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Spontaneous Symmetry Breaking and Higgs Mode: Comparing Gross-Pitaevskii and Nonlinear Klein-Gordon Equations

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Abstract: We discuss the mechanism of spontaneous symmetry breaking and the elementary excitations for a weakly-interacting Bose gas at a finite temperature. We consider both the non-relativistic case, described by the Gross-Pitaevskii equation, and the relativistic one, described by the cubic nonlinear Klein-Gordon equation. We analyze similarities and differences in the two equations and, in particular, in the phase and amplitude modes (i.e., Goldstone and Higgs modes) of the bosonic matter field. We show that the coupling between phase and amplitude modes gives rise to a single gapless Bogoliubov spectrum in the non-relativistic case. Instead, in the relativistic case the spectrum has two branches: one is gapless and the other is gapped. In the non-relativistic limit we find that the relativistic spectrum reduces to the Bogoliubov one. Finally, as an application of the above analysis, we consider the Bose-Hubbard model close to the superfluid-Mott quantum phase transition and we investigate the elementary excitations of its effective action, which contains both non-relativistic and relativistic terms.

Keywords: superfluidity; Gross-Pitaevskii equation; nonlinear Klein-Gordon equation; Higgs mode; Bose-Hubbard model

1. Introduction

The mechanism of spontaneous symmetry breaking is widely used to study phase transitions [1]. Usually the approach introduced by Landau [2,3] for second-order phase transitions is adopted, where an order parameter is identified and its acquiring a non-zero value corresponds to a transition from a disordered phase to an ordered one. In other words, when a nonlinearity is added to the symmetric problem, and its strength exceeds a critical value, there is a loss of symmetry in the system, that is called spontaneous symmetry breaking, alias self-trapping into an asymmetric state [4]. In particular, for weakly-interacting Bose gases, the spontaneous breaking of the $U(1)$ group leads to the transition to a superfluid phase [1]. In this normal-to-superfluid phase transition the order parameter is the mean value of the bosonic matter field both in the non-relativistic case [5,6] and the relativistic one [7]. In the last years, symmetry breaking with the subsequent self-trapping has been investigated intensively by our group in the case of non-relativistic bosonic and fermionic superfluids made of alkali-metal atoms under the action of an external double-well potential [8,9] or in the presence of Josphson junctions [10–12].

In this review paper we compute and study the spectrum of elementary excitations for both non-relativistic and relativistic Bose gases in the ordered phase of the normal-to-superfluid phase transition. We calculate the elementary excitations by expanding the bosonic matter field as the sum of its mean value and fluctuations around it. We first study the Euclidean action of the bosonic gases and derive the elementary excitations as an intermediate step in the computation of the grand canonical potential. Then we consider the equations of motion of the bosonic field, which are the Gross-Pitaevskii equation [13,14] in the non-relativistic case and the cubic nonlinear Klein-Gordon equation [15,16] in the relativistic case, and we derive the linear equations of motion for fluctuations. We show that the complex fluctuating field around the symmetry-breaking uniform and constant solution can be written in terms of the angle field of the phase, the so-called Goldstone field [17] and an amplitude field, the so-called Higgs field [18]. For a discussion of the Goldstone and Higgs fields in Condensed Matter Physics see the recent review Ref. [19]. We then compare the results found for non-relativistic and relativistic cases. In recent years, the interplay between the spectrum of spontaneously broken ground state of the relativistic and the non-relativistic theories has thoroughly been studied (see, for instance, [20–23]). Here we show that while the non-relativistic Bose gas is characterized only by a gapless (Goldstone-like) mode, the relativistic Bose gas has also a gapped (Higgs-like) mode, whose energy gap goes to infinity as the non-relativistic limit is approached. In fact, the appearance of the gapless spectrum is the direct consequence of the general Goldstone theorem, which says that the number of gapless modes is equal to number of the number of broken generators [17]. More precisely, as shown by Nielsen and Chadha [24], in general there are two types of Goldstone bosons: those with an energy proportional to an even power of the momentum and those with a dispersion relation that is an odd power of the momentum. Within this context, a generalized Goldstone theorem holds [24]: the sum of twice the number of Goldstone modes of the first type and the number of Goldstone modes of the second type is at least equal to the number of independent broken symmetry generators. In our case we find one gapless mode since the broken symmetry group is $U(1)$, which has only one generator.

In the last section, the methods used for the weakly-interacting Bose gas are used to investigate the Bose-Hubbard model [25], which describes the non-relativistic dynamics of bosons on a lattice. Quite remarkably, close to the superfluid-Mott quantum phase transition, the Bose-Hubbard model is captured by an effective action which contains both non-relativistic and relativistic terms. We calculate and analyze the spectrum of elementary excitations of this effective action.

2. Spontaneous Symmetry Breaking: Non-Relativistic Case

2.1. Elementary Excitations from Non-Relativistic Partition Function

Let us consider a non-relativistic gas of weakly-interacting bosons in a volume V at absolute temperature T. The Eucidean action (imaginary time formalism) of the system is given by [5,6]:

$$S = \int_0^{\hbar\beta} d\tau \int_V d^D\vec{r} \left\{ \psi^* \left(\hbar \frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu \right) \psi + \frac{g}{2}|\psi|^4 \right\} \tag{1}$$

where $\psi(\vec{r},\tau)$ is the bosonic matter field, m is the mass of each bosonic particle and μ is the chemical potential which fixes the thermal average number of bosons in the system. We assume that the gas is dilute, such that we can approximate the interaction potential $V(\vec{r})$ with a contact interaction, i.e., setting $V(\vec{r}) = g\delta(\vec{r})$, where the coupling g by construction reads:

