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Weyl invariance for generalized supergravity backgrounds from the doubled formalism

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It has recently been shown that a set of the generalized type IIB supergravity equations follows from the requirement of kappa symmetry of the type IIB Green–Schwarz superstring theory defined on an arbitrary background. In this paper, we show that the whole bosonic part of the generalized type II supergravity equations can be reproduced from the T -duality covariant equations of motion of the double field theory by choosing a non-standard solution of the strong constraint. Then, by using the doubled formalism, we show the Weyl invariance of the bosonic string sigma model on a generalized gravity background. According to the dual-coordinate dependence of the dilaton, the Fradkin–Tseytlin term nicely removes the Weyl anomaly. This result seems likely to support that string theories can be consistently defined on arbitrary generalized supergravity backgrounds.
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1. Introduction

A generalization of the type IIB supergravity recently proposed in Ref. [1] is a fascinating subject in string theory.¹ This generalized system includes extra vector fields as well as the standard component fields of the type IIB supergravity. The classical action has not been revealed yet, and only the equations of motion are presented. Hereafter, we will refer to them as the generalized supergravity equations of motion (GSE) for simplicity.

It is well known that the on-shell condition of type IIB supergravity ensures the kappa invariance of the Green–Schwarz string theory [5,6]. Conversely, in a recent paper [7], the GSE have been reproduced by solving the kappa-symmetry constraints, generalizing the well-known fact. So far, it is considered that type IIB string theories on generalized supergravity backgrounds would not be Weyl invariant, though still scale invariant. Therefore, it has not been clear whether string theory is consistently defined on such backgrounds.

It is worth noting that a T -duality transformation rule from a solution of the GSE to a solution of standard supergravity is given in Ref. [1]. On the other hand, one can map a solution of standard supergravity with a linear dilaton to a solution of the GSE by performing a formal T -duality transformation along a direction for which the dilaton is not isometric [8]. These results indicate that solutions of standard supergravity and the GSE should be treated on an equal footing in the context of string theory, because the T -duality is a symmetry of string theory. However, there is a puzzle for

¹ This was originally proposed to support a q -deformed $\text{AdS}_5 \times S^5$ background [2–4] as a solution.

the Weyl invariance. The Weyl invariance of string theories defined on solutions of the GSE may be broken, as discussed in Ref. [1], though string theories on solutions of standard supergravity are Weyl invariant. Due to the T -duality symmetry, it seems likely that Weyl invariance should be preserved even for the solutions of the GSE in a certain manner.²

In order to clarify the issue of Weyl invariance under T -duality transformations, it is useful to utilize the manifestly T -duality-covariant formulations of supergravity and string theory: double field theory (DFT) [9–15]³ and the double sigma model (DSM) [16–21]. In the previous work [22], a modification of the DFT was studied so as to incorporate the GSE (in the context of the DFT for the NS–NS sector).⁴ The modified DFT (mDFT) allows us to reproduce the GSE by choosing a section under which all of the fields do not depend on the dual coordinates. In this paper, we will show that the extra generalized vector X^M of the mDFT can always be removed with a redefinition of the dilaton. In this sense, the mDFT is no more than the usual DFT, and it should rather be called the (m)DFT. The point is that this redefinition introduces the dual-coordinate dependence into the dilaton (while keeping the strong constraint intact), and this dual-coordinate dependence is the origin of the modification of the supergravity equations of motion. Moreover, we extend the (m)DFT by including the R–R fields and show that the GSE are definitely reproduced from the $O(D, D)$ covariant equations of motion of the (m)DFT. We then find the relation between the R–R potentials and their strengths, the $O(D, D)$ transformation rule, and the generalized type IIA supergravity equations of motion. We also argue that the usual DFT action is nothing but the action for the GSE.

After formulating the GSE from the perspective of the DFT, we will consider the string sigma model defined on the doubled target space, namely the DSM. As is well known, the Weyl anomaly of the string sigma model is canceled if the NS–NS background satisfies the supergravity equations of motion [24]. Here, the Weyl anomaly coming from the background metric G_{mn} and the Kalb–Ramond field B_{mn} is canceled by adding a counterterm to the string action, the Fradkin–Tseytlin term [25]. For a more general case in which the background satisfies the GSE, we can no longer find an appropriate counterterm to cancel the Weyl anomaly in the usual consideration, and hence the Weyl symmetry is broken to the scale symmetry [1] (see also Refs. [26–29]). This has been the common understanding so far. In this paper, we consider the DSM defined on a general solution of the GSE, and elucidate that the Weyl anomaly can always be canceled by introducing a linear dual-coordinate dependence into the dilaton of the Fradkin–Tseytlin term.⁵ In this sense, the usual supergravity backgrounds and solutions of the GSE can be treated on an equal footing in string theory.

This paper is organized as follows. In Sect. 2, after giving a short review of the mDFT [22], we reinterpret the mDFT as the usual DFT with a modified section. In Sect. 3, we review the R–R sector of the DFT and then reproduce the generalized type IIA and IIB supergravity equations by employing the modified section. We also present the T -duality transformation rule for solutions of the generalized type II supergravities. In Sect. 4, we discuss the Weyl invariance of the string sigma model defined on a solution of the GSE. Section 5 is devoted to conclusions and discussion.

² We are grateful to A. A. Tseytlin for useful discussions on this point.

