

Title	Universal description of viscoelasticity with foliation preserving diffeomorphisms
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Citation	Physics Letters B (2009), 681(3): 290-295
Issue Date	2009-11-02
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Type	Journal Article
Textversion	publisher



Universal description of viscoelasticity with foliation preserving diffeomorphisms

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ARTICLE INFO

Article history:

Received 9 September 2009

Accepted 9 October 2009

Available online 14 October 2009

Editor: T. Yanagida

PACS:

47.10.-g

47.10.A-

11.10.-z

ABSTRACT

A universal description is proposed for generic viscoelastic systems with a single relaxation time. Foliation preserving diffeomorphisms are introduced as an underlying symmetry which naturally interpolates between the two extreme limits of elasticity and fluidity. The symmetry is found to be powerful enough to determine the dynamics in the first order of strains.

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1. Introduction

Viscoelasticity is a notion that unifies solids and fluids (see [1] for example), and is applied to a wide range of materials as in rheology [2]. Viscoelastic materials behave as solids during short intervals of time, while they do as ordinary viscous fluids at long time scales [3,4].

In order to get a concrete image for viscoelastic materials, let us imagine something like a chewing gum. We can see many particles bonding in it to each other (see Fig. 1). When the material is stressed, the bonds produce an elastic force and try to make the particles back to the original configuration. That is, the system exhibits elasticity during short intervals of time. However, if we keep the deformation for a long time, then the bonding structure changes to reduce the free energy as in Fig. 1, and the shear stress vanishes. We further assume that the material is elastic for compressions even for long time scales, so that the bulk stress does not undergo such relaxation (see Fig. 2). Thus the system exhibits fluidity at long intervals of time.

In this Letter, we propose a framework in which the characteristics of elasticity and fluidity can be both dealt with on equal footings. We introduce two key ingredients. One is a new variable (to be called the intrinsic metric) which represents the bonding structures. The other is the gauge symmetry of foliation preserving

diffeomorphisms (FPDs),¹ which we find to interpolate between the two extreme limits of elasticity and fluidity in a natural manner. We show that requiring the invariance under FPDs uniquely determines the dynamics of viscoelastic systems in the first order of strains.

2. Geometrical setup

For an elastic material, a sufficiently small region around any point can always be regarded as being deformed from the configuration with no strains. We call the shape before the deformation the natural shape (see Fig. 3). This is a straightforward generalization of the notion of the “natural length” for a spring or a rubber string. For a viscoelastic material, we assume that it exhibits the elasticity during short time intervals, so that we can define the natural shape around a given point at each moment. Since the natural shape is constant in time for elastic materials, its time dependence represents the plasticity (i.e. non-elasticity) of the material. In order to describe the natural shape quantitatively, we introduce a new dynamical variable to be called the intrinsic metric, and discuss its properties in this section.

At each time t , we introduce coordinates $\xi = (\xi^a)$ ($a = 1, 2, 3$) arbitrarily on the material.² We then define the intrinsic metric of the material (denoted by $g_{ab}(\xi, t)$) such that the distance between

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¹ An interesting application to quantum gravity was recently found by Hořava [5]. For mathematical details of FPDs, see also [6].

² Here we assume that the coordinates ξ move smoothly on the material as the time t varies.

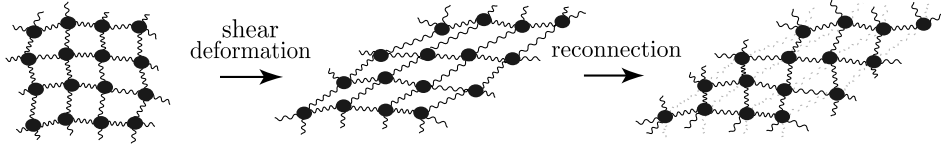


Fig. 1. Shear deformation of viscoelastic material. If we keep the deformation for a long time, the bonds are reconnected and then the stress due to the deformation vanishes.

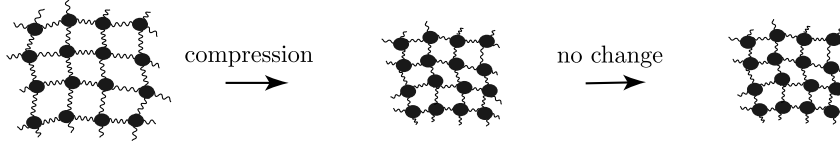


Fig. 2. Compression of viscoelastic material. The material we consider here is elastic for compressions even for long time scales, and thus the bulk stress does not undergo relaxation.

