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Comments on T-Dualities of Ramond-Ramond Potentials

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The type IIA/IIB effective actions compactified on T^d are known to be invariant under the T-duality group $SO(d,d;\boldsymbol{Z})$, although the invariance of the R-R sector cannot be seen so directly. Inspired by a work of Brace, Morariu and Zumino, we introduce new potentials, which are mixtures of R-R potentials and the NS-NS 2-form, in order to make the invariant structure of R-R sector more transparent. We give a simple proof that if these new potentials transform as a Majorana-Weyl spinor of $SO(d,d;\boldsymbol{Z})$, the effective actions are indeed invariant under the T-duality group. The argument is made in such a way that it can apply to Kaluza-Klein forms of arbitrary degree. We also demonstrate that these new fields simplify all the expressions, including the Chern-Simons term.

§1. Introduction

Recent developments in string theory have been based on various kinds of duality symmetries. Among these, the T-duality was found first.[1),2)] This symmetry changes the size of the compactified space into its inverse in string unit. Although this symmetry was first recognized in the spectra of perturbative strings, it came to be believed that it should hold as an exact symmetry, not simply as a perturbative one.[3)] Later, at the level of the low energy effective action, the T-duality invariance of the type IIA/IIB theory was identified with part of an already known, much larger, and hidden set of symmetries of type II supergravities.[4)-7)] It was actually conjectured that the duality group of the full string theory can be extended to the U-duality group $E_{d+1(d+1)}(\boldsymbol{Z})$ when compactified on a d-dimensional torus.[8)]

Being a subgroup of the U-duality group, the T-duality group $SO(d,d;\boldsymbol{Z})$ has a special property: It is the maximum subgroup that consists of the elements that transform NS-NS and R-R fields into themselves. On the other hand, we sometimes encounter situations where NS-NS and R-R fields are better treated in a separate manner. This is often the case when classical black-hole solutions of string theory are considered. Another example may be given by study of classical configurations based on the Born-Infeld action. Thus, it would be useful if one can know in a simple manner how NS-NS and R-R fields transform under the T-duality group, without resorting to embedding the whole structure once into the vast U-duality group.

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The T-duality invariance can actually be seen very easily for the NS-NS sector of the supergravity action. [9] There, the kinetic term of the Kaluza-Klein (KK) scalars (G_{ij}, B_{ij}) $(i, j = 1, \cdots, d)$ can be written as[*]

$$\mathcal{L}_{\rm NS} = \frac{1}{8}\, e^{-2\phi}\, {\rm tr}\left(\partial_\mu M^{-1} \partial^\mu M\right), \tag{1·1}$$

with the $2d \times 2d$ matrix

$$M = (M_{rs}) = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{1·2}$$

and the $(10-d)$-dimensional dilaton ϕ. Thus the kinetic term is manifestly invariant under T-duality transformations $\Lambda \in O(d,d;\boldsymbol{Z})$ if the dilaton does not change and $M = (M_{rs})$ $(r, s = 1, \cdots, 2d)$ transforms as

$$\overline{M} = \left(\Lambda^{-1}\right)^T \cdot M \cdot \Lambda^{-1}. \tag{1·3}$$

The KK 1-forms $(G_{\mu i}, B_{\mu i})$ give a vector representation of $O(d,d;\boldsymbol{Z})$ and also have an invariant kinetic term. [9] These facts will be reviewed later in more detail.

On the other hand, the invariance of the sector including R-R potentials under the T-duality group $SO(d,d;\boldsymbol{Z})$[**] is not so manifest as that for the NS-NS sector is. There have actually been many works in which T-duality was studied as a subgroup of the U-duality group $E_{d+1(d+1)}(\boldsymbol{Z})$. [10] However, in order to write down the action in a manifestly U-invariant form, one needs to make a non-trivial mapping from the original fields to some other fields, which usually makes the T-duality symmetry for the original fields indirect. As for the works based on the T-duality itself, results have been obtained [11] only for Nahm transformations which generate a subgroup of $O(d,d;\boldsymbol{Z})$.

By decomposing representations of $E_{d+1(d+1)}(\boldsymbol{Z})$ with respect to $SO(d,d;\boldsymbol{Z})$, it has also been shown that Majorana-Weyl representations of $SO(d,d;\boldsymbol{Z})$ should appear in the R-R sector (see, for example, Ref. 12)). However, as was discussed in detail for type IIA with $d = 3$ in Refs. 13) and 14), the R-R potentials themselves do not give Majorana-Weyl spinors directly. Instead, one needs to combine them with the NS-NS 2-form to get new fields that have such simple transformation properties under $SO(d,d;\boldsymbol{Z})$. Although a prescription for arranging these fields is known for each d, by starting from 11-dimensional supergravity, [15] it is rather complicated due to the field redefinitions which depend strongly on the dimensionality. The main aim of this article is to present the prescription of constructing the new fields and to demonstrate the T-duality invariance of the R-R sector with the Chern-Simons term in a simple form. We proceed by investigating solely the structure of the effective action of type IIA/IIB strings with all fermionic fields set to zero. Inclusion of

[*] This B_{ij} will be denoted $B^{(0)}_{ij}$ in the following sections to indicate that this is a scalar for the noncompact $(10-d)$-dimensional space-time with coordinates x^μ $(\mu = 0, 1, \cdots, 9-d)$. We also take the string unit so that $\alpha' = 1$.

[**] Each of type IIA and type IIB is only invariant under the subgroup $SO(d,d;\boldsymbol{Z})$ of $O(d,d;\boldsymbol{Z})$, as we see below.

fermions with analysis of supersymmetry will be discussed elsewhere. This work was inspired by analysis made by Brace, Morariu and Zumino.[13),14)]

The main result can be summarized as follows. First, we introduce new potentials $D_{p+1} = (1/(p+1)!)D_{\hat{\mu}_1\cdots\hat{\mu}_{p+1}}dx^{\hat{\mu}_1}\wedge\cdots\wedge dx^{\hat{\mu}_{p+1}}$ $(\hat{\mu}_1,\cdots,\hat{\mu}_{p+1}=0,1,\cdots,9)$, which are mixtures of R-R potentials and the NS-NS 2-form as

$$\begin{aligned} D_0 &\equiv C_0, & D_1 &\equiv C_1, \\ D_2 &\equiv C_2 + \widehat{B}_2\wedge C_0, & D_3 &\equiv C_3 + \widehat{B}_2\wedge C_1, \\ D_4 &\equiv C_4 + \tfrac{1}{2}\widehat{B}_2\wedge C_2 + \tfrac{1}{2}\widehat{B}_2\wedge\widehat{B}_2\wedge C_0, \end{aligned} \tag{1·4}$$

where C_{p+1} is the original $(p+1)$-form R-R potential, and $\widehat{B}_2$ is the NS-NS 2-form in 10 dimensions. We further introduce potentials of higher degree, D_{p+1} $(p+1=5,\cdots,8)$, as their electromagnetic duals. More precisely, we introduce the sum of field strengths

$$F \equiv e^{-\widehat{B}_2}\wedge\sum_{p+1=0}^{8} dD_{p+1} = \sum_{p+2=1}^{9} F_{p+2}, \tag{1·5}$$

and require the following relations in their equations of motion:

$$\begin{aligned} &*F_1 = F_9, & &*F_2 = -F_8, \\ &*F_3 = -F_7, & &*F_4 = F_6, \\ &*F_5 = F_5, & &*F_6 = -F_4, \\ &*F_7 = -F_3, & &*F_8 = F_2, \\ &*F_9 = F_1. \end{aligned} \tag{1·6}$$

Note that $*^2 F_n = (-1)^{n+1}F_n$ in 10-dimensional Minkowski space. The existence of the fields $D_5,\cdots,D_8$ is allowed by the equations of motion for $D_0,\cdots,D_4$.

Our first claim is that, as far as the equations of motion are concerned, the R-R action with the Chern-Simons term can be rewritten into the simple form

$$\begin{aligned} S^{(\mathrm{IIA})}_{\mathrm{R+CS}} &\equiv \frac{1}{8\kappa_{10}^2}\int d^{10}x\,\sqrt{-\widehat{g}}\sum_{p+2=2,4,6,8} F_{p+2}\wedge *F_{p+2}, \\ S^{(\mathrm{IIB})}_{\mathrm{R+CS}} &\equiv \frac{1}{8\kappa_{10}^2}\int d^{10}x\,\sqrt{-\widehat{g}}\sum_{p+2=1,3,5,7,9} F_{p+2}\wedge *F_{p+2}, \end{aligned} \tag{1·7}$$

with all the $D_0,\cdots,D_8$ being regarded as independent variables and with (1·6) being the constraints imposed after the equations of motion are derived.