$$g = \int_V d^D\vec{r}\, V(\vec{r}) \tag{2}$$

The constant β is related to the absolute temperature T by:

$$\beta = \frac{1}{k_B T} \tag{3}$$

where k_B is the Boltzmann constant. By using functional integration can now define the partition function Z, and the grand canonical potential Ω, as follows [5,6]:

$$Z = \int D[\psi, \psi^*]\, exp\left\{-\frac{S[\psi,\psi^*]}{\hbar}\right\} \tag{4}$$

$$\Omega = -\frac{1}{\beta} ln(Z) \quad . \tag{5}$$

In the Lagrangian we can now consider the effective potential, let us call it V_{eff} defined as:

$$V_{eff} = -\mu|\psi|^2 + \frac{g}{2}|\psi|^4 \tag{6}$$

The phase transition correspond to a spontaneous symmetry breaking process and for this reason we need to find the minima of this potential [5,6]. We impose the conditions of stationarity on the first derivative:

$$\frac{\partial V_{eff}}{\partial \psi^*} = \psi(g|\psi|^2 - \mu) = 0 \tag{7}$$

and the minimum is given by:

$$|\psi_0| = \begin{cases} 0 & \text{if } \mu < 0 \\ \sqrt{\frac{\mu}{g}} & \text{if } \mu > 0 \end{cases} \tag{8}$$

The superfluid regime corresponds to the lower case. It is a condition on the modulus of ψ and therefore we have a circle of minima of radius $|\psi_0|$. The choice of a particular minimum breaks the $U(1)$ symmetry of the Lagrangian. For the superfluid phase we will take the real-valued vacuum expectation value, i.e., $\psi_0 = \psi_0^* = |\psi_0|$.

To maintain full generality in the following calculations however we leave the value of ψ_0 implicit. We can expand ψ as follows:

$$\psi(\vec{r},\tau) = \psi_0 + \eta(\vec{r},\tau) \tag{9}$$

where η is the complex fluctuation field. We can now expand the Lagrangian to the second-order (i.e., Gaussian) in the fluctuations:

$$S = \int_0^{\hbar\beta} d\tau \int_V d^D\vec{r} \quad \{ -\mu\psi_0^2 + \frac{1}{2}g\psi_0^4 + \psi_0\hbar\frac{\partial}{\partial\tau}\psi - \mu\psi_0(\eta+\eta^*) + g\psi_0^3(\eta+\eta^*) + \eta^*\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu + 2g\psi_0\right)\eta + \frac{g}{2}\psi_0^2(\eta\eta + \eta^*\eta^*)\} \tag{10}$$

The linear terms are written for the sake of completeness but they do not contribute. Indeed the linear terms in the fluctuations cancel out. Instead the linear terms in the derivatives give no contribution to the equation of motion.

The next step is to expand the fluctuation field in the Fourier space as:

$$\eta = \sqrt{\frac{1}{V\hbar\beta}} \sum_{n,\vec{q}} \eta_{n,\vec{q}} e^{i(\omega_n\tau + \vec{q}\vec{r})} \tag{11}$$

where ω_n are the Matsubara frequencies:

$$\omega_n = \frac{2\pi n}{\hbar\beta} \quad n \in \mathbb{Z} \tag{12}$$

Let S_0 be the part of the action which does not depend on η and η^*. The grand canonical potential for the constant term results:

$$\Omega_0 = -V\frac{\mu^2}{2g} \tag{13}$$

For the quadratic part instead, S_2, we will use the Fourier transform defined above and the fact that:

$$\int_V d^D\vec{r} \quad \frac{1}{V}e^{i(\vec{q}-\vec{q}')\vec{r}} = \delta_{\vec{q},\vec{q}'} \tag{14}$$

$$\int_0^{\hbar\beta} d\tau \quad \frac{1}{\hbar\beta}e^{i(\omega_n-\omega_{n'})\tau} = \delta_{n,n'} \tag{15}$$

where $\delta_{\vec{q}\vec{q}'}$ is the Kroenecker delta. We can write S_2 as:

$$\begin{aligned} S_2 = \frac{1}{2}\sum_{n,\vec{q}}\sum_{n',\vec{q}'}\frac{1}{V\hbar\beta}\int_0^{\hbar\beta} d\tau \int_V d^D\vec{r}\,\{ & e^{i(\omega_n-\omega_{n'})\tau+i(\vec{q}-\vec{q}')\vec{r}}\eta^*_{n',\vec{q}'}[i\hbar\omega_n + \frac{\hbar^2q^2}{2m} - \mu + 2g\psi_0^2]\eta_{n,\vec{q}} + \\ & e^{-i(\omega_n-\omega_{n'})\tau-i(\vec{q}-\vec{q}')\vec{r}}\eta^*_{-n',-\vec{q}'}[-i\hbar\omega_n + \frac{\hbar^2q^2}{2m} - \mu + 2g\psi_0^2]\eta_{-n,-\vec{q}} + \\ & \frac{g}{2}\psi_0^2(e^{i(\omega_n+\omega_{n'})\tau+i(\vec{q}+\vec{q}')\vec{r}}\eta_{n,\vec{q}}\eta_{n',\vec{q}'} + e^{-i(\omega_n+\omega_{n'})\tau-i(\vec{q}+\vec{q}')\vec{r}}\eta_{-n,-\vec{q}}\eta_{-n',-\vec{q}'} + \\ & e^{-i(\omega_n+\omega_{n'})\tau-i(\vec{q}+\vec{q}')\vec{r}}\eta^*_{n,\vec{q}}\eta^*_{n',\vec{q}'} + e^{i(\omega_n+\omega_{n'})\tau+i(\vec{q}+\vec{q}')\vec{r}}\eta^*_{-n,-\vec{q}}\eta^*_{-n',-\vec{q}'})\} \end{aligned} \tag{16}$$

