ON THE SLOPE OF FOURGONAL SEMISTABLE FIBRATIONS

BEORCHIA VALENTINA AND ZUCCONI FRANCESCO

ABSTRACT. We bound the slope of sweeping curves B in the 4-gonal locus $\overline{\mathcal{M}}^1_{g,4}$. Our results follow from some Bogomolov-type inequalities for nef rank two vector bundles on ruled surfaces.

CONTENTS

1. INTRODUCTION

In this paper we are concerned with semistable fibrations $f\colon S \to B$, that is flat surjective morphisms between a smooth surface S and a smooth curve B such that, for every $b \in B$, the fiber $F_b := f^{-1}(b)$ is a semistable curve and if $b \in B$ is general then F_b is smooth and of genus g.

Following Xiao [X], we can associate with $f\colon S \to B$ a rational number $s(f)$, called the slope of $f\colon S \to B$, and defined as follows:

$$s(f) := \frac{K_f^2}{\chi_f}$$

1991 *Mathematics Subject Classification.* Primary 14D22, 14H51; Secondary 14H10.
Key words and phrases. fibration, Chern classes, Viehweg weak positivity, moduli of curves, gonality.

where $K_f = K_S - f^\star K_B$ is the relative canonical divisor, $\chi_f := \deg f_\star\omega_f$, and $\omega_f := \mathcal{O}_S(K_f)$.

The starting point of our study is the celebrated Cornalba-Harris *slope inequality*; see: [CH], which asserts that if $f\colon S \to B$ is a family of semistable curves of genus g over an integral complete curve and with smooth general fiber, then $s(f) \geq 4 + \frac{4}{g}$. Moreover, if $s(f) = 4 + \frac{4}{g}$ and $g \geq 3$, then the modular image of B is contained in the hyperelliptic locus, c.f. [ACG, Theorem 8.4 page 391]. The same inequality for not necessarily semistable families has been proved by Xiao: [X, Sec. 3 Theorem 2, p. 459].

A nice introduction of the main known results related to slope inequalities can be found in [ACG, page 438]. Here we briefly recall that in [S] Stankova showed that if $f\colon S \to B$ is a semistable fibration such that the general fiber is a smooth trigonal curve, then $s(f) \geq \frac{24(g-1)}{5g+1}$, and equality holds if and only if all fibers are irreducible, S is a triple cover of a ruled surface Y over B and the ramification divisor, R, of the cover $\rho\colon S \to Y$ satisfies $R \equiv \frac{1}{3}\rho^\star\rho_\star R$. Under certain restrictive assumptions on the singular fibres, she also gave the better bound $s(f) \geq 5 - \frac{6}{g}$ if g is even and the Maroni invariant of the general fiber is zero. Recently Barja-Stoppino [BS], and Fedorchuk-Jensen [FJ] showed that if g is even the bound $s(f) \geq 5 - \frac{6}{g}$ holds for zero Maroni invariant, without the hypothesis of semistability and with no assumptions on the singular fibers. Finally, the trigonal odd genus case for sweeping families in the trigonal locus of the moduli space has been considered in [DP].

1.1. **Our result.** Concerning the slope of fourgonal fibrations the bounding problem is widely open. There are estimates on $s(f)$ by Barja-Zucconi [BZ]) and by Cornalba-Stoppino [CS], [St] for fibrations, which factorise through a double cover of a hyperelliptic fibration. In this paper we establish some bounds on the slope of semistable fourgonal fibrations $f\colon S \to B$ with smooth general fiber, whose modular image is not contained in some specific loci. For instance, if g is odd and the modular image in not contained in three divisors, then we prove that

$$s(f) \geq \frac{16(g-1)}{3g+1}.$$

For the definition of such loci and for a precise statement of our Main Theorem we need to describe our construction and to introduce some notation.

Given $f\colon S \to B$, it is well known that, up to a finite base-change, f factorises through a rational map $\rho\colon S \dashrightarrow Y$, where $\pi_B\colon Y \to B$ is a ruled surface; see Theorem 4.1. We resolve $\rho\colon S \dashrightarrow Y$ and we get a commutative diagram:

(1.1)
$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\pi_B} \\ S & \xrightarrow{f} & B, \end{array}$$

where $\pi\colon X \to Y$ is a *generically finite* morphism of degree 4 and branched over a divisor $B(\pi)$ of Y. Now let $X \to \widehat{X} \to Y$ be the Stein factorisation of $\pi\colon X \to Y$ and let $\widehat{\pi}\colon \widehat{X} \to Y$ be the *finite* factorisation morphism. It turns out that if the modular image of f is a curve $B \subset \overline{\mathcal{M}}^1_{g,4}$, which does not interesect a certain proper subscheme $\Xi \subset \overline{\mathcal{M}}^1_{g,4}$ (see Section 4.), the Stein factorisation of $\pi\colon X \to Y$ is a Gorenstein cover (see Definition 2.1). Nevertheless, our theory allows to estimate $s(f)$ under the weaker assumption that $\pi\colon X \to Y$ is finite over any point $p \in B(\pi)$ such that p is *not* a simple node; and in this case we will say that $B \subset \overline{\mathcal{M}}^1_{g,4}$ is a curve with *good Gorenstein factorisation*; see Definition 4.2 and Subsection 4.2. We remark that there exist sweeping families satisfying such a condition, like for instance the Harris - Morrison families constructed in [HMo, Theorem 2.5].

With such an assumption, we find a partial resolution $\widetilde{\tau}\colon \widetilde{X} \to \widehat{X}$ and obtain the following diagram:

(1.2)
$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\pi}} & \widetilde{Y} \\ \downarrow{\scriptstyle \widetilde{\tau}} & & \downarrow{\scriptstyle \sigma\circ\pi_B} \\ \widehat{X} & \xrightarrow{\pi_B\circ\widehat{\pi}} & B, \end{array}$$

where $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{Y}$ is a degree 4 Gorenstein cover, $\sigma\colon \widetilde{Y} \to Y$ is the blow up of Y at suitable points, and if we set $\widetilde{f} := \pi_B \circ \widehat{\pi} \circ \widetilde{\tau}$, then $\widetilde{f}\colon \widetilde{X} \to B$ is a (not necessarily semistable) fibration, whose slope can be bounded. Finally we show that $s(f) \geq s(\widetilde{f})$.

The bound on $s(\widetilde{f})$ relies on the theory of Gorenstein covers. Indeed, there is a projective bundle $\mathbb{P}(\widetilde{\mathcal{E}})$ over $\widetilde{Y}$ and an embedding $\widetilde{j}\colon \widetilde{X} \to \mathbb{P}(\widetilde{\mathcal{E}})$, which allows to express $s(\widetilde{f})$ in terms of the Chern classes of $\widetilde{\mathcal{E}}$ and of the rank two Casnati-Ekedahl *bundle of conics* $\widetilde{\mathcal{F}} := \widetilde{\pi}_\star \mathcal{I}_{\widetilde{X},\mathbb{P}(\widetilde{\mathcal{E}})}(2)$.

Next we define the *Casnati-Ekedahl locus* $\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$, which is the subscheme of $\overline{\mathcal{M}}^1_{g,4}$ given by:

$$\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4}) := \overline{\{[F] \in \overline{\mathcal{M}}^1_{g,4} |\ \mathcal{F}_F \text{ is not balanced}\}}$$

where a rank r vector bundle $\mathcal{H}$ on $\mathbb{P}^1$ is called *balanced* if for the sequence of integers $(a_1, \ldots, a_r)$ given by its splitting into the direct sums of line bundles, $\mathcal{H} \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$, it holds that: $|a_i - a_j| \leq 1$ where $i \neq j$ and $i, j = 1, ..., r$.

If $g \geq 10$, the codimensions in $\overline{\mathcal{M}}^1_{g,4}$ of the Casnati-Ekedahl loci are 1 if g is odd, and 2 if g is even, see Theorem 3.10.

Finally, we set $T \subset \overline{\mathcal{M}}^1_{g,4}$ to be divisor of curves with a triple ramification, $D \subset \overline{\mathcal{M}}^1_{g,4}$ the divisor of double simple ramification, and $\Upsilon \subset \overline{\mathcal{M}}^1_{g,4}$ the subscheme corresponding to points of the boundary divisor $\delta \subset \overline{\mathcal{M}}^1_{g,4}$ given by stable curves with a ramification in a singular point, or by stable curves with two or more nodes (see Definition 3.11). We will say that the fibration satisfies the condition (†) if

(†) $B \cap \Upsilon = \emptyset$ and if Y has an irreducible negative section contained in the branch divisor, there are no triple nor total ramification points for $\pi : X \to Y$ over any of its points.

In the present paper we prove that:

Main Theorem: *Let $f : S \to B$ be a semistable fourgonal fibration with good Gorenstein factorisation, and assume that the general fiber F has genus $g \geq 10$. Denote again by $B \subset \overline{\mathcal{M}}^1_{g,4}$ its modular image and assume that $B \not\subset T$ and $B \not\subset D$. Then:*

(1) if g is odd and $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$, then $s(f) \geq \frac{16(g-1)}{3g+1}$;

(2) if g is even, $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$ and the condition $(\dagger)$ is satisfied, then $s(f) \geq \frac{16(g-1)}{3g+2}$;

(3) if the 4-gonal morphism $h\colon F \to \mathbb{P}^1$ does not factorise and the condition $(\dagger)$ is satisfied, then $s(f) \geq \frac{24(g-1)}{(5g+3)}$, with equality if and only if $f\colon S \to B$ factorises through a finite degree four cover $\pi\colon S \to Y$ of a ruled surface $Y \to B$, whose ramification divisor R satisfies $R \equiv \frac{1}{4}\pi^\star\pi_\star R$;

(4) if $h\colon F \to \mathbb{P}^1$ factorises through a double cover of a hyperelliptic curve of genus $\gamma < \frac{(g-3)}{6}$ and the condition $(\dagger)$ is satisfied, then $s(f) \geq \frac{4(g-1)}{(g-\gamma)}$.

Remark. We remark that A. Patel, in his Ph. D. thesis, has proved that for $g \equiv 3 \mod 6$ the slope of a sweeping 4-gonal family not contained entirely in the divisors T, D, $\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$ and in the Maroni divisor $M(\overline{\mathcal{M}}^1_{g,4})$ (see Remark 4.3) is bounded below by $s(f) \geq \frac{11}{2} - \frac{15}{2g}$. We prove in Remark 4.3 that such a statement is consistent with our results.

1.2. **The contents of each section.** In Section 2, we recall some basic results of the theory of finite Gorenstein covers. For a fibration $f : S \to B$, which factors through a finite Gorenstein cover $\pi : S \to Y$ of a ruled surface over B, we express the slope of f in terms of the π-relative canonical divisor and the Chern classes of the reduced direct image sheaf.

We end Section 2 with an important result (Theorem 2.10), establishing some Bogomolov-type inequalities between the Chern classes of a rank two vector bundle on a ruled surface, under the assumption that the vector bundle is weakly positive outside a zero-dimensional subscheme.

In Section 3 we introduce the Casnati-Ekedahl bundle of conics associated with a degree four cover. Using the Viehweg Weak Positivity Theorem [V1, 3.4] we show that both the reduced direct image sheaf and the bundle of conics are weakly positive outside the branch locus of the cover and a zero-dimensional subscheme. Theorem 2.10 and some additional arguments allow then to conclude when S is a Gorenstein cover of a ruled surface.

In Section 4 we define the property of having a good Gorenstein factorisation. Under such an assumption we can estimate the invariants of $f\colon X \to B$ in terms of those of a family $\widetilde{f}\colon \widetilde{X} \to B$ where $\widetilde{X}$ is Gorenstein; see Theorem 4.3. Then in Theorem 4.4 we show that the bounds claimed in our Main Theorem hold for $s(\widetilde{f})$. We end Section 4 with the proof of the Main Theorem, which is now a consequence of the fact that S is a minimal model of $\widetilde{X}$.

Acknowledgment. This research is supported by MIUR funds, PRIN project *Geometria delle varietà algebriche* (2015), coordinator A. Verra, and by GNSAGA of INdAM.

The first author is also supported by funds of the Università degli Studi di Trieste - Finanziamento di Ateneo per progetti di ricerca scientifica - FRA 2015.

The second author is also supported by funds of the Università degli Studi di Udine - Finanziamento di Ateneo RICERCA LIBERA di ATENEO 2015.

The authors are grateful to M. Á. Barja, L. Stoppino and E. Tenni for fruitful discussions had in September 2010, when they visited the Department of Mathematics and Informatics in Trieste. The authors thank also P. Hawking, M. Manetti, A. Verra, B. Fantechi and A. Alzati.