Second, for d-dimensional toroidal compactification, we assemble the set of KK scalars into the form (D_α) with 2^{d-1} entries:

$$\begin{array}{lll} & d=1: & (D_\alpha)=(D_1) \\ \underline{\mathrm{IIA:}} & d=2: & (D_\alpha)=(D_1,D_2) \\ & d=3: & (D_\alpha)=(D_1,D_2,D_3,D_{123}) \\ & d=4: & (D_\alpha)=(D_1,D_2,D_3,D_4,D_{123},D_{124},D_{134},D_{234}) \\ & \quad\vdots & \qquad\vdots \end{array} \tag{1·8}$$

$$
\underline{\text{IIB :}}\qquad
\begin{array}{ll}
d=1: & (D_\alpha)=(D)\\
d=2: & (D_\alpha)=(D,D_{12})\\
d=3: & (D_\alpha)=(D,D_{12},D_{13},D_{23})\\
d=4: & (D_\alpha)=(D,D_{12},D_{13},D_{14},D_{23},D_{24},D_{34},D_{1234})\\
\vdots & \qquad\vdots
\end{array}
\tag{1·9}
$$

Here $D_\alpha = D_{i_1\cdots i_{p+1}}$ is the component of D_{p+1} in the compact directions $y^{i_1},\cdots,y^{i_{p+1}}$ $(1\le i_1<\cdots<i_{p+1}\le d)$. Similarly, we also assemble the set of KK 1-forms $D_{\mu\, i_1\cdots i_p}$ $(\mu=0,1,\cdots,9-d)$:

$$
\underline{\text{IIA :}}\qquad
\begin{array}{ll}
d=1: & (D_{\mu\,\alpha})=(D_\mu)\\
d=2: & (D_{\mu\,\alpha})=(D_\mu,D_{\mu\,12})\\
d=3: & (D_{\mu\,\alpha})=(D_\mu,D_{\mu\,12},D_{\mu\,13},D_{\mu\,23})\\
d=4: & (D_{\mu\,\alpha})=(D_\mu,D_{\mu\,12},D_{\mu\,13},D_{\mu\,14},D_{\mu\,23},D_{\mu\,24},D_{\mu\,34},D_{\mu\,1234})\\
\vdots & \qquad\vdots
\end{array}
\tag{1·10}
$$

$$
\underline{\text{IIB :}}\qquad
\begin{array}{ll}
d=1: & (D_{\mu\,\alpha})=(D_{\mu\,1})\\
d=2: & (D_{\mu\,\alpha})=(D_{\mu\,1},D_{\mu\,2})\\
d=3: & (D_{\mu\,\alpha})=(D_{\mu\,1},D_{\mu\,2},D_{\mu\,3},D_{\mu\,123})\\
d=4: & (D_{\mu\,\alpha})=(D_{\mu\,1},D_{\mu\,2},D_{\mu\,3},D_{\mu\,4},D_{\mu\,123},D_{\mu\,124},D_{\mu\,134},D_{\mu\,234})\\
\vdots & \qquad\vdots
\end{array}
\tag{1·11}
$$

This assembling may continue to KK forms of higher degree when d is sufficiently small.

Our second claim is that the dimensionally-reduced action of the R-R sector with the Chern-Simons term can be rewritten for type IIA and IIB, respectively, as*)

$$
\mathcal{L}_{\mathrm{R+CS}} = \frac{1}{4}\,\partial_\mu D_\alpha\, S^{\mp}_{\alpha\beta}(M)\,\partial^\mu D_\beta, + \frac{1}{16}\,\partial_{[\mu}D_{\nu]\,\alpha} S^{\pm}_{\alpha\beta}(M)\,\partial^{[\mu}D^{\nu]}{}_\beta + \cdots, \tag{1·12}
$$

where $S^{\pm}_{\alpha\beta}(M)$ $(\alpha,\beta=1,\cdots,2^{d-1})$ is a representation matrix of M in the Majorana-Weyl representation of $SO(d,d;\boldsymbol{R})$ with chirality $\pm$. The invariance of the action thus now becomes apparent by assuming that both D_α and $D_{\mu\,\alpha}$ transform as Majorana-Weyl spinors:

$$
\begin{aligned}
\overline{D}_\alpha &= S^{\mp}_{\alpha\beta}(\Lambda) D_\beta,\\
\overline{D}_{\mu\,\alpha} &= S^{\pm}_{\alpha\beta}(\Lambda) D_{\mu\,\beta}.
\end{aligned}
\tag{1·13}
$$

We will prove the identity (1·12) for arbitrary d, including KK forms of arbitrary degree. We simplify the argument with the use of the fermionic oscillator construction of the Majorana representation given in Refs. 13) and 14).

This paper is organized as follows. In §2, in order to fix our convention, we first give a brief review of the invariance of the NS-NS sector and then introduce

*) The precise form is given by (4·25)–(4·27).

new potentials D_{p+1}. In §3, we explicitly construct the spinor representations of $O(d,d;\boldsymbol{Z})$, closely following Refs. 13) and 14), and then we rewrite the R-R action plus the Chern-Simons term into a manifestly T-duality invariant form in §4. Section 5 is devoted to discussion. The existence of the fields $D_5,\cdots,D_8$ is proved in the Appendix, with a demonstration that our new fields D_{p+1} greatly simplify all the expressions including the Chern-Simons term.

§2. Type IIA/IIB effective actions

The action of 10-dimensional type IIA/IIB supergravity in the string metric can be split into three parts: [16)]

$$S = S_{\rm NS} + S_{\rm R} + S_{\rm CS}. \tag{2·1}$$

The first term is the action for the NS-NS sector,

$$S_{\rm NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-\widehat{g}}\;e^{-2\widehat{\phi}}\left(\widehat{R} + 4\,|d\widehat{\phi}|_{\widehat{g}}^2 - \frac{1}{2}\,|\widehat{H}_3|_{\widehat{g}}^2\right), \tag{2·2}$$

where the $x^{\hat{\mu}}$ ($\hat{\mu}=0,1,\cdots,9$) are 10-dimensional coordinates, and $\widehat{g}_{\hat{\mu}\hat{\nu}}$, $\widehat{B}_{\hat{\mu}\hat{\nu}}$ and $\widehat{\phi}$ denote the 10-dimensional metric, NS-NS 2-form and dilaton, respectively. The NS-NS field strength is written as $\widehat{H}_3 = d\widehat{B}_2$ with $\widehat{B}_2 = (1/2)\widehat{B}_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}\wedge dx^{\hat{\nu}}$. We adopt the rule that the subscript of a form indicates its degree when it has a definite meaning in 10-dimensions. We also often consider a sum of forms of various degrees, like $\varOmega = \sum_K \varOmega_K = \sum_K (1/K!)\varOmega_{\hat{\mu}_1\cdots\hat{\mu}_K}dx^{\hat{\mu}_1}\wedge\cdots\wedge dx^{\hat{\mu}_K}$, and for this we introduce the invariant norm as

$$|\,\varOmega\,|_{\widehat{g}}^2 \equiv \sum_K \frac{1}{K!}\,\widehat{g}^{\hat{\mu}_1\hat{\nu}_1}\cdots\widehat{g}^{\hat{\mu}_K\hat{\nu}_K}\,\varOmega_{\hat{\mu}_1\cdots\hat{\mu}_K}\varOmega_{\hat{\nu}_1\cdots\hat{\nu}_K}. \tag{2·3}$$

The action for the R-R sector, $S_{\rm R}$, can be written for IIA and IIB, respectively, as

$$\begin{aligned} S_{\rm R}^{\rm (IIA)} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-\widehat{g}}\,\left(|F_2|_{\widehat{g}}^2 + |F_4|_{\widehat{g}}^2\right), \\ S_{\rm R}^{\rm (IIB)} &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-\widehat{g}}\,\left(|F_1|_{\widehat{g}}^2 + |F_3|_{\widehat{g}}^2 + \frac{1}{2}\,|F_5|_{\widehat{g}}^2\right), \end{aligned} \tag{2·4}$$

where the R-R field strengths F_{p+2} are defined from the $(p+1)$-form R-R potentials $C_{p+1} = (1/(p+1)!)C_{\hat{\mu}_1\cdots\hat{\mu}_{p+1}}dx^{\hat{\mu}_1}\wedge\cdots\wedge dx^{\hat{\mu}_{p+1}}$ as

$$\begin{aligned} F_1 &= dC_0, & F_2 &= dC_1, \\ F_3 &= dC_2 + \widehat{H}_3\wedge C_0, & F_4 &= dC_3 + \widehat{H}_3\wedge C_1, \\ F_5 &= dC_4 + \tfrac{1}{2}\widehat{H}_3\wedge C_2 - \tfrac{1}{2}\,\widehat{B}_2\wedge dC_2. \end{aligned} \tag{2·5}$$

The Chern-Simons term $S_{\rm CS}$ is given by

$$\begin{aligned} S_{\rm CS}^{\rm (IIA)} &= \frac{1}{4\kappa_{10}^2}\int \widehat{B}_2\wedge dC_3\wedge dC_3, \\ S_{\rm CS}^{\rm (IIB)} &= \frac{1}{4\kappa_{10}^2}\int \widehat{B}_2\wedge dC_4\wedge dC_2. \end{aligned} \tag{2·6}$$

We use the convention that an NS-NS field wears a hat ($\widehat{\ }$) in 10-dimensions, while R-R fields do not. This is because NS-NS fields generally need to be redefined after toroidal compactification in order to behave nicely as fields existing in the noncompact $(10-d)$-dimensional space-time [see, for example, (2·7), (2·16), (2·17) and (2·20)].

After toroidal compactification on T^d, there will appear various KK forms both from the NS-NS and the R-R sectors. We first review the NS-NS case, closely following Ref. 9).