Hence using the relations involving the Kroenecker deltas written above we can write the precedent equation in a different (and far more simple) form, namely involving a matrix formalism:

$$S_2 = \frac{1}{2}\sum_{n,\vec{q}}\begin{bmatrix}\eta^*_{n,\vec{q}} & \eta_{-n,-\vec{q}}\end{bmatrix} M \begin{bmatrix}\eta_{n,\vec{q}} \\ \eta^*_{-n,-\vec{q}}\end{bmatrix} \tag{17}$$

where M is the matrix given by:

$$M = \begin{bmatrix} i\hbar\omega_n + \frac{\hbar^2q^2}{2m} - \mu + 2g\psi_0^2 & g\psi_0^2 \\ g\psi_0^2 & -i\hbar\omega_n + \frac{\hbar^2q^2}{2m} - \mu + 2g\psi_0^2 \end{bmatrix} \tag{18}$$

The second-order correction contribution to the partition function is given by:

$$Z_2 = \int D[\eta,\eta^*] \quad e^{-S_2} \tag{19}$$

and the second-order correction to the grand canonical potential is given by:

$$\Omega_2 = -\frac{1}{\beta}lnZ_2 = \frac{1}{2\beta}\sum_{n,\vec{q}}ln(detM) = \frac{1}{2\beta}\sum_{n,\vec{q}}ln[\hbar^2\omega_n^2 + (E_q)^2] \tag{20}$$

where E_q reads:

$$E_q = \sqrt{\frac{\hbar^2q^2}{2m}\left(\frac{\hbar^2q^2}{2m} - 2\mu + 4g\psi_0^2\right) + \mu^2 + 3g^2\psi_0^4 - 4g\psi_0^2\mu} \tag{21}$$

After the summation over Matsubara frequencies we can finally write:

$$\Omega_2 = \sum_{\vec{q}}\left\{\frac{E_q}{2} + \frac{1}{\beta}ln(1 - e^{-\beta E_q})\right\} \tag{22}$$

Putting all parts of the grand canonical potential together:

$$\Omega = \Omega_0 + \Omega_2^{(0)} + \Omega_2^{(T)} \tag{23}$$

where Ω_0 is the mean-field potential while $\Omega_2^{(0)}$, the zero-point energy, and $\Omega_2^{(T)}$, the thermodynamic fluctuation term, are given by:

$$\Omega_2^{(0)} = \sum_{\vec{q}} \frac{E_q}{2} \tag{24}$$

$$\Omega_2^{(T)} = \sum_{\vec{q}} \frac{1}{\beta} ln(1 - e^{-\beta E_q}) \,. \tag{25}$$

For further details on the derivation of these equations and the renormalization of the divergent Gaussian grand potential $\Omega_2^{(0)}$ see Ref. [26].

It is clear that these calculations hold for both superfluid and normal phases: in all calculations we left implicit the value of ψ_0. For the superfluid phase $\psi_0^2 = \frac{\mu}{g}$ and therefore:

$$E_q = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\frac{\hbar^2 q^2}{2m} + 2\mu\right)} \tag{26}$$

This spectrum is known as the Bogoliubov spectrum [27] of elementary excitations of the non-relativistic Bose gas. This is a gapless spectrum and, at small momenta, it becomes the phonon mode $E_q \simeq \sqrt{\mu/m}\hbar q$, which can be identified as the Goldstone mode that appears necessarily in models exhibiting a spontaneous breakdown of continuous symmetries [17]. Thus, the Goldstone mode is only an approximation of the Bogoliubov mode. One finds a pure Goldstone mode only freezing amplitude fluctuations.

2.2. Elementary Excitations from Non-Relativistic Equation of Motion

2.2.1. Non-Relativistic Complex Fluctuations

By imposing the stationarity condition on the non-relativistic action (1), after having performed a Wick rotation from imaginary time to real time, one gets the Gross-Pitaevskii equation [13,14] for a weakly interacting bosonic gas which is given by:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m}\psi - \mu\psi + g|\psi|^2\psi \tag{27}$$

Let ψ_0 be the value of the $\psi(\vec{r},\tau)$ field, which satisfies the condition (8). Let $\eta(\vec{r},\tau)$ be the fluctuation around that value. If we expand the Gross-Pitaevskii equation to the first order in the fluctuations we obtain:

$$i\hbar\frac{\partial}{\partial t}\eta = -\frac{\hbar^2\nabla^2}{2m}\eta - \mu\eta + g\psi_0^2(2\eta + \eta^*) \tag{28}$$

If we now perform a Fourier Transform we obtain:

$$(\hbar\omega + \frac{\hbar^2 q^2}{2m} - \mu + 2g\psi_0^2)\eta_{\omega,\vec{q}}e^{-i\omega t} + g\psi_0^2\eta^*_{\omega,-\vec{q}}e^{+i\omega t} = 0 \tag{29}$$

$$(-\hbar\omega + \frac{\hbar^2 q^2}{2m} - \mu + 2g\psi_0^2)\eta^*_{\omega,-\vec{q}}e^{+i\omega t} + g\psi_0^2\eta_{\omega,\vec{q}}e^{-i\omega t} = 0 \tag{30}$$

which in turns gives:

$$\hbar^2\omega^2 = \frac{\hbar^4 q^4}{4m^2} + 2\frac{\hbar^2 q^2}{2m}(2g\psi_0^2 - \mu) + \mu^2 - 4g\psi_0^2\mu + 3g^2\psi_0^4 \quad (31)$$

In the superfluid regime, where $\psi_0^2 = \frac{\mu}{g}$, this equation gives exactly Equation (26). Thus, the spectrum obtained from the equation of motion is the same one derived from the partition function.