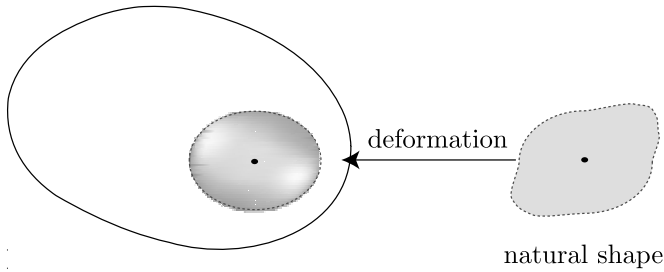


Fig. 3. A sufficiently small region around any point can always be regarded as being deformed from the natural shape.

two points at ξ and $\xi + d\xi$ at fixed time t is given by the distance in the natural shape (see Fig. 4):

$$ds^2 = g_{ab}(\xi, t) d\xi^a d\xi^b \equiv (\text{length in the natural shape})^2. \quad (2.1)$$

We let $X^i(\xi, t)$ be the spatial Cartesian coordinates of the point with ξ at t . We emphasize that the intrinsic metric $g_{ab}(\xi, t)$ is independent of $X^i(\xi, t)$, and can differ from the induced metric,³

$$h_{ab}(\xi, t) \equiv \partial_a X^i(\xi, t) \partial_b X^i(\xi, t), \quad (2.2)$$

which measures the distance in the real three-dimensional space (see Fig. 4). Note that their discrepancies

$$\varepsilon_{ab}(\xi, t) \equiv \frac{1}{2}(h_{ab}(\xi, t) - g_{ab}(\xi, t)) \quad (2.3)$$

represent the strain tensor of the system. In order to see this, let us consider an elastic material. We take the coordinates ξ such that they move with the atoms and coincide with their Cartesian coordinates in the mechanical equilibrium. When the material is deformed slightly, the atom labeled with ξ has the coordinates $X^a(\xi, t) = \xi^a + u^a(\xi, t)$, where $u^a(\xi, t)$ is the displacement vector, and we have

$$h_{ab} = \delta_{ab} + \partial_a u_b + \partial_b u_a + \mathcal{O}(u^2). \quad (2.4)$$

Since the intrinsic metric keeps taking the values in the mechanical equilibrium, $g_{ab} = \delta_{ab}$, we have that $\varepsilon_{ab} = (1/2)(h_{ab} - g_{ab}) \simeq (1/2)(\partial_a u_b + \partial_b u_a)$. This shows that ε_{ab} is certainly the strain tensor for elastic systems. In the following discussions, we make the linear approximation with respect to ε_{ab} . This implies that for the quantities of $\mathcal{O}(\varepsilon_{ab})$ one can raise or lower vector indices by either of g_{ab} or h_{ab} .

We denote by ρ_0 the mass density in the absence of strains. In this Letter, we assume that ρ_0 is a constant independent of ξ and t . Then the mass contained in a volume element $d^3\xi = d\xi^1 d\xi^2 d\xi^3$ is given by $\rho_0 \sqrt{g(\xi, t)} d^3\xi$, and the mass density in the real three-dimensional space is given by

$$\rho(\xi, t) = \frac{\rho_0 \sqrt{g(\xi, t)} d^3\xi}{\sqrt{h(\xi, t)} d^3\xi} = \frac{\sqrt{g(\xi, t)}}{\sqrt{h(\xi, t)}} \rho_0. \quad (2.5)$$

As an example, let us consider a squeeze deformation of a two-dimensional viscoelastic material formed by bonding particles as in Fig. 5. The wavy lines stand for the bonds. The left figure represents the material before the deformation. Here we take ξ to be Cartesian coordinates, and then both of the induced and intrinsic metrics have the same form: $h_{ab} = g_{ab} = \delta_{ab}$. Therefore the mass density ρ is equal to ρ_0 . We assume that ξ are attached to the particles, so that they comove with the material under the deformation. The middle figure represents the material just after the deformation. The induced metric and the mass density change according to the deformation, while the intrinsic metric does not. The induced metric is evaluated as $ds^2 = (d\xi^1)^2 + (d\xi^2)^2 - 2 d\xi^1 d\xi^2 \cos(\pi/2 + \theta)$ by using the cosine formula. The right figure represents the material after a sufficiently long time. In the process of relaxation, the induced metric and the mass density are preserved, while the intrinsic metric g_{ab} becomes proportional to h_{ab} , $g_{ab} = f(\xi, t) h_{ab}$, as will be discussed around (4.7). The factor $f(\xi, t)$ can be determined from ρ through (2.5).

