \;=\; -\log\frac{1}{q}$$
$$-\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma(-B_t) \;=\; \log\frac{1}{q}$$

But B_t and $-B_t$ have exactly the same distribution and $\log \frac{1}{q} > 0$. Hence, we substitute (D.2.7) to obtain

$$\mathbb{E}\left[e^{-r\tau_q}\right] \;=\; \exp\left[\log\frac{1}{q}\cdot\left[-\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)-\sqrt{2\,r+\left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)^2}\right]\right] \;=\; q^c \tag{D.2.8}$$

where c is as defined in (4.2). In addition, note that

$$\mathbb{E}_Z\left[\int_0^\infty e^{-rt}Z_t\,\mathrm{d}t\right] \;=\; \mathbb{E}_Z\left[\int_0^\infty e^{-rt}Z_t\,\mathrm{d}t\,\middle|\,\tau\leq t\right]+\mathbb{E}_Z\left[\int_0^\infty e^{-rt}Z_t\,\mathrm{d}t\,\middle|\,\tau\geq t\right] \tag{D.2.9}$$

But

$$\mathbb{E}_z\left[\int_0^\infty e^{-rt}Z_t\,\mathrm{d}t\right] \;=\; \int_0^\infty e^{-rt}\mathbb{E}_Z\left[Z_t\right]\,\mathrm{d}t \;=\; \int_0^\infty e^{-rt}e^{\mu t}\,\mathrm{d}t \;=\; \frac{1}{r-\mu}$$

and

$$\mathbb{E}_Z\left[\int_0^\infty e^{-rt}Z_t\,\mathrm{d}t\,\middle|\,\tau\geq t\right] \;=\; \mathbb{E}_Z\left[e^{-r\tau}\cdot\frac{q}{r-\mu}\right] \;=\; \frac{q^{1+c}}{r-\mu} \tag{D.2.10}$$

where the first equality follows from noticing that on each realisation of τ the value of Z_t will be equal to q. Hence the expected value of the continuation value after each τ is $q/(r-\mu)$. Thus, we can substitute back in (D.2.9) to obtain that

$$\mathbb{E}_Z\left[\int_0^\infty e^{-rt}Z_t\,\mathrm{d}t\,\middle|\,\tau\leq t\right] \;=\; \frac{1-q^{1+c}}{r-\mu} \tag{D.2.11}$$

D.3 Model with two principals

Our main analysis assumes an exogenous functional form for the managerial labour market. In this section, we want to demonstrate that our results can partially be extended to a model in which the agent's post-termination payoff is generated by a new employment offer from a second principal.

In particular, suppose that at the point of his termination the agent is offered a tenure contract from a second principal, that is the agent's employment is up to infinity. Production under the second principal is given by

$$y_t \;=\; \sqrt{b\cdot\theta_t}\cdot e_t$$

In addition, in order to simplify our analysis we will assume that the second principal can observe directly the agent's reports to the first. In some sense, we require that the communication between the first principal and the agent is public. This assumption, together with the introduction of the second principal, creates an even more sophisticated version of the managerial labour market than the one we considered in the previous section.

Finally, we will restrict the first principal to only use direct and incentive compatibly contracts[1]. This will result on the agent receiving no rents from his employment from the second principal, since his reports on his ability will be public. Nevertheless, we will endow the agent with some market power by assuming that he captures $1 - \lambda$ of the expected surplus that his employment under the second principal creates. For simplicity, we set the outside options of the agent and the second principal to zero.

Our analysis will proceed as follows. In Section D.3.1 we will derive the optimal termination time of this model under consistent deviations. Subsequently, in Section D.3.2 we will present a contract that implements this termination times even when the agent's action space is unrestricted.

D.3.1 Optimal effort and termination

We start by first considering the interaction between the second principal and the agent. Since there is no information asymmetry between the two it follows that the implemented level of effort is the first best one

$$e_t^b(\theta_t) \;=\; \sqrt{b \cdot \theta_t} \tag{D.3.1}$$

Therefore, the flow surplus of his employment under the second principal is $b \cdot \theta_t / 2$. Hence, it follows from the properties of the Geometric Brownian motion that the expected discounted surplus at the beginning of this employment is $b \cdot \theta_\tau / [2(r - \mu)]$.

Now consider the post-termination payoff of an agent that uses the natural extension of consistent deviations in the current setup. In particular, the agent is restricted to continue to hide his misreport at time zero even under the second principal. Let $\widehat{V}^b(\hat{\theta}_0, \theta_0)$ denote the agent's payoff under such a deviation, and $V^b(\theta_0) = \widehat{V}^b(\theta_0, \theta_0)$ the associated on path payoff.