NS-NS sector:

We parametrize the 10-dimensional metric as

$$\begin{aligned} d\widehat{s}^2 &\equiv \widehat{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} \\ &= g_{\mu\nu} dx^\mu dx^\nu + G_{ij}(dy^i + A^i_\mu dx^\mu)(dy^j + A^j_\nu dx^\nu). \end{aligned} \tag{2·7}$$

Here, the 10-dimensional coordinates are decomposed as $(x^{\hat{\mu}}) = (x^\mu, y^i)$ $(\mu = 0, 1, \cdots, 9-d\,;\ i = 1, 2, \cdots, d)$, and we assume that all the fields depend only on the noncompact coordinates x^μ. With this parametrization, the kinetic term for potentials will take a complicated form, since the KK 1-form

$$A^{(1)\,i} \equiv A^i_\mu dx^\mu \tag{2·8}$$

will appear when contracting the indices in the compact directions. To simplify this, we follow the prescription of Ref. 9), which we found can be restated as follows. First, given a sum of forms $\Omega = \sum_K \Omega_K$, we decompose it as

$$\Omega = \sum_q \sum_n \frac{1}{n!}\, \Omega^{(q)}_{i_1\cdots i_n}\, dy^{i_1} \wedge \cdots \wedge dy^{i_n}, \tag{2·9}$$

where the superscript (q) indicates that $\Omega^{(q)}_{i_1\cdots i_n}$ is a q-form for noncompact indices:

$$\Omega^{(q)}_{i_1\cdots i_n} = \frac{1}{q!}\, \Omega_{\mu_1\cdots\mu_q\, i_1\cdots i_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_q}. \tag{2·10}$$

Second, we introduce a new form Ω' by replacing dy^i in Ω with $dy^i - A^{(1)\,i}$ and reorganize it as in (2·9):

$$\begin{aligned} \Omega' &\equiv \Omega\,\big|_{dy^i \to\, dy^i - A^{(1)\,i}} \\ &= \sum_q \sum_n \frac{1}{n!}\, \Omega'^{(q)}_{i_1\cdots i_n}\, dy^{i_1} \wedge \cdots \wedge dy^{i_n}. \end{aligned} \tag{2·11}$$

Then the kinetic term can be expressed in such a way that all the indices are contracted only with $g^{\mu\nu}$ and G^{ij}:

$$|\,\Omega\,|^2_{\hat{g}} = |\,\Omega'\,|^2_{g,G} \equiv \sum_q \sum_n \Big|\, \Omega'^{(q)}_n \Big|^2_{g,G}, \tag{2·12}$$

where we have defined

$$\left| \Omega'^{(q)}_n \right|^2_{g,G} \equiv \frac{1}{n!} G^{i_1 j_1} \cdots G^{i_n j_n} \frac{1}{q!} g^{\mu_1 \nu_1} \cdots g^{\mu_q \nu_q} \Omega'_{\mu_1 \cdots \mu_q i_1 \cdots i_n} \Omega'_{\nu_1 \cdots \nu_q j_1 \cdots j_n}. \tag{2·13}$$

For example, the NS-NS field strength $\widehat{H}_3$ is rewritten as

$$\begin{aligned} \widehat{H}'^{(1)}_{ij} &= dB^{(0)}_{ij}, \\ \widehat{H}'^{(2)}_{i} &= dB^{(1)}_{i} - B^{(0)}_{ij} dA^{(1)\,j}, \\ \widehat{H}'^{(3)} &= dB^{(2)} - \frac{1}{2} \left(B^{(1)}_i dA^{(1)\,i} + dB^{(1)}_i A^{(1)\,i} \right), \end{aligned} \tag{2·14}$$

where we have introduced

$$\begin{aligned} B^{(0)}_{ij} &\equiv \widehat{B}^{(0)}_{ij}, \\ B^{(1)}_{i} &\equiv \widehat{B}^{(1)}_{i} + \widehat{B}^{(0)}_{ij} A^{(1)\,j}, \\ B^{(2)} &\equiv \widehat{B}^{(2)} - \frac{1}{2} \widehat{B}^{(1)}_i A^{(1)\,i}. \end{aligned} \tag{2·15}$$

Conversely, we have

$$\begin{aligned} \widehat{B}^{(0)}_{ij} &\equiv B^{(0)}_{ij}, \\ \widehat{B}^{(1)}_{i} &\equiv B^{(1)}_{i} - B^{(0)}_{ij} A^{(1)\,j}, \\ \widehat{B}^{(2)} &\equiv B^{(2)} + \frac{1}{2} B^{(1)}_i A^{(1)\,i} + \frac{1}{2} B^{(0)}_{ij} A^{(1)\,i} A^{(1)\,j}, \end{aligned} \tag{2·16}$$

which give the original $\widehat{B}_2$ as

$$\begin{aligned} \widehat{B}_2 &= \frac{1}{2} \widehat{B}^{(0)}_{ij} dy^i \wedge dy^j + \widehat{B}^{(1)}_i dy^i + \widehat{B}^{(2)} \\ &= \frac{1}{2} B^{(0)}_{ij} \left(dy^i + A^{(1)\,i} \right) \left(dy^j + A^{(1)\,j} \right) + B^{(1)}_i \left(dy^i + A^{(1)\,i} \right) + B^{(2)} - \frac{1}{2} B^{(1)}_i A^{(1)\,i}. \end{aligned} \tag{2·17}$$

Then the NS-NS part of the action can be rewritten [9] as

$$S_{\mathrm{NS}} = \frac{1}{2\kappa^2_{10-d}} \int d^{10-d}x \sqrt{-g}\, \mathcal{L}_{\mathrm{NS}}, \tag{2·18}$$

where

$$\frac{1}{2\kappa^2_{10-d}} = \frac{1}{2\kappa^2_{10}} \int d^d y. \tag{2·19}$$

By introducing the $(10-d)$-dimensional dilaton ϕ as

$$e^{-2\phi} \equiv e^{-2\widehat{\phi}} \sqrt{G}, \tag{2·20}$$

the factor $\mathcal{L}_{\rm NS} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$ is given by

$$\begin{aligned}
\mathcal{L}_1 &= e^{-2\phi}\left[R + 4\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right],\\
\mathcal{L}_2 &= \frac{1}{8}\, e^{-2\phi}\, g^{\mu\nu}\,\mathrm{tr}\left(\partial_\mu M^{-1}\partial_\nu M\right),\\
\mathcal{L}_3 &= -\,\frac{1}{4}\, e^{-2\phi}\, g^{\mu_1\nu_1}g^{\mu_2\nu_2}F^{r}_{\mu_1\mu_2}\, M_{rs}\, F^{s}_{\nu_1\nu_2},\\
\mathcal{L}_4 &= -\,\frac{1}{12}\, e^{-2\phi}\, g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}\widehat{H}'_{\mu_1\mu_2\mu_3}\widehat{H}'_{\nu_1\nu_2\nu_3}\,,
\end{aligned} \tag{2·21}$$

where

$$\begin{aligned}
M \;&=\; (M_{rs}) \;\equiv\; \begin{pmatrix} G^{-1} & -G^{-1}B^{(0)} \\ B^{(0)}G^{-1} & G - B^{(0)}G^{-1}B^{(0)} \end{pmatrix}, \qquad \left(B^{(0)} \equiv \left(B^{(0)}_{ij}\right)\right)\\
\frac{1}{2}\,F^{r}_{\mu\nu}\,dx^\mu \wedge dx^\nu \;&\equiv\; \begin{pmatrix} dB^{(1)}_i \\ dA^{(1)\,i} \end{pmatrix}. \qquad (r,s = 1,\cdots,2d;\; i,j = 1,\cdots,d)
\end{aligned} \tag{2·22}$$

This form of the action makes manifest its invariance under the T-duality group $O(d,d;\boldsymbol{Z})$, provided that the fields transform as

$$\overline{M} \;=\; \left(\Lambda^{-1}\right)^T \cdot M \cdot \Lambda^{-1}\,, \quad \begin{pmatrix} \overline{B}^{(1)}_i \\ \overline{A}^{(1)\,i} \end{pmatrix} \;=\; \Lambda \begin{pmatrix} B^{(1)}_i \\ A^{(1)\,i} \end{pmatrix}, \quad \overline{B}^{(2)} \;=\; B^{(2)}, \tag{2·23}$$

for $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d,\boldsymbol{Z})$ satisfying $\Lambda^T J\,\Lambda = J$ with $J = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}$. The first transformation rule is equivalent to $\overline{E} \;=\; (aE+b)(cE+d)^{-1}$ for $E_{ij} = G_{ij} + B^{(0)}_{ij}$. [3)]

R-R sector with the Chern-Simons term:

The R-R potentials $C_{p+1} = (1/(p+1)!)\, C_{\hat\mu_1\cdots\hat\mu_{p+1}}\, dx^{\hat\mu_1}\wedge\cdots\wedge dx^{\hat\mu_{p+1}}$ also produce KK forms of various degrees after toroidal compactification. To simplify all the expressions, we first combine the R-R potentials with the NS-NS 2-form in 10 dimensions as follows:[*)]

$$\begin{aligned}
&D_0 \;\equiv\; C_0, && D_1 \;\equiv\; C_1,\\
&D_2 \;\equiv\; C_2 + \widehat{B}_2\wedge C_0, && D_3 \;\equiv\; C_3 + \widehat{B}_2\wedge C_1,\\
&D_4 \;\equiv\; C_4 + \tfrac{1}{2}\widehat{B}_2\wedge C_2 + \tfrac{1}{2}\widehat{B}_2\wedge\widehat{B}_2\wedge C_0. &&
\end{aligned} \tag{2·24}$$

The R-R field strengths are then expressed with these D_{p+1} as

$$\begin{aligned}
&F_1 \;=\; dD_0, && F_2 \;=\; dD_1,\\
&F_3 \;=\; dD_2 - \widehat{B}_2\wedge dD_0, && F_4 \;=\; dD_3 - \widehat{B}_2\wedge dD_1,\\
&F_5 \;=\; dD_4 - \widehat{B}_2\wedge dD_2 + \tfrac{1}{2}\widehat{B}_2\wedge\widehat{B}_2\wedge dD_0. &&
\end{aligned} \tag{2·25}$$

These can be written in the simple form

*) For the type IIA, the potentials D_1 and D_3 can be found in Ref. 14).

$$F = e^{-\widehat{B}_2} \wedge dD \tag{2·26}$$

if we introduce

$$D \equiv \sum_{p+1=0}^{4} D_{p+1}, \qquad F \equiv \sum_{p+2=1}^{5} F_{p+2}. \tag{2·27}$$