2.2.2. Non-Relativistic Amplitude and Phase Fluctuations

We will now compute the spectrum in a slightly different way. We now consider separately the phase and the amplitude fluctuations. We thus write the boson field $\psi(\vec{r}, t)$ this time as:

$$\psi(\vec{r}, t) = (\psi_0 + \sigma(\vec{r}, t))exp(i\theta(\vec{r}, t)) \quad (32)$$

i.e., including both the amplitude fluctuation field, σ, and the phase fluctuation field, θ, the Gross-Pitaevskii Equation (27) becomes (using the value of ψ_0 for the superfluid phase) at the first order in θ and σ:

$$i\hbar(\frac{\partial}{\partial t}\sigma + \psi_0 i\theta) = -\frac{\hbar^2}{2m}\nabla^2(\sigma + i\theta) + 2\mu\sigma \quad (33)$$

This equation can be split in its real and imaginary parts. The resulting equations are coupled for θ and σ:

$$-\hbar\frac{\partial}{\partial t}\psi_0\theta + \frac{\hbar}{2m}\nabla^2\sigma - 2\mu\sigma = 0 \quad (34)$$

$$\hbar\frac{\partial}{\partial t}\sigma + \frac{\hbar^2\psi_0}{2m}\nabla^2\theta = 0 \quad (35)$$

By performing now a Fourier transform we obtain:

$$i\hbar\omega\theta_{\omega,\vec{q}} + \left(\frac{\hbar^2 q^2}{2m} + 2\mu\right)\sigma_{\omega,\vec{q}} = 0 \quad (36)$$

$$i\hbar\omega\sigma_{\omega,\vec{q}} + \frac{\hbar^2 q^2}{2m}\theta_{\omega,\vec{q}} = 0 \quad (37)$$

where $\theta_{\omega,\vec{q}}$ and $\sigma_{\omega,\vec{q}}$ are the Fourier transforms of the fluctuation fields. If we substitute, for example, the expression of $\sigma_{\omega,\vec{q}}$ obtained by second equation in the first we get:

$$\left[\hbar^2\omega^2 - \left(\frac{\hbar^2 q^2}{2m} + 2\mu\right)\frac{\hbar^2 q^2}{2m}\right]\theta_{\omega,\vec{q}} = 0 \quad (38)$$

solving for ω we find again the Bogoliubov spectrum, Equation (26), that is the same results obtained with the other two methods. Note that if we consider only the phase fluctuations, i.e., we impose $\sigma = 0$, the Gross-Pitaevskii equation in the first order in θ becomes:

$$i\hbar\frac{\partial}{\partial t}\theta = \frac{\hbar}{2m}\nabla^2\theta \quad (39)$$

and if we consider the Fourier transform we obtain the following spectrum:

$$\hbar\omega = \frac{\hbar q^2}{2m} \quad (40)$$

this is a gapless spectrum which has the form of a free particle spectrum. Conversely if we consider the case $\theta = 0$, i.e., we consider only the amplitude fluctuations, we get the equation:

$$i\hbar\frac{\partial}{\partial t}\sigma = -\frac{\hbar^2}{2m}\nabla^2\sigma + 2\mu\sigma \tag{41}$$

which gives the spectrum:

$$\hbar\omega = 2\mu + \frac{\hbar^2 q^2}{2m} \tag{42}$$

this time we have a gapped spectrum, the gap being 2μ. This is consistent with what we would expect by the spontaneous symmetry mechanism: the breaking of the $U(1)$ symmetry in fact produces always a gapless mode, which is usually called Goldstone mode [17], and a gapped mode, which in Condensed Matter Physics is referred as Higgs mode [18,19].

3. Spontaneous Symmetry Breaking: Relativistic Case

3.1. Elementary Excitations from Relativistic Partition Function

Working with the same approximation for the dilute gas as in the previous section, for a weakly-interacting relativistic gas the Euclidean action is given by [7,28–30]:

$$S = \int_0^{\hbar\beta} d\tau \int_V d^D\vec{r} \quad \Big(\frac{\hbar^2}{mc^2}|\frac{\partial}{\partial\tau}\psi|^2 + 2\hbar\frac{\mu_r}{mc^2}\psi^*\frac{\partial}{\partial\tau}\psi + \frac{\hbar^2}{m}|\nabla\psi|^2 + (\frac{\mu_r^2}{mc^2} - mc^2)|\psi|^2 + \frac{g}{2}|\psi|^4\Big) \tag{43}$$

where $\psi(\vec{r},\tau)$ is the bosonic matter field and we have introduced the relativistic chemical potential, μ_r which is given by:

$$\mu_r = \mu + mc^2 \tag{44}$$

If we define again an effective potential V_{eff} such as:

$$V_{eff} = -(\frac{\mu_r^2}{mc^2} - mc^2)|\psi|^2 + \frac{g}{2}|\psi|^4 \tag{45}$$

Clearly if $\mu_r^2 - m^2c^4 > 0$ we have the superfluid phase: the $U(1)$ symmetry is broken and therefore we can proceed as we have done in the previous section. In particular the minima are given by:

$$|\psi_0| = \begin{cases} 0 & if \quad \mu_r^2 - m^2c^4 < 0 \\ \sqrt{\frac{\frac{\mu_r^2}{mc^2} - mc^2}{g}} & if \quad \mu_r^2 - m^2c^4 > 0 \end{cases} \tag{46}$$