³ For earlier observations along this direction, see footnotes 1 and 23 of Ref. [8].

⁴ More recently, it was proposed in Ref. [23] that the whole bosonic sector of the GSE can be reproduced from the manifestly U -duality-covariant formulation of supergravity, the exceptional field theory, by choosing a non-standard section and considering a certain Scherk–Schwarz-type ansatz. See Sect. 5 for more details.

⁵ For an earlier argument on the recovery of Weyl invariance in the doubled formalism, see, for example, footnote 23 of Ref. [1].

2. NS–NS sector of (m)DFT

In this section, we introduce the mDFT proposed in Ref. [22], in which only the NS–NS sector was studied and the R–R fields have not been included yet. We first give a short introduction to the mDFT. Then, we show that the mDFT can be regarded as the conventional DFT with a non-standard solution of the strong constraint.

2.1. A brief review of mDFT

Let us give a short introduction to the mDFT. In the absence of the R–R fields, the set of GSE in D dimensions takes the following form:

$$\begin{aligned} R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} + D_m X_n + D_n X_m &= 0, \\ \frac{1}{2} D^k H_{kmn} - (X^k H_{kmn} + D_m X_n - D_n X_m) &= 0, \\ R - \frac{1}{2} |H_3|^2 + 4 D_m X^m - 4 X^m X_m &= 0, \quad X_m \equiv I_m + Z_m. \end{aligned} \quad (2.1)$$

Here we have defined $|\alpha_p|^2 \equiv \frac{1}{p!} \alpha_{m_1 \dots m_p} \alpha^{m_1 \dots m_p}$, and the D -dimensional indices m, n, \dots are raised or lowered with the metric G_{mn} . The covariant derivative D_m is the conventional Levi–Civita connection associated with G_{mn} , and $H_{kmn} \equiv 3 \partial_{[k} B_{mn]}$. A vector field I^m and a one-form Z_m are defined so as to satisfy

$$D_m I_n + D_n I_m = 0, \quad I^k H_{kmn} + D_m Z_n - D_n Z_m = 0, \quad I^m Z_m = 0. \quad (2.2)$$

The conventional dilaton is included in Z_m as follows:

$$Z_m = \partial_m \Phi + U_m. \quad (2.3)$$

Note that the equations of motion in Eq. (2.1) reduce to the conventional supergravity ones if $I^m = 0$ and $U_m = 0$ are satisfied. Since the GSE depend on Φ and U_m only through the combination Z_m , there is an ambiguity in the decomposition of $Z = d\Phi + U$ into $d\Phi$ and U . Namely, at the level of the equations of motion, there is a local symmetry,

$$\Phi(x) \rightarrow \Phi(x) + \omega(x), \quad U(x) \rightarrow U(x) - d\omega(x). \quad (2.4)$$

Therefore, for a given solution of the GSE, we can always choose the dilaton to satisfy

$$\mathfrak{L}_I \Phi = I^m \partial_m \Phi = 0. \quad (2.5)$$

In Ref. [22], by using techniques developed in the DFT, the above equations of motion have been reformulated in a manifestly $O(D, D)$ T -duality-covariant form,

$$\hat{S}_{MN} = 0, \quad \hat{S} = 0, \quad \hat{\mathfrak{L}}_X \mathcal{H}_{MN} = 0, \quad \hat{\mathfrak{L}}_X d = 0, \quad X^M X_M = 0. \quad (2.6)$$

In order to explain these equations, let us begin with some basics (see Ref. [22] for more details). We consider the equations of motion in a $2D$ -dimensional doubled spacetime with the local coordinates $(x^M) = (x^m, \tilde{x}_m)$, where \tilde{x}_m are called the dual coordinates while x^m are the conventional coordinates in the supergravity. We then introduce the generalized metric \mathcal{H}_{MN} on the doubled spacetime, which can be parameterized as

$$\mathcal{H}(x) = (\mathcal{H}_{MN}) = \begin{pmatrix} G_{mn} - B_{mp} G^{pq} B_{qn} & B_{mk} G^{kn} \\ -G^{mk} B_{kn} & G^{mn} \end{pmatrix} \quad (2.7)$$

in terms of the conventional metric G_{mn} and the Kalb–Ramond field B_{mn} . The T -duality-invariant dilaton $d(x)$, often called the DFT dilaton, can be related to the conventional dilaton $\Phi(x)$ as

$$e^{-2d} = e^{-2\Phi} \sqrt{|G|}. \tag{2.8}$$

A generalized vector field \mathbf{X}^M , which is absent in the conventional DFT, is parameterized as

$$(\mathbf{X}^M) = \begin{pmatrix} I^m \\ U_m + B_{mn} I^n \end{pmatrix}, \tag{2.9}$$

where I^m and U_n here are identified with the ones appearing in the GSE. The $2D$ -dimensional indices M, N, \dots are raised or lowered with the $O(D, D)$ metric,

$$(\eta_{MN}) \equiv \begin{pmatrix} 0 & \delta_m^n \\ \delta_n^m & 0 \end{pmatrix}, \quad (\eta^{MN}) \equiv \begin{pmatrix} 0 & \delta_n^m \\ \delta_m^n & 0 \end{pmatrix}. \tag{2.10}$$