[1]This is with loss of generality because the information that the first principal receives also becomes available to the second. For example, it could be profitable for the first principal to receives less information, which would decrease the efficiency of his production, in order to force the second one to leave some rents for the agent.

Then

$$\begin{aligned}
\widehat{V}^b(\hat{\theta}_0, \theta_0) \;&=\; \mathbb{E}_\theta\left[\int_\tau^\infty e^{-r(t-\tau)} \cdot \left(w_t^b(\hat{\theta}^t) - \frac{e_t^b(\hat{\theta}^t)^2}{2} \cdot \frac{\hat{\theta}_0}{\theta_0}\right) \mathrm{d}t \,\middle|\, \theta_\tau\right] \\
&=\; V^b(\hat{\theta}_0) \,+\, \mathbb{E}_\theta\left[\int_\tau^\infty e^{-r(t-\tau)} \cdot \frac{e_t^b(\hat{\theta}^t)^2}{2} \cdot \left(1 - \frac{\hat{\theta}_0}{\theta_0}\right) \mathrm{d}t \,\middle|\, \theta_\tau\right] \\
&=\; (1-\lambda) \cdot \frac{b \cdot z_{\hat{\tau}} \cdot \hat{\theta}_0}{2\,(r-\mu)} \,+\, \frac{b \cdot z_{\hat{\tau}} \cdot \hat{\theta}_0}{2\,(r-\mu)} \cdot \left(1 - \frac{\hat{\theta}_0}{\theta_0}\right)
\end{aligned}$$

where as in the main analysis $z_{\hat{\tau}}$ denotes the proportional change of the manager's initial ability at the point of his termination, under the misreport $\hat{\theta}_0$.

Lemma D.3.1 (Envelop Theorem)**.** *The agent's on path payoff is absolutely continuous and has the weak derivative*

$$V'(\theta_0) \;=\; \mathbb{E}_z\left[\int_0^\tau e^{-rt} \cdot \frac{e_t(\theta^t)^2}{2\,\theta_0}\,\mathrm{d}t \,+\, e^{-r\tau} \cdot \widehat{V}_2^b(\theta_0, \theta_0)\right] \tag{D.3.2}$$

Proof. Identical with that of of Lemma 4.2. □

But then repeating the analysis of our main section we obtain the following representation of the first principal's revenue

$$\begin{aligned}
\max_{e,c} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_z\Bigg[\int_0^\tau e^{-rt} \cdot \left(\sqrt{a\,\theta_0\,z_t} \cdot e_t(\theta^t) - \left[1 + \eta(\theta_0)/\theta_0\right] \cdot \frac{e_t(\theta^t)^2}{2}\right)\mathrm{d}t \\
+\, e^{-r\tau} \cdot \left(\omega_p + \widehat{V}^b(\theta_0, \theta_0) - \,\eta(\theta_0)\,\widehat{V}_2^b(\theta_0, \theta_0)\right)\Bigg]\,\mathrm{d}F(\theta_0) \,+\, F(\theta^*) \cdot \omega_p \quad (\mathcal{P}_a)
\end{aligned}$$

which after substituting the expression of $\widehat{V}^b(\hat{\theta}_0)$ and its derivative becomes

$$\begin{aligned}
\max_{e,c} \int_{\theta^*}^{\overline{\theta}} \mathbb{E}_z\Bigg[\int_0^\tau e^{-rt} \cdot \left(\sqrt{a\,\theta_0\,z_t} \cdot e_t(\theta^t) - \left[1 + \eta(\theta_0)/\theta_0\right] \cdot \frac{e_t(\theta^t)^2}{2}\right)\mathrm{d}t \\
+\, e^{-r\tau} \cdot \left(\omega_p + \frac{b \cdot z_\tau}{2\,(r-\mu)} \cdot \left[(1-\lambda) \cdot \theta_0 - \eta(\theta_0)\right]\right)\Bigg]\,\mathrm{d}F(\theta_0) \,+\, F(\theta^*) \cdot \omega_p \quad (\mathcal{P}_a')
\end{aligned}$$

Therefore, maximising the first line of the above gives the point-wise optimal level of effort, which is the same as that of our main analysis.

$$e_t^*(\theta^t) \;=\; \frac{\sqrt{a\,\theta_0}}{1 + \eta(\theta_0)/\theta_0} \cdot \sqrt{z_t} \tag{D.3.3}$$