The first case correspond to the normal phase, characterized by a mean value of order parameter equal to zero. For both phases we choose the real-valued vacuum, let us call it ψ_0. Let us now call $\eta(\vec{r},\tau)$ the fluctuations around the minimum. We expand now the action to the second order in the fluctuations, maintaining for generality the value of ψ_0 implicit. We obtain:

$$\begin{aligned} S = V\hbar\beta(-(\frac{\mu_r^2}{mc^2} - mc^2)\psi_0^2 + \frac{g}{2}\psi_0^4) + \\ \int_0^{\hbar\beta} d\tau \int_V d^D\vec{r} \quad \{\hbar\frac{\mu_r}{mc^2}(\eta^*\frac{\partial}{\partial\tau}\eta - \eta\frac{\partial}{\partial\tau}\eta^*) + \frac{\hbar^2}{mc^2}|\frac{\partial}{\partial\tau}\eta|^2 + \frac{\hbar^2}{m}|\nabla\eta|^2 - \\ (\frac{\mu_r^2}{mc^2} - mc^2)|\eta|^2 + \frac{g}{2}\psi_0^2(\eta\eta + \eta^*\eta^* + 4|\eta|^2)\} \end{aligned} \tag{47}$$

The constant term:

$$S_0 = V\hbar\beta(-(\frac{\mu_r^2}{mc^2} - mc^2)\psi_0^2 + \frac{g}{2}\psi_0^4) \tag{48}$$

gives a contribution to the grand canonical potential:

$$\Omega_0 = V(-(\frac{\mu_r^2}{mc^2} - mc^2)\psi_0^2 + \frac{g}{2}\psi_0^4) \quad (49)$$

whereas the second-order correction of the action can be written in a matrix form in the Fourier space (the sum over the index n refers to the sum over the Matsubara frequencies):

$$S_2 = \frac{1}{2}\sum_{n,\vec{q}} \begin{bmatrix} \eta^*_{n,\vec{q}} & \eta_{-n,-\vec{q}} \end{bmatrix} \frac{1}{mc^2} M \begin{bmatrix} \eta_{\vec{q}} \\ \eta^*_{-\vec{q}} \end{bmatrix} \quad (50)$$

where M is given by:

$$M = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \quad (51)$$

where:

$$A = \hbar^2\omega_n^2 + 2\hbar\omega_n\mu_r + \hbar^2c^2q^2 - (\mu_r^2 - m^2c^4) + 2g\psi_0^2mc^2 \quad (52)$$

$$B = g\psi_0^2mc^2 \quad (53)$$

$$C = \hbar^2\omega_n^2 - 2\hbar\omega_n\mu_r + \hbar^2c^2q^2 - (\mu_r^2 - m^2c^4) + 2g\psi_0^2mc^2 \quad (54)$$

The second order contribution to the grand canonical potential then results:

$$\Omega_2 = \frac{1}{2\beta}\sum_{n,\vec{q}} ln(\frac{1}{m^2c^4}detM) = \frac{1}{2\beta}\sum_{n,\vec{q}}\sum_{j=\pm} ln[\frac{1}{m^2c^4}(\hbar^2\omega_n^2 + E_{j,q}^2)] \quad (55)$$

where $E_{\pm,q}$ is given by:

$$E^2_{\pm,q} = \hbar^2c^2q^2 + (\mu_r^2 + m^2c^4 + 2g\psi_0^2mc^2) \pm \sqrt{4\mu_r^2(\hbar^2c^2q^2 + m^2c^4 + 2g\psi_0^2mc^2) + g^2\psi_0^4m^2c^4} \quad (56)$$

Summing over the Matsubara frequencies we can finally write:

$$\Omega_2 = \sum_{\vec{q}}\sum_{j=\pm}\left\{\frac{E_{q,j}}{2} + \frac{1}{\beta}ln(1 - e^{-\beta E_{q,j}})\right\} \quad (57)$$

Putting all terms of the grand canonical potential density together we obtain:

$$\Omega = \Omega_0 + \Omega_2^{(0)} + \Omega_2^{(T)} \quad (58)$$

where $\Omega_2^{(0)}$, the zero-point Gaussian grand canonical potential density, and $\Omega_2^{(T)}$ is the fluctuation term, defined respectively as:

$$\Omega_2^{(0)} = \sum_{\vec{q}}\sum_{j=\pm}\frac{E_{q,j}}{2} \quad (59)$$

$$\Omega_2^{(T)} = \sum_{\vec{q}}\sum_{j=\pm}\frac{1}{\beta}ln(1 - e^{-\beta E_{q,j}}) \quad (60)$$

For the superfluid phase $\psi_0^2 = \frac{\frac{\mu_r^2}{mc^2} - mc^2}{g}$ and therefore the spectrum becomes:

$$E^2_{\pm,q} = \hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4) \pm \sqrt{4\mu_r^2\hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4)^2} \quad (61)$$

3.2. Elementary Excitations from Nonlinear Klein-Gordon Equation

3.2.1. Relativistic Complex Fluctuations

By extremizing the relativistic action (43), after having performed the Wick rotation as in the non-relativistic case, we find the cubic nonlinear Klein-Gordon equation [15,16] for a bosonic gas with relativistic chemical potential μ_r [7,28–30]:

$$(\frac{\hbar^2}{m}D_\nu D^\nu + mc^2 + g|\psi|^2)\psi = 0 \tag{62}$$

where D_ν is the covariant derivative defined by:

$$D_0 = \frac{1}{c}\frac{\partial}{\partial t} - i\frac{\mu_r}{\hbar c} \tag{63}$$

$$D_i = \partial_i \tag{64}$$

so the nonlinear Klein-Gordon equation can be written:

$$(\hbar^2\partial_t^2 - 2i\hbar\mu_r\partial_t - \hbar^2c^2\nabla^2 - (\mu_r^2 - m^2c^4) + gmc^2|\psi|^2)\psi = 0 \tag{65}$$