Finally, the authors thank the organizers of the Conference ”New Trends in Algebraic Geometry”, Università della Calabria, June 12-14 (2013), where the second author had the opportunity to present some of the results of this work.

We thank, also, the anonymous referee for the useful comments.

2. Preliminary results

In this paper a *fibration* $f\colon S \to B$ is a flat proper surjective morphism, with smooth connected general fiber, from a surface with canonical singularities S to a smooth curve B. Note that two dimensional canonical singularities are the same as du Val singularities, and they are analytically isomorphic to quotients of $\mathbb{C}^2$ by finite subgroups of $SL2(\mathbb{C})$. In particular the canonical divisor is a Cartier divisor since they are rational Gorenstein singularities. We denote by F_b the fiber over a point $b \in B$ and we denote by g the genus of a general fiber. We need to recall some results from the theory of Gorenstein covers.

Let X and Y be schemes. An affine morphism $\pi\colon X \to Y$ is called a *cover of degree* n if $\pi_\star\mathcal{O}_X$ is a locally free sheaf of rank n; observe that $\pi\colon X \to Y$ is a cover if and only if it is flat and finite. If Y is smooth and X is locally Cohen-Macaulay, then every finite surjective morphism is a cover.

There exists an exact sequence of the form $0 \to \mathcal{O}_Y \to \pi_\star\mathcal{O}_X \to \mathcal{E}^\vee \to 0$, where $\mathcal{E}^\vee$ is a locally free $\mathcal{O}_Y$ sheaf of rank $n-1$, called the *Tschirnahausen sheaf* of $\pi\colon X \to Y$. The above sequence splits and $\pi_\star\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}^\vee$; see [CE].

2.1. **Gorenstein Covers.** From now on we assume Y to be a smooth surface.

Definition 2.1. A *Gorenstein cover* $\pi\colon S \to Y$ is a finite surjective morphism from a normal surface S to a smooth surface Y such that all π-fibers are Gorenstein schemes.

The above definition is given in [CE] in a more general set up. If the cover $\pi\colon S \to Y$ has Gorenstein fibers, then S is Gorenstein and we have $(\pi_\star\mathcal{O}_S)^\vee = \pi_\star\omega_{S/Y}$ [Ha, exercise III 6.10], hence $\pi_\star\omega_{S/Y} = \mathcal{O}_Y \oplus \mathcal{E}$. We shall call $\mathcal{E}$ the *reduced direct image* sheaf. We will indicate by R the π-relative canonical divisor.

Finally, we shall denote by $\pi_Y : \mathbb{P}(\mathcal{E}) \to Y$ the projective bundle associated with $\mathcal{E}$. We recall the Casnati-Ekedahl Theorem:

Theorem 2.2. *Let Y be a smooth surface and let $\pi\colon S \to Y$ be a Gorenstein cover of degree $n \geq 3$. There exists a unique $\mathbb{P}^{n-2}$-bundle $\pi_Y : \mathbb{P} \to Y$ and an embedding $j\colon X \to \mathbb{P}$ such that $\pi = \pi_Y \circ j$. Moreover $\mathbb{P} \cong \mathbb{P}(\mathcal{E})$ and R satisfies:*

$$\mathcal{O}_S(R) \cong j^\star\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Proof. See [CE, Theorem 1.3]. □

For the rest of this and the next subsection we assume that

$$n \leq 4$$

and that S is a normal Gorenstein surface. In this case, as $\pi\colon S \to Y$ factorises through the closed embedding $j\colon S \to \mathbb{P}$ and the projective morphism $\pi_Y : \mathbb{P} \to Y$, and since a codimension 1 or 2 Gorenstein subscheme of a smooth variety is a local complete intersection (see, for instance, [E, 21.10, p. 537]), then $\pi\colon X \to Y$ is a local complete intersection (l.c.i. to short) morphism. Then we can apply the Grothendieck-Riemann-Roch theorem for singular varieties and proper l.c.i. morphisms (see: [Fu, Corollary 18.3.1 (c), page 354]) to write the invariants of S in terms of the invariants of Y and of the Chern classes of $\mathcal{E}$.

Let us denote by $A(Y)$ the Chow ring of Y and by $\equiv$ the numerical equivalence.

Lemma 2.3. *Let S be a normal Gorenstein surface and let Y be a smooth surface. Let $\pi\colon S \to Y$ be a finite morphism of degree n, and let $\mathcal{E}$ be its reduced direct image sheaf. Then in $A(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ we have:*

(1) $\pi_\star R \equiv 2c_1(\mathcal{E})$,
(2) $\chi(\mathcal{O}_S) = n\chi(\mathcal{O}_Y) + \frac{1}{2}c_1(\mathcal{E}) \cdot K_Y + \frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E})$;
(3) $c_1(\pi_\star \mathcal{O}_S(2R)) \equiv 3c_1(\mathcal{E})$,
(4) $c_2(\pi_\star \mathcal{O}_S(2R)) = 4c_1(\mathcal{E})^2 + c_2(\mathcal{E}) - R^2$.

Proof. By the Grothendieck-Riemann-Roch theorem for $\pi\colon S \to Y$ applied to the sheaf $\mathcal{O}_S$ we can write: $\mathrm{ch}(\pi_! \mathcal{O}_S) \cdot \mathrm{td}\mathcal{T}_Y = \pi_\star(\mathrm{ch}\mathcal{O}_S \cdot \mathrm{td}\mathcal{T}_S)$. In our case this means that $\mathrm{ch}(\pi_\star \mathcal{O}_S) \cdot (1 - \frac{1}{2}K_Y + \chi(\mathcal{O}_Y)) = \pi_\star(1 - \frac{1}{2}K_S + \chi(\mathcal{O}_S))$, that is

$$(n - c_1(\mathcal{E}) + \frac{1}{2}(c_1^2(\mathcal{E}) - 2c_2(\mathcal{E})) \cdot (1 - \frac{1}{2}K_Y + \chi(\mathcal{O}_Y)) = \pi_\star(1 - \frac{1}{2}K_S + \chi(\mathcal{O}_S)).$$

The divisorial part of the above equation in $A(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ gives: $-c_1(\mathcal{E}) = -\frac{1}{2}(\pi_\star K_S - nK_Y)$ and as $K_S \sim \pi^\star K_Y + R$, where $\sim$ denotes the linear equivalence, we have $\pi_\star K_S \equiv nK_Y + \pi_\star R$ and (1) follows. The equality between the codimension two cycles gives formula (2). Formulae (3), (4) follow by the same argument applied to the sheaf $\mathcal{O}_S(2R)$ and by (1) and (2). □

2.2. **First formula for the slope.** We shall now express the slope of a fibration, which factorises through a Gorenstein cover π, in terms of the π-relative canonical divisor and the Chern classes of the reduced direct image sheaf.

Observe that since S is Gorenstein, it admits a Cartier canonical divisor K_S. It follows that for the fibration $f\colon S \to B$, there exists a Cartier relative canonical divisor

$$K_f := K_S - f^\star K_B.$$

Furthermore, we set

$$\chi_f := \chi(f_\star \mathcal{O}_S(K_f)).$$

So the slope of f is defined as

$$s(f) := \frac{K_f^2}{\chi_f},$$

where the intersection number is taken in the sense of [K].

Remark. If S is a normal, Gorenstein surface with canonical singularities, then

$$K_f^2 = K_S^2 - 8(g-1)(g(B)-1), \qquad \chi_f = \chi(\mathcal{O}_S) - (g-1)(g(B)-1). \tag{2.1}$$

The formulae are well known when S is a smooth surface. If S is singular and $\tau : \widetilde{S} \to S$ is a minimal resolution of the singularities, then

$$\tau_\star \omega_{\widetilde{S}} \cong \omega_S, \qquad K_{\widetilde{S}}^2 = K_S^2, \qquad \chi(\mathcal{O}_{\widetilde{S}}) = \chi(\mathcal{O}_S),$$

hence the smooth case formulae (2.1) still hold.

Definition 2.4. We say that $f\colon S \to B$ factorises through the Gorenstein cover $\pi\colon S \to Y$ if there exists a fibration $\pi_B\colon Y \to B$ such that $f = \pi_B \circ \pi$.

In the rest of the paper, unless otherwise stated, Y is a ruled surface and $\pi_B\colon Y\to B$ is the ruling morphism. It is well-known that the $\mathbb{Q}$-Neron-Severi group $\mathrm{NS}(Y)_{\mathbb{Q}}:=\mathrm{NS}(Y)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a 2-dimensional vector space over $\mathbb{Q}$ and that $\mathrm{NS}(Y)_{\mathbb{Q}}=[T_Y]\mathbb{Q}\oplus[L]\mathbb{Q}$ where $[T_Y]$ is the numerical class of a section of $\pi_B\colon Y\to B$ and $[L]$ is the class of a ruling.

Let T_0 be the following $\mathbb{Q}$-divisor on Y:

$$T_0:=T_Y-\frac{1}{2}T_Y^2L.$$

Next lemma gives an expression for the first Chern class of the reduced direct image sheaf $\mathcal{E}$, which will be particularly useful in the sequel.

Lemma 2.5. *Let $f\colon S\to B$ be a genus-g fibration which factorises through a degree n Gorenstein cover $\pi\colon S\to Y$ such that $\pi_B\colon Y\to B$ is a ruled surface. Then*

$$c_1(\mathcal{E})\equiv(g+n-1)T_0+\left(\frac{c_1(\mathcal{E})^2}{2(g+n-1)}\right)L.$$

Proof. Let $b\in B$ a general point. We know that $\mathrm{NS}(Y)_{\mathbb{Q}}=[T_0]\mathbb{Q}\oplus[L]\mathbb{Q}$ where in this proof we set $L:=\pi_B^{-1}(b)$. We consider the restriction $\mathcal{E}_L$ of $\mathcal{E}$ to L. Then $c_1(\mathcal{E})\equiv\deg c_1(\mathcal{E}_L)T_0+\delta L$ for some $\delta\in\mathbb{Q}$.

We first show that $\deg c_1(E_L)=g+n-1$. Indeed, by definition $\pi^*L=F_b$. Hence by projection formula and by Lemma 2.3 (1) it holds that $\deg c_1(\mathcal{E}_L)=\frac{1}{2}R\cdot F_b$. The fiber F_b is general, hence it is transversal to R and the ramification divisor of the induced morphism $\pi_{|F}\colon F\to L$ is given by $R_{|F_b}$. By the Riemann-Hurwitz formula it follows that $g+n-1=\frac{1}{2}R\cdot F_b=\deg c_1(\mathcal{E}_L)$.

Now, since $T_0^2=0,\ L^2=0,\ T_0\cdot L=1$, we get $c_1(\mathcal{E})^2=2(g+n-1)\delta$ and the statement follows. □

Next proposition expresses the slope in terms of R^2 and the Chern classes of $\mathcal{E}$.

Proposition 2.6. *Let $f:S\to B$ be a fibration with S normal and with canonical singularities, which factorises through a Gorenstein degree $n\le 4$ cover $\pi\colon S\to Y$ of a ruled surface Y. Let R be the π-relative canonical divisor and let $\mathcal{E}$ be the reduced direct image sheaf of $\pi\colon S\to Y$. Let $c_1(\mathcal{E})$, $c_2(\mathcal{E})$ be respectively the first Chern class and the second Chern class of $\mathcal{E}$. Then*

$$s(f)=\frac{R^2-\frac{4}{g+n-1}c_1(\mathcal{E})^2}{\frac{g+n-2}{2(g+n-1)}c_1(\mathcal{E})^2-c_2(\mathcal{E})}. \tag{2.2}$$

Proof. By formula (2.1) we have

$$K_f^2=(R+\pi^\star K_Y)^2-8(g-1)(g(B)-1)=R^2+4c_1(\mathcal{E})\cdot K_Y+nK_Y^2-8(g-1)(g(B)-1),$$

where we applied Lemma 2.3 (1) and the projection formula for $\pi\colon S\to Y$.

In the $\mathbb{Q}$-basis T_0,L we have $K_Y\equiv-2T_0+(2g(B)-2)L$. Using Lemma 2.5 we obtain $c_1(\mathcal{E})\cdot K_Y=-\frac{c_1(\mathcal{E})^2}{(g+n-1)}+2(g+n-1)(b-1)$, hence $K_f^2=R^2-\frac{4}{(g+n-1)}c_1(\mathcal{E})^2$.

To show that $\chi_f=\frac{g+n-2}{2(g+n-1)}c_1(\mathcal{E})^2-c_2(\mathcal{E})$, we use the formula given in (2.1) $\chi_f=\chi(\mathcal{O}_S)-(g-1)(g(B)-1)$, and the equality given in Lemma 2.3 (2), taking into account the relation $\chi(\mathcal{O}_Y)=1-g(B)$, which holds for any ruled surface. □

2.3. **Weakly positive vector bundles on a ruled surface.** The aim of the next results is to find some suitable bounds on the invariants appearing in the formula for the slope given in (2.2).

