Generically nef vector bundles on ruled surfaces

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Abstract
The present paper concerns the invariants of generically nef vector bundles on ruled surfaces. By Mehta–Ramanathan Restriction Theorem and by Miyaoka characterization of semistable vector bundles on a curve, the generic nefness can be considered as a weak form of semistability. We establish a Bogomolov-type inequality for generically nef vector bundles with nef general fiber restriction on ruled surfaces with no negative section, see Theorem 3.1. This gives an affirmative answer in this case to a problem posed by Peternell [17]. Concerning ruled surfaces with a negative section, we prove a similar result for generically nef vector bundles, with nef and balanced general fiber restriction and with a numerical condition on first Chern class, which is satisfied, for instance, if in its class there is a reduced divisor, see Theorem 3.5. Finally, we use such results to bound the invariants of curve fibrations, which factor through finite covers of ruled surfaces.

Keywords Vector bundles · Chern classes · Fibrations · Finite covers

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1 Introduction

The present paper concerns the invariants of generically nef vector bundles on ruled surfaces, see Definition 2.1. By Mehta–Ramanathan Restriction Theorem 2.2 and by Miyaoka char-

Beorchia Valentina and Zucconi Francesco are members of GNSAGA of INdAM. The examples contained in Sect. 6.1 give an answer to a question posed by Ciro Ciliberto at the conference New Trends in Algebraic Geometry, Università della Calabria, June 12–14 (2013), about the sharpness of some slope estimates.

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acterization of semistable vector bundles on a curve, Theorem 2.3, the generic nefness can be considered as a weak form of semistability, see Corollary 2.4.

A question about a possible relation between generic nefness and Bogomolov-type inequality has been posed by Th. Peternell in the paper [17, remarks after Theorem 3.8]. By considering the Harder–Narasimhan filtration of a vector bundle $\mathcal{E}$ on a projective surface, Miyaoka proved the inequality $c_2(\mathcal{E}) \geq 0$, provided that $\mathcal{E}$ is generically nef and $c_1(\mathcal{E})$ is nef.

In the present paper, under the hypotheses of generic nefness and the nefness of the generic fiber restriction, we give an affirmative answer to Peternell's question for ruled surfaces with invariant $e = -C_0^2 \leq 0$ (see Theorem 3.1):

Theorem 1.1 *Let Y be a ruled surface on a smooth curve B with invariant $e = -C_0^2 \leq 0$. Let $\mathcal{E}$ be a generically nef vector bundle of rank r on Y with nef generic fiber restriction. Then*

$$c_2(\mathcal{E}) \geq \frac{\sum_{i=1}^{r-1} a_i}{2a} c_1(\mathcal{E})^2,$$

where $c_1(\mathcal{E}) \equiv aC_0 + \delta L$ and $(a_1, \ldots, a_r)$ is the generic splitting type of $\mathcal{E}$.

In the $e > 0$ case, we prove a similar bound under two additional assumptions, namely that the general fiber restriction is balanced, and $c_1(\mathcal{E}) \cdot C_0 \geq -\frac{e}{2}$, see Theorem 3.5. The last condition is satisfied, for instance, if $c_1(\mathcal{E})$ is nef or if $c_1(\mathcal{E})$ is effective and C_0 is not contained in the base locus of $|2\, c_1(\mathcal{E}) - C_0|$; this assumption is typically satisfied by the Tschirnhausen sheaf associated with a finite cover of smooth surfaces with reduced branch divisor.

Our results allow us to obtain some bounds on the invariants of fibered surfaces factoring through finite covers. In these specific cases, the bounds found are better than the recent bounds found by Lu and Zuo [13]. Moreover, in the case of primitive cyclic covers $\pi : S \to Y$, we obtain the same bound $\lambda_{g,0,n}$ given in [9, Remark 4.4], see Theorem 5.7.

The techniques involved concern vector bundles and algebraic surfaces techniques. We believe that our approach can be the starting point for further research in the theory of vector bundles on fibered surfaces in general.

2 Notation and preliminaries

Let us introduce the definitions involved in the notion of *generic nefness*, see [17, Definition 3.1]).

Definition 2.1 A vector bundle $\mathcal{E}$ on a smooth curve is *nef* if the tautological divisor of $\mathbb{P}(\mathcal{E})$ is nef.

A vector bundle $\mathcal{E}$ on a projective variety X of dimension $n \geq 2$ is called *generically nef* with respect to an ample divisor H if the restriction $\mathcal{E}_{|C}$ is nef for a general curve $C = D_1 \cap \cdots \cap D_{n-1}$, where $D_i \in |m_i\ H|$ are general and $m_i >> 0$; such a curve C is said to be *MR-general*, which means general in the sense of Mehta–Ramanathan, with respect to H (w.r.t. H).

A vector bundle $\mathcal{E}$ is *generically nef* if for every ample divisor H on X, the restriction $\mathcal{E}_{|C}$ is nef for a MR-general curve w.r.t. H.

We now recall Mehta–Ramanathan Restriction Theorem and Miyaoka characterization of semistable vector bundles on a curve, which imply that the generic nefness can be considered as a weak form of semistability.

Theorem 2.2 (Mehta–Ramanathan Restriction Theorem) *A locally free sheaf $\mathcal{E}$ on a projective variety X is semistable w.r.t. an ample divisor H if and only $\mathcal{E}_{|C}$ is semistable for C MR-general w.r.t. H.*

Theorem 2.3 (Miyaoka) *Let C be a smooth curve and E a vector bundle on C. Then E is semistable if and only if the $\mathbb{Q}$-bundle $E \otimes \frac{\det E^\vee}{\operatorname{rk} E}$ is nef, where $E^\vee = Hom(E, \mathcal{O}_C)$ is the dual vector bundle.*

Corollary 2.4 *If $\mathcal{E}$ is semistable w.r.t. H and if $c_1(\mathcal{E}) \cdot H \geq 0$, then $\mathcal{E}$ is generically nef w.r.t. H.*

Finally, since we shall consider vector bundles on ruled surfaces, we can talk of the general splitting type.

Definition 2.5 Let $p : Y \to B$ be a ruled surface over a smooth curve B, and let $\mathcal{E}$ be a rank r vector bundle on Y. We say that $(a_1, \ldots, a_r)$, with $a_1 \leq \cdots \leq a_r$ is the *generic splitting type* of $\mathcal{E}$ if for a general fiber L of p we have

$$\mathcal{E} \otimes \mathcal{O}_L \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

We say that a fiber L is a *jumping line* if

$$\mathcal{E} \otimes \mathcal{O}_L \ncong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Moreover, we say that $\mathcal{E}$ is *uniform* if it has no jumping lines.

Finally, let us recall that for a tensor product $\mathcal{V} \otimes \mathcal{L}$, where $\mathcal{V}$ is a rank r vector bundle and $\mathcal{L}$ is a line bundle on a smooth surface, we have:

$$c_1(\mathcal{V}\otimes\mathcal{L}) = c_1(\mathcal{V}) + r\, c_1(\mathcal{L}), \qquad c_2(\mathcal{V}\otimes\mathcal{L}) = c_2(\mathcal{V}) + (r-1)c_1(\mathcal{V})\cdot c_1(\mathcal{L}) + \frac{r(r-1)}{2} c_1(\mathcal{L})^2. \tag{2.1}$$

3 Bogomolov-type inequalities

Theorem 3.1 *Let Y be a ruled surface on a smooth curve B with invariant $e = -C_0^2 \leq 0$. Let $\mathcal{E}$ be a generically nef vector bundle of rank r on Y with nef generic fiber restriction.*

Then

$$c_2(\mathcal{E}) \geq \left(\frac{\sum_{i=1}^{r-1} a_i}{2a} \right) c_1(\mathcal{E})^2,$$

where $c_1(\mathcal{E}) \equiv aC_0 + \delta L$ and $(a_1, \ldots, a_r)$ is the generic splitting type of $\mathcal{E}$.

Proof We prove the statement by induction on $r \geq 2$. If $r = 2$, the claimed inequality has been proved in [5, Theorem 2.8], in the slightly different setting of blown up ruled surfaces, and under the assumption on $\mathcal{E}$ to be weakly positive and with nef general fiber restriction. We observe, however, that the weak positivity of a vector bundle implies its generic nefness. Moreover, since the weak positivity is preserved for any quotient bundle, the same properties hold for such quotients. Then it is simple to check that the proof of the statement in the rank 2 case can be done with some little adaptations.

So let us assume $r \geq 3$ and suppose that the claim holds for any vector bundle of rank $q \leq r - 1$ satisfying the assumptions of the statement.

We can consider the push–pull map

$$p^\star p_\star \mathcal{E}(-a_r\, C_0) \to \mathcal{E}(-a_r\, C_0);$$

such a map is generically injective; hence, it is an injective map of locally free sheaves. Moreover, the quotient sheaf is locally free outside a subscheme Z of codimension 2.

