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Holographic Renormalization Group Structure in Higher-Derivative Gravity

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Classical higher-derivative gravity is investigated in the context of the holographic renormalization group (RG). We parametrize the Euclidean time such that one step of time evolution in $(d+1)$-dimensional bulk gravity can be directly interpreted as that of block spin transformation of the d-dimensional boundary field theory. This parametrization simplifies the analysis of the holographic RG structure in gravity systems, and conformal fixed points are always described by AdS geometry. We find that higher-derivative gravity generically induces extra degrees of freedom, which acquire huge masses around stable fixed points and thus are coupled to highly irrelevant operators at the boundary. In the particular case of pure R^2-gravity, we show that some region of values of the coefficients of the curvature-squared terms allows us to have two fixed points (one is multicritical), which are connected by a kink solution. We further extend our analysis to Lorentzian metric to investigate a model of expanding universe described by the action with curvature-squared terms and a positive cosmological constant. We show that, in any dimensionality but four, there is a classical solution that describes the time evolution from one de Sitter geometry to another de Sitter geometry, along which the Hubble parameter changes significantly.

§1. Introduction

The AdS/CFT correspondence asserts, in its simplest form, that $(d+1)$-dimensional (super)gravity in an AdS background describes a d-dimensional CFT at the boundary.[1)−3)] (For a review, see Ref. 4).) One of the most important aspects of this correspondence is that it gives us a scheme to investigate the renormalization group (RG) structure of the d-dimensional field theory.[5)−14)] In this scheme, the *holographic RG*, the radial coordinate of the $(d+1)$-dimensional manifold is identified with the RG parameter of the corresponding boundary field theory, and a classical trajectory of bulk fields is interpreted as an RG flow of the corresponding coupling constants in the d-dimensional field theory. As an example, the Weyl anomaly of a four-dimensional field theory can be calculated using the holographic RG scheme and exactly reproduces the large N limit of the Weyl anomaly of the four dimensional $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory when the supergravity comes from type IIB supergravity on $\mathrm{AdS}_5 \times S^5$.[15)] For a field theory in any dimensionality, there is a systematic formulation of the holographic RG using the Hamilton-Jacobi equation of gravity systems[16)−18)] (see also Refs. 19) − 22)).

Classical Einstein gravity discussed above is actually the low energy limit of

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a string theory, and an important issue is whether this correspondence can be extended to the level of strings.[23)–28)] In Ref. 28), it was discussed that the AdS/CFT correspondence does hold even when α' corrections are taken into account, where α' is the square of the string length. The gravity system considered in Ref. 28) is R^2-gravity whose Lagrangian density contains curvature squared terms that appear after integrating over massive string excitation modes. (Such higher-derivative interactions also appear for matter fields.) In general, a higher-derivative system[*)] with the Lagrangian $L(q,\dot{q},\ddot{q})$ can be treated in the Hamilton formalism by introducing a new independent variable Q which equals $\dot{q}$ classically. (We call this new variable the *higher-derivative mode.*) Thus the Hamiltonian for this system is a function of (q,Q) and their conjugate momenta, (p,P). It was pointed out[28)] that the AdS/CFT correspondence in higher-dimensional gravity can be established if we use the mixed boundary conditions that set the Dirichlet boundary conditions for the light mode q and the Neumann boundary conditions for the higher-derivative mode Q (i.e., $P=0$ at the boundary). As a check of this proposal, the Weyl anomaly was calculated for the R^2-gravity system which is AdS/CFT dual to the $\mathcal{N}=2$ superconformal field theory in four dimensions,[**)] and the obtained result is the same as those of Refs. 25) and 26), which are consistent with the field theoretical calculation.[31)] A brief review of classical mechanics in higher-derivative systems is given in Appendix A. (For a review of higher-derivative gravity, see, e.g., Ref. 34).)

The main aim of the present paper is to further clarify the holographic RG structure in higher-derivative gravity, by investigating its classical solutions with the following steps. We first give a parametrization of the Euclidean time such that its evolution can be directly interpreted as the change of the unit length of the d-dimensional equal-time slice. We call this parametrization the *block spin gauge.* With the use of this gauge, we then investigate (1) a higher-derivative pure gravity system and (2) a system consisting of a scalar field with a higher-derivative interaction in Einstein gravity. For both systems, some regions of values of the coefficients of the higher-derivative terms allow us to have a stable AdS solution, around which the higher-derivative mode acquires a huge mass and thus is coupled to a highly irrelevant operator at the boundary. In the other regions of coefficient values, we show that any AdS solution becomes unstable, and the higher-derivative mode in the AdS background becomes tachyonic with a mass squared far below the unitarity bound, so that the holographic RG interpretation is not applicable. We also show, in the pure gravity case, that there are two AdS solutions in a certain region of values of the coefficients and there is also a solution that interpolates between these two AdS solutions. In the context of the holographic RG, this means that there are two fixed points in the phase diagram of the d-dimensional field theory, and that the solution that connects them corresponds to an RG flow from a multicritical point to another fixed point.

The organization of this paper is as follows. In §2 we introduce the block spin

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*) See Ref. 29) which also investigates higher-derivative systems in the context of string theory.

**) The gravity system is given by IIB supergravity on $\mathrm{AdS}_5 \times S^5/\boldsymbol{Z}_2$.[30)] The action contains an R^2-term, reflecting open-string excitations.

gauge. In §3 we investigate a higher-derivative pure gravity system, and in §4 we investigate a system consisting of a scalar field with a higher-derivative interaction in Einstein gravity. In §5, we extend our analysis to higher-derivative gravity with Lorentzian metric and investigate a model of expanding universe with a positive cosmological constant. There, we show that there is a solution for which one de Sitter space-time flows to another de Sitter space-time, along which the Hubble parameter changes greatly. Section 6 contains a conclusion and discussion about the meaning of the mixed boundary conditions proposed in Ref. 28).

§2. Block spin gauge

In this section we introduce a gauge in which (Euclidean) time evolution in a $(d+1)$-dimensional manifold is directly regarded as the change of the unit length in the d-dimensional equal-time slice. Although this gauge restricts the class of the geometry one can consider, it is actually sufficient for investigating the holographic RG structure in higher-derivative gravity.

We start by recalling the ADM decomposition that parametrizes a $(d+1)$-dimensional metric with Euclidean signature,

$$\begin{aligned} ds^2 &= \widehat{g}_{\mu\nu}\, dX^\mu dX^\nu \\ &= N(x,\tau)^2 d\tau^2 + g_{ij}(x,\tau)(dx^i + \lambda^i d\tau)(dx^j + \lambda^j d\tau), \end{aligned} \tag{2·1}$$

where $X^\mu = (x^i, \tau)$ with $i = 1, \cdots d$, and N and λ^i are the lapse and the shift function, respectively. In what follows, we exclusively consider the metric with d-dimensional Poincaré invariance by setting $g_{ij} = e^{-2q(\tau)}\delta_{ij}$, $N = N(\tau)$ and $\lambda^i = 0$:

$$ds^2 = N(\tau)^2 d\tau^2 + e^{-2q(\tau)}\delta_{ij} dx^i dx^j. \tag{2·2}$$

For this metric, the unit length in the d-dimensional equal-time slice at τ is given by $e^{q(\tau)}$.