The crucial fact we shall use is that both the reduced direct image sheaf $\mathcal{E}$ and the bundle of conics $\mathcal{F}$ turn out to be weakly positive outside the branch locus of the cover and outside a zero dimensional subscheme. Let us recall the definition of weak positivity (see [V2, Definition 2.11, Remark 2.12.2]).

Definition 2.7. A locally free sheaf $\mathcal{G}$ on a projective variety Y is *weakly positive* over Y if for every ample invertible sheaf $\mathcal{H}$ on Y and for every $r > 0$, the sheaf $Sym^r\mathcal{G} \otimes \mathcal{H}$ is ample.

We now consider the celebrated Viehweg's Weak Positivity Theorem [V1, 3.4], which states that the direct image of the relative canonical sheaf of a projective surjective morphism between projective varieties is weakly positive on the complement of the branch locus. As $\mathcal{E}$ is a quotient of $\pi_\star\omega_\pi$ and, as we shall see in Theorem 3.5, $\mathcal{F}$ coincides, outside a zero dimensional subscheme, with the reduced direct image sheaf associated with the discriminant morphism of the cover π, we will deduce that $\mathcal{E}$ and $\mathcal{F}$ are weakly positive outside the branch locus of the cover and outside a zero dimensional subscheme.

So we now study the Chern classes of weakly positive vector bundles on an open subscheme of a ruled surface, or more generally of a blow up of a ruled surface.

Let us fix the following:

Notation 2.8. Let $\sigma\colon \widetilde{Y} \to Y$ be the blow-up of a ruled surface $\pi_B\colon Y \to B$ in a finite number of points $q_1, ..., q_s \in Y$. With $\widetilde{L}$ we shall denote a fiber of $\widetilde{Y} \to B$, with $T_{\widetilde{Y}}$ the pull back of a section of $\pi_B : Y \to B$, and by E_i, $i = 1, ..., s$ the exceptional divisors.

Proposition 2.9. *Let $\mathcal{G}$ be a vector bundle on $\widetilde{Y}$, which is weakly positive outside a zero dimensional subscheme $\Theta \subset \widetilde{Y}$. Then $c_1(\mathcal{G})$ is nef.*

Proof. As $\mathcal{G}$ is weakly positive outside Θ, the line bundle $\det\mathcal{G} = \mathcal{O}_{\widetilde{Y}}(c_1(\mathcal{G}))$ satisfies the same property by [V2, Corollary 2.20]. By the definition of weak positivity, for any ample divisor H on $\widetilde{Y}$, $H \not\supset \Theta$, and for any integer $r > 0$, the bundle

$$\left(Sym^r\mathcal{O}_{\widetilde{Y}}(c_1(\mathcal{G}))\right) \otimes \mathcal{O}_{\widetilde{Y}}(H) = \mathcal{O}_{\widetilde{Y}}(rc_1(\mathcal{G}) + H)$$

is ample on $\widetilde{Y} \setminus \Theta$. Then for $m >> 0$, the divisor $m(rc_1(\mathcal{G}) + H)$ is very ample on $\widetilde{Y} \setminus \Theta$, so the base locus of the linear system $|m(rc_1(\mathcal{G}) + H)|$ is at most zero-dimensional. Therefore for any effective divisor Q on $\widetilde{Y}$ we have that $m(rc_1(\mathcal{G}) + H) \cdot Q \geq 0$, since otherwise Q would be contained in the base locus. It follows that $rc_1(\mathcal{G}) \cdot Q \geq 0$ and $c_1(\mathcal{G}) \cdot Q \geq 0$, which shows the nefness of $c_1(\mathcal{G})$ on $\widetilde{Y}$. □

The next result establishes some Bogomolov-type inequalities for rank two vector bundles on blows up of ruled surfaces, with the assumption that they are weakly positive outside a zero dimensional subscheme.

We need to recall the notion of *general splitting type*. Let $\mathcal{G}$ be a rank two vector bundle on a ruled surface $\pi_B\colon Y \to B$. A couple $(\alpha, \beta) \in \mathbb{Z} \oplus \mathbb{Z}$, where $\alpha \leq \beta$, is said to be the *general splitting type* of $\mathcal{G}$ if the restriction $\mathcal{G}_L$ of $\mathcal{G}$ to a general fiber L of $\pi_B\colon Y \to B$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$.

Observe that if $\widetilde{Y}$ is the blow-up of Y at a finite number of points, the general splitting type of a vector bundle $\mathcal{G}$ on $\widetilde{Y}$ with respect to the fibration $\widetilde{Y} \to B$ can be defined similarly.

Theorem 2.10. *Let $\sigma\colon \widetilde{Y} \to Y$ be the blow-up of a ruled surface $\pi_B\colon Y \to B$ at $q_1, ..., q_s \in Y$. If Y admits a negative section T_Y, assume that $q_i \notin T_Y$ for any $i = 1, \dots, s$.*

Let $\mathcal{G}$ be a rank two vector bundle on $\widetilde{Y}$, with (α, β), $0 < \alpha \leq \beta$ the general splitting type of $\mathcal{G}$. Let $c_1(\mathcal{G})$, $c_2(\mathcal{G})$ be respectively the first Chern class and the second Chern class of $\mathcal{G}$, and set $c_1(\mathcal{G}) \equiv (\alpha+\beta)T_{\widetilde{Y}} + \delta\widetilde{L} + \sum_{i=1}^{s} m_i E_i$, where we use the notations of 2.8.

Then

(1) $c_2(\mathcal{G}) \geq \frac{1}{4}(c_1(\mathcal{G})^2 + \sum_{i=1}^{s} m_i^2)$ *if* $\alpha = \beta$;

(2) $c_2(\mathcal{G}) \geq \frac{\alpha}{2(\alpha+\beta)}(c_1(\mathcal{G})^2 + \sum_{i=1}^{s} m_i^2)$ *if $\alpha < \beta$ and $\mathcal{G}$ is weakly positive outside a zero-dimensional subscheme $J \subset \widetilde{Y}$.*

Proof. We recall that $\mathrm{NS}(\widetilde{Y}) = [T_{\widetilde{Y}}]\mathbb{Z} \oplus [\widetilde{L}]\mathbb{Z} \oplus_{i=1}^{s} [E_i]\mathbb{Z}$. We shall adapt the construction of Brosius [B] to our more general case. We first show (2), that is we assume $\alpha < \beta$ and that $\mathcal{G}$ is weakly positive on $\widetilde{Y} \setminus J$. We set $\widetilde{\pi}_B := \pi_B \circ \sigma\colon \widetilde{Y} \to B$. Since $\alpha < \beta$, $(\widetilde{\pi}_B)_\star\mathcal{G}(-\beta T_{\widetilde{Y}})$ has rank one and it is a locally free sheaf on the curve B, since it is the direct image of a torsion free sheaf. Then by [Ha, prop. III.9.8] we can write:

$$(\widetilde{\pi}_B)_\star\mathcal{G}(-\beta T_{\widetilde{Y}}) = \mathcal{O}_B(N)$$

where N is a suitable divisor on B. By construction, the natural map $\widetilde{\pi}_B^\star\mathcal{O}_B(N) \to \mathcal{G}(-\beta T_{\widetilde{Y}})$ is generically injective, hence it is an injective map of locally free sheaves. It follows that the quotient sheaf is locally free outside a scheme Z of codimension 2. Denote by D the first Chern class of such a quotient sheaf. Then the canonical extension of Brosius ([B, Lemma 3, Proposition 2]) of the vector bundle $\mathcal{G}(-\beta T_{\widetilde{Y}})$ has the form

$$0 \to \mathcal{O}_{\widetilde{Y}}(\widetilde{\pi}_B^\star N) \to \mathcal{G}(-\beta T_{\widetilde{Y}}) \to \mathcal{O}_{\widetilde{Y}}((\alpha-\beta)T_{\widetilde{Y}} + \widetilde{\pi}_B^\star M + \sum_{i=1}^{s} m_i E_i) \otimes \mathcal{I}_Z \to 0,$$

where $\widetilde{\pi}_B^\star N + \widetilde{\pi}_B^\star M \equiv \delta\widetilde{L}$.

Twisting by $\beta T_{\widetilde{Y}}$ we get

$$0 \to \mathcal{O}_{\widetilde{Y}}(\beta T_{\widetilde{Y}} + \widetilde{\pi}_B^\star N) \to \mathcal{G} \to \mathcal{O}_{\widetilde{Y}}(\alpha T_{\widetilde{Y}} + \widetilde{\pi}_B^\star M + \sum_{i=1}^{s} m_i E_i) \otimes \mathcal{I}_Z \to 0. \tag{2.3}$$

Note that $\widetilde{\pi}_B^\star N \cdot E_i = 0$ for $i = 1, ..., s$. Hence we have:

$$c_2(\mathcal{G}) = (\beta T_{\widetilde{Y}} + \widetilde{\pi}_B^\star N)\cdot(\alpha T_{\widetilde{Y}} + \widetilde{\pi}_B^\star M + \sum_{i=1}^{s} m_i E_i)) + \mathrm{degZ} = \alpha\beta T_{\widetilde{Y}}^2 + \alpha\deg(\pi_B^\star N) + \beta\deg(\pi_B^\star M) + \mathrm{degZ}.$$

By using the expression

$$c_1(\mathcal{G}) \equiv (\alpha+\beta)T_{\widetilde{Y}} + \frac{(c_1(\mathcal{G})^2 - (\alpha+\beta)^2 T_{\widetilde{Y}}^2 + \sum_{i=1}^{s} m_i^2)}{2(\alpha+\beta)}\widetilde{L} + \sum_{i=1}^{s} m_i E_i,$$

we get $\deg(\widetilde{\pi}_B^\star N) = \frac{(c_1(\mathcal{G})^2 - (\alpha+\beta)^2 T_{\widetilde{Y}}^2 + \sum_{i=1}^{s} m_i^2)}{2(\alpha+\beta)} - \deg(\widetilde{\pi}_B^\star M)$ and therefore:

$$c_2(\mathcal{G}) = \frac{\alpha}{2(\alpha+\beta)}c_1^2 + \frac{\alpha}{2}(\beta-\alpha)T_{\widetilde{Y}}^2 + (\beta-\alpha)\deg(\pi_B^\star M) + \deg \mathrm{Z} + \alpha\frac{\sum_{i=1}^s m_i^2}{2(\alpha+\beta)}. \tag{2.4}$$

Finally, observe that as $\mathcal{G}$ is weakly positive on $\widetilde{Y}\setminus J$, and $\mathcal{O}_{\widetilde{Y}}(\alpha T_{\widetilde{Y}} + \widetilde{\pi}_B^\star M + \sum_{i=1}^s m_i E_i)\otimes \mathcal{I}_Z$ is a quotient line bundle of $\mathcal{G}$ on $\widetilde{Y}\setminus Z$, we deduce that $\mathcal{O}_{\widetilde{Y}}(\alpha T_{\widetilde{Y}} + \widetilde{\pi}_B^\star M + \sum_{i=1}^s m_i E_i)$ is weakly positive on $\widetilde{Y}\setminus (J\cup Z)$.

By Proposition 2.9 we have that $\alpha T_{\widetilde{Y}} + \widetilde{\pi}_B^\star M + \sum_{i=1}^s m_i E_i$ is nef on $\widetilde{Y}$.

In particular, if we choose an irreducible $T_{\widetilde{Y}}$ so that $T_{\widetilde{Y}}\cdot E_i = 0$, we have $(\alpha T_{\widetilde{Y}} + \pi_B^\star M)\cdot T_{\widetilde{Y}} \geq 0$, that is

$$\deg(\pi_B^\star M) \geq -\alpha T_{\widetilde{Y}}^2. \tag{2.5}$$

If there exists a section $T_{\widetilde{Y}}$ such that $T_{\widetilde{Y}}^2 = 0$ or $T_{\widetilde{Y}}^2 < 0$, by the hypothesis $q_i \notin T_Y$ we can choose such a section to get the bound (2.5). We get the statement (2) in this case from formula (2.4), as Z is an effective 0-dimensional cycle.