The generalized diffeomorphisms in the doubled spacetime are generated by the generalized Lie derivative $\hat{\mathfrak{L}}_V$, which acts on $\mathcal{H}_{MN}(x)$ and $d(x)$ as

$$\begin{aligned} \hat{\mathfrak{L}}_V \mathcal{H}_{MN} &= V^K \partial_K \mathcal{H}_{MN} + (\partial_M V^K - \partial^K V_M) \mathcal{H}_{KN} + (\partial_N V^K - \partial^K V_N) \mathcal{H}_{MK}, \\ \hat{\mathfrak{L}}_V e^{-2d} &= \partial_M (e^{-2d} V^M). \end{aligned} \tag{2.11}$$

We suppose that all of the fields and gauge parameters satisfy the so-called strong constraint,

$$\eta^{MN} \partial_M A(x) \partial_N B(x) = \partial_m A(x) \tilde{\partial}^m B(x) + \tilde{\partial}^m A(x) \partial_m B(x) = 0, \tag{2.12}$$

where $A(x)$ and $B(x)$ are fields or gauge parameters. Then, the generalized diffeomorphisms can be regarded as gauge symmetries of the DFT, and the associated gauge algebra is closed. In order to satisfy the strong constraint (2.12), we usually consider a solution in which all of the fields and gauge parameters are independent of \tilde{x}_m . On the other hand, we will take a different solution when we describe the GSE, as we will discuss in Sect. 2.2.

2.1.1. Equations of motion

Let us now explain the equations of motion (2.6). The last three equations in Eq. (2.6), which reproduce (2.2) and (2.5), indicate that the generalized vector \mathbf{X}^M is a null generalized Killing vector. On the other hand, the first two equations in Eq. (2.6) describe the dynamics of $\mathcal{H}_{MN}(x)$ and $d(x)$. In particular, the first equation reproduces the first two equations in Eq. (2.1) and the second equation leads to the last equation in Eq. (2.1). In fact, $\mathring{\mathcal{S}}_{MN}$ and $\mathring{\mathcal{S}}$ are the generalized Ricci tensor/scalar associated with the covariant derivative satisfying (see Ref. [22] for more details)

$$\mathring{\nabla}_K \eta_{MN} = 0, \quad \hat{\mathfrak{L}}_V = \hat{\mathfrak{L}}_V^\nabla, \quad \mathring{\nabla}_K \mathcal{H}_{MN} = 0, \quad \mathring{\nabla}_M d + \mathbf{X}_M = 0. \tag{2.13}$$

The explicit expressions of the modified quantities, the generalized connection $\mathring{\Gamma}_{MNK}$, the generalized Ricci tensor $\mathring{\mathcal{S}}_{MN}$, and the generalized Ricci scalar $\mathring{\mathcal{S}}$, in terms of $(\mathcal{H}_{MN}, d, \mathbf{X}^M)$ or $(G_{mn}, B_{mn}, \Phi, I^m, U_m)$, can be found in the preceding paper [22] (see Sects. 3.2, 4.1, and 4.2 therein). The corresponding quantities in the conventional DFT, Γ_{MNK} , \mathcal{S}_{MN} , and \mathcal{S} , can be reproduced from these modified quantities by setting $\mathbf{X}^M = 0$. Conversely, the modified quantities $(\mathring{\Gamma}_{MNK}, \mathring{\mathcal{S}}_{MN}, \mathring{\mathcal{S}})$ can be obtained from $(\Gamma_{MNK}, \mathcal{S}_{MN}, \mathcal{S})$ with the replacement

$$\partial_M d \rightarrow \partial_M d + \mathbf{X}_M. \tag{2.14}$$

The meaning of this shift will be clarified in the next subsection.

2.2. (m)DFT for “DFT on a modified section”

In this subsection, we show that the mDFT, which was reviewed in the previous subsection, is equivalent to the conventional DFT with a non-standard solution of the strong constraint. In this sense, the mDFT should rather be called the (m)DFT.

Let us first prove that by performing a certain generalized coordinate transformation, the null generalized Killing vector \mathbf{X}^M can always be brought into the following form:⁶

$$\mathbf{X}^M \equiv \begin{pmatrix} I^m \\ U_m + B_{mn} I^n \end{pmatrix} = \begin{pmatrix} I^m \\ 0 \end{pmatrix} \quad (I^m : \text{constant}). \quad (2.15)$$

This statement can be regarded as a generalization of the well-known fact in the Riemannian geometry that we can always find a certain coordinate system where the components of a Killing vector are constant (see, for example, Ref. [31]).

From the strong constraint, we can always find a section where all of the fields (\mathcal{H}_{MN} , d , \mathbf{X}^M) are independent of the dual coordinates. With this choice of section, the null and the generalized Killing properties lead to the conditions (2.2) and (2.5). Since I^m is a Killing vector field, we can always find a certain coordinate system $(x^m) = (x^\mu, y)$ in which the Killing vector is a coordinate basis: $I^m = c \delta_y^m$, where c is a constant. In such a coordinate system, both G_{mn} and Φ are independent of y . The three-form H_3 is also independent of y , as we can easily show $\mathcal{L}_I H_3 = 0$ from Eq. (2.2). Thus, we can generally expand H_3 as

$$H_3 = h_3 + c^{-1} \iota_I H_3 \wedge dy = h_3 - c^{-1} dZ \wedge dy \quad (\iota_I h_3 = 0), \quad (2.16)$$