It follows that we have a Brosius-type sequence (see [4]):

$$0 \to p^\star p_\star \mathcal{E}(-a_r\, C_0) \to \mathcal{E}(-a_r\, C_0) \to \mathcal{G} \otimes \mathcal{I}_Z \to 0. \tag{3.1}$$

By setting $\mathcal{A} := p_\star(\mathcal{E}(-a_r\, C_0))$, $A := c_1(\mathcal{A}) =$ and $\alpha := \deg(A)$, and by tensoring by $\mathcal{O}_Y(a_r\, C_0)$, we get:

$$0 \to (p^\star \mathcal{A})(a_r\, C_0) \to \mathcal{E} \to \mathcal{G}(a_r\, C_0) \otimes \mathcal{I}_Z \to 0, \tag{3.2}$$

where $\mathcal{G}$ is a vector bundle of rank $q \leq r - 1$ and $\mathcal{A}$ is a vector bundle of rank $r - q \geq 1$. We observe that $r - q$ is equal to the number of integers in the general splitting type $(a_1, \ldots, a_r)$, which are equal to a_r, so that we have

$$a = \sum_{i=1}^{r} a_i = \sum_{i=1}^{q} a_i + (r - q)a_r. \tag{3.3}$$

The sequence (3.2) gives

$$c_2(\mathcal{E}) = c_1(p^\star A\ (a_r\, C_0)) \cdot c_1(\mathcal{G}(a_r\, C_0)) + c_2(p^\star A\ (a_r\, C_0)) + c_2(\mathcal{G}(a_r\, C_0)) + Z. \tag{3.4}$$

Let us compute each term appearing in (3.4). By the first relation in (2.1), we have $c_1(p^\star A\ (a_r\, C_0)) \equiv p^\star A + (r - q)a_r\, C_0$. Next, we set $\mathcal{M} := \mathcal{G}(a_r\, C_0)$. From the sequence (3.2), we get

$$c_1(\mathcal{M}) \equiv (c_1(\mathcal{E}) - p^\star A - (r - q)a_r C_0) \equiv a_{\mathcal{M}}\, C_0 + (\delta - \alpha)L \tag{3.5}$$

where we have set

$$a_{\mathcal{M}} := \sum_{i=1}^{q} a_i,$$

and

$$\begin{aligned} c_1(p^\star A\ (a_r\, C_0)) \cdot c_1(\mathcal{M}) &= (p^\star A + (r - q)a_r\, C_0) \cdot (a_{\mathcal{M}}\, C_0 + (\delta - \alpha)L) = \\ &= (r - q)a_r\, a_{\mathcal{M}}\, C_0^2 + (\alpha\, a_{\mathcal{M}} + (r - q)a_r(\delta - \alpha). \end{aligned} \tag{3.6}$$

By taking into account the second relation (2.1), we get

$$\begin{aligned} c_2(p^\star A\ (a_r\, C_0)) &= (r - q - 1)p^\star A \cdot (a_r\, C_0) + \tfrac{(r-q)(r-q-1)}{2} a_r^2\, C_0^2 = \\ &= (r - q - 1)\alpha\, a_r + \tfrac{(r-q)(r-q-1)}{2} a_r^2\, C_0^2. \end{aligned} \tag{3.7}$$

Moreover, since $\mathcal{M}$ is a quotient of $\mathcal{E}$ away from the zero-dimensional scheme Z, $\mathcal{M}$ is a generically nef vector bundle of rank $q < r$. Now we analyze the general fiber restriction of $\mathcal{M}$. Since $\mathcal{E}$ has nef general fiber restriction, we have that the general splitting type $(a_1, \dots, a_r)$ satisfies

$$0 \leq a_1 \leq \cdots \leq a_r. \tag{3.8}$$

Moreover, by construction, the generic splitting type of $\mathcal{M}$ is

$$(a_1, \dots, a_q)$$

where the integers a_i are the first q integers appearing in (3.8); hence, also $\mathcal{M}$ is nef on the generic fiber restriction.

It follows that $\mathcal{M}$ satisfies the assumptions of the theorem and we can apply the induction hypothesis, which gives the inequality

$$c_2(\mathcal{M}) \geq \left(\frac{\sum_{i=1}^{q-1} a_i}{2\, a_{\mathcal{M}}} \right) c_1(\mathcal{M})^2. \tag{3.9}$$

By (3.5), we get

$$c_1(\mathcal{M})^2 = a_{\mathcal{M}}^2\, C_0^2 + 2\, a_{\mathcal{M}}(\delta - \alpha), \tag{3.10}$$

and by observing that

$$\sum_{i=1}^{q-1} a_i = a_{\mathcal{M}} - a_q,$$

the inequality (3.9) becomes

$$c_2(\mathcal{M}) \geq \left(\frac{a_{\mathcal{M}} - a_q}{2} \right) \left(a_{\mathcal{M}}\, C_0^2 + 2\,(\delta - \alpha) \right). \tag{3.11}$$

By taking into account (3.6), (3.7), (3.11) and the fact that Z is effective, from (3.4), we get

$$\begin{aligned} c_2(\mathcal{E}) \geq\ & (r-q)a_r\, a_{\mathcal{M}}\, C_0^2 + \alpha a_{\mathcal{M}} + (r-q)a_r(\delta - \alpha) \\ & + (r-q-1)\, a_r\, \alpha + \tfrac{(r-q)(r-q-1)}{2} a_r^2\, C_0^2 + \left(\tfrac{(a_{\mathcal{M}} - a_q)}{2} \right) a_{\mathcal{M}}\, C_0^2 \\ & + (a_{\mathcal{M}} - a_q)(\delta - \alpha). \end{aligned} \tag{3.12}$$

We can rewrite the inequality above in the form

$$c_2(\mathcal{E}) \geq d C_0^2 + (a_q - a_r)\alpha + c\delta, \tag{3.13}$$

where

$$\begin{aligned} d &:= (r-q)a_r\, a_{\mathcal{M}} + \frac{1}{2}(r-q)(r-q-1)a_r^2 + \frac{1}{2}(a_{\mathcal{M}} - a_q)\, a_{\mathcal{M}}, \\ c &:= (r-q)a_r + a_{\mathcal{M}} - a_q. \end{aligned}$$

In particular, the coefficient of α in the expression (3.13) is

$$a_q - a_r < 0.$$

Now we shall bound the integer α. Recall that a generically nef vector bundle has a generically nef first Chern class, since exterior products of nef vector bundles are nef (see, for instance, [12, Theorem 6.2.12, (iv)]). Hence, $c_1(\mathcal{M})$ is generically nef. So for any nef divisor $xC_0 + yL$ with $x \geq 1$ and $y \geq -\frac{1}{2}x\, C_0^2$, we have

$$c_1(\mathcal{M}) \cdot (xC_0 + yL) \geq 0,$$

that is

$$(a_{\mathcal{M}}\, C_0 + (\delta - \alpha)L) \cdot (xC_0 + yL) = a_{\mathcal{M}} \left(xC_0^2 + y\right) + x(\delta - \alpha) \geq 0,$$

which gives

$$\alpha \leq \delta + \frac{a_{\mathcal{M}} \left(xC_0^2 + y\right)}{x},$$

and since this holds for any $x \geq 1$ and any $y \geq -\frac{1}{2}x\, C_0^2$, we get

$$\alpha \leq \delta + \frac{1}{2} a_{\mathcal{M}}\, C_0^2.$$

By substituting the right hand expression in (3.13), we get

$$c_2(\mathcal{E}) \geq \left(d + \frac{1}{2}(a_q - a_r)\, a_{\mathcal{M}}\right)\, C_0^2 + (c + a_q - a_r)\,\delta. \tag{3.14}$$

Next we shall suitably express the integer δ. As $c_1(\mathcal{E})^2 = a^2 C_0^2 + 2a\delta$, we can write

$$\delta = -\frac{a}{2} C_0^2 + \frac{c_1(\mathcal{E})^2}{2a}.$$

Then the inequality (3.14) becomes

$$c_2(\mathcal{E}) \geq (d + \frac{1}{2}(a_q - a_r)\, a_{\mathcal{M}} - \frac{a}{2}(c + a_q - a_r))C_0^2 + \frac{(c + a_q - a_r)}{2a} c_1(\mathcal{E})^2. \tag{3.15}$$

By computing the coefficients in (3.15), we get that the coefficient of C_0^2 is zero and we get

$$c_2(\mathcal{E}) \geq \left(\frac{a_{\mathcal{M}} + (r - q - 1)a_r}{2a}\right) c_1(\mathcal{E})^2 = \left(\frac{a - a_r}{2a}\right) c_1(\mathcal{E})^2,$$

which is the bound in the statement. □

As a particular case, we can consider generically nef vector bundles $\mathcal{E}$ with nef and *balanced* general fiber restriction, that is the restriction of $\mathcal{E}$ to a general fiber of Y is a balanced vector bundle with splitting type $(m, \ldots, m, m+1, \ldots, m+1)$.