Assume now that for any section T_Y of Y, its σ-pull-back $T_{\widetilde{Y}}$ satisfies $T_{\widetilde{Y}}^2 > 0$, which is equivalent to saying that $Y = \mathbb{P}(\mathcal{V})$ with $\mathcal{V}$ a stable rank two vector bundle on the curve B. Take T_Y to be the tautological divisor of $\mathbb{P}(\mathcal{V})$. By Miyaoka Theorem [Mi] the normalised tautological divisor $T_Y - \frac{1}{2}\pi_B^\star c_1(\mathcal{V}) \equiv T_Y - \frac{1}{2}T_Y^2 L$ is a nef $\mathbb{Q}$-divisor. Then for any rational number $\epsilon > 0$ we have that

$$T_Y - \frac{1}{2}T_Y^2 L + \epsilon T_Y$$

is $\mathbb{Q}$-ample. In particular we have that for $l >> 0$

$$l(T_{\widetilde{Y}} - \frac{1}{2}T_{\widetilde{Y}}^2\widetilde{L} + \epsilon T_{\widetilde{Y}})\cdot(\alpha T_{\widetilde{Y}} + \pi_B^\star M) \geq 0, \tag{2.6}$$

since we can avoid the points $q_1, .., q_s$.

By equation (2.6) we have that $(\alpha/2+\epsilon)T_{\widetilde{Y}}^2 + (1+\epsilon)\deg(\pi_B^\star M) \geq 0$. Since this holds for any $\epsilon > 0$, we have $\deg(\pi_B^\star M) \geq -\alpha/2\ T_{\widetilde{Y}}^2$. Using such inequality in equation (2.4) we conclude.

Now we show (1). We have $\alpha = \beta$ and the claim can be proved in the same way as above, by taking into account that $\widetilde{\pi}_B^\star(\widetilde{\pi}_B)_\star \mathcal{G}(-\beta T_{\widetilde{Y}}) =: \mathcal{H}$ has rank two and its Bogomolov discriminant $\Delta(\mathcal{H}) = 4c_2(\mathcal{H}) - c_1(\mathcal{H})^2$ is zero, since $\mathcal{H}$ is the pull back of a vector bundle on a curve. On the other hand, the exact sequence

$$0 \to \mathcal{H} \to \mathcal{G}(-\beta T_{\widetilde{Y}}) \to \mathcal{I}_Z \otimes_{\mathcal{O}_{\widetilde{Y}}} \mathcal{O}_{\widetilde{Y}} \to 0$$

gives $Z = c_2(\mathcal{G}(-\beta T_{\widetilde{Y}})) = c_2(\mathcal{G}) + c_1(\mathcal{G})\cdot(-\beta T_{\widetilde{Y}}) + \beta^2 T_{\widetilde{Y}}^2$; as Z is effective, we have $c_2(\mathcal{G}(-\beta T_{\widetilde{Y}})) \geq 0$ and by computing explicitly the last expression, the claimed bound follows. □

3. Slope and Chern classes of the the Casnati-Ekedahl bundle of conics

In this section we will show the claims of the Main Theorem for a semistable fibration $f\colon S \to B$ which factorises through a degree 4 Gorenstein cover $\pi\colon S \to Y$ and such that its general fiber F is a smooth fourgonal curve.

3.1. **The Casnati-Ekedahl bundle of conics.** If $\pi\colon S\to Y$ is a degree 4 Gorenstein cover by Theorem 2.2 there exists a unique $\mathbb{P}^2$-bundle $\pi_Y\colon \mathbb{P}\to Y$ and an embedding $j\colon S\to\mathbb{P}$ such that $\pi=\pi_Y\circ j$, $\mathbb{P}\cong\mathbb{P}(\mathcal{E})$ where $\mathcal{O}_S(R)\cong j^\star\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Moreover, by [CE, Proof of Step B); p. 445], the push forward of the exact sequence

$$0\to\mathcal{I}_{S,\mathbb{P}(\mathcal{E})}(2)\to\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)\to\mathcal{O}_S(2)\to 0$$

gives the following exact sequence of locally free sheaves on the surface Y:

$$0\to\pi_\star\mathcal{I}_{S,\mathbb{P}(\mathcal{E})}(2)\to Sym^2(\mathcal{E})\to\pi_\star\mathcal{O}_S(2R)\to 0. \tag{3.1}$$

The sheaf $\mathcal{F}:=\pi_\star\mathcal{I}_{S,\mathbb{P}(\mathcal{E})}(2)$ is called *Casnati-Ekedahl bundle of conics*. The next proposition shows how to write R^2 in terms of the classes $c_1(\mathcal{E})^2$, $c_2(\mathcal{E})$, $c_2(\mathcal{F})$.

Proposition 3.1. *Let $f\colon S\to B$ be a genus g fibration which factorises through a degree 4 Gorenstein cover $\pi\colon S\to Y$ where Y is a smooth surface. Then*

$$c_1(\mathcal{F})=c_1(\mathcal{E}), \tag{3.2}$$

and

$$R^2=2c_1(\mathcal{E})^2-4c_2(\mathcal{E})+c_2(\mathcal{F}). \tag{3.3}$$

Proof. By sequence (3.1) and by a standard computation we have:

$$c_1(Sym^2(\mathcal{E}))=4c_1(\mathcal{E}),\quad c_2(Sym^2(\mathcal{E}))=5c_2(\mathcal{E})+5c_1(\mathcal{E})^2. \tag{3.4}$$

By Lemma 2.3 (3) we have $c_1(\pi_\star\mathcal{O}_S(2R))\equiv 3c_1(\mathcal{E})$ and by Lemma 2.3 (4)

$$c_2(\pi_\star\mathcal{O}_S(2R))=c_2(\mathcal{E})+4c_1(\mathcal{E})^2-R^2.$$

Hence from the sequence (3.1) we get $c_1(\mathcal{F})=c_1(\mathcal{E})$ and $c_2(\mathcal{F})=R^2+4c_2(\mathcal{E})-2c_1(\mathcal{E})^2$, which gives our claim. □

Corollary 3.2. *Let $f\colon S\to B$ be a genus g fibration which factorises through a degree 4 Gorenstein cover $\pi\colon S\to Y$ where $\pi_B\colon Y\to B$ is a ruled surface. Then*

$$s(f)=\frac{\frac{2(g+1)}{(g+3)}c_1(\mathcal{E})^2-4c_2(\mathcal{E})+c_2(\mathcal{F})}{\frac{(g+2)}{2(g+3)}c_1(\mathcal{E})^2-c_2(\mathcal{E})}. \tag{3.5}$$

Proof. It follows from Proposition 3.1 and from Proposition 2.6. □

The next step consists in bounding the second Chern class of the bundle of conics $\mathcal{F}$. We shall actually apply Theorem 2.10, with an additional argument. Indeed, the results [C, Proposition 4.4, remarks on page 1364] allow to prove that $\mathcal{F}$ is weakly positive outside the branch locus $B(\pi)$ and a zero-dimensional subscheme.

The formulation of the statement requires some preliminaries.

Definition 3.3. Let $\pi\colon S\to Y$ be a Gorenstein degree four cover, with Y a smooth surface. We say that $y\in Y$ is π-*planar* (see [C, Definition 3.2])) if the fiber scheme $\pi^{-1}(y)$ is isomorphic to the scheme $Spec\left(\frac{k(y)[u,v]}{(u^2,v^2)}\right)$.

The local analysis given in [C, remarks on page 1364] yield the following:

Lemma 3.4. *If S is smooth, a point $y \in Y$ is π-planar iff y is at least a fourfold point of the branch divisor $B(\pi)$ of $\pi\colon S \to Y$.*

Let $\pi\colon S \to Y$, where Y is a smooth surface, be a Gorenstein degree four cover with reduced direct image sheaf $\mathcal{E}$ and Casnati-Ekedahl bundle of conics $\mathcal{F}$. Then the *discriminant scheme* $\Delta(S)$ and *discriminant map* $\Delta(\pi) : \Delta(S) \to Y$ are defined [C, Definition 4.1]. The discriminant scheme corresponds to the locus of degenerate conics in the pencils of conics associated with the fibers of π.

Theorem 3.5. *The discriminant map $\Delta(\pi) : \Delta(S) \to Y$ associated with a degree four Gorenstein cover $\pi : S \to Y$ is a generically finite morphism of degree three. Moreover, $\Delta(\pi)$ is not finite over a point $y \in Y$ iff $y \in Y$ is π-planar, and in this case $\Delta(\pi)^{-1}(y) \cong \mathbb{P}^1_{k(y)}$.*

Finally, if there are no π-planar points, then $\Delta(\pi) : \Delta(S) \to Y$ is a Gorenstein cover of degree three, the corresponding reduced direct image sheaf is $\mathcal{F} = (\Delta(\pi))_\star\omega_{\Delta(S)/Y}/\mathcal{O}_Y$ and the branch divisors of π and $\Delta(\pi)$ coincide.

Proof. See [C, Proposition 4.4], where a stronger version of the statement is shown. □

As a consequence we have the following:

Proposition 3.6. *Let S be a normal surface and Y a smooth surface. If $\pi\colon S \to Y$ is a Gorenstein cover of degree four with reduced branch divisor $B(\pi)$, then the Casnati-Ekedahl bundle of conics $\mathcal{F}$ is weakly positive outside $B(\pi)$ and a zero-dimensional scheme J.*

Proof. By Theorem 3.5, $y \in Y$ is a π-planar point if $\pi^{-1}(y)$ is supported on a singular point of S or it is at least a fourfold point of $B(\pi)$. Since S is normal, Y is smooth and $B(\pi)$ is reduced, there is at most a finite number of π-planar points by Lemma 3.4, hence the discriminant morphism $\Delta(\pi) : \Delta(S) \to Y$ is Gorenstein and finite outside a zero-dimensional subscheme J, supported on the π-planar points, by Theorem 3.5.

Moreover, we observe that, by a similar argument to the one in [C, Proposition 4.5 i), Corollary 4.11], one can deduce that $\Delta(S)$ is integral since S is normal.

Therefore $\mathcal{F}$ concides with the $\Delta(\pi)$ reduced direct image outside J, and by the Viehweg Weak Positivity Theorem [V1, 3.4] $\mathcal{F}$ is weakly positive outside $B(\pi) \cup J$. □

Next we need to determine the general splitting type of $\mathcal{F}$. This will be done using Schreyer's results on fourgonal curves.

3.2. **Chern classes of the Casnati-Ekedahl bundle of conics.** Let F be a fourgonal curve of genus $g \geq 10$. By the Geometric Riemann-Roch Theorem, the span of any gonal divisor on the canonical model of F is two dimensional, and the union of such spans determine the three-dimensional gonal scroll $W \subset \mathbb{P}^{g-1}$, containing the canonical model of F (c.f. [DZ, Theorem 5]). Set H to be the hyperplane divisor on W and let Π be a fiber of the natural projection $W \to \mathbb{P}^1$. With these notations we have:

Theorem 3.7. *There exist $b_1, b_2 \in \mathbb{N}$ such that the canonical model C of a fourgonal curve F of genus $g \geq 5$ is the complete intersection $C = Q_1 \cap Q_2$, where $Q_1 \in |2H - b_1\Pi|$ and $Q_2 \in |2H - b_2\Pi|$. The two integers b_1 and b_2 satisfy the following relations:*

$$0 \leq b_2 \leq b_1 \leq g - 5, \quad b_1 + b_2 = g - 5.$$

Moreover $\pi_F\colon F \to \mathbb{P}^1$ *factorises through a double cover of a curve of genus* $\gamma < \frac{(g-3)}{6}$ *if and only if* $b_1 > \frac{2}{3}(g-3)$ *and* $\gamma = \frac{b_2}{2}+1$.

Proof. See [Sch, Sections 6.2, 6.3, 6.4, 6.5, 6.6]. □

Corollary 3.8. *The generic splitting types* (α,β) *of the Casnati-Ekedahl bundle for a fourgonal family satisfy the following:*

$$4 \le \alpha \le \beta \le g-1, \qquad \alpha+\beta = g+3.$$

Moreover, $\pi_F\colon F \to \mathbb{P}^1$ *factorises through a double cover of a curve of genus* $\gamma < \frac{(g-3)}{6}$ *if and only if* $\beta > \frac{2g+3}{3}$ *and* $\gamma = \frac{\alpha}{2}-1$.

Proof. Let L be a general fiber of Y, set $V = \mathbb{P}(\mathcal{E}\otimes\mathcal{O}_L)$ and let T_V be the tautological divisor of V. By projection formula $\pi_\star\mathcal{I}_{S,\mathbb{P}}(2T_\mathbb{P})\otimes\mathcal{O}_L \cong \pi_\star(\mathcal{I}_{S,\mathbb{P}}(2T_\mathbb{P})\otimes\pi^\star\mathcal{O}_L)$, so $\mathcal{F}\otimes\mathcal{O}_L = \pi_\star(\mathcal{I}_{F,V}(2T_{\mathbb{P}|V}))$.

We can determine the minimal free resolution of $\mathcal{I}_{F,V}$ using Theorem 3.7.