We consider two kinds of gauge fixings (or parametrizations of time). One is the temporal gauge which is obtained by setting $N(\tau) = 1$:

$$ds^2 = d\tau^2 + e^{-2q(\tau)}\delta_{ij} dx^i dx^j. \tag{2·3}$$

The other is a gauge fixing that can be made only when the condition

$$\frac{dq(\tau)}{d\tau} > 0 \qquad (-\infty < \tau < \infty) \tag{2·4}$$

is satisfied. Then q can be regarded as a new time coordinate. We call this parametrization the *block spin gauge*.*) By writing $q(\tau)$ as t, the metric in this gauge is expressed as**)

$$ds^2 = Q(t)^{-2} dt^2 + e^{-2t}\delta_{ij} dx^i dx^j. \tag{2·5}$$

*) In this gauge, the unit length in the d-dimensional equal-time slice at t is given by $a(t) = a_0 e^t$ with a positive constant a_0. If we consider the time evolution $t \to t + \delta t$, the unit length changes as $a \to e^{\delta t} a$. In other words, one step of time evolution directly describes that of block spin transformation of the d-dimensional field theory.

**) This form of metric sometimes appears in literature (see, e.g., Ref. 35)).

Since two parametrizations of time (temporal and block spin) are related as

$$t = q(\tau), \tag{2·6}$$

together with the condition (2·4), the coefficient $Q(t)$ is given by

$$Q(t) = \left.\frac{dq(\tau)}{d\tau}\right|_{\tau = q^{-1}(t)} (>0). \tag{2·7}$$

Note that constant Q ($\equiv 1/l$) gives the AdS metric of radius l,

$$\begin{aligned} ds^2 &= d\tau^2 + e^{-2\tau/l}\, dx_i^2 \quad \text{(temporal gauge)} \\ &= l^2 dt^2 + e^{-2t}\, dx_i^2, \quad \text{(block spin gauge)} \end{aligned} \tag{2·8}$$

with the boundary at $\tau = -\infty$ (or $t = -\infty$).

Here we show that the condition (2·4) sets a restriction on the possible geometry, by solving the Einstein equation both in the temporal and block spin gauge. In the temporal gauge, the Einstein-Hilbert action

$$\boldsymbol{S}_E = \int_{M_{d+1}} d^{d+1}X \sqrt{\widehat{g}} \left[2\Lambda - \widehat{R}\right] \tag{2·9}$$

becomes

$$\boldsymbol{S}_E = -d(d-1)\mathcal{V}_d \int d\tau e^{-dq(\tau)} \left(\dot{q}(\tau)^2 + \frac{1}{l^2}\right), \tag{2·10}$$

up to total derivative. Here we have parametrized the cosmological constant as $\Lambda = -d(d-1)/2l^2$, and $\mathcal{V}_d$ is the volume of the d-dimensional space. The general classical solutions for this action are

$$\frac{dq}{d\tau} = \frac{1}{l}\,\frac{1 - Ce^{d\tau/l}}{1 + Ce^{d\tau/l}}. \qquad (C \geq 0) \tag{2·11}$$

This shows that geometry with nonvanishing, finite C ($C \neq 0$ or ∞) cannot be described in the block spin gauge, since $\dot{q}$ vanishes at $\tau = -\frac{l}{d}\ln C$, violating the condition (2·5). In fact, in the block spin gauge (2·5), the action (2·9) becomes

$$\boldsymbol{S}_E = -d(d-1)\mathcal{V}_d \int dt e^{-dt} \left(\frac{1}{l^2 Q} + Q\right), \tag{2·12}$$

which readily gives the classical solution as

$$Q(t) = \pm\frac{1}{l}. \tag{2·13}$$

This actually reproduces only the AdS solution in the temporal gauge with $C = 0$ or ∞.

§3. Higher-derivative pure gravity in the block spin gauge

In this section we investigate classical R^2-gravity in the block spin gauge and give a holographic RG interpretation to higher-derivative modes. A brief review of classical mechanics in higher-derivative systems is given in Appendix A.

The action of pure R^2-gravity in a $(d+1)$-dimensional manifold M_{d+1} with a boundary Σ_d is generally given by

$$\begin{aligned}\boldsymbol{S} = &\int_{M_{d+1}} d^{d+1}X\sqrt{\widehat{g}}\left(2\Lambda - \widehat{R} - a\widehat{R}^2 - b\widehat{R}_{\mu\nu}^2 - c\widehat{R}_{\mu\nu\rho\sigma}^2\right)\\ &+\int_{\Sigma_d} d^dx\sqrt{g}\left(2K + x_1\,RK + x_2\,R_{ij}K^{ij} + x_3\,K^3 + x_4\,KK_{ij}^2 + x_5\,K_{ij}^3\right),\end{aligned} \tag{3·1}$$

with some given constants a, b and c. Here, K_{ij} is the extrinsic curvature of Σ_d given by

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \nabla_i\lambda_j - \nabla_j\lambda_i\right), \qquad \left(\cdot \equiv \frac{d}{dt}\right) \tag{3·2}$$

and $K = g^{ij}K_{ij}$. Here, ∇_i and R_{ijkl} are, respectively, the covariant derivative and the Riemann tensor defined by g_{ij} in the ADM decomposition (2·1). The first terms in the boundary terms in (3·1) are those for Einstein gravity given in Ref. 36), and the remaining terms are the most general ones which are invariant under the $(d+1)$-dimensional diffeomorphism that does not change the position of the boundary. (For details, see Ref. 28). Other studies of boundary terms in higher-derivative gravity can be found in Refs. 37) and 38).)

Substituting the block spin gauge metric (2·5) into the action (3·1), we obtain

$$\boldsymbol{S}[Q(t)] = \mathcal{V}_d\int_{t_0}^{\infty} dt\ L(Q,\dot{Q}), \tag{3·3}$$

where

$$\begin{aligned}L(Q,\dot{Q}) = &e^{-dt}\left(\frac{2\Lambda}{Q} - d(d-1)Q - \frac{A}{2}Q\dot{Q}^2 + BQ^3\right)\\ &+\left[\frac{4d}{3}\big(d(d+1)a + db + 2c\big) + d\left(d^2x_3 + dx_4 + x_5\right)\right]\frac{d}{dt}\left(e^{-dt}Q^3\right),\end{aligned} \tag{3·4}$$

with

$$A = 2d\big(4da + (d+1)b + 4c\big), \quad B = \frac{d(d-3)}{3}\big(d(d+1)a + db + 2c\big). \tag{3·5}$$

We have set t to run from t_0 to ∞. The Lagrangian (3·4) gives the Euler-Lagrange equation for Q as

$$Q\ddot{Q} + \frac{1}{2}\dot{Q}^2 - dQ\dot{Q} = \frac{1}{A}\left(\frac{2\Lambda}{Q^2} + d(d-1) - 3BQ^2\right). \tag{3·6}$$

The classical action S is obtained by substituting into $\boldsymbol{S}$ the classical solution $Q(t)$ with the boundary condition $Q(t_0) = Q_0$ and the regularity of $Q(t)$ in the limit $t \to \infty$. It is a function of the boundary value, $\boldsymbol{S}[Q(t)] \equiv S(Q_0, t_0)$.