We now write the field as the sum of the vacuum expectation value, ψ_0, and a fluctuation field, let us call it η. The nonlinear Klein-Gordon equation in the first-order of the fluctuation is given by:

$$(\hbar^2\partial_t^2 - 2i\hbar\mu_r\partial_t - \hbar^2c^2\nabla^2 - (\mu_r^2 - m^2c^4) + 2gmc^2\psi_0^2)\eta + gmc^2\psi_0^2\eta^* = 0 \tag{66}$$

We now perform a Fourier transform for this equation and its complex conjugate, obtaining:

$$(-\hbar^2\omega^2 - 2\mu_r\hbar\omega + \hbar^2c^2q^2 - (\mu_r^2 - m^2c^4) + 2gmc^2\psi_0^2)\eta_{\omega,\vec{q}}e^{-i\omega t} + gmc^2\psi_0^2\eta^*_{\omega,-\vec{q}}e^{+i\omega t} = 0 \tag{67}$$

$$(-\hbar^2\omega^2 + 2\mu_r\hbar\omega + \hbar^2c^2q^2 - (\mu_r^2 - m^2c^4) + 2gmc^2\psi_0^2)\eta^*_{\omega,-\vec{q}}e^{+i\omega t} + gmc^2\eta_{\omega,\vec{q}}e^{-i\omega t} = 0 \tag{68}$$

These equation give the following solutions:

$$\hbar^2\omega_\pm^2 \;=\; \hbar^2c^2q^2 \;+\; m^2c^4 \;+\; \mu_r^2 \;+\; 2gmc^2\psi_0^2 \pm \sqrt{4\mu_r^2(\hbar^2c^2q^2 + m^2c^4 + 2gmc^2\psi_0^2) + g^2m^2c^4\psi_0^4} \tag{69}$$

and substituting the value of ψ_0 for the superfluid phase given by Equation (46) we obtain:

$$\hbar^2\omega_\pm^2 = \hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4) \pm \sqrt{4\mu_r^2\hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4)^2} \tag{70}$$

Also, in the relativistic case, we have the same spectrum found using the partition function.

3.2.2. Relativistic Amplitude and Phase Fluctuations

We show now that we can find the spectrum also by expanding the matter field ψ as:

$$\psi = (\psi_0 + \sigma(\vec{r},t))exp(i\theta(\vec{r},t)) \tag{71}$$

where σ is the Higgs amplitude field and $\theta(\vec{r},t)$ is the Goldstone angle field. Using again the value of ψ_0 for the superfluid phase given by the condition (46), we obtain by expanding the cubic nonlinear Klein-Gordon Equation (62) in the first order of the fluctuations:

$$(\hbar^2\partial_t^2 - 2i\hbar\mu_r\partial_t - \hbar^2c^2\nabla^2)(\sigma + i\psi_0\theta) + 2(\mu_r^2 - m^2c^4)\sigma = 0 \tag{72}$$

which like the non-relativistic case can be decoupled in its imaginary and real parts. The equations are however coupled:

$$(\hbar^2\partial_t^2 - \hbar^2c^2\nabla^2 + 2(\mu_r^2 - m^2c^4))\sigma + 2\hbar\mu_r\partial_t\psi_0\theta = 0 \quad (73)$$
$$(\hbar^2\partial_t^2 - \hbar^2c^2\nabla^2)\psi_0\theta - 2\hbar\mu_r\partial_t\sigma = 0 \quad (74)$$

By performing the Fourier transform we obtain:

$$(-\hbar^2\omega^2 - \hbar^2c^2q^2 + 2(\mu_r^2 - m^2c^4))\sigma_{\omega,\vec{q}} - 2\omega\hbar\mu_r\psi_0\theta_{\omega,\vec{q}} = 0 \quad (75)$$
$$(-\hbar^2\omega^2 - \hbar^2c^2q^2)\psi_0\theta_{\omega,\vec{q}} + 2\omega\hbar\mu_r\sigma_{\omega,\vec{q}} = 0 \quad (76)$$

where $\theta_{\omega,\vec{q}}$ and $\sigma_{\omega,\vec{q}}$ are the Fourier transforms of the fluctuation fields. By using the expression of $\sigma_{\omega,\vec{q}}$ found by solving the second equation and substituting it in the first we obtain:

$$\{[(-\hbar^2\omega^2 - \hbar^2c^2q^2 + 2(\mu_r^2 - m^2c^4))(-\hbar^2\omega^2 - \hbar^2c^2q^2 + 2(\mu_r^2 - m^2c^4)] - 4\omega^2\hbar^2\mu_r^2\}\psi_0\theta_{\omega,\vec{q}} = 0 \quad (77)$$

which gives:

$$\hbar^4w^4 - 2\hbar^2\omega^2(\hbar^2c^2q^2 + 3\mu_r^2 - m^2c^4) + \hbar^2c^2q^2[\hbar^2c^2q^2 - 2(\mu_r^2 - m^2c^4)] = 0 \quad (78)$$

and ω is therefore given by:

$$\hbar\omega_{\pm} = \sqrt{\hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4) \pm \sqrt{\hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4)^2}} \quad (79)$$

which is exactly the same result found with the other methods.

It is important to observe that the cubic nonlinear Klein-Gordon equation is used to describe not only a relativistic Bose gas but also the dynamics of Cooper pairs in superconductors described by the Bardeen-Cooper-Schrifer (BCS) theory [19,31].