Indeed, let $W=\mathbb{P}(\mathcal{W})\subset\mathbb{P}^{g-1}$ be the scroll containing the canonical model C of F, and let T_W be the tautological divisor on W.

By Theorem 3.7 a minimal free resolution of $\mathcal{I}_{C,W}(2T_W)$ has the following form:

$$0\to\mathcal{O}_W(-2T_W+(g-5)\Pi)\to\mathcal{O}_W(b_1\Pi)\oplus\mathcal{O}_W(b_2\Pi)\to\mathcal{I}_{C,W}(2T_W)\to 0.$$

As the pull back of T_V to F is the ramification divisor, we have

$$\mathcal{E}\otimes\mathcal{O}_L\cong\mathcal{W}(-K_{\mathbb{P}^1}).$$

Now consider the isomorphism between the two scrolls: $\phi: V\to W\subset\mathbb{P}^{g-1}$. By construction it follows that $H\sim\phi^\star(T)-2\Pi$. Then a minimal free resolution of $\mathcal{I}_{F,V}(2T_V)$ is given by:

$$0\to\mathcal{O}_V(-2T_V+(g+3)\Pi)\to\mathcal{O}_V((b_1+4)\Pi)\oplus\mathcal{O}_V((b_2+4)\Pi)\to\mathcal{I}_{F,V}(2T_V)\to 0.$$

Since ${\pi_{|V}}_\star\mathcal{O}_V(-2T_V+(g+3)\Pi) = R^1{\pi_{|V}}_\star\mathcal{O}_V(-2T_V+(g+3)\Pi)=0$, we have

$${\pi_{|V}}_\star\mathcal{I}_{F,V}(2T)\cong\mathcal{O}_{\mathbb{P}^1}(b_1+4)\oplus\mathcal{O}_{\mathbb{P}^1}(b_2+4),$$

so we get the claim by setting $b_2=\alpha-4$, $b_1=\beta-4$ in Theorem 3.7. □

We shall see that if we consider curves $B\subset\overline{\mathcal{M}}^1_{g,4}$ not contained in some specific closed subschemes, then the bundle of conics is in fact balanced, so we will get some better bounds on the general splitting type. To this purpose let us introduce the following locus:

Definition 3.9. The *Casnati-Ekedahl locus* $\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$ is the closure of the locus in $\overline{\mathcal{M}}^1_{g,4}$ corresponding to curves with non balanced bundle of conics:

$$\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4}) := \overline{\{[F]\in\overline{\mathcal{M}}^1_{g,4} |\ \mathcal{F}_F\ \text{is not balanced}\}}$$

Such a locus turns out to be a proper closed subscheme by the following result.

Theorem 3.10. *If* $g\ge 10$, *the codimension in* $\overline{\mathcal{M}}^1_{g,4}$ *of the Casnati-Ekedahl locus is given by:*

$$\operatorname{codim}\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4}) = \begin{cases} 1 & \text{if} \quad g \text{ odd,}\\ 2 & \text{if} \quad g \text{ even}\end{cases}$$

Proof. The dimensions of such subschemes can be computed using the results in [BSa]. Since the authors use different notation and define some loci, which are in some cases slightly different from the one we are considering, we briefly sketch the computation.

The authors introduce the invariant λ, which is in general the minimum degree of a linear series distinct from the g^1_4 [BSa, Theorem 6.10]. We observe that the locus $\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$ is equal, in the authors' notation, to the stratum $\overline{\mathcal{M}}_g^{\lceil \frac{g}{2} \rceil}$, given by the closure of the curves satisfying $\lambda \le \lceil \frac{g}{2} \rceil$.

Indeed, the relation between λ and β, where $(g+3-\beta, \beta)$, $\beta \ge (g+3)/2$, is the splitting type of the bundle of conics of a given fourgonal curve, can be obtained by the formula

$$\deg Z = g + \lambda - 5$$

given in [BSa, Theorem 4.4] for $t = 0$, which expresses the minimum degree of a surface Z ruled by conics containing the canonical model of the curve. By Schreyer's' Theorem 3.7, the class of such a surface in the gonal scroll is given by $2H - b_1\Pi$, where H is the tautological divisor of the gonal scroll $\mathcal{E}(-2)$. Since $b_1 = \beta - 4$ (see the proof of Corollary 3.8), we also have:

$$\deg Z = 2(g-3) - \beta + 4.$$

So we finally get $\beta = g + 3 - \lambda$.

The dimensions of the strata are given by the formula in the Main Theorem of [BSa], which states that

$$\dim \overline{\mathcal{M}}_g^{\lambda} = g + 2\lambda + 1,$$

if $\lambda \le \lceil \frac{g}{2} \rceil$, which corresponds to the conditions $\beta \ge (g+5)/2$ if g is odd, or $\beta \ge (g+4)/2$ if g is even. We get

$$\dim \overline{\mathcal{M}}_g^{\lceil \frac{g}{2} \rceil} = \begin{cases} 2g+2 & \text{if} \quad g \text{odd}, \\ 2g+1 & \text{if} \quad g \text{ even}, \end{cases}$$

while $\dim \overline{\mathcal{M}}^1_{g,4} = 2g + 3$.

Finally, we observe that the condition $t \ge 1$ on the invariant t introduced in [BSa, Definition of page 13] defines a proper subscheme of $\overline{\mathcal{M}}^1_{g,4}$ by [BSa, Theorem 11.1]. □

Next we consider the divisors in $\overline{\mathcal{M}}^1_{g,4}$ given by the closures of the loci corresponding to smooth fourgonal curves with one triple ramification point or two simple ramification points on the same fiber of the cover of $\mathbb{P}^1$.

Definition 3.11. We set

$$T := \overline{\{[C] \in \overline{\mathcal{M}}^1_{g,4} | 3p + q \in g^1_4\}},$$

the divisor of triple ramification,

$$D := \overline{\{[C] \in \overline{\mathcal{M}}^1_{g,4} | 2p + 2q \in g^1_4\}},$$

the divisor of double simple ramifications.

Moreover, we introduce the proper subscheme

$$\Upsilon \subset \overline{\mathcal{M}}^1_{g,4}$$

corresponding to points of the boundary divisor $\delta \subset \overline{\mathcal{M}}^1_{g,4}$ given by stable curves with a ramification in a singular point, or by stable curves with two or more nodes.

In the next theorem we finally present our bounds of $c_2(\mathcal{F})$ in terms of $c_1(\mathcal{E})^2$. We denote by F_b and by L_b the fiber of $f\colon S \to B$ and respectively of $\pi_B\colon Y \to B$ over the point $b \in B$. We also denote by $\pi_b\colon F_b \to L_b \cong \mathbb{P}^1$ the restriction to F_b of the degree four cover $\pi\colon S \to Y$.

We recall that the condition (†) has been defined in section 1.1.

Theorem 3.12. *Let $g \geq 10$. Let $B \subset \overline{\mathcal{M}}^1_{g,4}$, and assume that $B \nsubseteq T$, $B \nsubseteq D$. Let $f\colon S \to B$ be a semistable fibration which factorises through a degree 4 Gorenstein cover.*

Then it holds:

1. *if condition (†) is satisfied, then $c_2(\mathcal{F}) \geq \frac{2}{(g+3)}c_1(\mathcal{E})^2$;*
2. *if condition (†) is satisfied and if $\pi_b\colon F_b \to L_b$ factorises through a double cover of a hyperelliptic curve of genus $\gamma < \frac{(g-3)}{6}$, then $c_2(\mathcal{F}) \geq \frac{\gamma+1}{(g+3)}c_1(\mathcal{E})^2$;*
3. *if condition (†) is satisfied and $\pi_b\colon F_b \to L_b$ does not factorise, then $c_2(\mathcal{F}) \geq \frac{1}{6}c_1(\mathcal{E})^2$;*

Moreover, if g is even and $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$ and if condition (†) is satisfied, then

$$c_2(\mathcal{F}) \geq \frac{(g+2)}{4(g+3)}c_1(\mathcal{E})^2. \tag{3.6}$$

If g is odd and $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$, then

$$c_2(\mathcal{F}) \geq \frac{1}{4}c_1(\mathcal{E})^2. \tag{3.7}$$

Proof. By the assumptions $B \nsubseteq T$, $B \nsubseteq D$, the branch locus of π is reduced, so $\mathcal{F}$ is weakly positive outside a zero dimensional scheme and outside the branch locus by Proposition 3.6.

If Y admits no negative section, then the statement of Theorem 2.10 (2) holds. Indeed, for any rational positive number ϵ, the linear system $T_Y + (\epsilon - T_Y^2)L$ is $\mathbb{Q}$-ample, so a general $\mathbb{Q}$-divisor in such a system does not contain any component of the branch divisor, hence the proof of Theorem 2.10 (2) can be applied. The statements (1), (2) and (3) follow by applying the results on the general splitting type of $\mathcal{F}$ given in Corollary 3.8.

If Y admits a negative section T_0, and if $\mathcal{F}$ is nef over T_0, then any quotient of $\mathcal{F}$ is nef over T_0, and the proof of Theorem 2.10 together with the results of Corollary 3.8 yield the desired inequalities.

Finally, in the case when Y admits a negative section T_0, and if $\mathcal{F}$ is not nef over T_0, we claim that our additional assumption (†) guarantees that $\mathcal{F}(-T_0)$ is nef over T_0.

First notice that if $\mathcal{F}$ is not nef over T_0, then T_0 is contained in the branch locus $B(\pi)$ of π. By the assumption $B \nsubseteq T$, $B \nsubseteq D$, we have that the branch locus is reduced, hence we have a first inequality:

$$0 < (B(\pi) - T_0) \cdot T_0 = (2c_1(\mathcal{F}) - T_0) \cdot T_0. \tag{3.8}$$

Moreover, as $B(\pi)$ and the branch divisor of the discriminant morphism $\Delta(\pi)$ coincide outside a zero dimensional subscheme, T_0 is also in the branch of $\Delta(\pi)$, so denoting by $p_{\mathcal{F}} : \mathbb{P}(\mathcal{F}) \to Y$ the natural projection, the divisor $\Delta(S) \subset \mathbb{P}(\mathcal{F})$ satisfies:

$$\Delta(S) \cdot p_{\mathcal{F}}^{\star}T_0 \sim 2C + C', \tag{3.9}$$

where C and C' are distinct irreducible sections of $p_{\mathcal{F}}^{\star}T_0 = \mathbb{P}(\mathcal{F} \otimes \mathcal{O}_{T_0})$. We also point out that $\Delta(S) \cdot p_{\mathcal{F}}^{\star}T_0$ contains no fibers of the ruled surface $p_{\mathcal{F}}^{\star}T_0$ as such fibers would correspond to π-planar points. Since the π-fibers over such points are isomorphic to $Spec\left(\frac{k(y)[u,v]}{(u^2,v^2)}\right)$, we see that any curve containing such a subscheme as a fiber has at least a node at such a point, with a double ramification. It follows that its stable model belongs either to the boundary of the divisor D of double simple ramifications, or it has two or more singularities, hence this would contradict the condition (†).

Now we observe that

$$C \cap C' = \emptyset. \tag{3.10}$$

Indeed a point in $C \cap C'$ would imply a total ramification point for the discriminant cover $\Delta(\pi)$, hence the cover π would have either a total ramification point, or a triple ramification point, or a planar point. Indeed, the only pencils of irreducible conics admitting only one reducible conic are the hyperosculating and the osculating ones. Furthermore, the only pencil of reducible conics yielding a zero dimensional Gorenstein base locus is the one corresponding to a planar point. The condition (†) implies that all this cases can not occur.

Now we show that by the relations (3.8), (3.9), (3.10) it follows that $\mathcal{F}(-T_0)$ is nef over T_0. Since $\mathcal{F} \otimes \mathcal{O}_{T_0}$ is not nef, denoting by H the tautological divisor of $\mathbb{P}(\mathcal{F} \otimes \mathcal{O}_{T_0})$, and by H_0 a section of minimal selfintersection, we have

$$H \cdot H_0 < 0,$$

so H contains H_0. But as the restriction of H to the discriminant Δ gives the ramification divisor of the discriminant morphism, and since the only section contained in the ramification divisor over T_0 is C, we have $C = H_0$.

Our claim is equivalent to show that $H + (-T_0^2)L$ is nef where L is a ruling of $\mathbb{P}(\mathcal{F} \otimes \mathcal{O}_{T_0})$. In other words, we want to show that $(H - p_{\mathcal{F}}^{\star}T_0) \cdot A \geq 0$ for any irreducible reduced curve $A \subset \mathbb{P}(\mathcal{F} \otimes \mathcal{O}_{T_0})$.