3. Foliation preserving diffeomorphisms

Since we have introduced a different coordinate system ξ at each time, in order to describe the actual motion of a material one needs to specify how the fluid particles move relatively to the coordinate system ξ . This can be realized by introducing a vector field $N^a(\xi, t)$, with which the fluid particle located at ξ at time t is supposed to move to the position $\xi^a + N^a(\xi, t) \delta t$ after the time interval δt (see Fig. 6).

In fact, by using N^a , the time derivative along the path of a fluid particle (the material derivative) of a scalar quantity $\varphi(\xi, t)$ can be expressed as

$$\frac{D\varphi}{Dt} = \dot{\varphi} + N^a \partial_a \varphi. \quad (3.1)$$

For example, when measured in the real space, the velocity $v^i(\xi, t)$ and the acceleration $a^i(\xi, t)$ of the fluid particle located at ξ are given by

$$v^i = \frac{DX^i}{Dt} = \dot{X}^i + N^a \partial_a X^i, \quad (3.2)$$

³ For a function $F(\xi, t)$ we write $\dot{F} = \partial F / \partial t$ and $\partial_a F = \partial F / \partial \xi^a$.

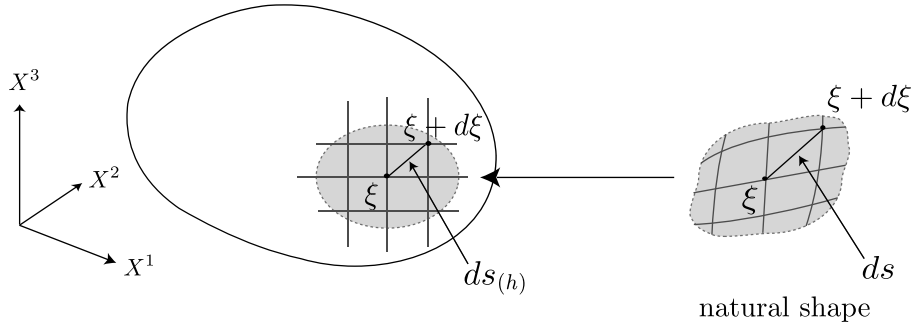


Fig. 4. The intrinsic metric is defined at each time t as the distance in the natural shape, $ds^2 = g_{ab}(\xi, t) d\xi^a d\xi^b$. The real distance is measured with the induced metric, $ds_{(h)}^2 = h_{ab}(\xi, t) d\xi^a d\xi^b$.

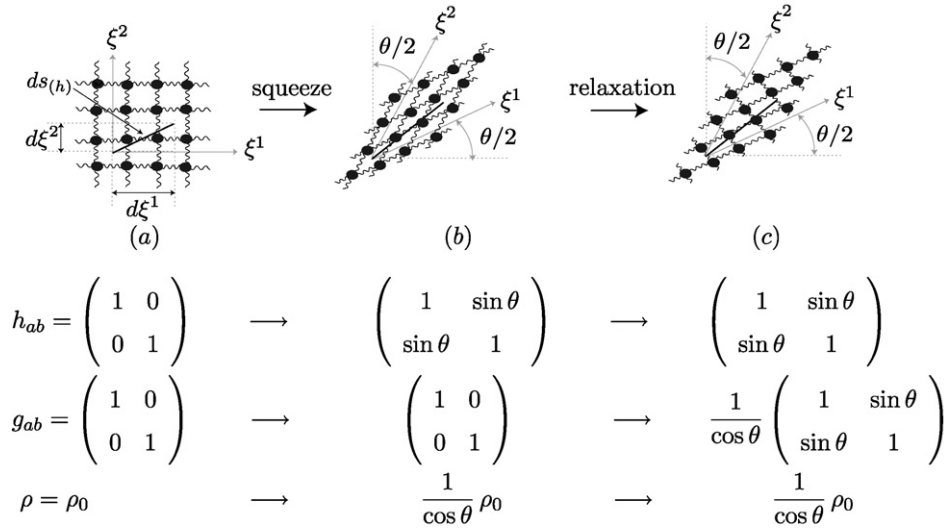


Fig. 5. Squeeze with the angle θ . This can be obtained by combining a shear deformation and a compression (plus a rotation).