In the holographic RG, this classical action would be interpreted as the bare action of a d-dimensional field theory with the bare coupling Q_0 at the UV cutoff $\Lambda = \exp(-t_0)$.[2),3),5)] Thus, the strategy of our analysis is as follows. We first find the solutions that converge to $Q=$const as $t \to \infty$ in order to have a finite classical action. We then examine the stability of the solution to read off the form of the general classical solutions. Since the solution $Q =$ const gives an AdS geometry, the fluctuation of Q around the solution is regarded as describing the motion of the higher-derivative mode in the AdS background, which leads to a holographic RG interpretation of the higher-derivative mode.

Following the above strategy, we first look for AdS solutions (i.e., $Q(t) =$ const). By parametrizing the cosmological constant as

$$\Lambda = -\frac{d(d-1)}{2l^2} + \frac{3B}{2l^4}, \tag{3·7}$$

the equation of motion (3·6) gives two AdS solutions,

$$Q^2 = \begin{cases} \dfrac{1}{l^2} & \equiv \dfrac{1}{l_1^2}\,, \\[2ex] \dfrac{d(d-1)}{3B} - \dfrac{1}{l^2} & \equiv \dfrac{1}{l_2^2}\,, \end{cases} \tag{3·8}$$

where the solution $Q = 1/l_2$ exists only when $B > 0$.*) They have radii l_i $(i = 1, 2)$, respectively, and we call them AdS$^{(i)}$ $(i = 1, 2)$. We assume that we can take the limit $a, b, c \to 0$ smoothly, in which the system reduces to Einstein gravity on AdS of radius l. We also assume that this AdS gravity comes from the low-energy limit of a string theory, so that its radius $l_1 = l$ should be sufficiently larger than the string length. On the other hand, the AdS$^{(2)}$ solution, if it exists, appears only when the higher-derivative terms are taken into account. As the coefficients of the higher-derivative terms are thought to stem from string excitations, their coefficients a, b and c (and hence A and B) are $\mathcal{O}(\alpha')$. Thus the radius of the AdS$^{(2)}$ is of the order of the string length, as can be seen from the solution (3·8).

Next, we examine the perturbation of classical solutions around (3·8), writing

$$Q(t) = \frac{1}{l_i} + X_i(t). \tag{3·9}$$

The equation of motion (3·6) is then linearized as

$$\ddot{X}_i - d\dot{X}_i - l_i^2 m_i^2 X_i = 0, \tag{3·10}$$

with

$$m_i^2 \equiv -\frac{2}{A}\left(2\Lambda l_i^2 + \frac{3B}{l_i^2}\right). \tag{3·11}$$

*) We consider only the case $Q > 0$ because of the condition (2·4).

Equation (3·10) is nothing but the equation of motion for a scalar field with mass squared m_i^2 in the background of the AdS$_{d+1}$ geometry, $ds^2 = l_i^2 dt^2 + e^{-2t}\sum_k dx_k^2$ (block spin gauge), and the general solution is given by a linear combination of the functions

$$f_i^{\pm}(t) \equiv \exp\left[\left(\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + l_i^2 m_i^2}\right) t\right]. \qquad (3\cdot12)$$

Here $l_i^2 m_i^2$ can be easily calculated from (3·8) and (3·11) as

$$\begin{cases} l_1^2 m_1^2 &= \dfrac{2}{A}\left(d(d-1)l^2 - 6B\right), \\ l_2^2 m_2^2 &= -\dfrac{6B}{A} \cdot \dfrac{d(d-1)l^2 - 6B}{d(d-1)l^2 - 3B}. \end{cases} \qquad (3\cdot13)$$

In the following, we investigate these solutions for $i = 1$ and 2, to understand the behavior of general classical solutions.

perturbation around AdS$^{(1)}$

From (3·12) and (3·13), we see that the behavior of $f_1^{\pm}(t)$ depends on the signature of A. For $A > 0$, recalling that A is $\mathcal{O}(\alpha')$, $f_1^{+}(t)$ grows and $f_1^{-}(t)$ damps very rapidly. On the other hand, for $A < 0$, the value in the square root in (3·12) becomes negative, and thus both $f_1^{\pm}(t)$ grow as $e^{dt/2}$ being oscillating rapidly.

perturbation around AdS$^{(2)}$

We assume $B > 0$ because, as mentioned above, AdS$^{(2)}$ exists only in this region. For $A > 0$, both of $f_2^{\pm}(t)$ grow exponentially, because $l_2^2 m_2^2 < 0$. On the other hand, for $A < 0$, $f_2^{+}(t)$ grows and $f_2^{-}(t)$ damps exponentially.

Besides, as we explained before, the solution that is of interest to us is a solution that converges to either AdS$^{(1)}$ or AdS$^{(2)}$ as $t \to \infty$, satisfying the condition that $Q(t)$ be positive for the entire region of t [see (2·7)]. After all, we can see that the classical solutions behave as in Figs. 1 and 2. The numerical calculation with the proper boundary condition at $t = +\infty$ actually exhibits these figures and shows that the branch $f_i^{-}(t)$ is selected around $Q = 1/l_i$. The result of the numerical calculation for $A > 0$ and $B > 0$ is shown in Fig. 3.

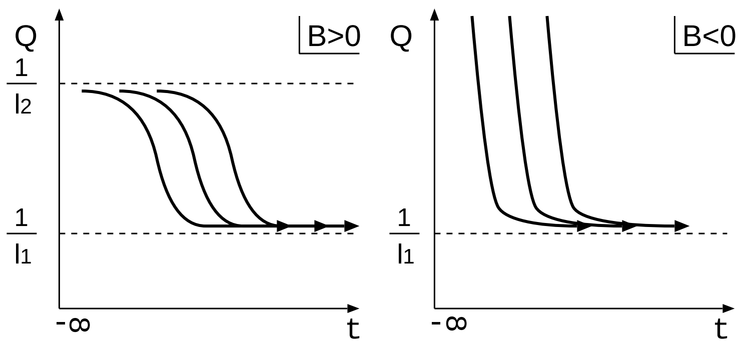

Fig. 1. Classical solutions $Q(t)$ for $A > 0$.