where we used Eq. (2.2), and h_3 should satisfy $\mathcal{L}_I h_3 = 0$ that follows from $\mathcal{L}_I H_3 = 0$ and $\mathcal{L}_I Z = 0$. From this expansion, we find an expansion of the B -field satisfying $H_3 = dB_2$,

$$B_2 = b_2 - c^{-1} U \wedge dy \quad (\iota_I b_2 = 0, \quad h_3 = db_2), \quad (2.17)$$

where we used $dZ = dU$, and b_2 can always be chosen such that $\mathcal{L}_I b_2 = 0$ is satisfied. This shows that we can always take a gauge (for generalized diffeomorphisms) so that B_{mn} is also independent of y (i.e., $\mathcal{L}_I B_2 = 0$), and all of the NS–NS fields are now independent of y . From Eq. (2.17) and $\iota_I U = 0$, which comes from Eqs. (2.2) and (2.5), we also find the relation

$$\iota_I B_2 - U = 0. \quad (2.18)$$

This completes the proof that a null generalized Killing vector \mathbf{X}^M can always be brought into the form (2.15).

Then, since all of the fields are independent of y , the \tilde{y} dependence can sneak in without violating the strong constraint. Indeed, in a coordinate system where Eq. (2.15) is realized, the shift (2.14) from the DFT to the mDFT can be interpreted as an implicit introduction of the linear \tilde{y} dependence into the dilaton $d_*(x)$ or $\Phi_*(x)$:

$$(\partial_M d) = \begin{pmatrix} \partial_m d \\ 0 \end{pmatrix} \rightarrow (\partial_M d + \mathbf{X}_M) = \begin{pmatrix} \partial_m d \\ I^m \end{pmatrix} = (\partial_M d_*), \quad d_* \equiv d + c \tilde{y}. \quad (2.19)$$

⁶ Our proof partially follows the discussion given in Sect. 3.1 of Ref. [30].

When the dilaton does not depend on \tilde{y} (i.e., $c = 0$), I^m vanishes and the modification disappears. From the above argument, solutions of the mDFT can always be described as solutions of the DFT in which the dilaton has a linear dual-coordinate dependence.

Conversely, let us consider a solution of the DFT where the DFT dilaton has a linear dual-coordinate dependence, $d_* = d(x) + c^m \tilde{y}_m$ with constant c^m . From the identification

$$\partial_M d + X_M = \partial_M d_*, \quad (2.20)$$

we obtain that

$$I^m = c^m, \quad U_m + B_{mn} I^n = 0. \quad (2.21)$$

Note here that d_* , say the (m)DFT dilaton, can have only linear dependence on the dual coordinates with constant coefficients if we prefer to avoid the explicit appearance of the dual coordinates in I^m and $\partial_M d$. Suppose that all of the fields, collectively denoted by φ , are independent of \tilde{y} . Then the strong constraint requires that

$$0 = \partial^M d \partial_M \varphi = I^m \partial_m \varphi = \mathcal{L}_I \varphi. \quad (2.22)$$

Here, in the last equality we have used the fact that I^m is constant in order to express the condition in a covariant form. Then, the following conditions, namely Eqs. (2.2) and (2.5), are automatically satisfied:

$$\mathcal{L}_I G_{mn} = 0, \quad I^m \partial_m \Phi = 0, \quad \iota_I H_3 + dU = 0, \quad I^m U_m = 0. \quad (2.23)$$

Here, the dilaton Φ is defined through the relation

$$e^{-2d} = e^{-2\Phi} \sqrt{|G|}, \quad (2.24)$$

and it is independent of \tilde{y} . When the R–R fields are also introduced, they should also satisfy

$$\mathcal{L}_I \mathcal{F}_p = 0. \quad (2.25)$$

From the above viewpoint, it is not necessary to look for the action for the GSE. The DFT action supplies the $2D$ -dimensional equations of motion of the DFT. If the DFT dilaton has the dual-coordinate dependence, the equations of motion take the form of the GSE.

3. Ramond–Ramond sector of (m)DFT

In this section, we introduce the R–R fields by following the well-established formulation of the DFT [32–36].⁷ The whole bosonic part of the GSE is reproduced from the equations of motion of the DFT by choosing a modified section.

3.1. Ramond–Ramond sector of DFT

3.1.1. Gamma matrices and $O(D, D)$ spinors

It is convenient to introduce the gamma matrices $\{\gamma^M\} = \{\gamma^m, \gamma_m\}$ satisfying the $O(D, D)$ Clifford algebra,⁸

$$\{\gamma^M, \gamma^N\} = \eta^{MN}. \quad (3.1)$$

⁷ The approaches in Refs. [32,33], [35], and [34,36], respectively, are slightly different from each other. In this paper, we basically follow the approach of Refs. [32,33]. It is also useful to follow [35] when we consider the type II supersymmetric DFT [37].

⁸ In this paper, we call the $\text{Pin}(D, D)$ group simply $O(D, D)$.