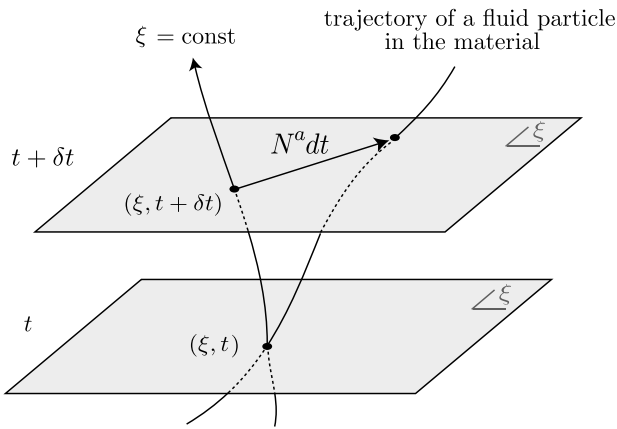


Fig. 6. Definition of N^a .

$$a^i = \frac{Dv^i}{Dt} = \dot{v}^i + N^a \partial_a v^i. \tag{3.3}$$

Then we consider the foliation preserving diffeomorphism (FPD):

$$t' = t, \quad \xi'^a = \xi'^a(\xi, t). \tag{3.4}$$

We say that a quantity $T^{a\dots b\dots}(\xi, t)$ is a covariant tensor, if it transforms as a three-dimensional tensor at each time:

$$T'^{a\dots b\dots}(\xi', t) = \frac{\partial \xi'^a}{\partial \xi^c} \dots \frac{\partial \xi^d}{\partial \xi'^b} \dots T^{c\dots d\dots}(\xi, t), \tag{3.5}$$

where the transition functions $\partial \xi'^a / \partial \xi^c$ and $\partial \xi^d / \partial \xi'^b$ should be evaluated at time t . For example, X^i , v^i and a^i are covariant scalars, and g_{ab} , h_{ab} and ε_{ab} are rank two covariant tensors.

On the other hand, N^a is not a covariant vector but transforms with an inhomogeneous term⁴:

$$N'^a(\xi', t) = \frac{\partial \xi'^a(\xi, t)}{\partial \xi^b} N^b(\xi, t) + \frac{\partial \xi'^a(\xi, t)}{\partial t}. \tag{3.6}$$

To see this, let us consider a fluid particle in two different coordinate systems, and assume that it has the coordinates ξ^a and ξ'^a , respectively, at time t . Then, by definition, at time $t + \delta t$, they become $\xi^a + N^a(\xi, t)\delta t$ and $\xi'^a + N'^a(\xi', t)\delta t$, respectively. Because these are related by the transformation (3.4) at $t + \delta t$, we have

$$\begin{aligned}
 \xi'^a + N'^a(\xi', t)\delta t &= \xi'^a(\xi + N(\xi, t)\delta t, t + \delta t) \\
 &= \xi'^a + \frac{\partial \xi'^a(\xi, t)}{\partial \xi^b} N^b(\xi, t)\delta t + \frac{\partial \xi'^a(\xi, t)}{\partial t} \delta t,
 \end{aligned} \tag{3.7}$$

and thus we obtain (3.6).

⁴ This is naturally understood, if one recognizes that N^a can be regarded as a gauge field that appears when the three-dimensional diffeomorphism is gauged in the time direction.

Note that the time derivative of a covariant tensor $T^{a\dots b\dots}$ is no longer covariant. We, however, can make a covariant tensor by considering the derivative along fluid particles as in (3.2) and (3.3):

$$\frac{D}{Dt} T^{a\dots b\dots} = \dot{T}^{a\dots b\dots} + \mathcal{L}_{\mathcal{N}} T^{a\dots b\dots}, \quad (3.8)$$

where $\mathcal{L}_{\mathcal{N}}$ is the Lie derivative with respect to the vector field $\mathcal{N} = N^a \partial_a$:

$$\begin{aligned} \mathcal{L}_{\mathcal{N}} T^{a\dots b\dots} &= N^c \partial_c T^{a\dots b\dots} - \partial_c N^a T^{c\dots b\dots} - \dots \\ &+ \partial_b N^c T^{a\dots c\dots} + \dots \end{aligned} \quad (3.9)$$