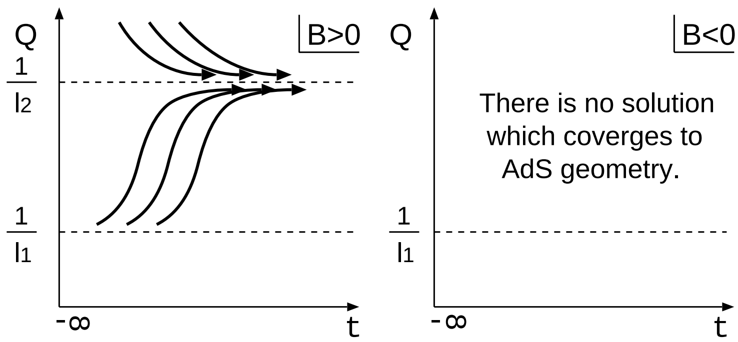

Fig. 2. Classical solutions $Q(t)$ for $A<0$.

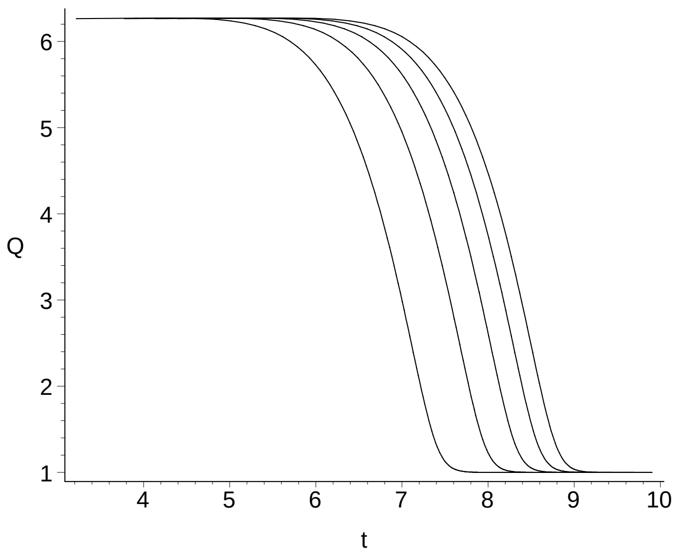

Fig. 3. Result of the numerical calculation of classical solutions $Q(t)$ for the values $d=4$, $A=0.1$, $B=0.1$ and $l=1$ ($1/l_1=1$ and $1/l_2=6.24$).

Now we give a holographic RG interpretation to the above results. We first consider the $\mathrm{AdS}^{(1)}$ solution. Equation (3·10) is the equation of motion of a scalar field in the AdS background of radius l, with mass squared given by

$$m_1^2 = -\frac{2}{A}\left(2\Lambda l^2 + \frac{3B}{l^2}\right) = \frac{2}{A}\left(d(d-1) - \frac{6B}{l^2}\right). \tag{3·14}$$

Thus for $A>0$, the higher-derivative mode Q is interpreted as a very massive scalar mode, and thus it is coupled to a highly irrelevant operator around the fixed point, since its scaling dimension is given by[2),3)]

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^4}{4} + l^2 m_1^2} \gg d. \tag{3·15}$$

This can also be understood from Fig. 1 which depicts a rapid convergence of the RG flow to the fixed point $Q(t) = 1/l$. On the other hand, for $A < 0$, the mass squared of the higher-derivative mode is far below the unitary bound $-d^2/4l^2$ for a scalar mode in the AdS$^{(1)}$ geometry,[3] and the scaling dimension becomes complex. Thus, in this case, the higher-derivative mode makes the AdS$^{(1)}$ geometry unstable, and a holographic RG interpretation cannot be given to such a solution.

We next consider the AdS$^{(2)}$. For $A > 0$ and $B > 0$ in Fig. 1, it can be seen that classical trajectories begin from AdS$^{(2)}$ to AdS$^{(1)}$. In the context of the holographic RG, this means that the AdS$^{(2)}$ solution $Q(t) = 1/l_2$ corresponds to a multicritical point in the phase diagram of the boundary field theory. From (3·8) and (3·11), the mass squared of the mode Q around the AdS$^{(2)}$ can be calculated as

$$m_2^2 = -\frac{2}{A}\left(d(d-1) - \frac{6B}{l^2}\right), \tag{3·16}$$

and if this mass squared is above the unitarity bound,

$$l_2^2 m_2^2 = -\frac{6B}{A}\frac{d(d-1)l^2 - 6B}{d(d-1)l^2 - 3B} > -\frac{d^2}{4}, \tag{3·17}$$

the scaling dimension of the corresponding operator is given by

$$\begin{aligned} \varDelta &= \frac{d}{2} + \sqrt{\frac{d^2}{4} + l_2^2 m_2^2} \\ &\cong \frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{6B}{A}}. \end{aligned} \tag{3·18}$$

For example, we consider the case in which $d = 4$, $a = b = 0$ and $c > 0$.*) In this case, $A = 32c > 0$ and $B = 8c/3 > 0$, and thus the scaling dimension of Q around the AdS$^{(2)}$ is $\varDelta \cong 2 + \sqrt{7/2}$. It would be interesting to investigate which conformal field theory describes this fixed point.

We conclude this section with a comment on the c-theorem. In the block spin gauge, the function $Q^{1-d}(t)$ can be regarded as the c-function of the d-dimensional field theory.[8] Figure 1 shows that it increases when $A > 0$, but this does not contradict the assertion of the c-theorem, because in this case, the kinetic term of $Q(t)$ in the bulk action has a negative sign [see (3·4)].

§4. Scalar field with higher-derivative interaction in Einstein gravity

In this section, we consider a scalar field with a higher-derivative interaction in Einstein gravity.

To simplify the discussion below, we consider the action

$$\boldsymbol{S} = \int_{M_{d+1}} d^{d+1}X\sqrt{\widehat{g}}\left[V(\phi) - \widehat{R} + \frac{1}{2}\widehat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{c}{2}\left(\widehat{\nabla}^2\phi\right)^2\right] + 2\int_{\Sigma_d} d^d x K, \tag{4·1}$$

*) This includes IIB supergravity on $AdS_5 \times S^5/\boldsymbol{Z}_2$ which is AdS/CFT dual to $\mathcal{N} = 2$ $USp(N)$ SYM$_4$.[30),26)]

where $\widehat{\nabla}$ is the covariant derivative defined by $\widehat{g}_{\mu\nu}$, and c is a given small constant of the order of α'. Substituting the block spin gauge metric (2·5) into (4·1), $\boldsymbol{S}$ becomes

$$\begin{aligned}\boldsymbol{S} &= \mathcal{V}_d \int_{t_0} dt L(\phi, \dot{\phi}, \ddot{\phi}; Q) \\ &= \mathcal{V}_d \int_{t_0} dt e^{-dt} \left\{ \frac{1}{Q} V(\phi) - d(d-1)Q + \frac{Q}{2}\dot{\phi}^2 + \frac{c}{2} e^{2dt} Q \left[\left(e^{-dt} Q \dot{\phi} \right)^{\cdot} \right]^2 \right\}. \end{aligned} \tag{4·2}$$