4. Analysis and Comparison of Spectra

We now proceed to study the spectra we have found. In the non-relativistic case we have found a gapless Bogoliubov spectrum, given by:

$$\hbar\omega = \sqrt{\frac{\hbar^2q^2}{2m}\left(\frac{\hbar^2q^2}{2m} + 2\mu\right)} \quad (80)$$

This spectrum for small momenta gives:

$$\hbar\omega \simeq \sqrt{\frac{\mu}{m}}\hbar q \quad if \quad \frac{\hbar^2q^2}{2m} \ll \mu \quad (81)$$

Therefore for small momenta we obtain a phonon-like linear spectrum. For sufficiently large momenta we instead get:

$$\hbar\omega_q \simeq \frac{\hbar^2q^2}{2m} \quad if \quad \frac{\hbar^2q^2}{2m} \gg \mu \quad (82)$$

In this case we obtained a free-particle quadratic spectrum. This shows that the contact interaction does not affect the spectrum in the limit of high energies, whereas in the opposite limit we get a linear spectrum.

We now consider the relativistic spectrum. In the calculations we found two modes, namely:

$$\hbar\omega_{\pm} = \sqrt{\hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4) \pm \sqrt{4\mu_r^2\hbar^2c^2q^2 + (3\mu_r^2 - m^2c^4)^2}} \quad (83)$$

At high energies we have that the term involving the higher-degree momentum becomes dominant and modes are given by:

$$\hbar\omega_{\pm} = \hbar c q \tag{84}$$

In this limit we have two free relativistic particles spectra: as in the non-relativistic case we obtained that at high energies the spectra are unaffected by the contact interaction. We note that we have two modes corresponding to a particle and its antiparticle. For small momenta the situation is different. In fact, using the relation between the relativistic chemical potential and the non-relativistic one (44), and the fact that $\mu \ll mc^2$ the Taylor expansion around $q \to 0$ yields:

$$\hbar\omega_{-} = \sqrt{\frac{\mu}{m}}\hbar q \tag{85}$$

$$\hbar\omega_{+} = 2mc^2 + \frac{\hbar^2 q^2}{2m} \tag{86}$$

We note that we have two modes: a gapless, i.e., Goldstone for the relativistic case mode, which is linear for small momenta like the Bogoliubov spectrum in the same limit, and a gapped mode, i.e., the Higgs mode for the relativistic case. Therefore, as expected, from the spontaneous symmetry breaking of the $U(1)$ symmetry we find the presence of both Goldstone and Higgs modes. The gapped mode for small energies is given by the sum between the gap and a quadratic term in the momenta which has the form of the spectrum of a non-relativistic free particle (which corresponds to the high momenta limit in the non-relativistic case), whereas the gapless mode in the same limit is actually the same.

Until now, however, we have not recovered the Bogoliubov spectrum. Let us now consider again the Goldstone (relativistic) mode, $\hbar\omega_{-}$. For small momenta it can be written as:

$$\hbar\omega_{-} = \sqrt{\hbar^2 c^2 q^2 - \frac{2\mu_r^2\hbar^2 c^2 q^2}{3\mu_r^2 - m^2 c^4} + \frac{4\mu_r^4\hbar^4 c^4 q^4}{(3\mu_r^2 - m^2 c^4)^3}} = \sqrt{\frac{\hbar^2 c^2 q^2}{3\mu_r^2 - m^2 c^4}\left(\frac{4\mu_r^2\hbar^2 c^2 q^2}{(3\mu_r^2 - m^2 c^4)^2} + (\mu_r^2 - m^2 c^4)\right)} \tag{87}$$

and now, since we are interested in the non-relativistic case, by imposing $\mu \ll mc^2$, we obtain the Bogoliubov spectrum (80). It should also be noted that the Bogoliubov mode is not, actually, the Goldstone mode of the non-relativistic case, as we have seen in Section 2.2.2. In that case the Goldstone mode coincides with the phase mode. Similarly the Higgs mode corresponds to the amplitude mode. In the relativistic case, however, both these modes are relative to the total fluctuation around the value of the minimum.

5. Application: The Bose-Hubbard Model

An interesting application of the above considerations is the Bose-Hubbard model. The model was first introduced by Gersch and Knollman [32] as a bosonic version of the Hubbard model for fermions on a lattice [33]. The Bose-Hubbard model is used to describe an interacting Bose gas confined in a periodic lattice by an external potential. We assume that for each site of the lattice the value of the potential is the same. With this assumption, the bosonic system is described by the Hamiltonian [25]:

$$\hat{H}_{BH} = -J\sum_{\langle ij\rangle}\hat{a}_i^{+}\hat{a}_j - (\mu - \epsilon)\sum_i \hat{a}_i^{+}\hat{a}_i + \frac{U}{2}\sum_i \hat{a}_i^{+}\hat{a}_i^{+}\hat{a}_i\hat{a}_i \tag{88}$$

where $\hat{a}_i$ is the annihilation operator for the site i, μ is the chemical potential of the gas, J is the coupling of the interaction between the nearest-neighbors (also called the "hopping" term), ϵ is the energy of the energy of each particle of every site due to its kinetic energy and to the confining potential, and finally U is proportional to the interaction strength of bosons. The Bose-Hubbard model has a phase transition between an insulating phase, called the Mott insulating phase, and a superfluid phase. In particular, for a system of $T = 0$ and volume V for the regions of phase space near the phase transitions, it can be shown that the behavior of the system is described, using an RPA approximation treating the hopping term as a perturbation, by the following action in the imaginary time formalism [25,34]:

$$S_{BH}^{(RPA)} = \int_{\mathbb{R}} d\tau \int_V d\vec{r} \{ K_1 \psi^* \frac{\partial}{\partial \tau} \psi + K_2 | \frac{\partial}{\partial \tau} \psi |^2 + K_3 |\nabla \psi|^2 + c_2 |\psi|^2 + c_4 |\psi|^4 \} \tag{89}$$