Corollary 3.2 *Let Y be a ruled surface on a smooth curve B with invariant $e = -C_0^2 \leq 0$. Let $\mathcal{E}$ be a generically nef vector bundle of rank r on Y, such that the restriction of $\mathcal{E}$ to a general fiber of Y is a nef and balanced vector bundle with splitting type $(m, \ldots, m, m+1, \ldots, m+1)$, and set $c_1(\mathcal{E}) \equiv aC_0 + \delta L$, $a = mr + k$ and $1 \leq k \leq r - 1$.*

Then

$$c_2(\mathcal{E}) \geq \left(\frac{a - (m+1)}{2a}\right) c_1(\mathcal{E})^2 = \left(\frac{r-1}{2r} - \frac{r-k}{2ar}\right) c_1(\mathcal{E})^2.$$

Moreover, the equality holds if and only if $\mathcal{E}$ is uniform and $\mathcal{E}$ is an extension sitting in an exact sequence of the type

$$0 \to p^{\star}\mathcal{B} \otimes \mathcal{O}_Y((m+1)C_0) \to \mathcal{E} \to p^{\star}\mathcal{V} \otimes \mathcal{O}_Y(mC_0) \to 0,$$

where $\mathcal{B}$ is a rank k vector bundle on B satisfying $\deg c_1(\mathcal{B}) = \delta$, *and $\mathcal{V}$ is a rank $r-k$ vector bundle on B satisfying $c_1(\mathcal{V}) \equiv 0$.*

Proof The bound is a direct consequence of the general bound. The characterization of the vector bundles attaining the equality can be directly obtained by imposing the equalities in all the bounds considered in the proof of Theorem 3.1. □

We remark that vector bundles with balanced general fiber restriction are the natural generalization of vector bundles with semistable general fiber restriction, in which case the Bogomolov discriminant is nonnegative by Moriwaki's Theorem [14, Theorem 2.2.1], which we recall.

Theorem 3.3 *Let $\varphi\colon Z \to C$ be a fibration from a smooth surface Z to a smooth curve C. Let $\mathcal{E}$ be a torsion-free sheaf on Z such that the restriction of $\mathcal{E}$ to a general fiber $F \subset Z$ is a μ-semistable locally free sheaf. Then the Bogomolov discriminant $\Delta(\mathcal{E})$ satisfies*

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{rk(\mathcal{E}) - 1}{2\, rk(\mathcal{E})} c_1(\mathcal{E})^2 \geq 0.$$

Since in Moriwaki Theorem the only assumption is the semistability of the general fiber restriction, one can wonder, if the generic balancedness condition could be sufficient in order to have a Bogomolov-type inequality. A negative answer is given in the following example.

Example 3.4 On $Y = \mathbb{P}^1 \times \mathbb{P}^1$, consider the rank two split vector bundle

$$\mathcal{E} = \mathcal{O}_Y(m, b_1) \oplus \mathcal{O}_Y(m+1, b_2).$$

We have

$$c_1(\mathcal{E})^2 = 2(2m+1)(b_1 + b_2), \qquad c_2(\mathcal{E}) = m(b_1 + b_2) + b_1,$$

so we see that $c_2(\mathcal{E})$ can be arbitrarily lowered by adjusting b_1, even with $c_1(\mathcal{E})^2$ fixed.

Let us conclude this section with a result in the $e > 0$ case. We will consider only the balanced case, and we will need to assume $c_1(\mathcal{E}) \cdot C_0 \geq -\frac{e}{2}$, which is satisfied, for instance, if $c_1(\mathcal{E})$ is nef, or if $c_1(\mathcal{E})$ is effective and C_0 is not contained in the base locus of $|2\, c_1(\mathcal{E}) - C_0|$. The last condition is typically satisfied by Tschirnhausen sheaves associated with surface covers with reduced branch divisor.

Theorem 3.5 *Let Y be a ruled surface on a smooth curve B with invariant $e = -C_0^2 > 0$. Let $\mathcal{E}$ be a generically nef vector bundle of rank r on Y, such that the restriction of $\mathcal{E}$ to a general fiber of Y is a nef and balanced vector bundle with splitting type $(m, \dots, m, m+1, \dots, m+1)$, $c_1(\mathcal{E}) \equiv aC_0 + \delta L$, $a = mr + k$ and $1 \leq k \leq r - 1$.*

Assume, moreover, that $c_1(\mathcal{E}) \cdot C_0 \geq -\frac{e}{2}$.

Then

$$c_2(\mathcal{E}) \geq \left(\frac{a - (m+1)}{2a} - \frac{a - k(m+1)}{2a(a-1)} \right) c_1(\mathcal{E})^2. \tag{3.16}$$

Proof The proof is similar to that of Theorem 3.1. By assumption, the restriction of $\mathcal{E}$ to a general fiber L of $p\colon Y \to \mathbb{P}^1$ is balanced. Since by hypothesis $c_1(\mathcal{E}) \cdot L = a = mr + k$, the general fiber restriction of $\mathcal{E}$ is of the type

$$\mathcal{E}_{|L} \cong \bigoplus^{k} \mathcal{O}_{\mathbb{P}^1}(m+1) \oplus \bigoplus^{r-k} \mathcal{O}_{\mathbb{P}^1}(m).$$

Then we have again a Brosius-type exact sequence:

$$0 \to p^\star p_\star \mathcal{E}(-(m+1)C_0) \to \mathcal{E}(-(m+1)C_0) \to \mathcal{G} \otimes \mathcal{I}_Z \to 0, \tag{3.17}$$

where $p^\star p_\star\mathcal{E}(-(m+1)C_0)$ has rank k and $\mathcal{G}$ has rank $(r-k)$.

Set $\mathcal{A} := p_\star\mathcal{E}(-(m+1)C_0)$, $A := c_1(\mathcal{A}) =$, $\alpha := \deg(A)$ and $\mathcal{M} := \mathcal{G}((m+1)C_0)$, so that (3.17) becomes

$$0 \to p^\star\mathcal{A}((m+1)C_0) \to \mathcal{E} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0, \tag{3.18}$$

and

$$c_1(\mathcal{M}) = c_1(\mathcal{E}) - p^\star A - k(m+1)C_0 \equiv (r-k)m\, C_0 + (\delta-\alpha)L.$$

The main difference in the present proof is the bound on $c_2(\mathcal{M})$. Since the restriction of the Brosius exact sequence (3.17) to the general fiber $L \cong \mathbb{P}^1$ gives

$$0 \to \bigoplus^{k} \mathcal{O}_{\mathbb{P}^1} \to \bigoplus^{k} \mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus^{r-k} \mathcal{O}_{\mathbb{P}^1}(-1) \to \bigoplus^{r-k} \mathcal{O}_{\mathbb{P}^1}(-1) \to 0,$$

the restriction of $\mathcal{G}$ to the general fiber of Y is μ-semistable. Since $\mathcal{M}$ is a twist of $\mathcal{G}$, the same holds for the general fiber of $\mathcal{M}$. Hence, $\mathcal{M}$ is Bogomolov semistable by Moriwaki Theorem 3.3, and we have

$$c_2(\mathcal{M}) \geq \frac{(r-k-1)}{2(r-k)} c_1(\mathcal{M})^2 = \frac{(r-k)(r-k-1)}{2} m^2\, C_0^2 + (r-k)m(\delta-\alpha).$$

By observing that with the notations of the proof of Theorem 3.1, we have

$$a_r = m+1, \qquad q = r-k, \qquad a_{\mathcal{M}} = (r-k)m,$$

the relation (3.12) becomes

$$\begin{aligned} c_2(\mathcal{E}) \geq\ & k(r-k)m(m+1)C_0^2 + (\delta-\alpha)k(m+1) + \alpha(r-k)m + (k-1)(m+1)\alpha \\ & + \frac{k(k-1)}{2}(m+1)^2 C_0^2 + \frac{(r-k)(r-k-1)}{2} m^2\, C_0^2 + (r-k)m(\delta-\alpha), \end{aligned} \tag{3.19}$$

which simplifies as

$$\begin{aligned} c_2(\mathcal{E}) \geq & \left(k(r-k)m(m+1) + \frac{k(k-1)}{2}(m+1)^2 + \frac{(r-k)(r-k-1)}{2} m^2 \right) C_0^2 \\ & -\alpha + (k + (r-1)m)\delta. \end{aligned}$$

Next we use the generic nefness of $c_1(\mathcal{E})$ to bound $\alpha = \deg(A)$. Let H be a very ample divisor of Y which avoids the points of Z arising in the Brosius sequence (3.17). Since $\mathcal{E}$ is generically nef, and since $\mathcal{M} \otimes \mathcal{I}_Z$ is a quotient of $\mathcal{E}$, $\mathcal{G}((m+1)C_0) \otimes \mathcal{O}_{m_0 H}$ is nef for $m_0 >> 0$, hence $c_1(\mathcal{M}) \otimes \mathcal{O}_{m_0 H}) \geq 0$.