Hence, for each θ_0 the first principal is facing the following stopping problem

$$\max_{\tau} \int_{\underline{\theta}}^{\overline{\theta}} \mathbb{E}_z \left[k(\theta_0) \cdot \int_0^{\tau} e^{-rt} \, z_t \, \mathrm{d}t + e^{-r\tau} \cdot \left(\omega_p + z_\tau \cdot b \cdot \frac{(1-\lambda) \cdot \theta_0 - \eta(\theta_0)}{2\,(r-\mu)} \right) \right] \mathrm{d}F(\theta_0) \qquad (\mathcal{T}_a)$$

Then the following proposition is implied by our analysis in Abbendix B.

Proposition D.3.1. *The solution of* ($\mathcal{T}_a$) *is*

$$\tau^*(\theta_0) \;=\; \inf\left\{ t \geq 0 \,:\, z_t \;\leq\; q(\theta_0) \right\} \qquad \text{(D.3.4)}$$

where

$$q(\theta_0) \;=\; \frac{c \cdot \omega_p}{1+c} \cdot \left[\frac{k(\theta_0)}{r-\mu} - b \cdot \frac{(1-\lambda) \cdot \theta_0 - \eta(\theta_0)}{2\,(r-\mu)} \right]^{-1} \qquad \text{(D.3.5)}$$

if the expression in the brackets is positive. Otherwise, $\tau_{\theta_0} = 0$.

Similarly to our analysis in the main part we are interested on how the cutoff $q(\theta_0)$ depends on θ_0 and other paparements.

Corollary D.1 (Terminal Abilities)**.** *Let* $Q(\theta_0) = \theta_0 \cdot q(\theta_0)$ *denote the level of ability on which the initial type* θ_0 *will be terminated. Then*

- *a sufficient condition for* $Q(\theta_0)$ *to be decreasing in* θ_0 *is that the firm is relatively productive:*

$$\frac{a}{b} \;\geq\; \frac{1}{2} \cdot \left[1 + \eta(\underline{\theta})/\underline{\theta}\right]^3 \qquad (\mathcal{A}')$$

- *When* $Q(\theta_0) < \theta_0$*, this is increasing in the productive of the first principal, a, for all* $\theta_0 \in [\underline{\theta}, \overline{\theta}]$*, and decreasing in that of the second if and only if*

$$1 - \lambda \;\geq\; \frac{\eta(\theta_0)}{\theta_0} \qquad \text{(D.3.6)}$$

Proof of Corollary D.1. Substituting the functional form of $q(\theta_0)$ we obtain that

$$Q(\theta_0) \;\propto\; \left[\frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} - b \cdot \frac{1-\lambda-\eta(\theta_0)/\theta_0}{2\,(r-\mu)} \right]^{-1}$$

where we have ignored the positive constants that multiply it. Hence, differentiating we obtain that

$$Q'(\theta_0) \;\propto\; -\frac{\partial\,[\eta(\theta_0)/\theta_0]}{\partial\,\theta_0} \cdot \left[\frac{a/[2(r-\mu)]}{[1+\eta(\theta_0)/\theta_0]^2} - b \cdot \frac{1-\lambda-\eta(\theta_0)/\theta_0}{2\,(r-\mu)} \right]^{-2} \cdot \frac{b - \frac{2 \cdot a}{[1+\eta(\theta_0)/\theta_0]^3}}{2\,(r-\mu)}$$

But $\eta(\theta_0)$ is non-increasing by assumption. Hence the above derivative is negative if and only if

$$\frac{a}{b} \;\geq\; \frac{1}{2} \cdot \left[1 + \eta(\theta_0)/\theta_0\right]^3$$

which is implied by condition ($\mathcal{A}'$). □

D.3.2 Implementation

Here we provide a contract that generates the policies and payoffs that we identified in the previous section. Because there are two principals we will have to even specify the contract of the second.

To maintain the analysis as short as possible we will simply assume that the second principal sells her firm to the agent for prince

$$p(\hat{\theta}_0) \;=\; \lambda \cdot \frac{b \cdot Q(\hat{\theta}_0)}{2\,(r-\mu)} \tag{D.3.7}$$

which on path leaves to the agent the correct surplus of production.