As the Lagrangian contains $\ddot{\phi}$, it is convenient to treat this system in the Hamilton formalism.[28)] Following the procedure given in Appendix A, we introduce a Lagrange multiplier π and rewrite the action in the following equivalent form

$$\boldsymbol{S} = \mathcal{V}_d \int_{t_0} dt \left[\pi \left(\dot{\phi} - e^{dt} \frac{\Phi}{Q} \right) + e^{-dt} \left(\frac{1}{Q} V(\phi) - d(d-1)Q + \frac{Q}{2} \dot{\phi}^2 \right) + \frac{c}{2} e^{dt} Q \dot{\Phi}^2 \right]. \tag{4·3}$$

Then, making the Legendre transformation from $\dot{\Phi}$ to the conjugate momentum

$$\Pi = c\, e^{dt} Q \dot{\Phi}, \tag{4·4}$$

we further rewrite the action into the first order form:

$$\boldsymbol{S} = \mathcal{V}_d \int_{t_0} dt \left[\pi\dot{\phi} + \Pi\dot{\Phi} - H(\phi, \Phi, \pi, \Pi; Q) \right], \tag{4·5}$$

where

$$H(\phi, \Phi, \pi, \Pi; Q) = d(d-1)e^{-dt} Q + \frac{1}{Q} \left[\frac{e^{-dt}}{2c} \Pi^2 + e^{dt} \pi \Phi - e^{-dt} V(\phi) - \frac{e^{dt}}{2} \Phi^2 \right]. \tag{4·6}$$

In (4·5), Q appears without a time derivative, and thus it can be easily solved. We obtain

$$Q^2(\phi, \Phi, \pi, \Pi) = \frac{1}{d(d-1)} \left[\frac{1}{2c} \Pi^2 - V(\phi) + e^{2dt} \left(\pi \Phi - \frac{1}{2} \Phi^2 \right) \right], \tag{4·7}$$

and substituting this into the Hamiltonian (4·6), we obtain the final form of the Hamiltonian:

$$H(\phi, \Phi, \pi, \Pi) = 2d(d-1)e^{-dt} Q(\phi, \Phi, \pi, \Pi). \tag{4·8}$$

Hamilton's equations are given by

$$Q\dot{\phi} = e^{dt}\Phi, \quad Q\dot{\Phi} = \frac{e^{-dt}}{c}\Pi, \quad Q\dot{\pi} = e^{-dt} V'(\phi), \quad Q\dot{\Pi} = e^{dt}\left(\Phi - \pi\right). \tag{4·9}$$

As in the pure gravity case, we first look for the AdS solution, which is given by $Q =$ const. If we set

$$V(\phi) \equiv -\frac{d(d-1)}{l^2} + \frac{\mu^2}{2}\phi^2, \tag{4·10}$$

the AdS solution that satisfies (4·7) and (4·9) is given by

$$Q=\frac{1}{l},\quad \phi=\Phi=\pi=\Pi=0. \tag{4·11}$$

We then expand Hamilton's equations (4·9) around the AdS solution (4·11) up to first order in the variables:

$$\frac{1}{l}\dot{\phi}=e^{dt}\Phi,\quad \frac{1}{l}\dot{\Phi}=\frac{e^{-dt}}{c}\Pi,\quad \frac{1}{l}\dot{\pi}=e^{-dt}\mu^2\phi,\quad \frac{1}{l}\dot{\Pi}=e^{dt}\left(\Phi-\pi\right). \tag{4·12}$$

These can be easily solved by performing the canonical transformation[28)]

$$\begin{pmatrix}\phi\\ \Phi\\ \pi\\ \Pi\end{pmatrix}=a_1\begin{pmatrix}1 & 0 & 0 & e^{dt}/M\\ 0 & e^{-dt}M & 1 & 0\\ 0 & e^{-dt}cm^2M & cM^2 & 0\\ cm^2 & 0 & 0 & e^{dt}cM\end{pmatrix}\begin{pmatrix}\widetilde{\phi}\\ \widetilde{\Phi}\\ \widetilde{\pi}\\ \widetilde{\Pi}\end{pmatrix}, \tag{4·13}$$

with

$$a_1^2\equiv\frac{1}{\sqrt{1-4c\mu^2}},\quad M^2\equiv\frac{1}{2c}\left(1+\sqrt{1-4c\mu^2}\right),\quad m^2\equiv\frac{1}{2c}\left(1-\sqrt{1-4c\mu^2}\right). \tag{4·14}$$

Then, the linearized Hamilton's equations (4·12) are decomposed into two sets of independent equations,

$$\begin{cases}\dot{\widetilde{\phi}} & =le^{dt}\widetilde{\pi},\\ \dot{\widetilde{\pi}} & =-lm^2e^{-dt}\widetilde{\phi},\end{cases}\qquad \begin{cases}\dot{\widetilde{\Phi}} & =le^{dt}\widetilde{\Pi},\\ \dot{\widetilde{\Pi}} & =-lM^2e^{-dt}\widetilde{\Phi},\end{cases} \tag{4·15}$$

which are equivalent to

$$\ddot{\widetilde{\phi}}-d\dot{\widetilde{\phi}}-l^2m^2\widetilde{\phi}=0, \tag{4·16}$$

$$\ddot{\widetilde{\Phi}}-d\dot{\widetilde{\Phi}}-l^2M^2\widetilde{\Phi}=0, \tag{4·17}$$

respectively.[*)] These are the equations of motion of two scalar fields with mass squared m^2 and M^2, respectively, in the AdS background $Q=1/l$. In particular, $\widetilde{\Phi}$ acquires a large mass when $c>0$, since its mass squared M^2 becomes $\sim 1/c\sim 1/\alpha'\gg m^2$. Thus, the bulk scalar field $\widetilde{\Phi}$ is coupled to a highly irrelevant operator at the boundary. If we assume that $\widetilde{\phi}$ is a relevant coupling, i.e. $-d^2/4l^2<m^2<0$, then the RG flow near the fixed point, $\phi=\Phi=0$, converges rapidly to the renormalized trajectory given by $\widetilde{\phi}=0$ [see Fig. 4]. On the other hand, when $c<0$, the mass squared of the scalar mode $\widetilde{\Phi}$ is far below the unitarity bound, and thus the AdS geometry becomes unstable. In this case, as in the pure gravity case with $A<0, B<0$, the holographic RG interpretation of the higher-derivative system is not possible.

*) When we add the higher-derivative term $\left(\widehat{\nabla}^2\phi\right)^2$ to the action, the scalar mode is not ϕ but $\widetilde{\phi}$, thus the mass of the observable field is not μ but m.