The equations of motion for $D_0, \cdots, D_4$ turn out to allow introduction of extra R-R potentials of higher degree, D_{p+1} $(p+1 = 5, \cdots, 8)$, that preserve the relation (2·26) with

$$D \equiv \sum_{p+1=0}^{8} D_{p+1}, \qquad F \equiv \sum_{p+2=1}^{9} F_{p+2}, \tag{2·28}$$

if we introduce the following identifications for the field strengths of higher degree:

$$\begin{aligned} &*F_1 = F_9, && *F_2 = -F_8, \\ &*F_3 = -F_7, && *F_4 = F_6, \\ &*F_5 = F_5, && *F_6 = -F_4, \\ &*F_7 = -F_3, && *F_8 = F_2, \\ &*F_9 = F_1 \end{aligned} \tag{2·29}$$

(see the Appendix). Interestingly, as far as the equations of motion are concerned, we can in turn regard all the R-R potentials, $D_0, \cdots, D_8$, as independent variables and choose

$$\begin{aligned} S^{(\mathrm{IIA})}_{\mathrm{R+CS}} &\equiv -\frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-\widehat{g}} \sum_{p+2=2,4,6,8} |F_{p+2}|^2_{\widehat{g}}, \\ S^{(\mathrm{IIB})}_{\mathrm{R+CS}} &\equiv -\frac{1}{8\kappa_{10}^2} \int d^{10}x \sqrt{-\widehat{g}} \sum_{p+2=1,3,5,7,9} |F_{p+2}|^2_{\widehat{g}}, \end{aligned} \tag{2·30}$$

as their action functional, with the understanding that the constraints (2·29) are imposed after (and only after) the equations of motion are derived. In fact, one can prove that this system gives the same equations of motion as those obtained from the sum of R-R and Chern-Simons terms $S_{\mathrm{R}} + S_{\mathrm{CS}}$, (2·4)–(2·6). We give a proof of this statement in the Appendix.

For d-dimensional toroidal compactification, we introduce the primed field for F as

$$F' \equiv F\,|_{dy^i \to dy^i - A^{(1)\,i}}\,. \tag{2·31}$$

Then the action for the R-R and Chern-Simons sector can be expressed as

$$S_{\mathrm{R+CS}} = \frac{1}{2\kappa_{10-d}^2} \int d^{10-d}x \sqrt{-g}\, \mathcal{L}_{\mathrm{R+CS}}, \tag{2·32}$$

with

$$\mathcal{L}_{\mathrm{R+CS}} = -\frac{1}{4} \sqrt{G}\, |\, F'\,|^2_{g,G}\,. \tag{2·33}$$

To show that $\mathcal{L}_{\rm R+CS}$ is invariant under $O(d,d;\boldsymbol{Z})$ when the set of KK fields coming from D transforms as a Majorana spinor of $O(d,d;\boldsymbol{Z})$, in the next section we explicitly construct the spinor representation of $O(d,d;\boldsymbol{R})$ by using fermionic operators. We mostly follow the convention of Refs. 13) and 14).

Before concluding this section, we would like to make a comment on the potentials D_1 and D_3 in the type IIA case. It is well known that type IIA supergravity can be obtained from 11-dimensional supergravity[5)] by dimensional reduction. The coordinate transformation along the 11-th direction $x^{10} \to x^{10} + \xi$ becomes a $U(1)$ symmetry in 10 dimensions:

$$\delta \widehat{B}_2 = 0, \qquad \delta C_1 = d\xi, \qquad \delta C_3 = -\widehat{B}_2 \wedge d\xi. \tag{2·34}$$

Thus, these D fields diagonalize the $U(1)$ symmetry: $D_1 \to D_1 + d\xi$, $D_3 \to D_3$. These are 10-dimensional analogues of the A' fields of Ref. 6).

§3. Spinor representation of $O(d,d;\boldsymbol{R})$

We first recall that the group $O(d,d;\boldsymbol{R})$ consists of $2d \times 2d$ matrices Λ satisfying

$$\Lambda^T\ J\ \Lambda = J, \qquad J = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}. \tag{3·1}$$

The group $O(d,d;\boldsymbol{Z})$ is defined as a subgroup that consists of matrices with integer-valued elements. It is known that both are generated by the following three types of matrices:[17)]

$$\Lambda_B = \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \qquad B^T = -B, \tag{3·2}$$

$$\Lambda_R = \begin{pmatrix} R^{-1} & 0 \\ 0 & R^T \end{pmatrix}, \qquad R \in GL(d;\boldsymbol{R}) \text{ or } GL(d;\boldsymbol{Z}), \tag{3·3}$$

$$\Lambda_i = -\begin{pmatrix} 1-e_i & -e_i \\ -e_i & 1-e_i \end{pmatrix}, \qquad (e_i)_{jk} = \delta_{ij}\delta_{ik}. \qquad (i=1,\cdots,d) \tag{3·4}$$

Note that $\det \Lambda_B = \det \Lambda_R = +1$ and $\det \Lambda_i = -1$. Thus one can construct a subgroup $SO(d,d;\boldsymbol{R})$ or $SO(d,d;\boldsymbol{Z})$ that is generated by Λ_B, Λ_R and $\Lambda_i\Lambda_j$.

The Dirac matrices $\Gamma_r = (\Gamma_{r\,\alpha\beta})$ with $2^d \times 2^d$ components are introduced as

$$\{\Gamma_r,\, \Gamma_s\} = 2J_{rs}, \qquad (r,s = 1,\cdots,2d) \tag{3·5}$$

and the spinor representation $S(\Lambda) = (S_{\alpha\beta}(\Lambda))$ is characterized by the property

$$S(\Lambda)\cdot\Gamma_s\cdot S(\Lambda)^{-1} = \sum_r \Gamma_r\, \Lambda^r{}_s. \tag{3·6}$$

To construct this representation, we introduce fermionic operators $\boldsymbol{\psi}^{i\dagger}$ and $\boldsymbol{\psi}_i$ with the anti-commutation relations

$$\{\boldsymbol{\psi}_i, \boldsymbol{\psi}^{j\dagger}\} = \delta_i{}^j\, \mathbf{1}, \qquad \{\boldsymbol{\psi}_i, \boldsymbol{\psi}_j\} = 0 = \{\boldsymbol{\psi}^{i\dagger}, \boldsymbol{\psi}^{j\dagger}\}. \qquad (i,j=1,\cdots,d) \tag{3·7}$$

We define hermitian conjugation as

$$(\boldsymbol{\psi}_i)^\dagger = \boldsymbol{\psi}^{i\dagger} \tag{3·8}$$

and introduce the vacuum $|\,0\,\rangle$ such that $\boldsymbol{\psi}_i\,|\,0\,\rangle = 0$ $(i = 1, \cdots, d)$ and $\langle\,0\,|\,0\,\rangle = 1$. Then the 2^d-dimensional fermion Fock space is spanned by the vectors

$$|\,\alpha\,\rangle = \boldsymbol{\psi}^{i_1\dagger} \cdots \boldsymbol{\psi}^{i_n\dagger}\,|\,0\,\rangle\,, \qquad (n = 0, \cdots, d) \tag{3·9}$$

where α is a multi-index $\alpha = (i_1, \cdots, i_n)$ $(i_1 {<} \cdots {<} i_n)$, and the Dirac matrices can be introduced with respect to this as

$$\begin{aligned} \boldsymbol{\psi}^{i\dagger}\,|\,\beta\,\rangle &= \sum_\alpha |\,\alpha\,\rangle\,\frac{1}{\sqrt{2}}\,(\Gamma_i)_{\alpha\beta}\,, \\ \boldsymbol{\psi}_i\,|\,\beta\,\rangle &= \sum_\alpha |\,\alpha\,\rangle\,\frac{1}{\sqrt{2}}\,(\Gamma_{d+i})_{\alpha\beta}\,. \end{aligned} \tag{3·10}$$

Thus, if we can always introduce an operator $\boldsymbol{\Lambda}$ to any element

$$\Lambda = \begin{pmatrix} (a_i{}^j) & (b_{ij}) \\ (c^{ij}) & (d^i{}_j) \end{pmatrix} \in O(d, d; \boldsymbol{R}) \tag{3·11}$$

such that

$$\begin{aligned} (\,\boldsymbol{\Lambda}\boldsymbol{\psi}^{j\dagger}\boldsymbol{\Lambda}^{-1}, \quad \boldsymbol{\Lambda}\boldsymbol{\psi}_j\boldsymbol{\Lambda}^{-1}\,) &= (\,\boldsymbol{\psi}^{i\dagger}a_i{}^j + \boldsymbol{\psi}_i c^{ij}, \quad \boldsymbol{\psi}^{i\dagger}b_{ij} + \boldsymbol{\psi}_i d^i{}_j\,) \\ &= (\,\boldsymbol{\psi}^{i\dagger}, \quad \boldsymbol{\psi}_i\,)\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{aligned} \tag{3·12}$$

then, introducing the matrix $S_{\alpha\beta}(\Lambda)$ with $\boldsymbol{\Lambda}\,|\,\beta\,\rangle = \sum_\alpha |\,\alpha\,\rangle\,S_{\alpha\beta}(\Lambda)$, we can establish the relation (3·6). For this, it is enough to construct the operators that correspond to the elements given in (3·2)–(3·4), and it is easy to show that the following are solutions: [13), 14)]

$$\begin{aligned} \boldsymbol{\Lambda}_B &= e^{-\boldsymbol{B}} \equiv \exp\left(-\frac{1}{2}B_{ij}\boldsymbol{\psi}^{i\dagger}\boldsymbol{\psi}^{j\dagger}\right), \\ \boldsymbol{\Lambda}_R &= (\det R)^{1/2}\exp\left(-\boldsymbol{\psi}^{i\dagger}A_i{}^j\boldsymbol{\psi}_j\right), \qquad \left(R = \left(R_i{}^j\right) = \exp\left(A_i{}^j\right)\right) \\ \boldsymbol{\Lambda}_i &= \boldsymbol{\psi}_i + \boldsymbol{\psi}^{i\dagger}. \qquad\qquad (i = 1, \cdots, d) \end{aligned} \tag{3·13}$$

Note that all of these operators give real-valued matrix elements, so that the resulting representation is automatically Majorana. Note also that the $\boldsymbol{\Lambda}_i$ do not give a faithful representation, so that there are always ambiguities in their orderings.