For example, from g_{ab} , we can make a covariant tensor K_{ab} to be called the extrinsic curvature,

$$K_{ab} = \frac{1}{2} \frac{Dg_{ab}}{Dt} = \frac{1}{2} (\dot{g}_{ab} + \partial_a N^c g_{cb} + \partial_b N^c g_{ac} + N^c \partial_c g_{ab}). \quad (3.10)$$

Similarly, we can define the extrinsic curvature $K_{ab}^{(h)}$ which corresponds to h_{ab} ,

$$K_{ab}^{(h)} = \frac{1}{2} \frac{Dh_{ab}}{Dt} = \frac{1}{2} (\dot{h}_{ab} + \partial_a N^c h_{cb} + \partial_b N^c h_{ac} + N^c \partial_c h_{ab}). \quad (3.11)$$

Note that

$$K_{ab}^{(h)} - K_{ab} = \frac{D\varepsilon_{ab}}{Dt}. \quad (3.12)$$

We then introduce the dreibeins $e_a^i(\xi, t)$ and their inverses $e_i^a(\xi, t)$:

$$e_a^i(\xi, t) = \partial_a X^i(\xi, t), \quad e_i^a e_b^i = \delta_b^a. \quad (3.13)$$

From (3.2) and (3.13), we obtain

$$\begin{aligned} h_{ab} &= e_a^i e_b^i, & \partial_a e_b^i &= \partial_b e_a^i, \\ \dot{e}_a^i &= \partial_a v^i - (\partial_a N^b) e_b^i - N^b \partial_b e_a^i, \end{aligned} \quad (3.14)$$

and thus

$$\frac{D e_a^i}{Dt} = \partial_a v^i. \quad (3.15)$$

By using the dreibeins, the scalars $v^i(\xi, t)$ and $a^i(\xi, t)$ can be transformed into vector fields as

$$v^a \equiv e_a^i v^i, \quad v_a \equiv e_a^i v_i, \quad a^a \equiv e_a^i a^i, \quad a_a \equiv e_a^i a_i. \quad (3.16)$$

We can calculate the acceleration a_a by using the Leibniz rule for D/Dt and (3.15) as

$$\begin{aligned} a_a &= e_a^i \frac{D v^i}{Dt} = \frac{D(e_a^i v^i)}{Dt} - v^i \frac{D e_a^i}{Dt} = \frac{D v_a}{Dt} - \partial_a \left(\frac{1}{2} (v^i)^2 \right) \\ &= \dot{v}_a + (\partial_a N^b) v_b + N^b \partial_b v_a - \partial_a \left(\frac{1}{2} v^b v_b \right). \end{aligned} \quad (3.17)$$

Furthermore, by using the first of (3.14) and (3.15), we can show

$$K_{ab}^{(h)} = \frac{1}{2} ((\partial_a v^i) e_b^i + (\partial_b v^i) e_a^i) = \frac{1}{2} (\nabla_a^{(h)} v_b + \nabla_b^{(h)} v_a). \quad (3.18)$$

The set of FPDs forms a gauge symmetry group of the system, which can be gauge-fixed arbitrarily according to convenience in describing the dynamics of a given system. Two of the useful gauge fixings are the following:

(A) Comoving frame:

We set $N^a(\xi, t) = 0$. In this frame, $X^i(\xi, t)$ describes the motion of the fluid particle attached to the coordinate ξ , and we have

$$\begin{aligned} v^i(\xi, t) &= \dot{X}^i(\xi, t), \\ a^i(\xi, t) &= \ddot{X}^i(\xi, t) \quad (\text{in the comoving frame}). \end{aligned} \quad (3.19)$$

This frame is useful for describing the dynamics of elastic materials. Note that the acceleration field (3.17) is expressed as

$$a_a = \dot{v}_a - \partial_a \left(\frac{1}{2} v^b v_b \right) \quad (\text{in the comoving frame}). \quad (3.20)$$

The last term gives the inertial force.