An ample divisor on a ruled surface admitting a negative section C_0 is of the type $H \in |xC_0 + yL|$, with $x > 0$ and $y > x\,e$. The condition $c_1(\mathcal{M}) \cdot m_0 H \geq 0$ gives

$$\alpha \leq \delta + m(r-k)\frac{y - xe}{x}; \tag{3.20}$$

in particular $\alpha \leq \delta + m(r-k)\frac{1}{x}$ for any $x > 0$, so

$$\alpha \leq \delta. \tag{3.21}$$

Moreover, using again the trick

$$\delta = -\frac{a}{2}C_0^2 + \frac{c_1(\mathcal{E})^2}{2a}, \tag{3.22}$$

we get

$$c_2(\mathcal{E}) \geq \Bigg(k(r-k)m(m+1) + \frac{k(k-1)}{2}(m+1)^2 + \frac{(r-k)(r-k-1)}{2}m^2 - \frac{a}{2}(k+(r-1)m-1) \Bigg) C_0^2 + \frac{(k+(r-1)m-1)}{2a} c_1(\mathcal{E})^2,$$

that is

$$c_2(\mathcal{E}) \geq \frac{(r-k)m}{2} C_0^2 + \frac{a-(m+1)}{2a} c_1(\mathcal{E})^2. \tag{3.23}$$

The last bound is not satisfactory, since $C_0^2 < 0$, so we finally use the assumption that $c_1(\mathcal{E}) \cdot C_0 \geq \frac{C_0^2}{2}$, which gives $\delta \geq \left(\frac{1}{2} - a\right) C_0^2$. The expression (3.22) yields $C_0^2 \geq -\frac{c_1(\mathcal{E})^2}{a(a-1)}$, and by (3.23), we get

$$c_2(\mathcal{E}) \geq \left(\frac{a-(m+1)}{2a} - \frac{a-k(m+1)}{2a(a-1)} \right) c_1(\mathcal{E})^2.$$

□

4 The normalized relative canonical divisor

In this section, we shall apply the Bogomolov-type inequalities to the Tschirnhausen sheaf of a finite cover of a Hirzebruch surface. Indeed, by the Viehweg Weak Positivity Theorem [21], the Tschirnhausen sheaf is weakly positive away from the branch locus, and hence nef on the complement of the branch locus (see also [11][1]), so it is in particular generically nef. This will allow us to bound the relative Euler characteristic χ_f of a fibration factoring through a finite cover.

Moreover, we shall introduce the *normalized relative canonical divisor* of a finite morphism π and we shall show that its self-intersection is related with the slope.

We first recall how to determine the invariants and the slope of a fibration factoring through a finite cover.

Definition 4.1 Let $\pi : S \to Y$ be a finite cover of degree n between smooth surfaces. Then the sheaf $\pi_\star \mathcal{O}_S$ is locally free of rank n. Similarly to the argument given in [15], we can consider the natural injective map $\mathcal{O}_Y \to \pi_\star \mathcal{O}_S$, which admits a splitting by $1/n$ times the trace map. Let $\mathcal{E}_1$ be the cokernel of such a map; $\mathcal{E}_1$ is a locally free sheaf of rank $n-1$ on Y and we have

$$\pi_\star \mathcal{O}_S \cong \mathcal{O}_Y \oplus \mathcal{E}_1.$$

Following [15], it is customary to call the dual sheaf $\mathcal{E} := \mathcal{E}_1^\vee$ the *Tschirnhausen sheaf* of the finite morphism π.

Finally, if $\omega_{S/Y}$ denotes the relative canonical sheaf, we have $(\pi_\star \omega_{S/Y})^\vee \cong \pi_\star \mathcal{O}_S$ by relative duality, hence

$$\pi_\star \omega_{S/Y} \cong \mathcal{O}_Y \oplus \mathcal{E}.$$

[1] Appendix to [18].

Lemma 4.2 *Let S, Y be smooth surfaces, and let $\pi : S \to Y$ be a finite cover of degree n with relative canonical divisor $K_{S/Y}$. Then in the rational Chow ring $A(Y) \otimes \mathbb{Q}$, we have*

(1) $\pi_\star K_{S/Y} \equiv 2c_1(\mathcal{E})$,
(2) $\chi(\mathcal{O}_S) = n\chi(\mathcal{O}_Y) + \frac{1}{2}c_1(\mathcal{E}) \cdot K_Y + \frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E})$;

Proof The first relation is well known in the case of flat finite morphisms. The result follows from Grothendieck–Riemann–Roch Theorem applied to the morphism $\pi : S \to Y$ and the sheaf $\omega_{S/Y}$.

The Grothendieck–Riemann–Roch Theorem asserts that for a proper morphism π of smooth varieties we have

$$\mathrm{ch}(\pi_!\, \omega_{S/Y}) \cdot \mathrm{td}\, \mathcal{T}_Y = \pi_\star(\mathrm{ch}\, \omega_{S/Y} \cdot \mathrm{td}\, \mathcal{T}_S).$$

As $R^1\pi_\star\omega_{S/Y} = 0$ since π is finite, we have

$$\pi_!\, \omega_{S/Y} = \pi_\star\omega_{S/Y} = \mathcal{E}.$$

This yields

$$\begin{aligned}(n + c_1(\mathcal{E}) + \frac{1}{2}\left(c_1^2(\mathcal{E}) - c_2(\mathcal{E})\right) \cdot \left(1 - \frac{1}{2}K_Y + \chi(\mathcal{O}_Y)\right)\\ = \pi_\star\left(1 + K_{S/Y} + \frac{1}{2}K_{S/Y}^2\right) \cdot \left(1 - \frac{1}{2}K_S + \chi(\mathcal{O}_S)\right).\end{aligned}$$

The divisorial part satisfies

$$c_1(\mathcal{E}) - \frac{n}{2}K_Y = \pi_\star\left(K_{S/Y} - \frac{1}{2}K_S\right).$$

As $K_S \sim \pi^\star K_Y + K_{S/Y}$, we have $\pi_\star K_S \equiv nK_Y + \pi_\star K_{S/Y}$ and the first claim follows.

The equality between the codimension two cycles gives formula (2). □

Definition 4.3 A *fibration* $f : S \to B$ is a flat surjective morphisms between a smooth surface S and a smooth curve B with connected fibers, such that if $x \in B$ is general then $F_x := f^{-1}(x)$ is a smooth curve.

Following Xiao [20], we can associate with $f : S \to B$ a rational number $s(f)$, called the *slope* of f, defined as:

$$s(f) := \frac{K_f^2}{\chi_f}$$

where $K_f = K_S - f^\star K_B$ is the relative canonical divisor, $\chi_f := \deg f_\star\omega_f$, and $\omega_f := \mathcal{O}_S(K_f)$.

Remark 4.4 We recall the well-known relations:

$$K_f^2 = K_S^2 - 8(g-1)(g(B)-1), \qquad \chi_f = \chi(\mathcal{O}_S) - (g-1)(g(B)-1). \tag{4.1}$$

Corollary 4.5 *Let $f : S \to B$ be a fibration, which factorizes through a finite cover $\pi : S \to Y$ of a ruled surface Y. Then*

$$K_f^2 = K_{S/Y}^2 - \frac{4}{(g+n-1)}c_1(\mathcal{E})^2, \qquad \chi_f = \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E}). \tag{4.2}$$

$$s(f) = \frac{K_f^2}{\chi_f} = \frac{K_{S/Y}^2 - \frac{4}{(g+n-1)}c_1(\mathcal{E})^2}{\frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E})}. \tag{4.3}$$

Proof We can write $K_S^2 = K_{S/Y}^2 + 2K_{S/Y} \cdot \pi^\star K_Y + nK_Y^2$, then by projection formula and Lemma 4.2 $K_S^2 = K_{S/Y}^2 + nc_1(\mathcal{E}) \cdot K_Y + nK_Y^2$. As $K_f^2 = K_S^2 - 8(g-1)(b-1)$, where $b = g(B)$, we have

$$K_f^2 = K_{S/Y}^2 + nc_1(\mathcal{E}) \cdot K_Y + nK_Y^2 - 8(g-1)(b-1).$$

Finally, by choosing the generators of the Neron-Severi group of Y to be the classes $[C_0]$ and $[L]$ where $L \in \mathrm{NS}(Y)$ is the class of a ruling and $C_0 \in \mathrm{NS}(Y)$ is the class of a section of minimal self-intersection, we may write

$$c_1(\mathcal{E}) \equiv (g+n-1)C_0 + \left(\frac{c_1(\mathcal{E})^2}{2(g+n-1)} + (g+n-1)\, C_0^2\right) L, \tag{4.4}$$

and the first formula follows.

Taking into account that $\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1)$, the formula for χ_f follows from Lemma 4.2, (2). □

Now we introduce the normalized relative canonical divisor of a finite cover, and we shall see that it is closely related to the slope of the induced fibration. Such a connection is not surprising, as a similar argument has already been used in such a context.