In order to construct Weyl representations, we define the matrix

$$\Gamma_{2d+1} \equiv \frac{1}{2^d}\prod_{i=1}^{d}(\Gamma_i + \Gamma_{d+i})(\Gamma_i - \Gamma_{d+i}), \tag{3·14}$$

which satisfies $\{\Gamma_{2d+1}, \Gamma_r\} = 0$ $(r = 1, \cdots, 2d)$. By looking at the correspondence (3·10), one can easily see that Γ_{2d+1} corresponds to $(-1)^{\boldsymbol{N_F}}$ with $\boldsymbol{N_F} = \sum_i \boldsymbol{\psi}^{i\dagger}\boldsymbol{\psi}_i$.

Thus, the projection to the subspace with $(-1)^{\boldsymbol{N_F}} = 1$ leads to a Majorana-Weyl representation $(2^{\,d-1})_{\boldsymbol{s}}$ and that to the subspace with $(-1)^{\boldsymbol{N_F}} = -1$ leads to $(2^{\,d-1})_{\boldsymbol{c}}$. Note that $\boldsymbol{\Lambda}_i$ is a linear function of fermions and thus changes the chirality. Therefore, in order for an operator to preserve the chirality it must correspond to an element in $SO(d,d;\boldsymbol{R})$.

We further introduce an operator $\boldsymbol{J}$ that corresponds to $J = (J_{rs})$ as

$$\boldsymbol{J} = i^{d(d-1)/2}\ \boldsymbol{\Lambda}_1 \cdots \boldsymbol{\Lambda}_d, \tag{3·15}$$

where the phase factor is chosen such that $\boldsymbol{J}^2 = 1$. One can actually prove that

$$\boldsymbol{J}\ \boldsymbol{\psi}^{i\dagger}\ \boldsymbol{J} = \boldsymbol{\psi}_i, \qquad \boldsymbol{J}\ \boldsymbol{\psi}_i\ \boldsymbol{J} = \boldsymbol{\psi}^{i\dagger}. \tag{3·16}$$

It is easy to check that for all the Λ in (3·13) [and thus for all elements in $O(d,d;\boldsymbol{R})$], their transposes $\Lambda^T = J\cdot\Lambda^{-1}\cdot J$ are mapped to $\boldsymbol{\Lambda}^\dagger$:

$$\boldsymbol{\Lambda}^\dagger = \boldsymbol{J}\boldsymbol{\Lambda}^{-1}\boldsymbol{J}. \tag{3·17}$$

In particular, we have

$$\boldsymbol{\Lambda}_B^\dagger = e^{-\boldsymbol{B}^\dagger} = \exp\left(\frac{1}{2}B_{ij}\,\boldsymbol{\psi}_i\boldsymbol{\psi}_j\right), \qquad \Lambda_B^T = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \tag{3·18}$$

Note also that the normalization of the operators (3·13) is correctly chosen such that they satisfy the condition (3·17).

We finally make the comment that this operator $\boldsymbol{J}$ is essentially the charge conjugation operator. In fact, the operators defined by

$$\boldsymbol{C}^\pm \equiv \boldsymbol{\Lambda}_1^\pm \cdots \boldsymbol{\Lambda}_d^\pm \tag{3·19}$$

with

$$\boldsymbol{\Lambda}_i^\pm \equiv \boldsymbol{\psi}^{i\dagger} \pm \boldsymbol{\psi}_i \tag{3·20}$$

can be easily seen to satisfy

$$\begin{aligned} &\boldsymbol{C}^\pm\ \left(\boldsymbol{C}^\pm\right)^\dagger = \boldsymbol{1}, && \left(\boldsymbol{C}^\pm\right)^2 = (-1)^{d(d\mp1)/2}\,\boldsymbol{1}, \\ &\boldsymbol{C}^\pm\,\boldsymbol{\psi}^{i\dagger}\ \left(\boldsymbol{C}^\pm\right)^{-1} = \mp\,(-1)^d\,\boldsymbol{\psi}_i, && \boldsymbol{C}^\pm\,\boldsymbol{\psi}_i\ \left(\boldsymbol{C}^\pm\right)^{-1} = \mp\,(-1)^d\,\boldsymbol{\psi}^{i\dagger}. \end{aligned} \tag{3·21}$$

This implies that the matrices $C^\pm = (C^\pm_{\alpha\beta})$ defined by $\boldsymbol{C}^\pm\,|\,\beta\,\rangle = |\,\alpha\,\rangle\ C^\pm_{\alpha\beta}$ satisfy the condition for the charge conjugation of $SO(d,d)$: [18]

$$\begin{aligned} &C^\pm\ \left(C^\pm\right)^\dagger = 1, \qquad \left(C^\pm\right)^T = (-1)^{d(d\mp1)/2}\,C^\pm, \\ &C^\pm\ \Gamma_r\ \left(C^\pm\right)^{-1} = \mp\,(-1)^d\ \left(\Gamma_r\right)^T. \end{aligned} \tag{3·22}$$

§4. R-R potentials and *T*-duality

In this section, we show that the R-R action plus the Chern-Simons term after toroidal compactification on T^d, (2·32)–(2·33), is actually invariant under $SO(d,d;\boldsymbol{Z})$ if a set of our R-R fields transform as a Majorana-Weyl spinor.

We first introduce a one-to-one correspondence between the set of forms and the space of creation operators by replacing the differential in the compact direction dy^i with the fermion creation operator $\boldsymbol{\psi}^{i\,\dagger}$. In this way, from*)

$$\Omega = \sum_n \frac{1}{n!}\ \Omega_{i_1\cdots i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n} = \sum_q \sum_n \frac{1}{n!}\ \Omega^{(q)}_{i_1\cdots i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n}, \tag{4·1}$$

we obtain

$$\boldsymbol{\Omega} \equiv \sum_n \frac{1}{n!}\ \Omega_{i_1\cdots i_n} \boldsymbol{\psi}^{i_1\dagger} \cdots \boldsymbol{\psi}^{i_n\dagger} = \sum_q \sum_n \frac{1}{n!}\ \Omega^{(q)}_{i_1\cdots i_n} \boldsymbol{\psi}^{i_1\dagger} \cdots \boldsymbol{\psi}^{i_n\dagger}. \tag{4·2}$$

This actually gives an algebra-isomorphism. We also extend our rule such that $\Omega^{(q)}_{i_1\cdots i_n}$ has $\boldsymbol{N_F} = q$, and thus it will anticommute with all the fermionic operators when q is odd. We define a state corresponding to Ω as

$$|\,\Omega\,\rangle \equiv \boldsymbol{\Omega}\,|\,0\,\rangle\,. \tag{4·3}$$

Note that the following holds for any two forms Ω and Ξ:

$$\boldsymbol{\Omega}\,|\,\Xi\,\rangle = |\,\Omega \wedge \Xi\,\rangle\,. \tag{4·4}$$

Now that we have the above isomorphism, we can introduce the operator corresponding to F in (2·26):

$$\boldsymbol{F} = e^{-\widehat{\boldsymbol{B}}_2}\, d\boldsymbol{D}. \tag{4·5}$$

Since the F_{p+2} are even (odd) forms for type IIA (IIB), we have $(-1)^{\boldsymbol{N_F}}\,|\,F\,\rangle = +\,|\,F\,\rangle$ for type IIA and $= -\,|\,F\,\rangle$ for type IIB. This implies that each state has a definite chirality and thus forms a Majorana-Weyl representation of $SO(d,d;\boldsymbol{Z})$. Noting that the replacement $dy^i \to dy^i - A^{(1)\,i}$ as in (2·31) is equivalent to the operation

$$\boldsymbol{\psi}^{i\dagger} \to e^{\boldsymbol{\psi}_i A^{(1)\,i}} \boldsymbol{\psi}^{i\dagger} e^{-\boldsymbol{\psi}_i A^{(1)\,i}} = \boldsymbol{\psi}^{i\dagger} - A^{(1)\,i}, \tag{4·6}$$

we can simply express the operator corresponding to F' as

$$\boldsymbol{F}' = e^{\boldsymbol{\psi}_i A^{(1)\,i}} \boldsymbol{F} e^{-\boldsymbol{\psi}_i A^{(1)\,i}}, \tag{4·7}$$

and thus the corresponding state can be written as

$$\begin{aligned} |\,F'\,\rangle &= \boldsymbol{F}'\,|\,0\,\rangle = e^{\boldsymbol{\psi}_i A^{(1)\,i}} \boldsymbol{F}\,|\,0\,\rangle \\ &= e^{\boldsymbol{\psi}_i A^{(1)\,i}} e^{-\widehat{\boldsymbol{B}}_2}\,|\,dD\,\rangle = e^{\boldsymbol{\psi}_i A^{(1)\,i}} e^{-\widehat{\boldsymbol{B}}_2} e^{-\boldsymbol{\psi}_i A^{(1)\,i}} \cdot e^{\boldsymbol{\psi}_i A^{(1)\,i}}\,|\,dD\,\rangle\,. \end{aligned} \tag{4·8}$$

*) Recall that the superscript (q) indicates that $\Omega^{(q)}_{i_1\cdots i_n}$ is a q-form for noncompact indices [see (2·10)].

