Ample canonical heights for endomorphisms on projective varieties

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Introduction

- We work over $\overline{\mathbb{Q}}$.
- An *endomorphism* means a dominant morphism from a variety to itself.
- For an endomorphism f on a smooth projective variety X, the (*first*) *dynamical degree of* f is
$$\delta_f = \lim_{n\to\infty}((f^*)^n H \cdot H^{\dim X - 1})^{\frac{1}{n}},$$
where H is an ample divisor.
- Let X be a smooth projective variety and D a divisor on X. Then h_D denotes the height function associated to D, which is determined up to the difference of a bounded function.
- For a smooth projective variety X, h_X denotes a fixed height function associated to an ample divisor with $h_X \geq 1$.

Theorem 1 (The canonical height for a polarized endomorphism, Call–Silverman [CaSi93])

Let X be a smooth projective variety and f an endomorphism on X with $f^*H \sim dH$ where H is an ample divisor and $d > 1$. Then the canonical height
$$\hat{h}_{H,f}(x) = \lim_{n\to\infty}\frac{h_H(f^n(x))}{d^n}$$
converges for every $x \in X(\overline{\mathbb{Q}})$. Moreover,
- $\hat{h}_{H,f}(x) \geq 0$ *for every* x,
- $\hat{h}_{H,f} \circ f = d\hat{h}_{H,f}$, *and*
- *(Northcott-type finiteness property)* $\{x \in X(K) \mid \hat{h}_{H,f}(x) = 0\}$ *is a finite set for any number field* K.

The canonical height is a new height function reflecting the dynamics of f. Our aim is to generalize the definition of the canonical heights to arbitrary endomorphisms.

Definition 2

Let X be a smooth projective variety and f an endomorphism on X with $\delta_f > 1$. Set
$$l_f = \min\left\{ l \in \mathbb{Z}_{\geq 0} \mid \left\{\frac{h_X(f^n(x))}{\delta_f^n n^l}\right\}_{n=0}^{\infty} \text{ is bounded for } \forall x \in X(\overline{\mathbb{Q}})\right\}.$$
The *upper*/*lower canonical heights* for f are defined as
$$\overline{h}_f(x) = \limsup_{n\to\infty}\frac{h_X(f^n(x))}{\delta_f^n n^{l_f}},$$
$$\underline{h}_f(x) = \liminf_{n\to\infty}\frac{h_X(f^n(x))}{\delta_f^n n^{l_f}}.$$

Immediately the following follows.

Proposition 3

Let X be a smooth projective variety and f an endomorphism on X with $\delta_f > 1$.

- $\overline{h}_f(x) \geq \underline{h}_f(x) \geq 0$ *for every* x *and*
- $\overline{h}_f \circ f = \delta_f \overline{h}_f$, $\underline{h}_f \circ f = \delta_f \underline{h}_f$.

Main results

Definition 4

Let X be a smooth projective variety and f an endomorphism on X. For a subfield $K \subset \overline{\mathbb{Q}}$, we set
$$Z_f(K) = \{x \in X(K) \mid \underline{h}_f(x) = 0\}.$$

When f is a polarized endomorphism, then $Z_f(K)$ is a finite set for every number field K (Northcott-type finiteness property). So we expect a finiteness property that $Z_f(K)$ is "small" for a general endomorphism f.

Conjecture 1

Let X be a smooth projective variety and f an endomorphism on X with $\delta_f > 1$. Take any number field K. Then $Z_f(K)$ is contained in a proper closed subset $V \subset X$ with $f(V) \subset V$.

We can prove Conjecture 1 for certain cases.

Theorem 5

Let X be a smooth projective variety and f an endomorphism on X with $\delta_f > 1$. Conjecture 1 holds in the following cases.
(i) $f^*H \equiv \delta_f H$ for an ample $\mathbb{R}$-divisor H on X. This contains the case when the Picard number of X is one.
(ii) $\rho(X) \leq 2$ and f is an automorphism.
(iii) X is an abelian variety which is isogenous to a product of elliptic curves and pairwise non-isogenous simple abelian varieties of dimension > 1. This includes endomorphisms on abelian varieties of dimension ≤ 3.
(iv) X is a smooth projective surface.

Sketch of proof.
(i) In this case, the ample canonical height is essentially equivalent to the canonical height due to Call–Silverman.
(ii) If $\rho(X) = 2$, we can take two nef $\mathbb{R}$-divisors $D_\pm$ which are eigenvectors of f^* in $N^1(X)_\mathbb{R}$ and the associated canonical heights $\hat{h}_{D_\pm,f}$, which help us to compute the ample canonical height.
(iii) **Step 1** Assume $X = E^r$ (E: an elliptic curve). Then $f \in \mathrm{End}(E^r)$ is represented by a $(r \times r)$-matrix in $\mathrm{End}(E)_\mathbb{Q}$: the rational number field or a imaginary quadratic field. Then we can compute the ample canonical height by the aid of the Jordan normal form of the matrix.
Step 2 Assume X is a simple abelian variety. Then it turns out that a *nef canonical height* introduced by Kawaguchi–Silverman [KaSi16a] is essentially equivalent to the ample canonical height. Moreover, the zero sets of nef canonical heights on abelian varieties were determined by Kawaguchi–Silverman [KaSi16b].
Step 3 A general f is split to a product of endomorphisms in **Step 1** or **Step 2**. Then we can prove the claim.
(iv) **Step 1** If f is an automorphism on a surface, it turns out that the ample canonical height is essentially equivalent to the the canonical height due to Kawaguchi [Kaw08].
Step 2 Any non-automorphic endomorphism on a minimal surface which is isomorphic to neither $\mathbb{P}^2$ nor abelian surfaces admits a certain fibration to a curve ([MSS17]). Then we can investigate the zero set of the ample canonical height by the aid of the fibration structure.

Applications

Theorem 6 (A dynamical Mordell–Lang type result)

Let X be a smooth projective variety and f, g endomorphisms on X such that $\delta_f = \delta_g > 1$ and $l_f = l_g$. We assume one of the following:
- $f^*H \equiv \delta_f H$ *and* $g^*H' \equiv \delta_g H'$ *for some ample $\mathbb{R}$-divisors H, H' on X,*
- $\rho(X) \leq 2$ *and f, g are automorphisms,*
- *X is an abelian variety which is isogenous to a product of elliptic curves and pairwise non-isogenous simple abelian varieties of dimension > 1, or*
- *X is a smooth projective surface.*

Take a dense f-orbit $O_f(x)$ and a dense g-orbit $O_g(y)$. Then the set $\{|n - m| \mid n, m \in \mathbb{Z}_{\geq 0},\ f^n(x) = g^m(y)\}$ is upper bounded. Furthermore, if both f and g are étale, then the set $\{(n, m) \in (\mathbb{Z}_{\geq 0})^2 \mid f^n(x) = g^m(y)\}$ is a finite union of sets of the form $\{(kn + i, kn + j)\}_{n=0}^{\infty}$ for some $k, i, j \in \mathbb{Z}_{\geq 0}$.

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