Next, we focus on the contract of the first principal. As in our main analysis, this will require from the agent to make a choice from a menu only at time zero.

$$\Big\{\, w(\hat{\theta}_0, y_t),\; [W_t(\hat{\theta}_0)]_{t>0},\; W_0(\hat{\theta}_0) \Big\} \tag{$\mathcal{W}$}$$

In addition, the principal allows the agent to decide on his own when it is the optimal time to leave. The first component of this payoff is the linear flow wage

$$w_t(\hat{\theta}_0, y_t) \;=\; w(\hat{\theta}_0) \cdot y_t \qquad \text{where} \qquad w(\hat{\theta}_0) \;=\; \frac{\sqrt{a}}{1 + \eta(\hat{\theta}_0)/\hat{\theta}_0} \tag{D.3.8}$$

which is identical to that of the main analysis. However, because the agent's post-termination payoff and the optimal stopping time are different the associated golden parachute will also differ.

$$W_t(\hat{\theta}_0) \;=\; \overline{W}(\hat{\theta}_0) + p(\hat{\theta}_0) \qquad \text{where} \qquad \overline{W}(\hat{\theta}_0) \;=\; \frac{\omega_p \cdot \hat{\theta}_0 \cdot \left(\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - b\right)}{(1-\lambda) \cdot \hat{\theta}_0 - \eta(\hat{\theta}_0)} \tag{D.3.9}$$

where $u(\theta_0)$, and $u(\hat{\theta}_0, \theta_0)$ which we will use later, is as in the main body. This is paid to the agent at any point $t > 0$ that he reports that his employment should be terminated.

The third and final component of the menu that the principal offers to the agent is the signing bonus $W_0(\hat{\theta}_0)$, then functional form of which is given by

$$W_0(\hat{\theta}_0) \;=\; \omega_a + \int_{\theta^*}^{\theta} \widehat{U}_2(x,x)\,\mathrm{d}x - \widehat{U}(\theta_0,\theta_0) \tag{D.3.10}$$

where $\widehat{U}(\hat{\theta}_0,\theta_0)$ denotes the continuation payoff (excluding the signing bonus of time zero) of a manager whose initial ability is θ_0, but his choice from the above menu was that corresponding to $\hat{\theta}_0$. The functional form of this function is given below.

As we mentioned before, our implementation result establishes that the analysis of the previous section is valid even when we do not impose any restrictions on the agent's action space.

Proposition D.3.2 (Implementation)**.** *Suppose that* ($\mathcal{A}'$) *holds. Then the menu of contracts* ($\mathcal{W}$) *implements the optimal policies* $e^*(\theta^t)$ *and* $\tau^*(\theta_0)$ *as identified in* (D.3.3) *and* (D.3.4)*, respectively. In addition, the implied payoff for the principal and the agent are also the same with those of the previous section.*

The rest of this section proves the above proposition. We have already shown in the main analysis that the given flow wage implements the correct level of effort in the initial ability has been truthfully reported. In addition, it implies that the agent's optimal stopping problem is

$$\max_{\tau}\;\; \mathbb{E}_z\Big[u(\hat{\theta}_0,\theta_0)\cdot\int_0^{\tau} e^{-rt}\,z_t\,\mathrm{d}t + e^{-r\tau}\cdot\Big(z_\tau\cdot\frac{b\cdot\theta_0}{2\,(r-\mu)} + \overline{W}(\hat{\theta}_0)\Big)\Big] \tag{$\mathcal{T}_a$}$$

To solve this we use again our generic analysis in Appendix D.1.

Lemma D.3.2 (Golden Parachutes)**.** *The solution of* ($\mathcal{T}_a$) *is*

$$\tau_a \;=\; \inf\big\{t\geq 0\,:\, z_t\,\leq\,\widehat{q}(\hat{\theta}_0,\theta_0)\big\} \tag{D.3.11}$$

where

$$\widehat{q}(\hat{\theta}_0,\theta_0) \;=\; \frac{c\cdot\overline{W}(\hat{\theta}_0)}{1+c}\cdot\left[\left(\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2}-b\right)\cdot\frac{\theta_0}{2\,(r-\mu)}\right]^{-1} \tag{D.3.12}$$

In addition, $\widehat{q}(\theta_0,\theta_0)=q(\theta_0)$ *and* $\widehat{q}(\hat{\theta}_0,\theta_0)=q(\hat{\theta}_0)\cdot\frac{\hat{\theta}_0}{\theta_0}$ *.*

Proof. The stopping time τ_a and the associated barrier $\widehat{q}(\hat{\theta}_0,\theta_0)$ follow immediately from the solution of the optimal stopping problem of Appendix D.1. In addition, we obtain that $\widehat{q}(\theta_0,\theta_0)=q(\theta_0)$ by substituting $\overline{W}(\theta_0)$ in the left hand side of this inequality, which can also be used to obtain that $\widehat{q}(\hat{\theta}_0,\theta_0)=\widehat{q}(\hat{\theta}_0,\hat{\theta}_0)\cdot\hat{\theta}_0/\theta_0$. □

Hence we have shown that if the initial type was truthfully reported, then the given flow wage and golden parachute implement the revenue maximising stopping time.