For instance, the Cornalba–Harris theory for bounding the slope of any fibration f relies on the study of the *normalized relative canonical* divisor of a fibration $f\colon S \to B$

$$\mathbb{K}_f := K_f - \frac{1}{g} f^\star c_1(f_\star \omega_f), \quad \widetilde{\omega}_f := \mathcal{O}_S(\mathbb{K}_f)$$

and on the *normalized Hodge bundle*

$$\mathbb{E}_f := f_\star \widetilde{\omega}_f.$$

Indeed, the pseudo-effectivity of $f_\star(\mathbb{K}_f^2)$, proved by Cornalba and Harris in [7, Theorem 1.1, Proposition 2.9 and Section 4], under the assumption that the Hilbert points of the general fiber are semistable, is a crucial step in their proof of the classical bound on the slope

$$s(f) \geq 4 - \frac{4}{g}.$$

A similar task has been used by Fedorchuk and Jensen [10], who obtained as a straight consequence of the positivity of $c_1(f_\star \widetilde{\omega}_f^{\otimes 2})$ the result that if $S \to B$ is a flat family of Gorenstein curves with the generic fiber a canonically embedded curve whose 2nd Hilbert point is semistable (e.g., with the generic fiber a general trigonal curve), then the slope satisfies the inequality $s(f) \geq 5 - \frac{6}{g}$.

Also in the context of projective vector bundles $\pi : \mathbb{P}(\mathcal{G}) \to Y$ fibered in $\mathbb{P}^{r-1}$ over a variety Y a similar divisor is studied, namely the so-called *normalized tautological divisor*

$$\mathbb{T}_{\mathcal{G}} := T_{\mathbb{P}(\mathcal{G})} - \frac{1}{r}\pi^\star c_1(\mathcal{G}) = -\frac{1}{r} K_{\mathbb{P}(\mathcal{G})/Y},$$

and the nefness of such a divisor has been investigated by Nakayama [16]. More precisely, Nakayama proved the following result:

Theorem 4.6 *Let $\mathcal{G}$ be a rank r vector bundle on a smooth complex projective variety Y of dimension $d \geq 2$. Then the following conditions are equivalent:*

- $\mathbb{T}_{\mathcal{G}}$ *is nef;*

- $\mathcal{G}$ *is* μ*-semistable and* $\left(c_2(\mathcal{G}) - \frac{(r-1)}{2r}c_1(\mathcal{G})^2\right) \cdot A^{d-2} = 0$ *for an ample divisor* A.

It turns out that in our context, as we are dealing with a finite morphism and a rank $(n-1)$ torsion-free sheaf $\mathcal{E}$ with $c_1(\mathcal{E}) = c_1(\pi_\star \omega_{S/Y})$, it is natural to give the following definition:

Definition 4.7 Let $\pi : S \to Y$ be a finite morphism of degree n. The $\mathbb{Q}$-divisor

$$\Lambda_\pi := K_{S/Y} - \frac{1}{(n-1)}\pi^\star(c_1(\pi_\star \omega_{S/Y})).$$

is called the *normalized relative canonical divisor* of π.

To explain the reason which leads to the definition of Λ_π, we need to recall the following well-known result (see [6]).

Theorem 4.8 *Let* Y *be an integral surface and let* $\pi : S \to Y$ *be a Gorenstein cover of degree* $n \geq 3$. *There exists a unique* $\mathbb{P}^{n-2}$*-bundle* $\pi_Y : \mathbb{P} \to Y$ *and an embedding* $i : S \to \mathbb{P}$ *such that* $\pi = \pi_Y \circ i$. *Moreover,* $\mathbb{P} \cong \mathbb{P}(\mathcal{E})$ *and the ramification divisor* R *satisfies:*

$$\mathcal{O}_S(R) \cong i^\star \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Remark 4.9 Consider the embedding $S \subseteq \mathbb{P}(\mathcal{E})$. Since $(T_{\mathbb{P}(\mathcal{E})})_{|S} = K_{S/Y}$, we get that

$$\Lambda_\pi = (\mathbb{T}_\mathcal{E})_S.$$

The connection between the slope and Λ_π is given by the following:

Proposition 4.10 *Let* $f : S \to B$ *be a fibration, with general fiber* F *a smooth curve of genus* g.

Assume that f *factorizes through a finite degree* n *cover* $\pi : S \to Y$, *where* Y *is a ruled surface, and assume that the general fiber restriction of the Tschirnhausen sheaf* $\mathcal{E}$ *is a twist of the trivial sheaf.*

By setting

$$\chi_0^{\max} := \frac{g}{2(n-1)(g+n-1)}c_1(\mathcal{E})^2, \tag{4.5}$$

we have

$$\chi_0^{\max} \geq \chi_f$$

and the following equality holds:

$$K_f^2 = \mathcal{F}(n,g)\chi_0^{\max} + \Lambda_\pi^2, \tag{4.6}$$

where

$$\mathcal{F}(n,g) = 6 - \frac{2}{n-1} - \frac{2n}{g}$$

is the conjectural bound of Stankova (see [19, Conjecture 13.3]*).*

In particular,

$$s(f) \geq \mathcal{F}(n,g) + \frac{\Lambda_\pi^2}{\chi_0^{\max}}. \tag{4.7}$$

Proof By Theorem 3.3, we have $c_2(\mathcal{E}) \geq \frac{n-2}{2(n-1)}c_1(\mathcal{E})^2$, and since by Corollary 4.5 it holds $\chi_f = \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E})$, it follows that $\chi_f \leq \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - \frac{n-2}{2(n-1)}c_1(\mathcal{E})^2 = \chi_0^{\max}$.
With a direct computation, one can obtain the equality (4.6), and the inequality (4.7) follows immediately. □

As a consequence, the problem of bounding the slope can be rephrased in a problem of bounding Λ_π^2.

The Bogomolov-type inequalities given in Corollary 3.2 and Theorem 3.5 allow us to derive a similar result also in the non-divisible case $(n-1)\not| g$. Indeed, the Tschirnhausen sheaf is nef outside the branch locus by the Weak Positivity Theorem of Viehweg [21, 3.4]. If we assume that the general fiber restriction of $\mathcal{E}$ is balanced, which can be rephrased with the assumption that the general fiber of the fibration corresponds to a point outside the Maroni locus of a suitable Hurwitz scheme or of the moduli space, and that the branch divisor is reduced, which corresponds to impose the open condition that π has generically simple ramifications, then Corollary 3.2 and Theorem 3.5 apply.

Therefore, recalling that $\chi_f = \frac{(g+n-2)}{2(g+n-1)}c_1^2(\mathcal{E}) - c_2(\mathcal{E})$ and setting $g+n-1=(n-1)m+k$, we have

$$\chi_f \leq \begin{cases} \left(\frac{(g+n-2)}{2(g+n-1)} - \frac{(n-2)}{2(n-1)} + \frac{n-1-k}{2(n-1)(g+n-1)}\right) c_1(\mathcal{E})^2 & \text{if } C_0^2 \geq 0 \\ \left(\frac{(g+n-2)}{2(g+n-1)} - \frac{(g+n-1)-(m+1)}{2(g+n-1)} + \frac{(g+n-1)-k(m+1)}{2(g+n-1)(g+n-2)}\right) c_1^2(\mathcal{E}) & \text{if } C_0^2 < 0. \end{cases}$$

that is

$$\chi_f \leq \chi_k^{\max} := \begin{cases} \frac{m}{2(g+n-1)} c_1(\mathcal{E})^2 & \text{if } C_0^2 \geq 0 \\ \left(\frac{m}{2(g+n-1)} + \frac{(n-1-k)m}{2(g+n-1)(g+n-2)}\right) c_1^2(\mathcal{E}) & \text{if } C_0^2 < 0. \end{cases} \tag{4.8}$$

Then we can write

$$K_f^2 = K_{S/Y}^2 - \frac{4}{g+n-1} c_1(\mathcal{E})^2 = \Lambda_\pi^2 + \mathcal{F}(n,g,k)\chi_k^{\max}$$

where

$$\mathcal{F}(n,g,k) := \begin{cases} \frac{6g-2(n-1)}{(g+n-1-k)} - \frac{2}{n-1} - \frac{2k}{(n-1)(g+n-1-k)} & \text{if } C_0^2 \geq 0 \\ \frac{(6g-2(n-1))(g+n-2)}{(g+n-1-k)(g+2n-3-k)} - \frac{2(g+n-1)(g+n-2)}{(n-1)(g+n-1-k)(g+2n-3-k)} & \text{if } C_0^2 < 0. \end{cases}$$

This shows that the function $\mathcal{F}(n,g,k)$ can be used to replace the function $\mathcal{F}(n,g)$ in the non-divisible case; more precisely, we have:

Proposition 4.11 *Let $f: S \to B$ a semistable fibration over a rational curve B. Assume that f factorizes through a finite cover of degree n of ruled surface and that the Tschirnhausen sheaf is balanced on the general fiber. If the genus of the general fiber of f is $g=(m-1)(n-1)+k$, where $1 \leq k \leq n-2$, then*

$$s(f) \geq \mathcal{F}(n,g,k) + \frac{\Lambda_\pi^2}{\chi_k^{\max}}. \tag{4.9}$$

Remark 4.12 The bounds (4.7) and (4.9) given in Propositions 4.10 and 4.11 hold also for fibrations $f: S \to B$, which are the relatively minimal model of fibrations satisfying the given hypotheses. Indeed, it is enough to observe that the self-intersection of the relative canonical divisor of a relatively minimal model of a given fibration cannot decrease, and the relative Euler characteristic is unchanged.