Here one can use (2·17) to show that

$$e^{\boldsymbol{\psi}_i A^{(1)\,i}}\widehat{\boldsymbol{B}}_2 e^{-\boldsymbol{\psi}_i A^{(1)\,i}} = \frac{1}{2}\,B^{(0)}_{ij}\boldsymbol{\psi}^{i\dagger}\boldsymbol{\psi}^{j\dagger} + B^{(1)}_i\boldsymbol{\psi}^{i\dagger} + B^{(2)} - \frac{1}{2}\,B^{(1)}_i A^{(1)\,i}. \quad (4·9)$$

Therefore we have

$$\begin{aligned} |\,F'\,\rangle &= e^{-\boldsymbol{B}^{(0)}} e^{-B^{(2)}} e^{(1/2)B^{(1)}_i A^{(1)\,i}} e^{\boldsymbol{\psi}^{i\dagger} B^{(1)}_i} e^{\boldsymbol{\psi}_i A^{(1)\,i}}\,|\,dD\,\rangle \\ &= e^{-\boldsymbol{B}^{(0)}} e^{-B^{(2)}} e^{\boldsymbol{V}}\,|\,dD\,\rangle\,, \end{aligned} \quad (4·10)$$

where

$$\begin{aligned} \boldsymbol{B}^{(0)} &\equiv \frac{1}{2}\,B^{(0)}_{ij}\boldsymbol{\psi}^{i\dagger}\boldsymbol{\psi}^{j\dagger}, \\ \boldsymbol{V} &\equiv \boldsymbol{\psi}^{i\dagger}B^{(1)}_i + \boldsymbol{\psi}_i A^{(1)\,i}. \end{aligned} \quad (4·11)$$

Since $(B^{(1)}_i,\, A^{(1)\,i})^T$ transforms as a vector of $O(d,d;\boldsymbol{Z})$, one can see that $\boldsymbol{V}$ transforms as

$$\overline{\boldsymbol{V}} = \boldsymbol{\Lambda}\ \boldsymbol{V}\ \boldsymbol{\Lambda}^{-1} \quad (4·12)$$

for $\Lambda \in SO(d,d;\boldsymbol{Z})$. In fact,

$$\begin{aligned} \overline{\boldsymbol{V}} &= (\boldsymbol{\psi}^\dagger, \boldsymbol{\psi}) \begin{pmatrix} \overline{B}^{(1)} \\ \overline{A}^{(1)} \end{pmatrix} = (\boldsymbol{\psi}^\dagger, \boldsymbol{\psi})\ \Lambda \begin{pmatrix} B^{(1)} \\ A^{(1)} \end{pmatrix} \\ &= \boldsymbol{\Lambda}\ (\boldsymbol{\psi}^\dagger, \boldsymbol{\psi})\ \boldsymbol{\Lambda}^{-1} \begin{pmatrix} B^{(1)} \\ A^{(1)} \end{pmatrix} = \boldsymbol{\Lambda}\ \boldsymbol{V}\ \boldsymbol{\Lambda}^{-1}. \end{aligned} \quad (4·13)$$

On the other hand, if we make a block-wise Gauss decomposition of M as

$$\begin{aligned} M &= \begin{pmatrix} 1_d & 0 \\ B^{(0)} & 1_d \end{pmatrix} \cdot \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \cdot \begin{pmatrix} 1_d & -B^{(0)} \\ 0 & 1_d \end{pmatrix} \\ &= \Lambda^T_{B^{(0)}} \cdot \Lambda_G \cdot \Lambda_{B^{(0)}}, \end{aligned} \quad (4·14)$$

then the corresponding operator $\boldsymbol{M}$ can be written as

$$\boldsymbol{M} = e^{-\boldsymbol{B}^{(0)\dagger}}\ \boldsymbol{\Lambda}_G\ e^{-\boldsymbol{B}^{(0)}}, \quad (4·15)$$

with

$$\boldsymbol{\Lambda}_G \equiv \sqrt{G}\,e^{-\boldsymbol{\psi}^{i\dagger}h_i{}^j\boldsymbol{\psi}_j}. \qquad \Big((G_{ij}) = e^{\left(h_i{}^j\right)}\Big) \quad (4·16)$$

This operator $\boldsymbol{\Lambda}_G$ has a special property. In fact, suppose that for a given state

$$|\,\Omega\,\rangle = \sum_q \sum_n \frac{1}{n!}\,\Omega^{(q)}_{i_1\cdots i_n}\,\boldsymbol{\psi}^{i_1\dagger}\cdots\boldsymbol{\psi}^{i_n\dagger}\,|\,0\,\rangle\,, \quad (4·17)$$

we introduce its hermitian conjugate as

$$\langle\,\Omega\,| = \sum_q \sum_n \frac{1}{n!}\,\langle\,0\,|\,\boldsymbol{\psi}_{i_n}\cdots\boldsymbol{\psi}_{i_1}\ *_{10-d}\,\Omega^{(q)}_{i_1\cdots i_n}, \quad (4·18)$$

where $*_{10-d}$ is the Hodge-star in the noncompact $(10-d)$ dimensions. Then the following identity holds:

$$d^{10-d}x\sqrt{-g}\,\sqrt{G}\,|\,\Omega\,|^2_{g,G} = -\,\langle\,\Omega\,|\,\boldsymbol{\Lambda}_G\,|\,\Omega\,\rangle\,. \tag{4·19}$$

In fact, using

$$\begin{aligned}\boldsymbol{\Lambda}_G\,|\,\Omega\,\rangle &= \sum_q\sum_n \frac{\sqrt{G}}{n!}\,\left(e^{-h}\right)_{i_1}^{\ \ j_1}\cdots\left(e^{-h}\right)_{i_n}^{\ \ j_n}\,\Omega^{(q)}_{j_1\cdots j_n}\boldsymbol{\psi}^{i_1\dagger}\cdots\boldsymbol{\psi}^{i_n\dagger}\,|\,0\,\rangle \\ &= \sum_q\sum_n \frac{\sqrt{G}}{n!}\,G^{i_1j_1}\cdots G^{i_nj_n}\,\Omega^{(q)}_{j_1\cdots j_n}\,\boldsymbol{\psi}^{i_1\dagger}\cdots\boldsymbol{\psi}^{i_n\dagger}\,|\,0\,\rangle\,,\end{aligned} \tag{4·20}$$

we can show

$$\begin{aligned}\langle\,\Omega\,|\,\boldsymbol{\Lambda}_G\,|\,\Omega\,\rangle &= \sum_q\sum_n\frac{\sqrt{G}}{n!}\,\left(*_{10-d}\Omega^{(q)}_{i_1\cdots i_n}\wedge\Omega^{(q)}_{j_1\cdots j_n}\right)\,G^{i_1j_1}\cdots G^{i_nj_n} \\ &= -\,d^{10-d}x\sqrt{-g}\,\sqrt{G}\,|\,\Omega\,|^2_{g,G}\,.\end{aligned} \tag{4·21}$$

Setting $\Omega = F'$ in (4·19), we have

$$d^{10-d}x\sqrt{-g}\,\sqrt{G}\,|\,F'\,|^2_{g,G} = -\,\langle\,F'\,|\,\boldsymbol{\Lambda}_G\,|\,F'\,\rangle\,. \tag{4·22}$$

Since this F' has the form given by (4·10), the R-R action with the Chern-Simons term can be expressed as

$$\begin{aligned}S_{\mathrm{R+CS}} &= -\,\frac{1}{8\kappa^2_{10-d}}\int d^{10-d}x\sqrt{-g}\,\sqrt{G}\,|\,F'\,|^2_{g,G} \\ &= \frac{1}{8\kappa^2_{10-d}}\int_{10-d}\langle\,F'\,|\,\boldsymbol{\Lambda}_G\,|\,F'\,\rangle \\ &= \frac{1}{8\kappa^2_{10-d}}\int_{10-d}\langle\,K\,|\,\boldsymbol{M}\,|\,K\,\rangle\,,\end{aligned} \tag{4·23}$$

with

$$|\,K\,\rangle = \exp\left(-B^{(2)}\right)\exp\left(\boldsymbol{V}\right)|\,dD\,\rangle\,. \tag{4·24}$$

This can also be written as

$$S_{\mathrm{R+CS}} = \frac{1}{8\kappa^2_{10-d}}\int_{10-d}S_{\alpha\beta}(M)\,K_\alpha\wedge *_{10-d}K_\beta, \tag{4·25}$$

where K_α is a sum of forms in noncompact directions:

$$K_\alpha = e^{-B^{(2)}}\wedge\left(e^{(1/\sqrt{2})\,\Gamma_r V^r}\right)_{\alpha\beta}\wedge dD_\beta, \tag{4·26}$$

with

$$\begin{aligned}B^{(2)} &= \frac{1}{2}\,B_{\mu\nu}\,dx^\mu\wedge dx^\nu, \qquad\qquad \text{(see (2·15))} \\ V^r &= \begin{pmatrix} B_{\mu\, i}\,dx^\mu \\ A^i_\mu\,dx^\mu\end{pmatrix}, \\ D_\alpha &= \sum_q\frac{1}{q!}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_q}\,D_{\mu_1\cdots\mu_q\,\alpha}\,.\end{aligned} \tag{4·27}$$

Since $\boldsymbol{M}$, $\boldsymbol{V}$ and $B^{(2)}$ transform as

$$\overline{\boldsymbol{M}} = \left(\boldsymbol{\Lambda}^{-1}\right)^{\dagger}\ \boldsymbol{M}\ \boldsymbol{\Lambda}^{-1}, \qquad \overline{\boldsymbol{V}} = \boldsymbol{\Lambda}\ \boldsymbol{V}\ \boldsymbol{\Lambda}^{-1}, \qquad \overline{B}^{(2)} = B^{(2)}, \tag{4·28}$$

we see that the action is invariant under the whole T-duality group $SO(d,d;\boldsymbol{Z})$ if $D=(D_\alpha)$ transforms as a Majorana-Weyl spinor:

$$\left|\,\overline{D}\,\right\rangle = \boldsymbol{\Lambda}\,|\,D\,\rangle\,. \tag{4·29}$$