(B) Laboratory frame:

We set $X^a(\xi, t) \equiv \xi^a$. In this frame, we have $\dot{X}^i(\xi, t) = 0$, $e_i^a = \delta_i^a$ and $h_{ab} = \delta_{ab}$. Then, from (3.2) and (3.16) we obtain

$$v^a(\xi, t) = N^a(\xi, t) \quad (\text{in the laboratory frame}). \quad (3.21)$$

This frame is useful for describing the dynamics of fluid. Note that the acceleration field (3.17) represents the material derivative of the velocity field

$$a_a = \dot{v}_a + v^b \partial_b v_a \quad (\text{in the laboratory frame}). \quad (3.22)$$

4. Fundamental equations

We are now in a position to write down a set of equations which determine the time evolution of $X^i(\xi, t)$, $g_{ab}(\xi, t)$ and $N^a(\xi, t)$ up to FPDs. The covariance under FPDs is found to be powerful enough to uniquely determine the dynamics in the first order of ε_{ab} .

4.1. $g_{ab}(\xi, t)$

We first consider the limiting case of elasticity. Due to our definition of the intrinsic metric, $g_{ab}(\xi, t)$ should be constant in time for elastic materials in the comoving frame ($N^a = 0$). The FPD-covariant expression for this statement is that the extrinsic curvature K_{ab} of (3.10) vanishes:

$$K_{ab} = 0 \quad (\text{elastic limit}). \quad (4.1)$$

This implies that the non-vanishing K_{ab} represents the genuinely plastic deformations.

In order to make further discussions, we need to separate the trace part from K_{ab} because it vanishes for any materials due to the mass conservation of the system:

$$K(\xi, t) \equiv g^{ab}(\xi, t) K_{ab}(\xi, t) = 0. \quad (4.2)$$

In fact, contracting (3.10) with g^{ab} , we obtain

$$\begin{aligned} K &= \frac{1}{2} g^{ab} (\dot{g}_{ab} + \partial_a N^c g_{cb} + \partial_b N^c g_{ac} + N^c \partial_c g_{ab}) \\ &= \frac{1}{\sqrt{g}} ((\sqrt{g}) \cdot + \partial_a (\sqrt{g} N^a)) \\ &= \frac{1}{\rho_{\text{int}}} (\dot{\rho}_{\text{int}} + \partial_a (\rho_{\text{int}} N^a)), \end{aligned} \quad (4.3)$$

where $\rho_{\text{int}}(\xi, t) = \rho_0 \sqrt{g(\xi, t)}$ is the mass density with respect to the intrinsic metric. This indicates that the vanishing of K is equivalent to the mass conservation, which can be easily seen in the comoving frame where $N^a = 0$ and $\dot{\rho}_{\text{int}} = 0$.

The traceless part of K_{ab} on the other hand describes the rate of the shear deformation of the intrinsic metric, and thus is expected to be proportional to the traceless part of the strain tensor

$$\tilde{K}_{ab}(\xi, t) = \frac{1}{\tau} \tilde{\varepsilon}_{ab}(\xi, t), \quad (4.4)$$

where τ is the relaxation time, and

$$\tilde{K}_{ab} \equiv K_{ab} - \frac{1}{3}K g_{ab}, \quad (4.5)$$

$$\tilde{\varepsilon}_{ab} \equiv \varepsilon_{ab} - \frac{1}{3}(g^{cd}\varepsilon_{cd})g_{ab}. \quad (4.6)$$

We call (4.4) the rheology equation hereafter. After a time interval much longer than τ , $\tilde{\varepsilon}_{ab}(\xi, t)$ vanishes and thus $g_{ab}(\xi, t)$ becomes proportional to $h_{ab}(\xi, t)$:

$$g_{ab}(\xi, t) = f(\xi, t)h_{ab}(\xi, t) \quad (f(\xi, t): \text{a scalar function}), \quad (4.7)$$

where $f(\xi, t)$ is determined by the mass conservation as in the example in Section 2.

Eq. (4.4) is the simplest and is expected to be universal. It is consistent with (4.1) because the elastic limit corresponds to $\tau = \infty$. The fluid limit is also realized correctly by taking $\tau = 0$, in which we have $\tilde{\varepsilon}_{ab} = 0$. This implies that g_{ab} is proportional to h_{ab} as in (4.7). The only remaining degree of freedom of g_{ab} becomes the density of the material, which means that the system corresponds to an isotropic fluid.