5 Positivity results on Λ_π

From the results of the previous section, it follows that a bound on Λ_π^2 gives a bound also on the slope. Therefore, we are going to establish some conditions, under which the normalized relative canonical divisor has nonnegative self-intersection. Some very similar problems have been studied in [2,3], but their results do not apply to our case.

We first analyze the restriction of the divisor Λ_π to the general fiber.

Proposition 5.1 *Let F be a general fiber of the fibration $f : S \to B$. If $g = (n-1)(m-1)$ and F is not contained in the Maroni locus, the restriction of Λ_π to F satisfies:*

(1) $(\Lambda_\pi)_{|F} \sim K_F - (m-2)\Gamma_F$, *where $\Gamma_F \in g^1_n$ is a gonal divisor;*
(2) $h^0(\mathcal{O}_F(K_F - (m-2)\Gamma_F)) = n-1$;
(3) *the linear system $|K_F - (m-1)\Gamma_F|$ is base point free.*

Proof We have

$$(\Lambda_\pi)_{|F} = (K_{S/Y} - \pi^\star(mC_0 + kL))_{|F} = (K_S - \pi^\star K_Y - \pi^\star(mC_0 + kL))_{|F}$$
$$\sim K_F - (m-2)\Gamma_F,$$

where $\Gamma_F \in g^1_n$ is a gonal divisor, which proves (1).

Let us compute $h^0(\mathcal{O}_F((m-2)\Gamma_F))$ using the Geometric Riemann Roch Theorem:

$$h^0(\mathcal{O}_F((m-2)\Gamma_F)) = (m-2)n - \dim\langle(m-2)\Gamma_F\rangle,$$

where $\langle(m-2)\Gamma_F\rangle \subset \mathbb{P}^{g-1}$ is the linear span on the canonical model of the curve F. Now recall that since F is Maroni general, the canonical model of F lies on $W \cong \mathbb{P}(\oplus^{n-1}\mathcal{O}_{\mathbb{P}^1}(m-2))$ embedded in $\mathbb{P}^{g-1}$ by the tautological linear system. It follows that

$$\dim\langle(m-2)\Gamma_F\rangle = (m-2)(n-2) + m - 3,$$

hence by the Geometric Riemann Roch Theorem we have

$$h^0(\mathcal{O}_F((m-2)\Gamma_F)) = (m-2)n - (m-2)(n-2) - (m-3) = m-1,$$

hence by Riemann Roch

$$h^0(\mathcal{O}_F(K_F - (m-2)\Gamma_F)) = h^0(\mathcal{O}_F((m-2)\Gamma_F)) - ((\deg(m-2)\Gamma_F) - g + 1)$$
$$= m - 1 - (m-2)n + g - 1 = n - 1,$$

which proves (2).

Finally, assume by contradiction that P is a base point of the linear system $|K_F - (m-2)\Gamma_F|$. Then by the Geometric Riemann Roch Theorem, we would have

$$\dim\langle K_F - (m-2)\Gamma_F\rangle = \dim\langle K_F - (m-2)\Gamma_F - P\rangle + 1,$$

and

$$\dim\langle(m-2)\Gamma_F\rangle = \dim\langle(m-2)\Gamma_F + P\rangle.$$

We claim that the last equality cannot be satisfied. Indeed, the subspace $\langle(m-2)\Gamma_F\rangle$ cuts on W exactly $m-2$ fibers; indeed, since the minimum degree of a unisecant curve on W is $m-2$, the subspace $\langle(m-2)\Gamma_F\rangle$ contains no horizontal component. It follows that the divisor cut out by $\langle(m-2)\Gamma_F\rangle$ on the canonical model of F is exactly $(m-2)\Gamma_F$. □

Corollary 5.2 *Let $f : S \to B$ be a fibration in irreducible curves, with general fiber F a smooth curve of genus g. Assume that f factorizes through a finite degree n cover $\pi : S \to Y$, where Y is a ruled surface, with Tschirnhausen sheaf generically a twist of the trivial sheaf.*

If the restriction map $H^0(\mathcal{O}_S(\Lambda_\pi)) \to H^0(\mathcal{O}_F(\Lambda_\pi))$ is surjective, then $\Lambda^2_\pi \geq 0$.

Proof We claim that the linear system $|\Lambda_\pi|$ has no horizontal base locus.

Assume by contradiction that Λ_π has a horizontal component $\varkappa$ in its base locus. Then $\varkappa_{|F}$ is contained in the base locus of $|(\Lambda_\pi)_{|F}|$. But the latter linear system is base point free by Proposition 5.1, (3). This proves that $|\Lambda_\pi|$ has no horizontal base locus.

Finally, since all the fibers of f are irreducible, Λ_π has no vertical base locus.

Summing up, as Λ_π is effective and $|\Lambda_\pi|$ has no positive dimensional base locus, we have $\Lambda_\pi^2 \geq 0$. □

5.1 Rational fibrations with uniform Tschirnhausen sheaf

In the following proposition, we shall prove that in the divisible case, the fibrations over a rational curve, with uniform and generically balanced Tschirnhausen sheaf and with semistable unisecant restriction, satisfy the assumption of Proposition 5.1. We remark that by a recent result given in [8], a sufficiently general curve in the Hurwitz scheme of degree n covers of curves of given genus $p \geq 0$ has a μ-semistable Tschirnhausen sheaf. Therefore, the assumption of semistability on unisecant restrictions can be read as a generality assumption concerning the family of pull-back curves of a family of general unisecant curves.

Proposition 5.3 *Let $f : S \to \mathbb{P}^1$ be a fibration with irreducible fibers, with general fiber F a smooth curve of genus g and gonality n, where $n \geq 5$, such that $(n-1)|g$ and with balanced reduced gonal direct image sheaf. Assume that f factorizes through a finite morphism $\pi : S \to Y$, where $p : Y \to \mathbb{P}^1$ is a Hirzebruch surface.*

If the restriction of $\mathcal{E}$ to some unisecant ample divisor is semistable and if $\mathcal{E}$ is uniform, then the restriction map $H^0(\mathcal{O}_S(\Lambda_\pi)) \to H^0(\mathcal{O}_F(\Lambda_\pi))$ is an isomorphism.

In particular, $s(f) \geq \mathcal{F}(n, g)$.

Proof We set $c_1(\mathcal{E}) = (g+n-1)C_0 + \delta L$, where C_0 is a section with $C_0^2 \leq 0$ and L a fiber of ruling on $Y = \mathbb{F}_e$.

Since $\mathcal{E}$ is uniform, the restriction to any fiber L of the ruling satisfies $\mathcal{E}_{|L} \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(m)$, where $m = \frac{g}{n-1} + 1$. Then the injective map of sheaves $p^\star p_\star \mathcal{E}(-mC_0) \to \mathcal{E}(-mC_0)$ is an isomorphism, so

$$\mathcal{E}(-mC_0) \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(p^\star A_i).$$

By the assumption that $\mathcal{E}$ is semistable with respect to an ample divisor $H \sim C_0 + (e+a)L$, we get $\deg \mathrm{A_i} = k$, for any i, and $k = \frac{\delta}{(n-1)} \geq 0$, so that

$$c_1(\mathcal{E}) \sim (n-1)(mC_0 + kL). \tag{5.1}$$

We finally get

$$\begin{aligned} h^0(\mathcal{O}_S(\Lambda_\pi)) &= h^0(\pi_\star \mathcal{O}_S(\Lambda_\pi)) \cong h^0(\mathcal{O}_Y(-(mC_0+kL)) \oplus \mathcal{E}(-(mC_0+kL))) \\ &= h^0(\oplus_{i=1}^{n-1} \mathcal{O}_Y) = n-1. \end{aligned}$$

On the other hand,

$$\begin{aligned} h^0(\mathcal{O}_S(\Lambda_\pi - F)) &= h^0(\pi_\star \mathcal{O}_S(\Lambda_\pi - F)) \cong h^0(\mathcal{E}(-(mC_0 + (k+1)L))) \\ &= h^0(\oplus_{i=1}^{n-1} \mathcal{O}_Y(-L)) = 0. \end{aligned}$$

Hence the restriction exact sequence

$$0 \to \mathcal{O}_S(\Lambda_\pi - F) \to \mathcal{O}_S(\Lambda_\pi) \to \mathcal{O}_F(\Lambda_\pi) \to 0$$

determines the isomorphism of the statement. □

Remark 5.4 We observe that in the case when $\mathcal{E}$ is a uniform vector bundle, that is $\Delta(\mathcal{E}) = 0$, and if $\mathcal{E}$ is μ-semistable with respect to some ample divisor, then the *normalized tautological divisor* $T_{\mathbb{P}(\mathcal{E})} - \frac{1}{(n-1)} p^\star c_1(\mathcal{E})$ where $p\colon \mathbb{P}(\mathcal{E}) \to Y$ is nef by Nakayama's Theorem 4.6. It follows that $\left(T_{\mathbb{P}(\mathcal{E})} - \frac{1}{(n-1)} p^\star c_1(\mathcal{E})\right)_S = K_{S/Y} - \frac{1}{(n-1)} \pi^\star c_1(\mathcal{E}) = \Lambda_\pi$ is also nef. Since the restriction map is surjective by Proposition 5.3, all this implies directly that in such a case $\Lambda_\pi^2 \geq 0$.