Furthermore, if we expand D with respect to noncompact indices as

$$D = \sum_q \frac{1}{q!}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_q}\,D_{\mu_1\cdots\mu_q}, \tag{4·30}$$

with

$$D_{\mu_1\cdots\mu_q} \equiv \sum_n \frac{1}{n!}\,D_{\mu_1\cdots\mu_q\,i_1\cdots i_n}\,dy^{i_1}\wedge\cdots\wedge dy^{i_n}, \tag{4·31}$$

then each coefficient $D_{\mu_1\cdots\mu_q}$ will also transform as a Majorana spinor:[*)]

$$\left|\,\overline{D}_{\mu_1\cdots\mu_q}\,\right\rangle = \boldsymbol{\Lambda}\,|\,D_{\mu_1\cdots\mu_q}\,\rangle\,, \tag{4·32}$$

or equivalently,

$$\overline{D}_{\mu_1\cdots\mu_q\,\alpha} = \sum_\beta S_{\alpha\beta}(\Lambda)\,D_{\mu_1\cdots\mu_q\,\beta}, \tag{4·33}$$

with multi-indices $\alpha=(i_1,\cdots,i_n)$ $(i_1<\cdots<i_n;\,n=0,\cdots,d)$. Since $D_{\mu_1\cdots\mu_q\,i_1\cdots i_n}$ vanishes if $q+n=$ even (odd) for type IIA (IIB), it has a definite chirality. This implies that $D_{\mu_1\cdots\mu_q}=(D_{\mu_1\cdots\mu_q\,\alpha})$ transforms as a Majorana-Weyl spinor for each set of noncompact indices $(\mu_1,\cdots,\mu_q)$.

§5. Discussion

In this article, we have given a simple proof that if the R-R potentials C_{p+1} are combined with the NS-NS 2-form as in (2·24), then their KK forms transform as Majorana-Weyl spinors under the T-duality group $SO(d,d;\boldsymbol{Z})$ in order to make the action invariant.

There should be various applications once transformation rules are obtained explicitly for the whole T-duality group. One application will be to establish relations among various classical solutions of type IIA/IIB supergravities by using the full T-duality group together with the S-duality of type IIB. Work in this direction is in progress and will be reported elsewhere.[19)]

We finally make a comment on the dilaton dependence in R-R potentials, assuming the case $\widehat{B}_2=0$, in which there is no distinction between the original R-R

*) To be more precise, the following discussion holds only when $q+d\le 10$.

potential C_{p+1} and our potential D_{p+1}. Usually we expect that another field strength defined by $\widetilde{F}_{p+2} = e^{\widehat{\phi}}\, dC_{p+1}$ corresponds to an R-R vertex operator of NSR strings in a flat background. To see this in our formulation, we first recall that we have introduced the $(10-d)$-dimensional dilaton ϕ as a singlet of $O(d,d;\boldsymbol{Z})$. This implies that the 10-dimensional dilaton $\widehat{\phi}$ should transform as $e^{\widehat{\phi}} \propto G^{1/4}$. On the other hand, we could have further decomposed the operator $\boldsymbol{\Lambda}_G$ as $\boldsymbol{\Lambda}_G = \boldsymbol{\Lambda}_E^\dagger \boldsymbol{\Lambda}_E$, where $E = (E_{ia})$ $(i, a = 1, \cdots, d)$ is a vielbein for G, $G = E\, E^T$. Then one might say that the state $\boldsymbol{\Lambda}_E \,|\, dC\,\rangle$ corresponds to an R-R vertex in a flat background. Thus, noting that the operator $\boldsymbol{\Lambda}_E$ will carry the factor $G^{1/4}$, we expect that $e^{\widehat{\phi}}\, dC$ will transform as in the flat case.

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Appendix A

——— *"Self-Dual" Formulation of Type II Effective Actions* ———

In this appendix, we prove that the original R-R action plus the Chern-Simon term [(2·4)–(2·6)]

$$
\begin{aligned}
S_{\mathrm{R}}^{(\mathrm{IIA})} + S_{\mathrm{CS}}^{(\mathrm{IIA})} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-\widehat{g}}\, \left(|F_2|_{\widehat{g}}^2 + |F_4|_{\widehat{g}}^2 \right) \\
&\quad + \frac{1}{4\kappa_{10}^2} \int \widehat{B}_2 \wedge dC_3 \wedge dC_3, \\
S_{\mathrm{R}}^{(\mathrm{IIB})} + S_{\mathrm{CS}}^{(\mathrm{IIB})} &= -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-\widehat{g}}\, \left(|F_1|_{\widehat{g}}^2 + |F_3|_{\widehat{g}}^2 + \frac{1}{2}\, |F_5|_{\widehat{g}}^2 \right) \\
&\quad + \frac{1}{4\kappa_{10}^2} \int \widehat{B}_2 \wedge dC_4 \wedge dC_2, \qquad \text{(A·1)}
\end{aligned}
$$

with C_1, C_3 (or D_1, D_3) and C_0, C_2, C_4 (or D_0, D_2, D_4) being independent variables, respectively, is equivalent to the new action, (2·30),

$$
\begin{aligned}
S_{\mathrm{R+CS}}^{(\mathrm{IIA})} &\equiv \frac{1}{8\kappa_{10}^2} \int \sum_{p+2=2,4,6,8} F_{p+2} \wedge *F_{p+2} = -\frac{1}{8\kappa_{10}^2} \int \sum_{p+2=2,4,6,8} |\, F_{p+2}\,|_{\widehat{g}}^2, \\
S_{\mathrm{R+CS}}^{(\mathrm{IIB})} &\equiv \frac{1}{8\kappa_{10}^2} \int \sum_{p+2=1,3,5,7,9} F_{p+2} \wedge *F_{p+2} = -\frac{1}{8\kappa_{10}^2} \int \sum_{p+2=1,3,5,7,9} |\, F_{p+2}\,|_{\widehat{g}}^2,
\end{aligned}
\qquad \text{(A·2)}
$$

with D_1, D_3, D_5, D_7 and D_0, D_2, D_4, D_6, D_8 being independent variables, in the sense that both give the same equations of motion when the constraints (2·29),

$$\begin{aligned} &*F_1 = F_9, && *F_2 = -F_8, \\ &*F_3 = -F_7, && *F_4 = F_6, \\ &*F_5 = F_5, && *F_6 = -F_4, \\ &*F_7 = -F_3, && *F_8 = F_2, \\ &*F_9 = F_1, && \end{aligned} \tag{A·3}$$

are imposed on the extra variables $D_5, \cdots, D_8$, after the equations of motion are derived from (A·2). Here their field strengths are defined by

$$F \equiv \sum_{p+2=1}^{9} F_{p+2} \equiv e^{-\widehat{B}_2} \wedge dD, \tag{A·4}$$

with

$$D \equiv \sum_{p+1=0}^{8} D_{p+1}, \tag{A·5}$$

and the 10-dimensional Hodge-star $*$ is defined by

$$\begin{aligned} &* \left(dx^{\hat{\mu}_1} \wedge \cdots \wedge dx^{\hat{\mu}_n} \right) \\ &\quad \equiv \frac{1}{(10-n)!} \frac{1}{\sqrt{-\widehat{g}}} \, \epsilon^{\hat{\mu}_1 \cdots \hat{\mu}_n \hat{\nu}_1 \cdots \hat{\nu}_{10-n}} \widehat{g}_{\hat{\nu}_1 \hat{\lambda}_1} \cdots \widehat{g}_{\hat{\nu}_{10-n} \hat{\lambda}_{10-n}} \, dx^{\hat{\lambda}_1} \wedge \cdots \wedge dx^{\hat{\lambda}_{10-n}}, \end{aligned} \tag{A·6}$$

with $\epsilon^{01\cdots 9} = +1$. Note that the K-forms satisfy $*^2 \, \Omega_K = (-1)^{K+1} \, \Omega_K$ and

$$\begin{aligned} d^{10}x \sqrt{-\widehat{g}} \mid \Omega_K \mid^2_{\widehat{g}} &\equiv d^{10}x \sqrt{-\widehat{g}} \, \frac{1}{K!} \, \widehat{g}^{\hat{\mu}_1 \hat{\nu}_1} \cdots \widehat{g}^{\hat{\mu}_K \hat{\nu}_K} \, \Omega_{\hat{\mu}_1 \cdots \hat{\mu}_K} \, \Omega_{\hat{\nu}_1 \cdots \hat{\nu}_K} \\ &= -\Omega_K \wedge *\Omega_K \end{aligned} \tag{A·7}$$

in 10-dimensional Minkowski space.