Here, the gamma matrices are real and satisfy $\gamma^m = (\gamma_m)^T$. The gamma matrices with multi-indices are defined as

$$\gamma^{M_1 \dots M_p} \equiv \gamma^{[M_1 \dots M_p]}, \quad \gamma^{m_1 \dots m_p} \equiv \gamma^{[m_1 \dots m_p]} = \gamma^{m_1} \dots \gamma^{m_p}. \quad (3.2)$$

From the anti-commutation relations,

$$\{\gamma^m, \gamma_n\} = \delta_n^m, \quad \{\gamma^m, \gamma^n\} = 0 = \{\gamma_m, \gamma_n\}, \quad (3.3)$$

the γ_m can be regarded as fermionic annihilation operators. The Clifford vacuum $|0\rangle$ is defined so as to satisfy

$$\gamma_m |0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad (3.4)$$

where $\langle 0| \equiv |0\rangle^T$. By acting γ^m matrices on $|0\rangle$, an $O(D, D)$ spinor can be constructed as

$$|\alpha_p\rangle \equiv \frac{1}{p!} \alpha_{m_1 \dots m_p} \gamma^{m_1 \dots m_p} |0\rangle, \quad (3.5)$$

and it is in one-to-one correspondence with a p -form, $\alpha_p \equiv \frac{1}{p!} \alpha_{m_1 \dots m_p} dx^{m_1} \wedge \dots \wedge dx^{m_p}$. A formal sum of $O(D, D)$ spinors,

$$|\alpha\rangle \equiv \sum_p \frac{1}{p!} \alpha_{m_1 \dots m_p} \gamma^{m_1 \dots m_p} |0\rangle, \quad (3.6)$$

corresponds to a poly-form $\alpha = \sum_p \alpha_p$.

The $O(D, D)$ transformations are generated by γ^{MN} satisfying

$$[\gamma^{MN}, \gamma_L] = \gamma_K (T^{MN})^K{}_L, \quad (T^{MN})^K{}_L \equiv 2 \eta^{K[M} \delta_L^{N]}. \quad (3.7)$$

By utilizing the generators, one can define the following quantities:

$$S_{e^T} \equiv e^{\frac{1}{2} h^m{}_n [\gamma_m, \gamma^n]} \quad [e_m{}^n \equiv (e^{h^T})_m{}^n], \quad e^{\mathbf{B}} \equiv e^{\frac{1}{2} B_{mn} \gamma^{mn}}, \quad e^{\boldsymbol{\beta}} \equiv e^{\frac{1}{2} \beta^{mn} \gamma_{mn}}. \quad (3.8)$$

We can easily show that they satisfy

$$S_{e^T} \gamma_N S_{e^T}^{-1} = \gamma_M (\Lambda_{e^T})^M{}_N, \quad e^{\mathbf{B}} \gamma_N e^{-\mathbf{B}} = \gamma_M (\Lambda_B)^M{}_N, \quad e^{\boldsymbol{\beta}} \gamma_N e^{-\boldsymbol{\beta}} = \gamma_M (\Lambda_\beta)^M{}_N, \\ \Lambda_{e^T} \equiv \begin{pmatrix} (e^T)^m{}_n & 0 \\ 0 & (e^{-1})_m{}^n \end{pmatrix}, \quad \Lambda_B \equiv \begin{pmatrix} \delta_n^m & 0 \\ B_{mn} & \delta_m^n \end{pmatrix}, \quad \Lambda_\beta \equiv \begin{pmatrix} \delta_n^m & \beta^{mn} \\ 0 & \delta_m^n \end{pmatrix}. \quad (3.9)$$

It is helpful to define the correspondent of the flat metric as

$$S_k \equiv \gamma_0 \gamma^0 - \gamma^0 \gamma_0 = S_k^{-1} = S_k^T, \quad S_k \gamma_N S_k^{-1} = \gamma_M k^M{}_N, \\ (k^M{}_N) \equiv \begin{pmatrix} k^m{}_n & 0 \\ 0 & k_m{}^n \end{pmatrix}, \quad (k^m{}_n) \equiv \text{diag}(-1, +1, \dots, +1) \equiv (k_m{}^n) \equiv (k_{mn}). \quad (3.10)$$

The correspondent of the B -untwisted metric,⁹

$$(\hat{\mathcal{H}}_{MN}) \equiv \begin{pmatrix} G_{mn} & 0 \\ 0 & G^{mn} \end{pmatrix}, \quad G_{mn} \equiv e_m{}^k e_n{}^l k_{kl}, \quad (G^{mn}) \equiv (G_{mn})^{-1}, \quad (3.11)$$

⁹ See Ref. [38] for discussions of the untwisted form of generalized tensors.

can also be defined as

$$S_{\hat{\mathcal{H}}} \equiv S_e S_k S_e^T = S_{\hat{\mathcal{H}}}^T, \quad S_e \equiv (S_e^T)^T, \quad S_{\hat{\mathcal{H}}} \gamma_N S_{\hat{\mathcal{H}}}^{-1} = (\gamma^M)^T \hat{\mathcal{H}}_{MN}. \quad (3.12)$$

This gives a natural metric,

$$\begin{aligned} \langle \alpha | S_{\hat{\mathcal{H}}} | \beta \rangle &= \sum_{p,q} \frac{1}{p! q!} \alpha_{m_1 \dots m_p} \beta_{n_1 \dots n_q} \langle 0 | \gamma_{m_p \dots m_1} S_{\hat{\mathcal{H}}} \gamma^{n_1 \dots n_q} | 0 \rangle \\ &= \sum_{p,q} \frac{1}{p! q!} \alpha_{m_1 \dots m_p} \beta_{n_1 \dots n_q} \langle 0 | \gamma_{m_p \dots m_1} (\gamma_{l_1 \dots l_q})^T S_{\hat{\mathcal{H}}} | 0 \rangle G^{l_1 n_1} \dots G^{l_q n_q} \\ &= \sqrt{|G|} \sum_p \frac{1}{p!} G^{m_1 n_1} \dots G^{m_p n_p} \alpha_{m_1 \dots m_p} \beta_{n_1 \dots n_p} = \langle \beta | S_{\hat{\mathcal{H}}} | \alpha \rangle. \end{aligned} \quad (3.13)$$