Note that (4.2) and (4.4) indicate that all the components of K_{ab} are of the order ε . Then from (3.12) we find that so are those of $K_{ab}^{(h)}$, and we have

$$\tilde{K}_{ab}^{(h)} - \tilde{K}_{ab} = \frac{D\tilde{\varepsilon}_{ab}^{(h)}}{Dt} + \mathcal{O}(\varepsilon^2), \quad (4.8)$$

$$K^{(h)} - K = \frac{D\varepsilon^{(h)}}{Dt} + \mathcal{O}(\varepsilon^2). \quad (4.9)$$

4.2. $X(\xi, t)$

The dynamics of $X^i(\xi, t)$ should be expressed as Euler's equation which is written with the stress tensor T_{ab} as

$$\rho a_a = -\nabla^{(h)b} T_{ba}, \quad (4.10)$$

where $\nabla^{(h)}$ is the covariant derivative with respect to h_{ab} . We show that the leading form of the stress tensor in the derivative expansion can be determined by the following requirements:

- T_{ab} is symmetric and covariant under FPDs.
- T_{ab} is linear in the strain ε_{ab} and the spatial derivative of the velocity, $\nabla_a^{(h)} v_b$.

The above requirements imply that T_{ab} is a linear combination of the irreducible components of ε_{ab} and $K_{ab}^{(h)} = (1/2)(\nabla_a^{(h)} v_b + \nabla_b^{(h)} v_a)$ (see (3.18)):

$$T_{ab} = -2\mu\tilde{\varepsilon}_{ab}^{(h)} - \frac{1}{\kappa}\varepsilon^{(h)}h_{ab} - 2\gamma\tilde{K}_{ab}^{(h)} - \zeta K^{(h)}h_{ab}, \quad (4.11)$$

where

$$\varepsilon^{(h)} \equiv h^{ab}\varepsilon_{ab}, \quad (4.12)$$

$$\tilde{\varepsilon}_{ab}^{(h)} \equiv \varepsilon_{ab} - \frac{1}{3}\varepsilon^{(h)}h_{ab}, \quad (4.13)$$

$$K^{(h)} \equiv h^{ab}K_{ab}^{(h)}, \quad (4.14)$$

$$\tilde{K}_{ab}^{(h)} \equiv K_{ab}^{(h)} - \frac{1}{3}K^{(h)}h_{ab}. \quad (4.15)$$

From the discussions around (4.8), we find that all the terms in (4.11) are of the order of ε .

In order to see the meaning of the coefficients in (4.11), we consider two extreme limits of elasticity and fluidity. We first consider the elastic limit. Since $K_{ab} = 0$ in the elastic limit (see (4.1)), the formulas (4.8) and (4.9) lead to

$$\tilde{K}_{ab}^{(h)} = \frac{D\tilde{\varepsilon}_{ab}^{(h)}}{Dt}(1 + \mathcal{O}(\varepsilon)), \quad (4.16)$$

$$K = \frac{D\varepsilon^{(h)}}{Dt}(1 + \mathcal{O}(\varepsilon)), \quad (4.17)$$

and thus we have

$$T_{ab} \simeq -2\mu\tilde{\varepsilon}_{ab}^{(h)} - \frac{1}{\kappa}\varepsilon^{(h)}h_{ab} - 2\gamma\frac{D\tilde{\varepsilon}_{ab}^{(h)}}{Dt} - \zeta\frac{D\varepsilon^{(h)}}{Dt}h_{ab} \quad (\text{elastic limit}). \quad (4.18)$$

This indicates that the parameters μ and $1/\kappa$ are the shear and bulk moduli, respectively. The last two terms express frictions.

On the other hand, the fluid limit is realized by considering the case where the time scale T of the variation of the shear strain $\tilde{\varepsilon}_{ab}$,

$$\frac{D}{Dt}\tilde{\varepsilon}_{ab} \sim \frac{1}{T}\tilde{\varepsilon}_{ab}, \quad (4.19)$$

is much longer than the relaxation time τ , $T \gg \tau$. We then can show that

$$\tilde{K}_{ab} = \tilde{K}_{ab}^{(h)} \left(1 + \mathcal{O}\left(\frac{\tau}{T}\right)\right), \quad (4.20)$$

because the following holds due to (4.8):

$$\begin{aligned} \tilde{K}_{ab}^{(h)} - \tilde{K}_{ab} &= \frac{D\tilde{\varepsilon}_{ab}^{(h)}}{Dt}(1 + \mathcal{O}(\varepsilon)) = \frac{D\tilde{\varepsilon}_{ab}^{(h)}}{Dt}(1 + \mathcal{O}(\varepsilon)) \sim \frac{1}{T}\tilde{\varepsilon}_{ab} \\ &= \frac{\tau}{T}\tilde{K}_{ab}. \end{aligned} \quad (4.21)$$