It remains to implement the truthful reporting of the initial type θ_0, which is achieved with the signing bonus $W_0(\hat{\theta}_0)$. Our analysis in Appendix D.1 gives that the agent's payoff at time zero, net of $W_0(\hat{\theta}_0)$, is given by

$$\widehat{U}(\hat{\theta}_0, \theta_0) \;=\; \begin{cases} \frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2}\frac{\theta_0}{2(r-\mu} \;+\; \left(\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - b\right)\cdot\frac{\theta_0}{2(r-\mu)}\cdot\frac{\widehat{q}(\hat{\theta}_0,\theta_0)^{1+c}}{c} & \text{, if } \;\widehat{q}(\hat{\theta}_0,\theta_0) \le 1 \\ \frac{b\cdot\theta_0}{2\,(r-\mu)} + \overline{W}(\hat{\theta}_0) & \text{, if } \;\widehat{q}(\hat{\theta}_0,\theta_0) > 1 \end{cases} \tag{D.3.13}$$

From which the following result follows.

Lemma D.3.3. *The signing bonus $W_0(\hat{\theta}_0)$ implements the truthful reporting of θ_0. In addition,*

$$\widehat{U}(\theta_0, \theta_0) \;+\; W_0(\theta_0) \;=\; \omega_a + \int_{\theta^*}^{\theta_0} V'(x)\,\mathrm{d}x \tag{D.3.14}$$

where $V'(x)$ is as given in (D.3.2), but calculated under the optimal effort and termination policies.

Proof. The agent's total payoff at time zero is

$$W_0(\hat{\theta}_0) \;+\; \widehat{U}(\hat{\theta}_0, \theta_0)$$

$\widehat{U}(\hat{\theta}_0, \theta_0)$ has two functional forms depending on if $\widehat{q}(\hat{\theta}_0, \theta_0) \le 1$ or not. However, the function is absolute continuous on θ_0. In particular, if $\widehat{q}(\hat{\theta}_0, \theta_0) > 1$, then

$$\widehat{U}(\hat{\theta}_0, \theta_0) \;=\; \frac{b\cdot\theta_0}{2\,(r-\mu)} + \overline{W}(\hat{\theta}_0) \quad\Rightarrow\quad \widehat{U}_2(\hat{\theta}_0, \theta_0) \;=\; \frac{b}{2\,(r-\mu)} \tag{D.3.15}$$

which is constant, and as a result non-decreasing in $\hat{\theta}_0$. Next, we demonstrate the same result for the case where $\widehat{q}(\hat{\theta}_0, \theta_0) \le 1$. Substituting $\widehat{q}(\hat{\theta}_0, \theta_0) = q(\hat{\theta}_0)\,\hat{\theta}_0/\theta_0$ and cancelling out some terms gives

$$\widehat{U}(\hat{\theta}_0, \theta_0) \;=\; \frac{1}{2\,(r-\mu)}\left[\frac{a\cdot\theta_0}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} + \left(\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - b\right)\cdot(\theta_0)^{-c}\cdot\frac{[\widehat{q}(\hat{\theta}_0)\cdot\hat{\theta}_0]^{1+c}}{c}\right]$$

Therefore, differentiating the above with respect to θ_0 we obtain that

$$\begin{aligned}\widehat{U}_2(\hat{\theta}_0,\theta_0) &= \frac{1}{2\,(r-\mu)}\left[\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \left(\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - b\right)\cdot\left[\frac{\hat{\theta}_0}{\theta_0}\cdot\widehat{q}(\hat{\theta}_0)\right]^{1+c}\right] \\ &= \frac{1}{2\,(r-\mu)}\left[\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - \left(\frac{a}{[1+\eta(\hat{\theta}_0)/\hat{\theta}_0]^2} - b\right)\cdot\widehat{q}(\hat{\theta}_0,\theta_0)^{1+c}\right]\end{aligned} \tag{D.3.16}$$