The charge conjugation matrix is defined as

$$\begin{aligned} C &\equiv (\gamma^0 \pm \gamma_0) \dots (\gamma^{D-1} \pm \gamma_{D-1}) \quad (D : \text{even/odd}), \\ C \gamma^M C^{-1} &= -(\gamma^M)^T, \quad C^{-1} = (-1)^{\frac{D(D+1)}{2}} C = C^T. \end{aligned} \quad (3.14)$$

By using C and $S_{\hat{\mathcal{H}}}$, the Hodge dual can be constructed as

$$\begin{aligned} C S_{\hat{\mathcal{H}}} \gamma^{m_1 \dots m_p} | 0 \rangle &= \sqrt{|G|} (-1)^p G^{m_1 n_1} \dots G^{m_p n_p} \gamma_{n_1 \dots n_p} C | 0 \rangle \\ &= \frac{1}{(D-p)!} (-1)^{\frac{p(p+1)}{2}} \varepsilon^{m_1 \dots m_p n_1 \dots n_{D-p}} \gamma^{n_1 \dots n_{D-p}} | 0 \rangle, \end{aligned} \quad (3.15)$$

where $\varepsilon_{01 \dots (D-1)} = +\sqrt{|G|}$ and indices are raised or lowered with G_{mn} .

The correspondent of the generalized metric is defined as

$$\begin{aligned} S_{\mathcal{H}} &\equiv e^{-B^T} S_{\hat{\mathcal{H}}} e^{-B} = S_{\mathcal{H}}^T, \quad S_{\mathcal{H}} \gamma_N S_{\mathcal{H}}^{-1} = (\gamma^M)^T \mathcal{H}_{MN}, \\ (\mathcal{H}_{MN}) &= \begin{pmatrix} \delta_m^k & B_{mk} \\ 0 & \delta_k^m \end{pmatrix} \begin{pmatrix} G_{kl} & 0 \\ 0 & G^{kl} \end{pmatrix} \begin{pmatrix} \delta_n^l & 0 \\ -B_{ln} & \delta_l^n \end{pmatrix}. \end{aligned} \quad (3.16)$$

As stressed in Ref. [33], $S_{\mathcal{H}}$ is a particular parameterization of the fundamental field \mathbb{S} that corresponds to the generalized metric \mathcal{H}_{MN} before providing a parameterization. If we take another parameterization of the generalized metric,

$$(\mathcal{H}_{MN}) = \begin{pmatrix} \delta_m^k & 0 \\ -\beta^{mk} & \delta_k^m \end{pmatrix} \begin{pmatrix} \tilde{g}_{kl} & 0 \\ 0 & \tilde{g}^{kl} \end{pmatrix} \begin{pmatrix} \delta_n^l & \beta^{kn} \\ 0 & \delta_l^n \end{pmatrix}, \quad (3.17)$$

then the field \mathbb{S} is parameterized as

$$S_{\tilde{\mathcal{H}}} \equiv e^{\beta^T} S_{\mathcal{H}} e^{\beta}, \quad S_{\tilde{\mathcal{H}}} \equiv S_{\tilde{e}} S_k S_{\tilde{e}}^T. \quad (3.18)$$

For later discussion, it is also convenient to define the following quantity,

$$\mathcal{K} \equiv -C \mathbb{S} = C \mathcal{K}^T C, \quad \mathcal{K} \gamma_M \mathcal{K}^{-1} = -\mathcal{H}_M^N \gamma_N, \quad (3.19)$$

and the chirality operator

$$\gamma^{D+1} \equiv (-1)^{N_F}, \quad N_F \equiv \gamma^m \gamma_m. \quad (3.20)$$

The chirality operator acts on an $O(D, D)$ spinor as

$$\gamma^{D+1} |\alpha\rangle = \sum_p (-1)^p |\alpha_p\rangle. \tag{3.21}$$

A chiral/anti-chiral spinor corresponds to a poly-form with even/odd degree.

3.1.2. *The classical action and equations of motion*

The dynamical fields that correspond to the R–R fields are introduced as $O(D, D)$ spinors,

$$|A\rangle \equiv \sum_p \frac{1}{p!} A_{m_1 \dots m_p} \gamma^{m_1 \dots m_p} |0\rangle, \tag{3.22}$$

which transform under generalized diffeomorphisms [32,33] as follows:

$$\delta_V |A\rangle = \hat{\mathfrak{L}}_V |A\rangle \equiv V^M \partial_M |A\rangle + \partial_M V_N \gamma^M \gamma^N |A\rangle. \tag{3.23}$$

Depending on the type IIA or IIB theory, $|A\rangle$ takes a definite chirality,

$$\gamma^{11} |A\rangle = \mp |A\rangle \quad (\text{IIA/IIB}). \tag{3.24}$$

It is easy to show that under the strong constraint, the R–R field strength,

$$|F\rangle \equiv \partial |A\rangle \equiv \gamma^M \partial_M |A\rangle, \tag{3.25}$$

transforms covariantly under generalized diffeomorphisms.