Since $\tilde{\varepsilon}_{ab}^{(h)} = \tilde{\varepsilon}_{ab}(1 + \mathcal{O}(\varepsilon)) = \tau\tilde{K}_{ab}(1 + \mathcal{O}(\varepsilon))$, we can rewrite the stress tensor (4.11) into

$$T_{ab} \simeq -2\eta\tilde{K}_{ab}^{(h)} - \zeta K^{(h)}h_{ab} - \frac{1}{\kappa}\varepsilon^{(h)}h_{ab} \quad (\text{fluid limit}), \quad (4.22)$$

where

$$\eta \equiv \gamma + \mu\tau. \quad (4.23)$$

By using (3.18), each term in (4.22) can be interpreted in terms of fluid mechanics if we take the laboratory frame ($X^a(\xi, t) = \xi^a$, $v^a = N^a$):

$$h_{ab} = \delta_{ab}, \quad (4.24)$$

$$\tilde{K}_{ab}^{(h)} = \frac{1}{2}(\partial_a v_b + \partial_b v_a) - \frac{1}{3}\partial_c v_c \delta_{ab}, \quad (4.25)$$

$$K^{(h)} = h^{ab}K_{ab}^{(h)} = \partial_c v_c, \quad (4.26)$$

which indicates that η and ζ represent the shear and bulk viscosities of the fluid, respectively.⁵ The third term in (4.22) can be interpreted as the pressure:

$$p = -\frac{1}{\kappa}\varepsilon^{(h)}. \quad (4.27)$$

To see this, we notice that $\varepsilon^{(h)} (\simeq (\sqrt{h} - \sqrt{g})/\sqrt{g})$ measures the deviation of the real volume element from that of the natural

⁵ Eq. (4.23) shows that the shear viscosity consists of two contributions. The first term γ reflects the friction which the material already has in the elastic limit, while the second term $\mu\tau$ represents the stress caused by the strain as the material undergoes plastic deformations [4].

shape,⁶ and that the pressure vanishes in the natural shape. Therefore p must be proportional to $-\varepsilon^{(h)}$. We then find that κ agrees with the coefficient of compression because $\kappa = \rho d\rho^{-1}/dp|_{\rho=\rho_0}$ ($\rho = \rho_0\sqrt{g} \simeq \rho_0(1 - \varepsilon^{(h)})$ and $p = -\kappa^{-1}\varepsilon^{(h)}$).

The set of Eqs. (4.2), (4.4), (4.10), (4.11) and (4.23) are the fundamental equations which govern the dynamics of a given viscoelastic material. We can see that macroscopic properties of such materials are characterized only by six parameters ($\rho_0, \tau, \eta, \zeta, \kappa, \gamma$) from which another parameter μ is obtained via (4.23).

5. Conclusion and discussion

We have proposed a set of fundamental equations to describe generic viscoelastic systems in a unified way. It is expressed as a world volume theory with the target space coordinates $X^i(\xi, t)$ and the intrinsic metric $g_{ab}(\xi, t)$. FPDs play an important role in interpolating the two extreme limits of elasticity and fluidity. We have shown that the covariance under FPDs uniquely determines the form of the equations in the first order of strains. We thus conclude that the set of equations gives a universal description of viscoelastic systems.

Here we have considered viscoelastic systems with a single relaxation time. It would be interesting to consider a generalization to the case with more than one relaxation time in order to describe more realistic materials realized in laboratories.

Acknowledgements

This work was supported by the Grant-in-Aid for the Global COE program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. T.A. is also supported by the Japan Society for the Promotion of Science (JSPS) for Young Scientists (No. 20-892). M.F. and H.K. are also supported by the Grant-in-Aid for Scientific Research Nos. 19540288 and 18540264, respectively, from MEXT.

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⁶ Note that in the linear order of $\varepsilon^{(h)}$ we have

$$\frac{\sqrt{\bar{h}} - \sqrt{g}}{\sqrt{g}} = \frac{\sqrt{\det(g_{ab} + 2\varepsilon_{ab}^{(h)})} - \sqrt{g}}{\sqrt{g}} \simeq \varepsilon^{(h)}.$$