where $\psi(\vec{r}, \tau)$ is an appropriately chosen order parameter (related to the mean value of the annihilation operator). The coefficients K_1, K_2, K_3, c_2, c_4 depend on the Bose-Hubbard parameters J, μ, ϵ and U. This dependence in shown and discussed in Ref. [34]. The form of the effective action (89) is strikingly similar to the one found for the relativistic case (43) due to the term $K_2 | \frac{\partial}{\partial \tau} \psi |^2$. Also, the non-relativistic term $K_1 \psi^* \frac{\partial}{\partial \tau} \psi$ has a correspondence in that action to the term linear in the relativistic chemical potential. The phase transition occurs at the change of sign of the coefficient of the quadratic term. Note that the transition is purely quantum since we are working at zero temperature. In fact following the same reasoning used in Section 3, we find that the minima of the effective potential are given by:

$$|\psi_0| = \begin{cases} 0 & \text{if} \quad c_2 > 0 \\ \sqrt{\frac{2c_2}{c_4}} & \text{if} \quad c_4 < 0 \end{cases} \tag{90}$$

and as before we choose the real-valued minimum for the superfluid phase. We now write the order parameter as a sum of its mean value and the fluctuations:

$$\psi(\vec{r}, \tau) = \psi_0 + \eta(\vec{r}, \tau) \tag{91}$$

and we expand the action up to the second order in the fluctuations and by following the same steps of Section 3, we find the spectrum:

$$E_{\pm} = \sqrt{K_3 q^2 + \left(\frac{K_1^2}{2K_2} + c_2 + 4c_4 \psi_0^2 \right) \pm \sqrt{\frac{K_1^2}{K_2} K_3 q^2 + \frac{K_1^4}{4K_2^2} + \frac{K_1^2}{K_2} (c_2 + 4c_4 \psi_0^2) + 4c_4^2 \psi_0^4}} \tag{92}$$

and substituting the value of ψ_0 for the superfluid phase we obtain:

$$E_{\pm} = \sqrt{K_3 q^2 + \left(\frac{K_1^2}{2K_2} - c_2 \right) \pm \sqrt{\frac{K_1^2}{K_2} K_3 q^2 + \left(\frac{K_1^2}{2K_2} - c_2 \right)^2}} \tag{93}$$

This spectrum has the same form of the one found for the superfluid phase for the relativistic gas (61). To better note the formal analogy the following identifications should be considered:

$$\hbar^2 c^2 \leftrightarrow K_3$$
$$\mu_r^2 \leftrightarrow \frac{K_1^2}{2K_2}$$
$$\mu_r^2 - m^2 c^4 \leftrightarrow c_2$$

As mentioned above, this formal analogy is possible thanks to the inclusion of the relativistic chemical potential in the relativistic action (43), that gives rise to the linear term in the time derivative and a correction to the quadratic term.

It is interesting now to study the behavior of the two modes of the spectrum (93) in the limits of low and high momenta. In particular in the first limit we find at leading order:

$$E_+ = \sqrt{2\left(\frac{K_1^2}{2K_2} - c_2\right)} + \frac{\frac{K_1^2}{2K_2} - c_2}{\sqrt{2}\left(\frac{K_1^2}{2K_2} - c_2\right)^{\frac{3}{2}}} K_3 q^2 \tag{94}$$

$$E_- = \frac{\sqrt{-c_2}}{\sqrt{\frac{K_1^2}{2K_2} - c_2}} K_3 q^2 \tag{95}$$

We have like in the relativistic gas a gapped Higgs mode ,which for low energies is quadratic in momenta, and a gapless Goldstone mode, which in the same limit is linear. If we expand up to the next to leading order the gapless mode, we obtain:

$$E_- = \sqrt{\frac{K_3 q^2}{2\left(\frac{K_1^2}{2K_2} - c_2\right)}\left[\frac{\left(\frac{K_1^2}{K_2}\right)^2 K_3 q^2}{2\left(\frac{K_1^2}{2K_2} - c_2\right)^2} - 2c_2\right]} \tag{96}$$

which is reminiscent of the Bogoliubov spectrum (26). In particular using the identifications written above, we obtain the result found in Equation (87). Finally we note that for high momenta we have:

$$E_\pm = \sqrt{K_3} q \tag{97}$$

which is analogous to the relativistic free particle spectrum found in Equation (84) for the relativistic superfluid.

6. Conclusions

In this brief review we have derived and studied the spectrum of the superfluid phase of both non-relativistic and relativistic bosonic gases. This phase is described by a spontaneous symmetry breaking process of the $U(1)$ group symmetry of the action. We have found, in agreement with the expectations, that in both cases there is indeed a gapless Goldstone mode due to phase fluctuation and a gapped Higgs mode due to amplitude fluctuations. However, while in the non-relativistic case the coupling between phase and amplitude gives rise to a total gapless Bogoliubov spectrum, in the relativistic case both modes are possible oscillations modes of the total fluctuation around the solution with broken symmetry. The difference between the Goldstone mode and the Bogoliubov mode in the non-relativistic case can be interpreted by noting that in this regime there is not the particle-antiparticle pair typical of the relativistic case. Then, we have verified that the Bogoliubov spectrum can be obtained as the non-relativistic limit of the relativistic Goldstone mode. Finally, we have analyzed the Bose-Hubbard model, that is characterized by the effective action close to the critical point of the Superfluid-Mott quantum phase transition which contains both non-relativistic and relativistic terms [25]. Apart some theoretical [34] and experimental [35] results for the Bose-Hubbard model, the properties of phase and amplitude fluctuations in this exotic effective action are not yet fully explored.

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