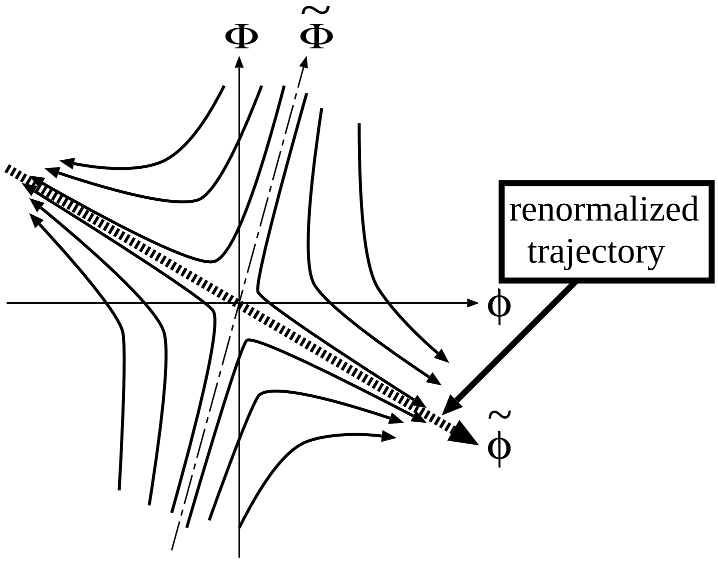

Fig. 4. The RG flow of the coupling constants $(\phi, \varPhi)$ near the fixed point $\phi = 0$, $\varPhi = 0$.

§5. Application to a model of universe with positive cosmological constant

In this section, we apply our analysis of higher-derivative pure gravity to systems of Lorentzian gravity with positive cosmological constant. There, as classical solutions, one can have de Sitter solutions instead of AdS solutions. We shall see that, in a certain region of values of coefficients of the higher-derivative terms, there are two de Sitter solutions as well as a kink solution which interpolates between these two de Sitter geometries.

We consider the following action of higher-derivative pure gravity in a $(d+1)$-dimensional Lorentzian manifold:

$$\boldsymbol{S} = \int d^{d+1}X\sqrt{-\widehat{g}}\Big(-2\varLambda + \widehat{R} - a\widehat{R}^2 - b\widehat{R}^2_{\mu\nu} - c\widehat{R}^2_{\mu\nu\rho\sigma}\Big). \tag{5·1}$$

Our analysis is completely parallel to that given in §3. We use the block spin gauge metric

$$ds^2 = -\frac{1}{Q^2}dt^2 + e^{2t}\delta_{ij}dx^i dx^j, \tag{5·2}$$

where we have flipped the sign of the exponent to describe the expanding universe. If $Q = 1/l$ =const, (5·2) represents a de Sitter space-time of radius l. With the metric (5·2), the action (5·1) becomes

$$\boldsymbol{S} = \mathcal{V}_d \int dt e^{dt}\left[-\frac{2\varLambda}{Q} - d(d-1)Q - \frac{A}{2}Q\dot{Q}^2 + BQ^3\right], \tag{5·3}$$

where A and B are again given by (3·5). This action gives the equation of motion for Q

$$Q\ddot{Q} + \frac{1}{2}\dot{Q}^2 + dQ\dot{Q} = -\frac{1}{A}\left(\frac{2\Lambda}{Q^2} - d(d-1) + 3BQ^2\right), \tag{5·4}$$

which is identical to (3·6) if we make the change $\Lambda \to -\Lambda$ and $t \to -t$. By parametrizing the cosmological constant as

$$\Lambda = \frac{d(d-1)}{2l^2} - \frac{3B}{l^4}, \tag{5·5}$$

the de Sitter solutions are obtained from (5·4) as

$$Q^2 = \begin{cases} \dfrac{1}{l^2} & \equiv \dfrac{1}{l_1^2}\,, \\ \dfrac{d(d-1)}{3B} - \dfrac{1}{l^2} & \equiv \dfrac{1}{l_2^2}\,, \end{cases} \tag{5·6}$$

where the solution $Q = 1/l_2$ exists only when $B > 0$. We call the solution $Q = 1/l_i$ the $\mathrm{dS}^{(i)}$ $(i = 1, 2)$ solution.

As we did in §3, we next examine the perturbation of solutions around these de Sitter solutions.*) By writing $Q(t)$ as

$$Q(t) = \frac{1}{l_i} + X_i(t), \tag{5·7}$$

the equation of motion (5·4) is linearized as

$$\ddot{X}_i + d\dot{X}_i - \lambda_i X_i = 0, \tag{5·8}$$

where

$$\begin{cases} \lambda_1 &= \dfrac{2}{A}\left(d(d-1)l^2 - 6B\right)\,, \\ \lambda_2 &= -\dfrac{6B}{A}\cdot\dfrac{d(d-1)l^2 - 6B}{d(d-1)l^2 - 3B}\,. \end{cases} \tag{5·9}$$

This equation is actually the time reversal of the linearized equation in the AdS case [see (3·10) and (3·13)], and thus we readily find from Figs. 1 and 2 that the general classical solutions behave as in Figs. 5 and 6.**) Note that we now can have a meaningful solution when $A < 0$, $B < 0$, since we no longer need to restrict our consideration to systems with finite classical action.

The interesting case is that of $B > 0$. Then there is a solution that describes the time evolution of space-time from one de Sitter geometry to another de Sitter

*) Discussion on the stability around de Sitter solutions in higher-derivative gravity was first given in Ref. 32). (See also Ref. 33).)

**) Actually, there exist solutions that converge to the unstable de Sitter geometry. However, we ignored them in Figs. 5 and 6 because such solutions form a measure-zero subspace in the space of classical solutions.

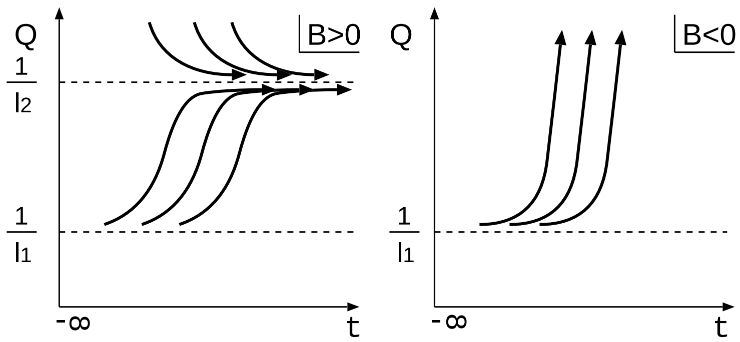

Fig. 5. Classical solutions $Q(t)$ for $A > 0$. The $\mathrm{dS}^{(1)}$ solution is unstable, and the space-time converges to the $\mathrm{dS}^{(2)}$ geometry, if it exists.