5.2 Upper bounds on Λ_π^2 and primitive cyclic covers

We recall that for finite covers, we have the following upper bound on R^2 in terms of $c_1(\mathcal{E})^2$, which is a consequence of the Hodge Index Theorem applied to the $\mathbb{Q}$ - divisor $R - \pi^\star \frac{2}{n} c_1(\mathcal{E})$:

Lemma 5.5 *Let $\pi : S \to Y$ be a Gorenstein cover degree n, and let $\mathcal{E}$ be the Tschirnhausen sheaf. Then*

$$K_{S/Y}^2 \leq \frac{4}{n} c_1(\mathcal{E})^2. \tag{5.2}$$

Proof See [5, Lemma 3.12]. □

Corollary 5.6 *With the assumptions of Lemma 5.5, we have*

$$\Lambda_\pi^2 \leq \frac{(n-2)^2}{n(n-1)^2} c_1(\mathcal{E})^2.$$

Proof Since $\Lambda_\pi = K_{S/Y} - \frac{1}{n-1} \pi^\star c_1(\mathcal{E})$, we have

$$\Lambda_\pi^2 = K_{S/Y}^2 - 2K_{S/Y} \cdot \frac{1}{n-1} \pi^\star c_1(\mathcal{E}) + \frac{1}{(n-1)^2} (\pi^\star c_1(\mathcal{E}))^2,$$

and by projection formula and by Lemma 4.2, (1), it follows

$$\begin{aligned} \Lambda_\pi^2 &= K_{S/Y}^2 - \frac{4}{n-1} c_1(\mathcal{E})^2 + \frac{n}{(n-1)^2} c_1(\mathcal{E})^2 = K_{S/Y}^2 - \frac{3n-4}{(n-1)^2} c_1(\mathcal{E})^2 \\ &\leq \frac{(n-2)^2}{n(n-1)^2} c_1(\mathcal{E})^2. \end{aligned}$$

□

We remark that the Hodge Index Theorem also implies that the equality holds in (5.2) if and only if

$$K_{S/Y} \sim \frac{1}{n} \pi^\star 2c_1\mathcal{E} \equiv \frac{1}{n} \pi^\star B_\pi.$$

Such a condition is satisfied, for instance, when all ramification points of π are total ramification points, that is of maximal ramification index n.

A typical context, when this happens, is the one of *primitive cyclic covers* $\pi : S \to Y$, that is covers such that there exist an effective divisor $A \subset Y$ and an effective divisor $D \subset S$ such that

$$S \cong \mathbf{Spec} \oplus_{i=0}^{n-1} \mathcal{O}_Y(iA),$$

and such that $\pi : S \to Y$ does not factorize through two covers of smaller degree. In this case, the following holds:

(1) $B_\pi \sim nA$ and $\pi^\star B_\pi = nD$;
(2) $K_{S/Y} \sim (n-1)D$;
(3) $\pi_\star \mathcal{O}_S(K_{S/Y}) \cong \bigoplus_{i=0}^{n-1} \mathcal{O}_Y(iA)$;
(4) $c_1(\mathcal{E}) = \frac{n(n-1)}{2} A$, $c_2(\mathcal{E}) = \frac{n(n-1)(n-2)(3n-1)}{12} A^2$.

From this, we obtain that

$$K_{S/Y} \sim \pi^\star \frac{2}{n} c_1(\mathcal{E}), \qquad K_{S/Y}^2 = \frac{4}{n} c_1(\mathcal{E})^2, \qquad \Lambda_\pi^2 = \frac{(n-2)^2}{n(n-1)^2} c_1(\mathcal{E})^2.$$

Since $c_1(\mathcal{E}) = \frac{n(n-1)}{2} A$, if $A^2 \geq 0$, we get $\Lambda_\pi^2 \geq 0$. This gives a bound, which is exactly the bound $\lambda_{g,0,n}$ given in [9, Remark 4.4]:

Theorem 5.7 *Let $f : S \to B$ be the relatively minimal model of a finite cyclic cover $\pi : \widetilde{S} \to Y$ of a ruled surface Y.*

Then

$$s(f) \geq \frac{24(g-1)(n-1)}{(n^2 + 4ng - 3n + 2 - 2g)} = 6 - \frac{6}{2n-1} - \frac{12n(n^2-1)}{2g(2n-1) + (n-1)(n-2)}.$$

Proof Recall that the self-intersection of the relative canonical divisor of a relatively minimal model of a given fibration cannot decrease, and the relative Euler characteristic is unchanged. Then we can apply formula (4.6). □

Remark 5.8 We observe that for cyclic covers the Tschirnhausen sheaf is uniform.

5.3 A Beniamino Segre's construction

Following closely [1, Chapter 21 Section 12], we shall show that for a general $[C] \in \overline{\mathcal{M}^1_{g,n}}$, where $\overline{\mathcal{M}^1_{g,n}}$ is the closure of the n-gonal locus in the moduli space $\overline{\mathcal{M}}_g$ of curves of genus g, if $p \colon C \to \mathbb{P}^1$ is the gonal covering, then the corresponding Tschirnhausen sheaf $\mathcal{E}_C$ is balanced.

Theorem 5.9 *Let $g \geq 3$. For any integer n such that $3 \leq n \leq \frac{g}{2}+1$, there exists a smooth curve C of genus g admitting a complete g^1_n without base points, and such that the corresponding Tschirnhausen sheaf $\mathcal{E}_C$ is balanced.*

Proof Let $\mathbb{F}_e$ be a Hirzebruch surface with invariant $e = -C_0^2 \geq 0$. Consider the complete linear system

$$\Sigma_{n,h} = |nC_0 + hL|.$$

Assume that $h > \frac{ne}{2}$. Then $\Sigma_{n,h}$ is very ample and the image of the morphism $\phi_{\Sigma_{n,h}} : \mathbb{F}_e \to \mathbb{P}^N$ associated with $\Sigma_{n,h}$ is a smooth surface; hence, by Bertini's theorem, the general member of $\Sigma_{n,h}$ is smooth. Then, by adjunction, the genus of the general element $\Gamma \in \Sigma_{n,h}$ is

$$g_{n,h} = (n-1)(h-1) - \frac{ne}{2}(n-1).$$

Moreover, by Riemann–Roch, we have

$$\dim\ \Sigma_{n,h} = g_{n,h} + 2n + 2h - 1 - ne.$$

Following [1, Theorem 12.16 see page 870], we immediately see that, given any fixed integer $n \geq 3$ and any fixed integer $e \geq 0$, the intervals

$$I_{n,h} = \left[g_{n,h} - n - h + 1 + \frac{ne}{2}, g_{n,h}\right]$$

cover the half line $[0, \infty)$. Then there exist an integer δ and an integer h such that $0 \leq \delta \leq h + n - 1 - \frac{ne}{2}$ and

$$g = g_{n,h} - \delta. \tag{5.3}$$

A simple computation shows that

$$\delta \leq g_{n,h} \leq \dim \Sigma_{n,h} - 2\delta - 1.$$

By Castelnuovo's theorem applied to $\Sigma_{n,h}$, see for instance [1, Theorem 12.6 page 865], it follows that given δ general points $a_1, \ldots, a_\delta \in \mathbb{F}_e$, there exists an irreducible curve $\Gamma \in \Sigma_{n,h} = |nC_0 + hl|$ having δ nodes at $a_1, \ldots, a_\delta$ and no other singularities. Let $\nu: Z \to \mathbb{F}_e$ be the blowup at $a_1, \ldots, a_\delta$ and let $E_i := \nu^{-1}(a_i)$, $i = 1, \ldots, \delta$. The normalization C of Γ is contained in Z and $\nu_{|C}: C \to \Gamma$ is the normalization morphism. Let $H_0 := \nu^{-1}(C_0)$ and $\widetilde{L} := \pi^{-1}(L)$. The smooth curve C has a g^1_n induced by the ruling of $\mathbb{F}_e$. Let us denote by D a divisor of the g^1_n. Then

$$D \sim L_{|C}, \qquad C \in |nH_0 + h\widetilde{L} - \sum_{i=1}^{\delta} E_i|.$$

By standard surface theory, we get

$$K_Z \sim -2H_0 - (2+e)L + \sum_{i=1}^{\delta} E_i.$$

Then $(n-2)H_0 + (h - e - 2 - v)L - \sum_{i=1}^{\delta} E_i \sim K_Z + C - vL$ and by adjunction theory on surfaces we have the following exact sequence

$$0 \to \mathcal{O}_Z(K_Z - vL) \to \mathcal{O}_Z(K_Z + C - vL) \to \omega_C(-vD) \to 0 \tag{5.4}$$

Notice now that by projection formula and by Serre duality we have

$$\begin{aligned} h^1(Z, \mathcal{O}_Z(K_Z - vL)) &= h^1(Z, \mathcal{O}_Z(vL)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(vP)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((-2-v)P)) = 0, \end{aligned}$$

since $v \geq 0$. As $h^0(Z, \mathcal{O}_Z(K_Z - vL)) = 0$, by considering the long cohomology sequence associated with the sequence 5.4, we obtain that the restriction morphism

$$H^0(Z, \mathcal{O}_Z(K_Z + C - vL)) \to H^0(C, \omega_C(-vD))$$

is an isomorphism. In particular $h^0(C, \omega_C(-vD)) = h^0(Z, \mathcal{O}_Z(K_Z + C - vL))$.