Since the Neron-Severi group of $\mathbb{P}(\mathcal{F} \otimes \mathcal{O}_{T_0})$ is $\mathbb{Z}[H_0] \oplus \mathbb{Z}[L]$, we have only to prove that $(H - p_{\mathcal{F}}^{\star}T_0) \cdot H_0 \geq 0$.

Note that by construction, the discriminant surface $\Delta(S) \subset \mathbb{P}(\mathcal{F})$ is a reduced effective divisor, with divisor class

$$\Delta(S) \sim 3T_{\mathbb{P}(\mathcal{F})} - \pi^{\star}c_1(\mathcal{F}),$$

where $T_{\mathbb{P}(\mathcal{F})}$ is the tautological divisor of $\mathbb{P}(\mathcal{F})$, and $\mathbb{P}(\mathcal{F} \otimes \mathcal{O}_{T_0})$ is not a subdivisor of $\Delta(S)$. In particular

$$\Delta(S)_{|\mathbb{P}(\mathcal{F}\otimes\mathcal{O}_{T_0})} = 2H_0 + C' \equiv 3H + (-c_1(\mathcal{F}) \cdot T_0)L,$$

so

$$3H \equiv 2H_0 + C' + (c_1(\mathcal{F}) \cdot T_0)L.$$

Assume that $H_0^2 \geq 0$. Then by the relation (3.8), we have

$$3(H - p_{\mathcal{F}}^{\star}T_0) \cdot H_0 = 2H_0^2 + C' \cdot H_0 + c_1(\mathcal{F}) \cdot T_0 - 3T_0^2 \geq 2H_0^2 - \frac{5}{2}T_0^2 > 0.$$

Finally we consider the case where $H_0^2 < 0$ and $H \cdot H_0 < 0$. We have

$$[H] = [H_0] + b[L]$$

where $b \in \mathbb{Z}$. Observe that by the relation (3.9) it follows that:

$$C' \sim 3H + (-c_1(\mathcal{F}) \cdot T_0)L - 2C = C + (3b + (-c_1(\mathcal{F}) \cdot T_0))L.$$

By our condition (3.10) we have that $H_0^2 = c_1(\mathcal{F}) \cdot T_0$. By the standard equality $c_1(\mathcal{F}) \cdot T_0 = H^2 = H_0^2 + 2b$ it follows that $b = 0$. This means that $H = H_0$, so

$$2(H + (-T_0^2)L) \cdot H_0 = 2(H_0^2 - T_0^2) = (2c_1(\mathcal{F}) - T_0) \cdot T_0 - T_0^2 \geq -T_0^2 > 0.$$

Now we use the fact $\mathcal{F}(-T_0)$ is nef over T_0 to get our bounds.

More precisely, by choosing $T_{\widetilde{Y}} = T_0$ in the sequence (2.3), by shifting it by $\mathcal{O}_Y(-T_0)$ and by restricting it to T_0, since any quotient of $\mathcal{F}(-T_0) \otimes \mathcal{O}_{T_0}$ is nef, instead of the inequality 2.5 we get the inequality

$$\deg(\pi_B^\star M) \geq -(\alpha - 1)T_0^2,$$

which allows to conclude in the same way as we have $\alpha \geq 4$ by Corollary 3.8.

Finally, assume that $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$. Then $\mathcal{F}$ is balanced. Again by Theorem 2.10 the claimed bounds (3.6) and (3.7) follow. Observe that for the proof of the odd balanced case we need not the assumption (†). □

The next result will allow to establish a bound on $c_2(\mathcal{E})$.

Lemma 3.13. *Let $\pi\colon S \to Y$ be a Gorenstein cover of arbitrary degree n, where S is a normal surface and let R be the π-relative canonical divisor. Let $\mathcal{E}$ be the reduced direct image sheaf. Then*

$$R^2 \leq \frac{4}{n} c_1(\mathcal{E})^2.$$

Proof. Let H be any ample divisor on Y. Since π is finite, the divisor $\pi^\star H$ is ample on S. Observe that the $\mathbb{Q}$-divisor $R - \frac{2}{n}\,\pi^\star c_1(\mathcal{E})$ satisfies

$$\left(R - \frac{2}{n}\,\pi^\star c_1(\mathcal{E})\right) \cdot \pi^\star H = (\pi_\star R - 2c_1(\mathcal{E})) \cdot H = 0.$$

By the Hodge Index Theorem we have

$$\left(R - \frac{2}{n}\,\pi^\star c_1(\mathcal{E})\right)^2 \leq 0,$$

which gives the claim.

□

Lemma 3.14. *Let $f\colon S \to B$ be a genus g fibration which factorises through a degree 4 Gorenstein cover $\pi\colon S \to Y$ where Y is a smooth surface. Then*

$$c_2(\mathcal{E}) \geq \frac{1}{4}\Big(c_1(\mathcal{E})^2 + c_2(\mathcal{F})\Big). \tag{3.11}$$

Proof. By Proposition 3.1 we have $R^2 = 2c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) + c_2(\mathcal{F})$. Since by Lemma 3.13 it holds $c_1(\mathcal{E})^2 \geq R^2$, the claim follows. □

Now we can apply Theorem 3.12 to a factorised fibration.

Corollary 3.15. *Let $f\colon S \to B$ be a genus g fibration which factorises through a degree 4 Gorenstein cover $\pi\colon S \to Y$ where $\pi_B\colon Y \to B$ is a ruled surface. Then*

$$s(f) \geq \frac{\frac{(g-1)}{(g+3)}c_1(\mathcal{E})^2}{\frac{(g+1)}{4(g+3)}c_1(\mathcal{E})^2 - \frac{1}{4}c_2(\mathcal{F})}. \tag{3.12}$$

Proof. Consider the following function obtained in Corollary 3.2:

$$s(c_1(\mathcal{E})^2, c_2(\mathcal{F}), c_2(\mathcal{E})) := \frac{\frac{2(g+1)}{(g+3)}c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) + c_2(\mathcal{F})}{\frac{(g+2)}{2(g+3)}c_1(\mathcal{E})^2 - c_2(\mathcal{E})}. \tag{3.13}$$

Observe that the partial derivative of $s(c_1(\mathcal{E})^2, c_2(\mathcal{F}), c_2(\mathcal{E}))$ with respect to $c_2(\mathcal{E})$ is positive if and only if $c_2(\mathcal{F}) \geq \frac{2}{(g+3)}c_1(\mathcal{E})^2$. This is always satisfied by Theorem 3.12 (1). Hence we can use the bound on $c_2(\mathcal{E})$ given in Lemma 3.14, and we conclude by applying Lemma 3.14. □

3.3. **Proof of the Main Theorem for factorised fibrations.** Now it is easy to see that if S is a *normal surface* and $f\colon S \to B$ is a semistable fibration which factorises through a finite Gorenstein cover, the claims of the main theorem are straight consequence of Theorem 3.10, of Theorem 3.12, of Corollary 3.15, and the fact that such a bound is an increasing function in $c_2(\mathcal{F})$. □

4. The main theorem

4.1. **A theorem on n- gonal semistable fibrations.** Let $B \subset \overline{\mathcal{M}}^1_{g,n}$ be a curve, which is not contained in the boundary divisor, and let $f : S \to B$ be the semistable fibration associated with B.

We recall the following facts, which are well-known to the experts:

Theorem 4.1. *Let $f\colon S \to B$ be a non isotrivial semistable fibration such that its general fiber is smooth non-hyperelliptic and of genus $g > 2$.*

Then, by possibly replacing B by a finite base-change, there exists a ruled surface $\pi_B\colon \mathbb{P}(\mathcal{B}) \to B$ and a rational degree n map $\rho\colon S \dashrightarrow \mathbb{P}(\mathcal{B})$, such that $f\colon S \to B$ is factorised through $\rho\colon S \dashrightarrow \mathbb{P}(\mathcal{B})$. Finally, S is a minimal surface of general type.

Proof. The proof follows by the theorem of semistable reduction and by standard facts of the theory of surfaces; see [HMo, Section 2, The basic construction, page 337]. □

By Theorem 4.1, in order to evaluate the slope of a non isotrivial semistable fibration with fiber an n-gonal curve it can be useful to study the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{P}(\mathcal{B}) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\pi_B} \\ S & \xrightarrow{f} & B, \end{array} \tag{4.1}$$

where S is a minimal surface of general type, in particular K_S is big and nef, and $\tau\colon X \to S$ is the minimal resolution of $\rho\colon S \dashrightarrow \mathbb{P}(\mathcal{B})$. We denote by Y the ruled surface $\mathbb{P}(\mathcal{B})$.

We observe now that the morphism $\widehat{\pi}\colon \widehat{X} \to Y$ arising from the Stein factorisation of $\pi\colon X \to Y$ is not necessarily a Gorenstein cover. The non Gorenstein zero dimensional schemes of degree four are isomorphic to one of the following (see [HaMi, Tables 6.1 and 6.2]:

$$\mathrm{Spec}\; K \oplus \frac{K[x,y]}{(x,y)^2}, \quad \mathrm{Spec}\; \frac{K[x,y]}{(x^3,xy,y^2)}, \quad \mathrm{Spec}\; \frac{K[x,y,z]}{(x,y,z)^2}.$$

Hence the possible non Gorenstein fibers are supported on total or triple ramification points for the cover $\widehat{\pi}$, and since they are not curvilinear, they are singular points of the corresponding fiber in the fibration $\widehat{X} \to B$. The stable model of such a fiber is either a curve with two or more nodes, or a curve with a node in P, and such that one or both branches of the curve are ramified in P. So such fibers occur on curves corresponding to non general points of the boundary divisor δ. It follows that if the starting curve $B \subset \overline{\mathcal{M}}^1_{g,4}$ does not interesect a certain proper subscheme $\Xi \subset \overline{\mathcal{M}}^1_{g,4}$, the Stein factorisation of $\pi\colon X \to Y$ is a Gorenstein cover.

Nevertheless, it is possible to treat a more general case, by imposing some condition on the possible non Gorenstein fibers of $\widehat{\pi} : \widehat{X} \to Y$.

From now on and with the notation of diagram 4.1 we assume that the following condition holds:

($\star$) the possible non Gorenstein points of $\widehat{X}$ occur only over simple nodes of $B(\pi)$.

We point out that the sweeping families constructed by Harris and Morrison in [HMo] satisfy our condition (cf. [BeZ]).

Definition 4.2. A *curve with good Gorenstein factorization* is a smooth curve $B \subset \overline{\mathcal{M}}^1_{g,n}$ such that the condition ($\star$) holds.

Next we shall determine the invariants of one dimensional families B of fourgonal curves with good Gorenstein factorization.

4.2. **The Gorenstein model.** We consider again the diagram (4.1). Let $X \to \widehat{X} \to Y$ be the Stein factorisation of $\pi\colon X \to Y$. The technical side of Definition 4.2 is that the singular points of $\widehat{X}$ which are not Gorenstein singularities occur only over the points of the branch locus $B(\pi)$ of $\pi\colon X \to Y$ where $B(\pi)$ has a node. We want to replace the finite Stein morphism $\widehat{\pi}\colon \widehat{X} \to Y$ with a Gorenstein morphism $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{Y}$ where $\widetilde{X}$ is a Gorenstein model of $\widehat{X}$ and $\widetilde{Y}$ is obtained by suitable blow-ups of the ruled surface Y.

Let us study closely what happens over a node p of $B(\pi)$ if the analytical germ of $\widehat{X}$ over a germ of Y centered at p has a non Gorenstein singularity. Essentially we need to understand the local behaviour of the cover $\widehat{\pi}\colon \widehat{X} \to Y$ over an analytical neighboorhood of p isomorphic to the polydisk $\Delta \subset \mathbb{C}^2$. We will follow [L]. We know that if $\widehat{\pi}\colon \widehat{X} \to Y$ is a finite and dominant morphism from a variety $\widehat{X}$ with isolated singularities to a smooth variety Y and there is a simple normal crossing divisor D in Y such that $\widehat{\pi}\colon \widehat{X} \to Y$ is smooth over $Y \setminus D$, then $\widehat{X}$ has algebraic abelian quotient singularities. In our case to analyse the behaviour at $p \in B(\pi) \subset Y$ we can locally identify $B(\pi) \subset Y$ with $\{xy = 0\} \subset \mathbb{C}^2_{x,y}$ and p with $(0,0)$. The local fundamental group of $\mathbb{C}^2_{x,y} \setminus \{xy = 0\}$ near the origin is free abelian of rank 2, generated by loops around the two boundary divisors. Thus, away from the branch locus, $\widehat{\pi}\colon \widehat{X} \to Y$ is locally analytically equivalent to a covering of the form $\mathbb{C}^2_{u,v} \setminus \{uv = 0\} \to \mathbb{C}^2_{x,y} \setminus \{xy = 0\}$ determined by the corresponding morphism of fundamental groups. Since the degree is 4, we have to consider subgroups of index ≤ 4. Explicitly, let $N = \mathbb{Z}^2$ be the lattice of 1-parameter subgroups of the torus $(\mathbb{C}^*)^2_{x,y}$.