From the strong constraint, the operator ∂ is nilpotent, and one can readily see that the field strength is invariant under the gauge transformations for the R–R fields,

$$\delta_\lambda |A\rangle = \partial |\lambda\rangle, \tag{3.26}$$

where $|\lambda\rangle$ is an arbitrary $O(D, D)$ spinor which respects the chirality (3.24). The Bianchi identity also follows from the nilpotency $\partial^2 = 0$,

$$\partial |F\rangle = \partial^2 |A\rangle = 0. \tag{3.27}$$

In the democratic formulation [39,40], the self-duality relation should be imposed at the level of the equations of motion. In our convention, it takes the form

$$|F\rangle = \mathcal{K} |F\rangle. \tag{3.28}$$

From this expression and the Bianchi identity, we obtain the following relation:

$$\partial \mathcal{K} |F\rangle = 0. \tag{3.29}$$

This is nothing but the equation of motion for the R–R field, as we will see below.

Now, let us write down the bosonic part of the (pseudo-)action for the type II supergravity,

$$\mathcal{L} = e^{-2d} \mathcal{S} - \frac{1}{4} \langle F | \mathbb{S} | F \rangle = e^{-2d} \mathcal{S} + \frac{1}{4} \overline{\langle F |} \mathcal{K} | F \rangle, \tag{3.30}$$

where we have defined $\overline{\langle F |} \equiv \langle F | C^T$. Taking a variation with respect to A , we obtain

$$\delta_A \mathcal{L} = \frac{1}{2} \overline{\langle \delta A |} \partial \mathcal{K} | F \rangle - \partial_M \left[\frac{1}{2} \overline{\langle \delta A |} \gamma^M \mathcal{K} | F \rangle \right], \tag{3.31}$$

and as expected, the equations of motion for A reproduce (3.29). The variation with respect to the DFT dilaton becomes

$$\delta_d \mathcal{L} = -2e^{-2d} \mathcal{S} \delta d + \partial_M (4e^{-2d} \mathcal{H}^{MN} \partial_N \delta d), \quad (3.32)$$

and there is no contribution from the R–R fields. Finally, the variation with respect to the generalized metric gives rise to

$$\begin{aligned} \delta_{\mathcal{H}} \mathcal{L} &= -\frac{1}{2} e^{-2d} \mathcal{S}^{MN} \delta \mathcal{H}_{MN} - \partial_M [e^{-2d} \nabla_N \delta \mathcal{H}^{MN}] + \frac{1}{4} \overline{\langle F |} \delta \mathcal{K} |F\rangle \\ &= -\frac{1}{2} \left(e^{-2d} \mathcal{S}^{MN} - \frac{1}{4} \mathcal{H}^{(M} \mathcal{K} \overline{\langle F |} \gamma^{N)K} \mathcal{K} |F\rangle \right) \delta \mathcal{H}_{MN} - \partial_M [e^{-2d} \nabla_N \delta \mathcal{H}^{MN}] \\ &= -\frac{1}{2} e^{-2d} (\mathcal{S}^{MN} + \mathcal{E}^{MN}) \delta \mathcal{H}_{MN} - \partial_M [e^{-2d} \nabla_N \delta \mathcal{H}^{MN}], \end{aligned} \quad (3.33)$$

where we have employed the identity [33]

$$\delta \mathcal{K} = \frac{1}{2} \mathcal{H}^{(M} \mathcal{K} \gamma^{N)K} \mathcal{K} \delta \mathcal{H}_{MN}, \quad (3.34)$$

in the second equality, and defined the “energy–momentum tensor” [32,33]

$$\mathcal{E}^{MN} \equiv \frac{1}{4} e^{2d} \left[\langle F | (\gamma^{(M})^T \mathbb{S} \gamma^{N)} |F\rangle - \frac{1}{2} \mathcal{H}^{MN} \langle F | \mathbb{S} |F\rangle \right]. \quad (3.35)$$

In summary, the equations of motion of the DFT are given by

$$\mathcal{S}_{MN} + \mathcal{E}_{MN} = 0, \quad \mathcal{S} = 0, \quad \mathcal{K} \mathcal{K} |F\rangle = 0. \quad (3.36)$$

3.1.3. The classical action and equations of motion in the conventional formulation

Next, let us show that the expressions we obtained above indeed reproduce the well-known expressions in conventional supergravity by choosing (2.7), (2.8), and $\tilde{\partial}^m = 0$.