First, we note that the original action (A·1) can be written as

$$\begin{aligned} S_{\rm R}^{\rm (IIA)} + S_{\rm CS}^{\rm (IIA)} &= \frac{1}{4\kappa_{10}^2} \int \left(F_2 \wedge *F_2 + F_4 \wedge *F_4 + \widehat{B}_2 \, F_4^2 + \widehat{B}_2^2 \, F_4 \, F_2 + \frac{1}{3} \, \widehat{B}_2^3 \, F_2^2 \right), \\ S_{\rm R}^{\rm (IIB)} + S_{\rm CS}^{\rm (IIB)} &= \frac{1}{4\kappa_{10}^2} \int \left(F_1 \wedge *F_1 + F_3 \wedge *F_3 + \frac{1}{2} \, F_5 \wedge *F_5 \right. \\ &\qquad \left. + \widehat{B}_2 \, F_5 \, F_3 + \frac{1}{2} \, \widehat{B}_2^2 \, F_5 \, F_1 + \frac{1}{6} \, \widehat{B}_2^3 \, F_3 \, F_1 \right). \end{aligned} \tag{A·8}$$

Combined with the NS-NS action $S_{\rm NS}$, (2·2), the equations of motion are thus

IIA

$$0 = d \left(*F_4 + \widehat{B}_2 \, F_4 + \frac{1}{2} \, \widehat{B}_2^2 \, F_2 \right),$$

$$
\begin{aligned}
0 &= d\left(- * F_2 + \widehat{B}_2 * F_4 + \frac{1}{2}\,\widehat{B}_2^2\, F_4 + \frac{1}{6}\,\widehat{B}_2^3\, F_2\right), \\
0 &= d\left(e^{-2\widehat{\phi}} * \widehat{B}_2\right) + F_2 * F_4 - \frac{1}{2}\, F_4^2,
\end{aligned} \tag{A·9}
$$

<u>IIB</u>

$$
\begin{aligned}
0 &= d\left(*F_5 + \widehat{B}_2\, F_3 + \frac{1}{2}\,\widehat{B}_2^2\, F_1\right), \\
0 &= d\left(- * F_3 + \widehat{B}_2 * F_5 + \frac{1}{2}\,\widehat{B}_2^2\, F_3 + \frac{1}{6}\,\widehat{B}_2^3\, F_1\right), \\
0 &= d\left(*F_1 - \widehat{B}_2 * F_3 + \frac{1}{4}\,\widehat{B}_2^2\,(F_5 + *F_5) + \frac{1}{6}\,\widehat{B}_2^3\, F_3 + \frac{1}{24}\,\widehat{B}_2^4\, F_1\right), \\
0 &= d\left(e^{-2\widehat{\phi}} * \widehat{B}_2\right) + F_1 * F_3 + \frac{1}{2}\, F_3\, F_5 + \frac{1}{2}\, F_3 * F_5.
\end{aligned} \tag{A·10}
$$

Also, the Einstein equation has an energy-momentum tensor of the R-R fields given by

$$
T^{(\mathrm{R})}_{\hat{\mu}\hat{\nu}} = \begin{cases} \mathcal{E}_{\hat{\mu}\hat{\nu}}(F_2) + \mathcal{E}_{\hat{\mu}\hat{\nu}}(F_4), & (\mathrm{IIA}) \\ \mathcal{E}_{\hat{\mu}\hat{\nu}}(F_1) + \mathcal{E}_{\hat{\mu}\hat{\nu}}(F_3) + \frac{1}{2}\,\mathcal{E}_{\hat{\mu}\hat{\nu}}(F_5), & (\mathrm{IIB}) \end{cases} \tag{A·11}
$$

where $\mathcal{E}_{\hat{\mu}\hat{\nu}}(F_n)$ is defined for the n-form $F_n = (1/n!)\, F_{\hat{\mu}_1\cdots\hat{\mu}_n} dx^{\hat{\mu}_1}\wedge\cdots\wedge dx^{\hat{\mu}_n}$ as

$$
\mathcal{E}_{\hat{\mu}\hat{\nu}}(F_n) \equiv \frac{1}{(n-1)!}\, F_{\hat{\mu}\,\hat{\mu}_1\cdots\hat{\mu}_{n-1}}\, F_{\hat{\nu}}{}^{\hat{\mu}_1\cdots\hat{\mu}_{n-1}} - \frac{1}{2}\,\widehat{g}_{\hat{\mu}\hat{\nu}}\, \mid F_n \mid^2_{\widehat{g}}. \tag{A·12}
$$

Equations (A·9) and (A·10) imply that $F_1, \cdots, F_5$ can be expressed in the following form with integration "constants" D_{p+1} $(p+1 \geq 5)$:
<u>IIA</u>

$$
\begin{aligned}
*F_2 &= -\left(e^{-\widehat{B}_2} \wedge dD\right)_8 \equiv -F_8, \\
*F_4 &= \left(e^{-\widehat{B}_2} \wedge dD\right)_6 \equiv F_6,
\end{aligned} \tag{A·13}
$$

<u>IIB</u>

$$
\begin{aligned}
*F_1 &= \left(e^{-\widehat{B}_2} \wedge dD\right)_9 \equiv F_9, \\
*F_3 &= -\left(e^{-\widehat{B}_2} \wedge dD\right)_7 \equiv -F_7, \\
*F_5 &= \left(e^{-\widehat{B}_2} \wedge dD\right)_5 \equiv F_5.
\end{aligned} \tag{A·14}
$$

For example, the first equation of (A·9) is solved as

$$
*F_4 + \widehat{B}_2\, F_4 + \frac{1}{2}\,\widehat{B}_2^2\, F_2 = dD_5, \tag{A·15}
$$

with some 5-form D_5. Then $*F_4$ can be written as

$$\begin{aligned}
*F_4 &= dD_5 - \widehat{B}_2 F_4 - \frac{1}{2}\,\widehat{B}_2^2\,F_2 \\
&= dD_5 - \widehat{B}_2(dD_3 - \widehat{B}_2 dD_1) - \frac{1}{2}\,\widehat{B}_2^2\,dD_1 \\
&= dD_5 - \widehat{B}_2 dD_3 + \frac{1}{2}\,\widehat{B}_2^2\,dD_1 \\
&= \left(e^{-\widehat{B}_2} \wedge dD\right)_6 \\
&\equiv F_6.
\end{aligned} \tag{A·16}$$

Next, we treat all the fields D_{p+1} $(p+1 = 0, \cdots, 8)$ as independent variables with field strengths (A·4), and adopt (A·2) plus $S_{\rm NS}$ as their action functional. The variation of the action with respect to these fields can be easily found to be

<u>IIA</u>

$$\begin{aligned}
0 &= d\,(*F_8)\,, \\
0 &= d\left(- * F_6 + \widehat{B}_2 * F_8\right), \\
0 &= d\left(*F_4 - \widehat{B}_2 * F_6 + \frac{1}{2}\,\widehat{B}_2^2 * F_8\right), \\
0 &= d\left(- * F_2 + \widehat{B}_2 * F_4 - \frac{1}{2}\,\widehat{B}_2^2 * F_6 + \frac{1}{6}\,\widehat{B}_2^3 * F_8\right),
\end{aligned} \tag{A·17}$$

<u>IIB</u>

$$\begin{aligned}
0 &= d\,(*F_9)\,, \\
0 &= d\left(- * F_7 + \widehat{B}_2 * F_9\right), \\
0 &= d\left(*F_5 - \widehat{B}_2 * F_7 + \frac{1}{2}\,\widehat{B}_2^2 * F_9\right), \\
0 &= d\left(- * F_3 + \widehat{B}_2 * F_5 - \frac{1}{2}\,\widehat{B}_2^2 * F_7 + \frac{1}{6}\,\widehat{B}_2^3 * F_9\right), \\
0 &= d\left(*F_1 - \widehat{B}_2 * F_3 + \frac{1}{2}\,\widehat{B}_2^2 * F_5 - \frac{1}{6}\,\widehat{B}_2^3 * F_7 + \frac{1}{24}\,\widehat{B}_2^4 * F_9\right).
\end{aligned} \tag{A·18}$$

These are identical to the set of Bianchi identities and the equations of motion for the original fields $D_0, \cdots, D_4$ if we identify $*F_{p+2} = \pm F_{8-p}$ as in (A·3). Furthermore, the variation with respect to $\widehat{B}_2$ gives

<u>IIA</u>

$$0 = d\left(e^{-2\widehat{\phi}} * \widehat{B}_2\right) + \frac{1}{2}\,F_2 * F_4 + \frac{1}{2}\,F_4 * F_6 + \frac{1}{2}\,F_6 * F_8, \tag{A·19}$$

<u>IIB</u>

$$0 = d\left(e^{-2\widehat{\phi}} * \widehat{B}_2\right) + \frac{1}{2}\,F_1 * F_3 + \frac{1}{2}\,F_3 * F_5 + \frac{1}{2}\,F_5 * F_7 + \frac{1}{2}\,F_7 * F_9, \tag{A·20}$$

which are the same as the last equations of (A·9) and (A·10), respectively, after the identification (A·3) is made.

The Einstein equation will be accompanied by the new energy-momentum tensor for the R-R fields,

$$T^{(\mathrm{R})}_{\hat\mu\hat\nu} = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{n=2,4,6,8} \mathcal{E}_{\hat\mu\hat\nu}(F_n), & (\mathrm{IIA}) \\ \dfrac{1}{2} \displaystyle\sum_{n=1,3,5,7,9} \mathcal{E}_{\hat\mu\hat\nu}(F_n). & (\mathrm{IIB}) \end{cases} \tag{A·21}$$

This agrees with the previous form (A·11), since the following identity holds for the dual field $\widetilde{F}_{10-n} \equiv *F_n$:

$$\mathcal{E}_{\hat\mu\hat\nu}(\widetilde{F}_{10-n}) = \mathcal{E}_{\hat\mu\hat\nu}(F_n). \tag{A·22}$$

This identity can be easily proved by using

$$\begin{gathered} \frac{1}{(9-n)!}\, \widetilde{F}_{\hat\mu\,\hat\lambda_1\cdots\hat\lambda_{9-n}}\, \widetilde{F}_{\hat\nu}{}^{\hat\lambda_1\cdots\hat\lambda_{9-n}} = \frac{1}{(n-1)!}\, F_{\hat\mu\,\hat\lambda_1\cdots\hat\lambda_{n-1}}\, F_{\hat\nu}{}^{\hat\lambda_1\cdots\hat\lambda_{n-1}} - \widehat{g}_{\hat\mu\hat\nu}\, \mid F_n \mid^2_{\widehat{g}}\,, \\ \left| \widetilde{F}_{10-n} \right|^2_{\widehat{g}} = - \mid F_n \mid^2_{\widehat{g}}\,. \end{gathered} \tag{A·23}$$

Since the equivalence for the variation with respect to the dilaton $\widehat{\phi}$ is obvious, we have completed the proof of the equivalence of the two actions (A·1) and (A·2).

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Note added: After the first version of the present paper was put on the bulletin board, some related works appeared,[20),21)] which also investigate the T-duality transformation of R-R fields from a different point of view.