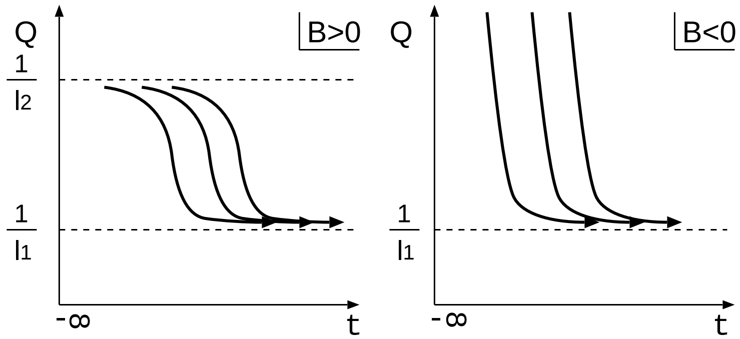

Fig. 6. Classical solutions $Q(t)$ for $A < 0$. The $\mathrm{dS}^{(1)}$ solution is stable in this case, and thus the space-time converges to the $\mathrm{dS}^{(1)}$ geometry. If the solution $\mathrm{dS}^{(2)}$ exists, there are solutions that describe the time evolution from $\mathrm{dS}^{(2)}$ to $\mathrm{dS}^{(1)}$.

geometry. Since the Hubble parameter is defined by $H(\tau) = \dot{R}(\tau)/R(\tau)$ for the metric $ds_{d+1}^2 = -d\tau^2 + R^2(\tau)ds_d^2$, we understand that the higher-derivative mode Q is the Hubble parameter:

$$H(\tau) = Q(t(\tau)). \tag{5·10}$$

Thus, the solutions for $B > 0$ in Figs. 5 and 6 describe a universe in which the Hubble parameter changes rapidly from one constant to another constant. Since we are assuming that the coefficients of the curvature squared terms are of the string scale, the difference between the two Hubble constants is extremely large. Such solutions always exist in all dimensionalities but four ($d = 3$), because $B \neq 0$ when $d \neq 3$. The absence of such solutions in four-dimensional space-time might be remedied by coupling an extra matter field to gravity.

§6. Conclusion

In this paper, we have investigated higher-derivative gravity systems. We introduced the block spin gauge (2·5) in which time evolution can be regarded directly as the change of the unit length in the d-dimensional time slice. We considered (1) higher-derivative pure gravity and (2) a scalar field with a higher-derivative interaction in Einstein gravity. We examined classical solutions in the block spin gauge and gave a holographic RG interpretation to the higher-derivative modes.

We showed the existence of AdS solutions for both systems (1) and (2), and discussed their stability. Under the condition that the bulk fields be regular in the region far from the boundary, we found that the stability of the AdS solutions depends on the values of the coefficients of higher-derivative terms. In the region of stable AdS, the higher-derivative mode can be interpreted as a very massive scalar field in the AdS background. Thus, in the context of the holographic RG, this mode is coupled to a highly irrelevant operator at the boundary. On the other hand, in the region of unstable AdS, the higher-derivative mode acquires a large negative mass squared that is far below the unitarity bound in AdS gravity. In this case, it is difficult to give a holographic RG interpretation.

For higher-derivative pure gravity, in particular, there is a region in which one can have two AdS solutions. In that region, one can also have a kink solution that describes a flow from one AdS geometry to another AdS geometry. (This is the case of $B > 0$ in Figs. 1 and 2.) In particular, for $A > 0$ and $B > 0$, the flow starts from the AdS geometry of much smaller radius (of the string scale). This describes an RG flow from a non-trivial multicritical point to another fixed point, the latter of which governs the universality class described by pure Einstein gravity. The appearance of such multicritical point is characteristic of the holographic RG for an R^2-gravity system.

As an application of our analysis, we investigated $(d+1)$-dimensional Lorentzian higher-derivative gravity with a positive cosmological constant. We found that there is a solution that describes the time evolution from one de Sitter geometry to another de Sitter geometry in a certain region of values of the coefficients of the curvature squared terms. Along this solution, the value of the Hubble parameter changes greatly.

Finally, we comment on the meaning of the mixed boundary conditions that were used in Ref. 28) (see also Appendix A below). As mentioned above, the higher-derivative mode near the stable AdS solution is coupled to a highly irrelevant operator at the boundary, and thus the RG flow around the corresponding fixed point converges rapidly to the renormalized trajectory on which the higher-derivative mode does not flow. We see below that one can actually pick up the renormalized trajectory by adopting the mixed boundary conditions.

In the case of pure gravity, the fixed point is given by the solution[*)]

$$Q = \frac{1}{l}. \tag{6·1}$$

On the other hand, from the Lagrangian (3·4), the conjugate momentum for Q is calculated as

$$P = -Ae^{-dt}\dot{Q} + \Big[4d\big(d(d+1)a + db + 2c\big) + 3d\big(d^2x_3 + dx_4 + x_5\big)\Big]Q^2. \tag{6·2}$$

*) We consider only the case in which the $\mathrm{AdS}^{(1)}$ is stable. In the presence of a scalar field that describes a relevant coupling, this solution corresponds to the renormalized trajectory.

Thus, the fixed point (6·1) can be picked up from the equation $P=0$ if we set the coefficients as in Ref. 28):

$$d^2x_3 + dx_4 + x_5 = -\frac{4}{3}\big(d(d+1)a + db + 2c\big). \tag{6·3}$$

In other words, using of the freedom to add total derivative terms to the action, the coefficients can be chosen such that the equation $P=0$ directly gives the fixed point. Note that the total derivative terms can be interpreted as the generating function of a canonical transformation that shifts the value of the conjugate momentum.

The situation does not change for a scalar field coupled to Einstein gravity with a higher-derivative interaction. When $\widetilde{\phi}$ is a relevant coupling, the renormalized trajectory is given by $\widetilde{\Phi}=0$, which is equivalent to $\widetilde{\Pi}=0$. On the other hand, from the canonical transformation (4·13), $\widetilde{\Pi}$ is expressed as

$$\widetilde{\Pi} = \sqrt{1-4c\mu^2}\left(\Pi - cm^2\phi\right). \tag{6·4}$$

Thus, if we add the term

$$\frac{d}{dt}F(\phi,\Phi) \equiv \frac{d}{dt}\left(cm^2\phi\Phi\right) \tag{6·5}$$

to the Lagrangian (4·2) (or equivalently $cm^2\widehat{\nabla}^\mu(\phi\,\partial_\mu\phi)$ to the Lagrangian density), we can shift the conjugate momenta as

$$\pi \to \pi + cm^2\Phi, \quad \Pi \to \Pi + cm^2\phi, \tag{6·6}$$

so that we have $\widetilde{\pi} \propto \pi$ and $\widetilde{\Pi} \propto \Pi$. This enables us to pick up the renormalized trajectory with the mixed boundary conditions ($\Pi = 0$).