The fact that the above is increasing in $\hat{\theta}_0$ follows from two observations. First, we are currently focusing on the case where $\widehat{q}(\hat{\theta}_0,\theta_0)\le 1$. Second, ($\mathcal{A}'$) together with Corollary D.1 imply that $\hat{\theta}_0 q(\hat{\theta}_0)$ is decreasing in $\hat{\theta}_0$. But we have already shown that $\widehat{q}(\hat{\theta}_0,\theta_0) = q(\hat{\theta}_0)\hat{\theta}_0/\theta_0$, where again we get from Corollary D.1 that the denominator is decreasing in $\hat{\theta}_0$. Therefore, $\widehat{q}(\hat{\theta}_0,\theta_0)$ is decreasing in $\hat{\theta}_0$. This implies that as $\hat{\theta}_0$ increases $\widehat{U}_2(\hat{\theta}_0,\theta_0)$ switches from (D.3.15) to (D.3.16). Moreover, on this unique switch it has to be that $\widehat{q}(\hat{\theta}_0,\theta_0)=1$. Hence, $\widehat{U}_2(\hat{\theta}_0,\theta_0)$ is continuous with respect to $\hat{\theta}_0$. Thus $\widehat{U}_2(\hat{\theta}_0,\theta_0)$ is increasing for all $\hat{\theta}_0$ and θ_0.

Therefore, it follows immediately from Lemma D.1.1 that the proposed contract implements the revenue maximising effort level and termination time. In addition, it follows from (D.3.10) that the agent's payoff at period zero is

$$\widehat{U}(\theta_0,\theta_0) + W_0(\theta_0) = \omega_a + \int_{\theta^*}^{\theta}\widehat{U}_2(x,x)\,\mathrm{d}x$$

where

$$\widehat{U}_2(\theta_0,\theta_0) = \begin{cases}\frac{1}{2\,(r-\mu)}\left[\frac{a}{[1+\eta(\theta_0)/\theta_0]^2} - \left(\frac{a}{[1+\eta(\theta_0)/\theta_0]^2}-b\right)\cdot q(\theta_0)^{1+c}\right] & \text{, if } q(\theta_0)\le 1\\ \frac{b}{2\,(r-\mu)} & \text{, if } q(\theta_0)\ge 1\end{cases} \tag{D.3.17}$$

But remember that the agent's payoff in Section 3.3 has been given from Lemma 4.2 to be

$$\begin{aligned}V'(\theta_0) &= \mathbb{E}_z\left[\int_0^\tau e^{-rt}\cdot\frac{e_t(\theta^t)^2}{2\,\theta_0}\,\mathrm{d}t + e^{-r\tau}\cdot\widehat{V}_2^b(\theta_0,\theta_0)\right] \\ &= \mathbb{E}_z\left[\frac{a/2}{[1+\eta(\theta_0)/\theta_0]^2}\cdot\int_0^\tau e^{-rt}\,z_t\,\mathrm{d}t + e^{-r\tau}\cdot\frac{b\cdot z_\tau}{2\,(r-\mu)}\right]\end{aligned}$$

where the second line follows from substituting the optimal effort of (D.3.3) and the functional form of $V_2^b(\cdot)$. Hence for $q(\theta_0)>1$ we get that $\widehat{U}_2(\theta_0,\theta_0)=V'(\theta_0)$.

It remains to prove the same for the case where $q(\theta_0)<1$. Since the optimal stopping time is a hitting time on a linear barrier we can use the results of Appendix D.2.2 to further simplify the above to the following expression.

$$\begin{aligned} V'(\theta_0) &= \frac{a/2}{[1+\eta(\theta_0)/\theta_0]^2} \cdot \frac{1-q(\theta_0)^{1+c}}{r-\mu} + \frac{b \cdot q(\theta_0)^{1+c}}{2\,(r-\mu)} \\ &= \frac{1}{2\,(r-\mu)} \left[\frac{a}{[1+\eta(\theta_0)/\theta_0]^2} - \left(\frac{a}{[1+\eta(\theta_0)/\theta_0]^2} - b \right) \cdot q(\theta_0)^{1+c} \right] \end{aligned} \tag{D.3.18}$$

where again comparing expressions we get that $\widehat{U}_2(\theta_0, \theta_0) = V'(\theta_0)$. This proves (D.3.14).

□

Therefore, the proposed contract implements the revenue maximising effort level and termination time. In addition, (D.3.14) gives that the agent's payoff is the same with that we calculated in the previous section. Therefore, the principal's revenue has to also be the same, which completes the proof of Proposition D.3.2.