Let $N' \subset N$ be a sublattice of index 4. Then the inclusion $N' \subset N$ corresponds to a degree 4 toric morphism $Z \to \mathbb{C}^2$ ramified along the toric boundary divisors. Since the degree is 4 the only case where Z is not Gorenstein is given by $Z = 1/4(1,1) = \mathbb{C}^2_{u,v}/(\mathbb{Z}/4\mathbb{Z})$ with $\mathbb{Z}/4\mathbb{Z}$-action $(u, v) \mapsto (iu, iv)$, and $Z \to \mathbb{C}^2$ given by $(u, v) \mapsto (x, y) = (u^4, v^4)$. Note that in this case, above p there exists a unique point $q \in \widehat{X}$ of total ramification. Now if $\widetilde{Z} \to Z$ is the minimal resolution and $\widetilde{Y} \to \mathbb{C}^2$ is the blowup at p, then $\widetilde{Z} \to \widetilde{Y}$ is finite flat. Now we can glue the local partial resolution of the quotient singularities q_1,..., q_s of type $1/4(1,1)$ inside $\widehat{X}$ to construct a partial resolution $\tau\colon \widetilde{X} \to \widehat{X}$ such $\widetilde{X}$ is Gorenstein over those singularities. Let $\widetilde{Y} \to Y$ be the germ of the blow-up of Y at the images of the point q_1,..., q_s. There exists a germ of a morphism $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{Y}$ such that if E_{q_i}, $i = 1, ..., s$ denote the corresponding exceptional divisors on $\widetilde{X}$ and if $E'_i := \widetilde{\pi}(E_{q_i})$ for $i = 1, \dots, s$, then it holds that

(1) $2K_{\widetilde{X}} = \tau^\star(2K_X) - \sum_{i=1}^s E_{q_i}$ where each E_{q_i} is a rational curve;
(2) $E^2_{q_i} = -4$ and $\widetilde{\pi} : E_{q_i} \to E$ is $4:1$ on E_{q_i}, $i = 1, \dots, s$;
(3) $E_{q_i} \cdot \widetilde{R} = 6$ and $E_{q_i} \not\subset \widetilde{R}$, where $\widetilde{R}$ denotes the ramification divisor of $\widetilde{\pi}$;
(4) $K_{\widetilde{X}} = \tau^\star K_X - \frac{1}{2}E_{q_i}$ as $\mathbb{Q}$- divisors and locally over q_i.

For the last equation above note that as $\mathbb{Q}$- divisors we can write locally $K_{\widetilde{X}} = \tau^\star K_X + aE_q$ and $-2 = E^2_q + aE^2_q = -4 - 4a$, hence $a = -\frac{1}{2}$.

Now we perform the analysis of a sublattice $N' \subset N$ of index 3. This means that over the node $p \in B(\pi)$ we have an open set of $\widehat{X}$ which is mapped 1-to-1 and another which is mapped 3-to-1. The only non-Gorenstein possibility is given by $Z = 1/3(1,1) = \mathbb{C}^2_{u,v}/(\mathbb{Z}/3\mathbb{Z})$. Denote by $E_a \subset \widetilde{X}$ the exceptional divisor over a non-Gorenstein point $a \in \widehat{X}$ of this kind. It follows that E_a is an irreducible reduced rational curve with selfintersection $E^2_a = -3$, and there is a 3-to-1 cover $E_a \to E''$ of the exceptional curve $E'' \subset \widetilde{Y}$ given by blowing-up at p. Moreover we have $E_a \cdot \widetilde{R} = 4$. Notice that to resolve the morphism we have to add a -1-curve $A_a \subset \widetilde{X}$ which is disjoint from E_a and which is mapped 1-to-1 to $E'' \subset \widetilde{Y}$. Notice that $\widetilde{R} \cdot A_a = 0$ but if we contract A_a we can't factorise the cover through $\widetilde{Y}$. We shall denote by $E''_1, \dots, E''_t$ all the exceptional curves on $\widetilde{Y}$ of this type obtained by the images of the point $a_1, ..., a_t \in \widetilde{X}$ of index 3.

Finally, if $N' \subset N$ is a sublattice of index 2, then we have that $\widehat{X}$ is Gorenstein above p.

The given description allows to pass from $\widehat{\pi}\colon \widehat{X} \to Y$ to a Gorenstein cover.

Theorem 4.3. *We use the above notation. Let $\pi\colon X \to Y$ be a generically finite degree four cover between a normal surface X and a ruled surface $\pi_B\colon Y \to B$ over a smooth curve B such that $f_X\colon X \to B$ is a genus $g \geq 10$ fibration where $f_X := \pi_B \circ \pi$. Let $\widehat{\pi}\colon \widehat{X} \to Y$ be the finite morphism of degree four given by the Stein factorisation $X \to \widehat{X} \to Y$ of $\pi\colon X \to Y$. Assume that the condition $(\star)$ holds for $\widehat{\pi}$.*

Then there exists a finite number of blow-ups $\sigma\colon \widetilde{Y} \to Y$ and a partial resolution $\widetilde{\tau}\colon \widetilde{X} \to \widehat{X}$ of $\widehat{X}$, such that $\widetilde{X}$ is Gorenstein and there exists an induced degree four Gorenstein cover $\widetilde{\pi}: \widetilde{X} \to \widetilde{Y}$ such that $\sigma \circ \widetilde{\pi} = \widehat{\pi} \circ \widetilde{\tau}$.

Moreover, the possible non Gorenstein points of $\widehat{X}$ are total ramification points or ramification points of index three for $\widehat{\pi}\colon \widehat{X} \to Y$. Let $q_1,\dots, q_s \in \widehat{X}$ be the non Gorenstein points of total ramification, and let $a_1,\dots,a_t \in \widehat{X}$ be the non Gorenstein index 3 ramification points. Then the $\widetilde{\pi}$-relative canonical divisor $\widetilde{R}$ satisfies

$$\widetilde{\pi}_\star \widetilde{R} \equiv 2\left((g+3)T_0 + \frac{c_1^2(\widetilde{\mathcal{E}})+9s+4t}{2(g+3)}\widetilde{L} - 3\sum_{i=1}^{s} E'_i - 2\sum_{j=1}^{t} E''_j\right), \tag{4.2}$$

where $T_0 = T_{\widetilde{Y}} - \frac{1}{2}T^2_{\widetilde{Y}}\widetilde{L}$, $T_{\widetilde{Y}}$ is any section of $\widetilde{Y}$, $\widetilde{L}$ is a general fiber of $\widetilde{Y} \to B$, E'_i, $i = 1,\dots,s$ are (-1)-curves arising from the images of total ramification non-Gorenstein points and E''_j, $j = 1,\dots,t$ are (-1)-curves arising from the images of index 3 non-Gorenstein points.

Proof. The analysis performed above on the non Gorenstein points of $\widehat{X}$ shows that $\sigma \circ \widetilde{\pi} = \widehat{\pi} \circ \widetilde{\tau}$.

We show the formula for $\widetilde{\pi}_\star\widetilde{R}$. By Theorem 2.3 (1) and by the fact that $\mathrm{NS}_{\mathbb{Q}}(\widetilde{Y}) = T_0\mathbb{Q} \oplus \widetilde{L}\mathbb{Q} \oplus E'_1\mathbb{Q} \oplus \dots \oplus E'_s\mathbb{Q} \oplus E''_1\mathbb{Q} \dots \oplus E''_t\mathbb{Q}$ we can write:

$$c_1(\widetilde{\mathcal{E}}) \equiv \frac{1}{2}\widetilde{\pi}_\star\widetilde{R} \equiv (g+3)T_0 + d\widetilde{L} + \sum_{i=1}^{s} a_i E'_i + \sum_{j=1}^{t} b_j E''_j.$$

Set $c_1^2 := c_1(\widetilde{\mathcal{E}})^2$. We get

$$d = \frac{c_1^2 + \sum a_i^2 + \sum b_j^2}{2(g+3)}.$$

We recall that $\widetilde{\pi}^\star E'_i = E_{q_i}$, $i = 1,\dots,s$ and $\widetilde{\pi}^\star E''_j = E_{a_j} + A_{a_j}$, $j = 1,\dots,t$. It follows that if $i \le k \le s$

$$6 = \widetilde{R}\cdot E_{q_k} = \widetilde{R}\cdot\widetilde{\pi}^\star E'_k = \widetilde{\pi}_\star\widetilde{R}\cdot E'_k =$$

$$= 2\left((g+3)T_0 + \frac{c_1^2 + \sum_{i=1}^s a_i^2 + \sum_{j=1}^t b_j^2}{2(g+3)}\widetilde{L} + \sum_{i=1}^{s} a_i E'_i + \sum_{j=1}^{t} b_j E''_j\right)\cdot E'_k = -2a_k,$$

hence $a_k = -3$ for any $k = 1,\dots,s$. Similarly if $1 \le l \le t$ we find

$$4 = \widetilde{R}\cdot(E_{a_l} + E'_{a_l}) = \widetilde{R}\cdot\widetilde{\pi}^\star E''_l = \widetilde{\pi}_\star\widetilde{R}\cdot E''_l =$$

$$= 2\left((g+3)T_0 + \frac{c_1^2 + \sum_{i=1}^s a_i^2 + \sum_{j=1}^t b_j^2}{2(g+3)}\widetilde{L} + \sum_{i=1}^{s} a_i E'_i + \sum_{j=1}^{t} b_j E''_j\right)\cdot E''_l = -2b_l,$$

hence $b_l = -2$ for any $l = 1,\dots,t$. □

We can apply Theorem 4.3 to the case given in diagram (4.1). Actually we can prove a slightly more general theorem.

Theorem 4.4. *With the same assumptions as in Theorem 4.3, the Gorenstein resolution* $\widetilde{\tau}\colon \widetilde{X} \to \widehat{X}$ *induces a fibration* $\widetilde{f}\colon \widetilde{X} \to B$*, which slope* $s(\widetilde{f})$ *satisfies the claims of the Main Theorem stated in the Introduction.*

Proof. Since $B(\pi)$ satisfies the $(\star)$-condition then we can apply Theorem 4.3 to $\widehat{\pi}\colon \widehat{X} \to Y$. Hence let $\widetilde{\tau}\colon \widetilde{X} \to \widehat{X}$ be the partial resolution of $\widehat{X}$, such that $\widetilde{X}$ is Gorenstein. Again by Theorem 4.3 there exists an induced degree four morphism $\widetilde{\pi} : \widetilde{X} \to \widetilde{Y}$ where $\widetilde{Y} \to Y$ is obtained by blowing up the images of the non Gorenstein index 3 points or of non Gorenstein total ramification points of $\widehat{X}$.

By Proposition 2.2, if $\widetilde{R} = K_{\widetilde{X}/\widetilde{Y}}$ there exists an embedding $\widetilde{j}\colon \widetilde{X} \to \mathbb{P}(\widetilde{\mathcal{E}})$ where $\mathcal{O}_{\widetilde{X}}(R) \cong j^\star\mathcal{O}_{\mathbb{P}(\widetilde{\mathcal{E}})}(1)$. We denote by $\widetilde{\mathcal{F}} = (\widetilde{\pi}_{\mathbb{P}})_\star\mathcal{I}_{\widetilde{X},\mathbb{P}(\widetilde{\mathcal{E}})}(2)$ the bundle of conics. We recall that by the proof of Proposition 2.6 it also holds:

$$\chi_{\widetilde{f}} = \chi(\mathcal{O}_{\widetilde{X}}) - (g-1)(g(B)-1) = 4\chi(\mathcal{O}_{\widetilde{Y}}) + \frac{1}{2}c_1(\widetilde{\mathcal{E}})\cdot K_{\widetilde{Y}} + \frac{1}{2}c_1(\widetilde{\mathcal{E}})^2 - c_2(\widetilde{\mathcal{E}}) - (g-1)(g(B)-1).$$

By the equation (4.2) obtained in Theorem 4.3 we have:

$$c_1(\widetilde{\mathcal{E}}) \equiv (g+3)T_0 + \frac{c_1^2(\widetilde{\mathcal{E}}) + 9s + 4t}{2(g+3)}\widetilde{L} - 3\sum_{i=1}^{s} E_i - 2\sum_{j=1}^{t} E_j'', \quad K_{\widetilde{Y}} \equiv -2T_0 + \widetilde{f}^\star K_B + \sum_{i=1}^{s} E_i' + \sum_{j=1}^{t} E_j''.$$

Then

$$c_1(\widetilde{\mathcal{E}})\cdot K_{\widetilde{Y}} = 2(b-1)(g+3) - \frac{c_1(\widetilde{\mathcal{E}})^2}{(g+3)} + 3\frac{g}{(g+3)}s + 2\frac{(g+1)}{(g+3)}t, \quad K^2_{\widetilde{Y}} = -8(g(\overline{B}-1) - s - t.$$

It follows that:

$$K^2_{\widetilde{f}} = 2\frac{(g+1)}{(g+3)}c_1(\widetilde{\mathcal{E}})^2 - 4c_2(\widetilde{\mathcal{E}}) + c_2(\widetilde{\mathcal{F}}), \tag{4.3}$$

and

$$\chi_{\widetilde{f}} = \frac{(g+2)}{2(g+3)}c_1(\widetilde{\mathcal{E}})^2 - c_2(\widetilde{\mathcal{E}}) + \frac{3g}{2(g+3)}s + \frac{(g+1)}{(g+3)}t \tag{4.4}$$

As $\widetilde{X}$ is normal, we may apply Lemma 3.13 to get

$$c_2(\widetilde{\mathcal{E}}) \geq \frac{1}{4}\Big(c_1(\widetilde{\mathcal{E}})^2 + c_2(\widetilde{\mathcal{F}})\Big), \tag{4.5}$$

which follows from the relation $\widetilde{R}^2 = 2c_1(\widetilde{\mathcal{E}})^2 - 4c_2(\widetilde{\mathcal{E}}) + c_2(\widetilde{\mathcal{F}})$ shown in Proposition 3.1.