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Appendix A
—— *General Theory of Higher-Derivative Systems* ——

In this appendix, we give a brief review of classical mechanics in higher-derivative systems with the action

$$\boldsymbol{S}[q] = \int_{t_0}^{t_1} dt\, L(q,\dot{q},\ddot{q}). \tag{A·1}$$

The variational principle gives the Euler-Lagrange equation

$$0 = \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q}. \tag{A·2}$$

This system can also be investigated in the Hamilton formalism. We first introduce a Lagrange multiplier p to treat $\dot{q}$ as a new canonical variable Q:

$$L\left(q,\dot{q},\ddot{q}\right) \to p\left(\dot{q}-Q\right) + L\left(q,Q,\dot{Q}\right). \tag{A·3}$$

We call Q the *higher-derivative mode*. Then, by making the Legendre transformation from $\dot{Q}$ to the conjugate momentum $P \equiv \partial L/\partial \dot{Q}$, this action can be rewritten into the first-order form

$$\boldsymbol{S}[q,Q,p,P] = \int_{t_0}^{t_1} dt\,\left[p\,\dot{q} + P\dot{Q} - H(q,Q;p,P)\right], \tag{A·4}$$

with the Hamiltonian

$$H(q,Q;q,P) \equiv p\,Q + P\cdot f(q,Q,P) - L(q,Q,f(q,Q,P)). \tag{A·5}$$

Here $f(q,Q,P)$ in (A·5) is obtained by solving $P = \partial L(q,Q,f)/\partial f = P(q,Q,f)$ in f. Again by the variational principle, we obtain Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{Q} = \frac{\partial H}{\partial P}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{P} = -\frac{\partial H}{\partial Q}, \tag{A·6}$$

together with the boundary conditions

$$p\,\delta q + P\delta Q = 0. \quad (t = t_0, t_1) \tag{A·7}$$

One can easily check that Hamilton's equations in (A·6) are equivalent to the Euler-Lagrange equation (A·2).

The boundary condition (A·7) is satisfied by the Dirichlet boundary conditions

$$\delta q = 0\,, \quad \delta Q = 0 \quad (t = t_0, t_1) \tag{A·8}$$

or the Neumann boundary conditions

$$p = 0\,, \quad P = 0, \quad (t = t_0, t_1) \tag{A·9}$$

for each variable q and Q. A choice of interest for us is the mixed boundary conditions, in which we set the Dirichlet conditions for q ($q(t_0) = q_0$ and $q(t_1) = q_1$) and the Neumann conditions for Q ($P(t_0) = P(t_1) = 0$). Then, if we substitute such classical solution into the bulk action $\boldsymbol{S}$, the resulting classical action is a function only of the boundary values of the light mode q; $\boldsymbol{S}[q(t),Q(t),p(t),P(t)] \equiv S(q_0,t_0;q_1,t_1)$.

In Ref. 28), the mixed boundary conditions were adopted to establish the holographic principle in higher-derivative gravity systems. In fact, if we set the mixed boundary conditions for a bulk field $\phi(x,t)$ as $\phi(x,t=t_a) = \phi_a(x)$ and $\Pi(x,t=t_a) = 0$ $(a=0,1)$,*) and carefully choose $\phi_1(x)$ such that the classical action is finite in the limit $t_1 \to +\infty$, then the classical action becomes a functional only of $\phi_0(x)$ and t_0, $S[\phi_0(x),t_0]$. This may be interpreted as the fixed-point action with the bare coupling ϕ_0 at the UV cutoff $\Lambda = \exp(-t_0)$, in the presence of an irrelevant operator corresponding to the higher-derivative mode of ϕ. In other words, the classical solution under the mixed boundary conditions may describe the RG flow of the coupling constant along the renormalized trajectory. The main text of the present paper supports this idea.

*) Π is the conjugate momentum of the higher-derivative mode Φ $(\sim \dot{\phi})$.

Appendix B
—— Higher-Derivative Pure Gravity without Gauge Fixing ——

In this appendix, we verify that $Q(t)$ in the block spin gauge metric is actually the higher-derivative mode in the sense given in Appendix A. We give a discussion by explicitly solving the equation of motion of (3·1) without assuming any particular form for the variables appearing in the metric (2·2).

Substituting (2·2) into (3·1), we obtain the Lagrangian of this system,*)

$$\begin{aligned} &L\left(q, \frac{\dot{q}}{N}, \left(\frac{\dot{q}}{N}\right)^{\cdot}; N\right) \\ &\quad = N e^{-dq(\tau)} \left\{ 2\Lambda - d(d-1)\left(\frac{\dot{q}(\tau)}{N}\right)^2 - \frac{A}{2N^2}\left[\left(\frac{\dot{q}(\tau)}{N}\right)^{\cdot}\right]^2 + B\left(\frac{\dot{q}(\tau)}{N}\right)^4 \right\}, \end{aligned} \tag{B·1}$$

where $\cdot \equiv d/d\tau$. Following the approach of Appendix A, we introduce a Lagrange multiplier p to set

$$\widetilde{Q}(\tau) = \frac{\dot{q}(\tau)}{N}. \tag{B·2}$$

The Lagrangian then becomes

$$L = p\left(\dot{q} - N\widetilde{Q}\right) - \frac{A}{2N} e^{-dq} \dot{\widetilde{Q}}^2 + N e^{-dq}\left(2\Lambda - d(d-1)\widetilde{Q}^2 + B\widetilde{Q}^4\right). \tag{B·3}$$

Since N is not dynamical, its classical value is easily found to be

$$N = \sqrt{\frac{A\dot{\widetilde{Q}}^2}{2p\widetilde{Q}e^{dq} - 2\left(2\Lambda - d(d-1)\widetilde{Q}^2 + B\widetilde{Q}^4\right)}}. \tag{B·4}$$

Substituting this into the Lagrangian, we obtain the action for this system:

$$\boldsymbol{S} = \int_{\tau_0} d\tau \left\{ p\dot{q} + 2\sqrt{\frac{A}{2} e^{-dq} \dot{\widetilde{Q}}^2 \left[p\widetilde{Q} - e^{-dq}\left(2\Lambda - d(d-1)\widetilde{Q}^2 + B\widetilde{Q}^4\right)\right]} \right\}. \tag{B·5}$$

Now we impose the condition (2·4) on $q(\tau)$, which allows us to change the integration variable from τ to q, giving

$$\boldsymbol{S} = \int_{q_0} dq \left\{ p + 2\sqrt{\frac{A}{2} e^{-dq} \dot{Q}^2 \left[pQ - e^{-dq}\left(2\Lambda - d(d-1)Q^2 + BQ^4\right)\right]} \right\}, \tag{B·6}$$

*) Here we ignore the boundary terms, because they do not affect the equation of motion.

where

$$Q(q) \equiv \widetilde{Q}(\tau(q)), \tag{B·7}$$

and $\cdot$ is now understood to represent d/dq. The action (B·6) can be further simplified by substituting the classical value of p, and we finally obtain the action

$$\boldsymbol{S} = \int_{q_0} dq\, e^{-dq}\left(-\frac{A}{2}\,Q\,\dot{Q}^2 + \frac{2\Lambda}{Q} - d(d-1)Q + BQ^3\right). \tag{B·8}$$

This is identical to the action (3·4) in the block spin gauge if we rewrite q as t. Thus we can conclude that $Q(t)$ in the block spin gauge metric (2·5) corresponds to the higher-derivative mode introduced in Appendix A, and it is related to the variable q in the temporal gauge ($N = 1$) by

$$Q(t) = \left.\frac{dq(\tau)}{d\tau}\right|_{\tau=q^{-1}(t)}. \tag{B·9}$$

Using the same procedure, we can also derive (4·8) from the temporal gauge metric (2·3) under the condition (2·4).

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