Since $a_1, \ldots, a_\delta$ are general points, we have

$$\max\{-1, h^0(Z, \mathcal{O}_Z(K_Z + C - vL)\} = \max\{-1, \dim \Sigma_{n-2,h-(e+2+v)} - \delta\}.$$

Hence if we assume that $h-(e+2+v) \geq \frac{(n-2)e}{2}$, then $\dim \Sigma_{n-2,h-(e+2+v)} - \delta = g-(n-1)v$.

We have shown that if $h - (e + 2 + v) \geq \frac{(n-2)e}{2}$ and if k is the unique integer such that $1 \leq k \leq n-2$ and $g = m(n-1) + k$ then

$$h^0(C, \omega_C(-vD)) = (m - v)(n - 1) + k \tag{5.5}$$

if $m \geq v$.

Finally, let $p\colon C \to \mathbb{P}^1$ be the gonal morphism; then $p_\star \omega_C = \omega_{\mathbb{P}^1} \oplus \mathcal{E}_C(-2)$, where

$$\mathcal{E}_C = \mathcal{O}_{\mathbb{P}^1}(m_1) \oplus \mathcal{O}_{\mathbb{P}^1}(m_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(m_{n-1}),$$

with $m_1 \leq m_2 \leq \cdots \leq m_{n-1}$. By projection formula and by Eq. 5.5, it follows that in fact $m_1 = \cdots = m_{n-1-k} = m - 1$ and $m_{n-k} = \cdots = m_{n-1} = m$, that is $\mathcal{E}_C$ is balanced. □

6 Sharpness results

In this section, we construct some examples that realize the bound on the slope given in Proposition 5.3.

6.1 Existence of fibrations with the required properties: rational basis

Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and $T = Y \times \mathbb{P}^1$. Let $\pi_i : T \to \mathbb{P}^1$ be the projection with respect to the i-th factor and set $\mathcal{L}_i := \pi_i^\star \mathcal{O}_{\mathbb{P}^1}(1)$ where $i = 1, 2, 3$. Let

$$S \in |nL_1 + nL_2 + nL_3|$$

be a general element, where $n \geq 3$. If $\langle x_0, x_1\rangle = H^0(T, \mathcal{L}_3)$, $\langle y_0, y_1\rangle = H^0(T, \mathcal{L}_2)$ and $\langle z_0, z_1\rangle = H^0(T, \mathcal{L}_1)$, then

$$S = V(F) \text{ where } F = \sum_{i=0}^{n} a_i((y_0 : y_1), (z_0 : z_1))x_0^{n-i} x_1^i.$$

Thus if $a_i((y_0 : y_1), (z_0 : z_1)) \in H^0(Y, \mathcal{O}_Y(n, n))$ for $i = 0, \ldots, n$ are general, the morphism $\pi : S \to Y$ is finite of degree n. Consider now the composition $f\colon S \to \mathbb{P}^1$ of the inclusion $j\colon S \hookrightarrow T$ with the natural morphism $\rho\colon T \to Y$ followed by the projection on the first factor $\pi_1'\colon Y \to \mathbb{P}^1$. The fiber over $z_0 = a, z_1 = b$ is the curve $C_{[a:b]} = V(F_{[a:b]})$ where $F_{[a:b]} = \sum_{i=0}^n a_i((y_0 : y_1), (a : b))x_0^{n-i} x_1^i$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, $f\colon S \to \mathbb{P}^1$ is a fibration in curves of genus $g = (n-1)^2$ and gonality n such that

$$s(f) = \mathcal{F}(n, g).$$

Moreover Λ_π is effective and it is induced on S by the linear system $(0, 0, n-2)$, hence $\Lambda_\pi^2 = 0$. Finally by performing the push-forward of the standard exact sequence

$$0 \to \mathcal{O}_T(K_{T|Y}) \to \mathcal{O}_T(K_{T|Y} + S) \to \omega_{S|Y} \to 0,$$

by projection formula and by relative duality it follows that the Tschirnhausen sheaf satisfies $\mathcal{E} \cong \mathcal{O}_Y(n, n)^{\oplus n-1}$, so it is a uniform and balanced vector bundle which is also semistable on the (0, 1)-sections of the projection $\pi_1' \colon Y \to \mathbb{P}^1$. This shows that the bound given in Proposition 5.3 is sharp. Note that instead of $S \in |nL_1 + nL_2 + nL_3|$ we can take $S \in |n_1L_1 + n_2L_2 + n_3L_3|$ where $n_i \geq 1$ to obtain similar results.

6.2 Existence of fibrations with the required properties: other cases

Let $T = C_1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where C_1 is a smooth curve of genus $g_1 > 0$, and let $\mathcal{L} := \pi^\star\mathcal{O}_{C_1}(L_1) \otimes \pi_2^\star\mathcal{O}_{\mathbb{P}^1}(n) \otimes \pi_3^\star\mathcal{O}_{\mathbb{P}^1}(n)$. As above we obtain a semistable n-gonal fibration over C_1 of genus $g = (n-1)^2$ such that:

$$s(f) = 6 - \frac{6}{(n-1)^2} = \mathcal{F}(n, g) + 4\frac{n-2}{(n-1)^2}$$

Also in this case Λ_π^2 is zero and the Tschirnhausen sheaf is uniform and with balanced fiber restriction.

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References

1. Arbarello, E., Cornalba, M., Griffiths, P.: Geometry of Algebraic Curves. A Series of Comprehensive Studies in Mathemstics 268, vol. II. Springer, Berlin (2011)
2. Barja, M.A., Stoppino, L.: Stability and singularities of relative hypersurfaces. Int. Math. Res. Not. IMRN **4**, 1026–1053 (2016)
3. Barja, M.A., Stoppino, L.: Positivity properties of relative complete intersections. arXiv:1410.3009 [math.AG] (2014)
4. Brosius, J.E.: Rank-2 vector bundles on a ruled surface. I. Math. Ann. **265**(2), 155–168 (1983)
5. Beorchia, V., Zucconi, F.: On the slope of fourgonal semistable fibrations. Math. Res. Lett. **25**(3), 723–757 (2018)
6. Casnati, G., Ekedahl, T.: Covers of algebraic varieties. I. A general structure theorem, covers of degree 3, 4 and Enriques surfaces. J. Algebr. Geom. **5**(3), 439–460 (1996)
7. Cornalba, M., Harris, J.: Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. Ann. Sci. Ec. Norm. Super. **21**(4), 455–475 (1988)
8. Deopurkar, A., Patel, A.: Vector bundles and finite covers, preprint arXiv:1608.01711
9. Enokizono, M.: Slopes of fibered surfaces with a finite cyclic automorphism. Mich. Math. J. **66**(1), 125–154 (2017)
10. Fedorchuk, M., Jensen, D.: Stability of 2nd Hilbert points of canonical curves. Int. Math. Res. Not. IMRN **2013**(22), 5270–5287 (2013)
11. Lazarsfeld, R.: Nefness modulo the branch locus of the bundle associated to a branched covering. Commun. Algebra **28**, 5598–5599 (2000)
12. Lazarsfeld, R.: Positivity in Algebraic Geometry, II, Positivity for Vector Bundles, and Multiplier Ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49. Springer, Berlin (2004)
13. Lu, X., Zuo, K.: On the gonality and the slope of a fibered surface. Adv. Math. **324**, 336–354 (2018)
14. Moriwaki, A.: A sharp slope inequality for general stable fibrations of curves. J. Reine Angew. Math. **480**, 177–195 (1996)
15. Miranda, R.: Triple covers in algebraic geometry. Am. J. Math. **107**(5), 1123–1158 (1985)
16. Nakayama, N.: Zariski-decomposition and abundance, MSJ Memoirs, vol. 14. Mathematical Society of Japan, Tokyo (2004)
17. Peternell, Th.: Generically nef vector bundles and geometric applications. In: Ebeling, W., Hulek, K., Smoczyk, K. (eds.) Complex and Differential Geometry: Conference held at Leibniz Universität Hannover, September 14–18, 2009, Springer, Berlin (2011)

18. Peternell, Th, Sommese, A.J.: Ample vector bundles and branched coverings. Commun. Algebra **28**(12), 5573–5599 (2000)
19. Stankova, Z.: Moduli of trigonal curves. J. Algebr. Geom. **9**(4), 607–662 (2000)
20. Xiao, G.: Fibred algebraic surfaces with low slope. Math. Ann. **276**, 449–466 (1987)
21. Viehweg, E.: Die Additivität der Kodaira Dimension für projektive Faserräume über Varietäten des allgemeinen Typs. J. Reine Angew. Math. **330**, 132–142 (1982)