By equations (4.3), (4.4) and by the inequality (4.5) we obtain:

$$s(\widetilde{f}) \geq 4 + \frac{c_2(\widetilde{\mathcal{F}}) - \frac{2}{(g+3)}c_1(\widetilde{\mathcal{E}})^2}{\frac{(g+1)}{4(g+3)}c_1(\widetilde{\mathcal{E}})^2 - \frac{1}{4}c_2(\widetilde{\mathcal{F}}) + \frac{3g}{2(g+3)}s + \frac{(g+1)}{(g+3)}t}. \tag{4.6}$$

Set

$$\upsilon(c_1(\widetilde{\mathcal{E}})^2, c_2(\widetilde{\mathcal{F}}), s, t) := 4 + \frac{c_2(\widetilde{\mathcal{F}}) - \frac{2}{(g+3)}c_1(\widetilde{\mathcal{E}})^2}{\frac{(g+1)}{4(g+3)}c_1(\widetilde{\mathcal{E}})^2 - \frac{1}{4}c_2(\widetilde{\mathcal{F}}) + \frac{3g}{2(g+3)}s + \frac{(g+1)}{(g+3)}t}.$$

We observe that $\upsilon(c_1(\widetilde{\mathcal{E}})^2, c_2(\widetilde{\mathcal{F}}), s, t)$ is an increasing function in $c_2(\widetilde{\mathcal{F}})$.

Now $c_2(\widetilde{\mathcal{F}})$ can be bounded using similar arguments as in the ruled case. The claim follows by next Proposition 4.5, as the resulting expressions are all bounded from below by the same expressions but evaluated in $s = t = 0$. $\square$

Proposition 4.5. *Let $B \subset \overline{\mathcal{M}}^1_{g,4}$, and assume that $B \nsubseteq T$, $B \nsubseteq D$.*

Provided that the cover $\widetilde{\pi} : \widetilde{X} \to \widetilde{Y}$ satisfies condition (†) in cases (1), (2) and (3) below, the second Chern class of the bundle of conics $\widetilde{\mathcal{F}}$ is bounded by:

(1) $c_2(\widetilde{\mathcal{F}}) \geq \frac{2}{(g+3)}\left(c_1(\widetilde{\mathcal{E}})^2 + 9s + 4t\right)$;

(2) $c_2(\widetilde{\mathcal{F}}) \geq \frac{1}{6}\left(c_1(\widetilde{\mathcal{E}})^2 + 9s + 4t\right)$ *if $\pi_b \colon F_b \to \mathbb{P}^1$ does not factorise, where F_b a general fiber of $\widetilde{f} \colon \widetilde{X} \to B$;*

(3) $c_2(\widetilde{\mathcal{F}}) \geq \dfrac{(g+2)}{4(g+3)}\left(c_1(\widetilde{\mathcal{E}})^2 + 9s + 4t\right)$ *if g is even and $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$;*

(4) $c_2(\widetilde{\mathcal{F}}) \geq \frac{1}{4}\left(c_1(\widetilde{\mathcal{E}})^2 + 9s + 4t\right)$ *if g is odd and $B \not\subset \mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$.*

Proof. We use Theorem 2.10. The proof is similar to the proof of Theorem 3.12, taking into account that we can still use for the case of the blow-up surface $\widetilde{Y}$ the bounds on β given in Corollary 3.8. $\square$

4.3. **Proof of the main theorem.** We can finally conclude with our main result. Indeed fix a curve $B \subset \overline{\mathcal{M}}^1_{g,4}$ as in the main theorem and consider the diagram 4.1. Let $\widehat{\pi} \colon \widehat{X} \to Y$ be the finite morphism to the ruled surface $\pi_B \colon Y \to B$ obtained by the Stein factorisation $X \to \widehat{X} \to Y$ of the generically finite morphism $\pi \colon X \to Y$. The surface $\widehat{X}$ is normal; see, for instance, [La, Example 2.1.15, page 126]. By assumption, B is a curve with good Gorenstein factorisation, hence the morphism $\widehat{\pi} \colon \widehat{X} \to Y$ satisfies the $(\star)$ condition. Then we can apply Theorem 4.4 to $\widehat{\pi} \colon \widehat{X} \to Y$ and we obtain a fibration $\widetilde{f} \colon \widetilde{X} \to B$ such that the claims hold for the slope of $\widetilde{f} \colon \widetilde{X} \to B$.

Now we have that $\widetilde{X}$ is a Gorenstein model of S and by Theorem 4.1, S is a minimal surface. Observe that $\widetilde{X}$ has only canonical singularities; indeed, $\widetilde{X}$ coincides with X outside the points over the nodes of the branch locus. Moreover, by our resolution process the surface $\widetilde{X}$ has only rational Gorenstein singularities, which are canonical.

Therefore $K_S^2 \geq K_{\widetilde{X}}^2$. This implies that $s(f) \geq s(\widetilde{f})$ and all the claims of the main theorem follow. $\square$

Remark. A. Patel recently proved in his Ph. D. thesis that $g \equiv 3 \mod 6$, the slope of a sweeping 4-gonal family not contained entirely in the divisors T, D, $\mathrm{CE}(\overline{\mathcal{M}}^1_{g,4})$ and the Maroni divisor $M(\overline{\mathcal{M}}^1_{g,4})$ consisting of curves with non balanced reduced direct image sheaf, is bounded below as $s(f) \geq \frac{11}{2} - \frac{15}{2g}$. In this particular numerical case, the reduced direct image sheaf of the general fiber has splitting type $(g+3)/3, (g+3)/3, (g+3)/3$, so the reduced direct image sheaf $\mathcal{E}$ arising in our construction has non negative Bogomolov discriminant, by [Mo, Theorem 2.2.1]. The inequalities

$$c_2(\mathcal{E}) \geq \frac{1}{3}c_1(\mathcal{E})^2,\ c_2(\mathcal{F}) \geq \frac{1}{4}c_1(\mathcal{E})^2,$$

together with our formula 3.5 give the desired bound.

References

[AC] E. Arbarello, M. Cornalba, *Footnotes to a paper of Beniamino Segre*, Math. Ann. 256 no. 3, (1981), 341-362.

[ACG] E. Arbarello, M. Cornalba, P. Griffiths, *Geometry of Algebraic Curves, Volume II*, A Series of Comprehensive Studies in Mathemtics 268, Springer, (2011).

[AW] M. Artin, G. Winters, *Degenerate fibres and stable reduction of curves*, Topology 10, (1971), 373-383.

[BS] M. A. Barja, L. Stoppino, *Slopes of trigonal fibred surfaces and of higher dimensional fibrations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), n. 4, (2009), 647-658.

[BZ] M. A. Barja, F. Zucconi, *On the slope of fibred surfaces*, Nagoya Math. J. 164, (2001), 103-131.

[BeZ] V. Beorchia, F. Zucconi, *A note on Harris Morrison sweeping families*, Collect. Math. 66 no. 2, (2015), 191-202.

[B] J.E. Brosius, *Rank-2 vector bundles on a ruled surface. I*, Math. Ann. 265, n. 2, (1983), 155-168.

[BSa] M. Brundu, G. Sacchiero, *Stratification of the moduli space of fourgonal curves*, Proc. Edinb. Math. Soc. (2) 57 no. 3, (2014), 631-686.

[CE] G. Casnati, T. Ekedahl, *Covers of algebraic varieties. I. A general structure theorem, covers of degree 3,4 and Enriques surfaces*, J. Algebraic Geom. 5, n. 3, (1996), 439-460.

[C] G. Casnati, *Covers of algebraic varieties III. The discriminant of a cover of degree 4 and the trigonal construction*, Trans. Amer. Math. Soc. 350, n. 4, (1998), 1359-1378.

[CH] M. Cornalba, J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sc. Ec. Norm. Sup. 21, n. 4, (1988), 455-475.

[CS] M. Cornalba, L. Stoppino, *A sharp bound for the slope of double cover fibrations*, Michigan Math. J. 56, n. 3, (2008), 551-561.

[DP] A. Deopurkar, A. Patel, *Sharp slope bounds for sweeping families of trigonal curves*, Math. Res. Lett. 20, no. 5, (2013), 869-884.

[DZ] P. De Poi, F. Zucconi, *Gonality, apolarity and hypercubics*, Bull. London Math. Soc. vol. 45 n. 2, (2011), 623-645.

[E] D. Eisenbud *Commutative Algebra with a View Toward Algebraic Geometry*, Series: Graduate Texts in Mathematics, Vol. 150, Springer (1995).

[FJ] M. Fedorchuk, D. Jensen, *Stability of 2nd Hilbert points of canonical curves*, Int. Math. Res. Not. n. 22, (2013), 5270-5287.

[Fr] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Universitext, Springer-Verlag, New York, (1998).

[Fu] W. Fulton, *Intersection Theory*, 2nd ed Springer, New York, (1998).

[HaMi] D. W. Hahn, R. Miranda, *Quadruple covers of algebraic varieties*, J. Algebraic Geom. 8 no. 1,(1999), 1-30.

[Ha] R. Hartshorne, *Algebraic geometry*, Grad. Texts Math., Vol. 52, Springer Verlag, New York-Heidelberg-Berlin, (1977).

[HMo] J. Harris, I. Morrison, *Slopes of effective divisors on the moduli space of stable curves*, Invent. Math. 99, (1990), 321-355.

[K] S. L. Kleiman, *Toward a Numerical Theory of Ampleness*, Annals of Mathematics, Second Series, Vol. 84 n. 3, (Nov., 1966), 293-344.

[L] H. Laufer, *Normal two dimensional singularities*, Ann. of Math. studies 71, Princeton University Press, (1971).

[La] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer-Verlag, Berlin, (2004).

[Mi] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic geometry, Sendai 1985, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam, (1987), 449-476.

[Mo] A. Moriwaki, *A sharp slope inequality for general stable fibrations of curves*, J. Reine Angew. Math. 480, (1996), 177-195.

[Sch] F. O. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. 275, n. 1, (1986), 105-137.

[S] Z. Stankova, *Moduli of trigonal curves*, J. Algebraic Geom. 9, n. 4, (2000), 607-662.

[St] L. Stoppino, *Slope inequalities for fibred surfaces via GIT*, Osaka J. Math. 45, n. 4, (2008), 1027-1041.

[V1] E. Viehweg, *Die Additivität der Kodaira Dimension für projektive Faserräume über Varietäten des allgemeinen Typs*, Journ. reine angew. Math. 330, (1982), 132-142.

[V2] E. Viehweg, *Quasi-Projective Moduli for Polarized Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Series of Modern Surveys in Mathematics, Springer (1995).

[X] G. Xiao, *Fibred algebraic surfaces with low slope,* Math. Ann. 276, (1987), 449-466.

DIPARTIMENTO DI MATEMATICA E GEOSCIENZE, UNIVERSITÀ DEGLI STUDI DI TRIESTE, VIA VALERIO 12/B, 34127 TRIESTE, ITALY, beorchia@units.it

DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIA DELLE SCIENZE 206, UNIVERSITÀ DEGLI STUDI DI UDINE, UDINE, 33100 ITALY, Francesco.Zucconi@uniud.it