From Eq. (3.23) and $\tilde{\partial}^m = 0$, the transformation of the R–R field under a generalized diffeomorphism becomes

$$\begin{aligned} \delta_V |A\rangle &= [v^m \partial_m + \partial_m v^n \gamma^m \gamma_n + \frac{1}{2} (\partial_m \tilde{v}_n - \partial_n \tilde{v}_m) \gamma^{mn}] |A\rangle \\ &= |\mathcal{L}_v A + d\tilde{v} \wedge A\rangle, \end{aligned} \quad (3.37)$$

and it is equivalent to a conventional diffeomorphism and a B -field gauge transformation. The field strength (3.25) and the Bianchi identity (3.27) take the following form:

$$|\mathcal{F}\rangle = \mathcal{K} |A\rangle = \gamma^m \partial_m |A\rangle = |dA\rangle, \quad \mathcal{K} |\mathcal{F}\rangle = |d\mathcal{F}\rangle = 0. \quad (3.38)$$

The self-duality relation for the R–R field strength becomes

$$|F\rangle = -C e^{-B^T} S_{\hat{\mathcal{H}}} e^{-B} |F\rangle = -e^B C S_{\hat{\mathcal{H}}} e^{-B} |F\rangle. \quad (3.39)$$

If the B -untwisted field strength is defined as

$$|\hat{F}\rangle \equiv e^{-B} |F\rangle, \quad (3.40)$$

which is invariant under the B -field gauge transformations, (3.39) can be rewritten as

$$|\hat{F}\rangle = -C S_{\hat{\mathcal{H}}} |\hat{F}\rangle. \tag{3.41}$$

From Eq. (3.15), we obtain the following relations:

$$-C S_{\hat{\mathcal{H}}} |\hat{F}\rangle = -\sum_p \frac{1}{p!} \hat{F}_{m_1 \dots m_p} C S_{\hat{\mathcal{H}}} \gamma^{m_1 \dots m_p} |0\rangle = \sum_p (-1)^{\frac{p(p+1)}{2}+1} |*\hat{F}_p\rangle, \tag{3.42}$$

and the self-duality relation (3.28) becomes

$$*\hat{F}_p = (-1)^{\frac{p(p+1)}{2}+1} \hat{F}_{10-p}, \quad \hat{F}_p = (-1)^{\frac{p(p-1)}{2}} * \hat{F}_{10-p}. \tag{3.43}$$

From this relation and the Bianchi identity for \hat{F} ,

$$d\hat{F} + H_3 \wedge \hat{F} = 0, \tag{3.44}$$

the equation of motion becomes

$$d * \hat{F} - H_3 \wedge * \hat{F} = 0, \tag{3.45}$$

where, compared to the Bianchi identity, the sign in front of H_3 is flipped, according to the sign in Eq. (3.43).

From Eq. (3.13), the action (3.30) becomes

$$\mathcal{L} = \sqrt{|G|} \left[e^{-2\Phi} \left(R + 4 D^m \partial_m \Phi - 4 |\partial \Phi|^2 - \frac{1}{2} |H_3|^2 \right) - \frac{1}{4} \sum_p |\hat{F}_p|^2 \right]. \tag{3.46}$$

In order to evaluate the equations of motion (3.36), let us recall that \mathcal{S}_{MN} takes the form [22]

$$(\mathcal{S}_{MN}) = \begin{pmatrix} 2 G_{(m|k} s^{[kl]} B_{l|n)} - s_{(mn)} - B_{mk} s^{(kl)} B_{ln} & B_{mk} s^{(kn)} - G_{mk} s^{[kn]} \\ s^{[mk]} G_{kn} - s^{(mk)} B_{km} & s^{(mn)} \end{pmatrix}, \tag{3.47}$$

$$s_{(mn)} \equiv R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} + 2 D_m \partial_n \Phi, \quad s_{[mn]} \equiv -\frac{1}{2} D^k H_{kmn} + \partial_k \Phi H^k{}_{mn}. \tag{3.47}$$

In fact, \mathcal{E}_{MN} also takes a similar form. From the rewriting

$$\mathcal{E}_{MN} = \frac{1}{4} e^{2d} \langle \hat{F} | [(\gamma_K)^\top S_{\hat{\mathcal{H}}} \gamma_L - \frac{1}{2} \hat{\mathcal{H}}_{KL} S_{\hat{\mathcal{H}}}] | \hat{F} \rangle (\Lambda_{-B^\top})^K{}_M (\Lambda_{-B})^L{}_N, \tag{3.48}$$

it is straightforward to derive

$$(\mathcal{E}_{MN}) = \begin{pmatrix} -2 G_{(m|k} \mathcal{K}^{kl} B_{l|n)} + T_{mn} + B_{mk} T^{kl} B_{ln} & -B_{mk} T^{kn} + G_{mk} \mathcal{K}^{kn} \\ -\mathcal{K}^{mk} G_{kn} + T^{mk} B_{km} & -T^{mn} \end{pmatrix}, \tag{3.49}$$

$$T_{mn} \equiv \frac{1}{4} e^{2\Phi} \sum_p \left[\frac{1}{(p-1)!} \hat{F}_{(m}{}^{k_1 \dots k_{p-1}} \hat{F}_{n)k_1 \dots k_{p-1}} - \frac{1}{2} G_{mn} |\hat{F}_p|^2 \right],$$

$$\mathcal{K}_{mn} \equiv \frac{1}{4} e^{2\Phi} \sum_p \frac{1}{(p-2)!} \hat{F}_{k_1 \dots k_{p-2}} \hat{F}_{mn}{}^{k_1 \dots k_{p-2}}. \tag{3.49}$$

Thus, the equations of motion (3.36) are summarized as

$$\begin{aligned} R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} + 2D_m \partial_n \Phi = T_{mn}, \quad R + 4D^m \partial_m \Phi - 4|\partial\Phi|^2 - \frac{1}{2}|H_3|^2 = 0, \\ -\frac{1}{2} D^k H_{kmn} + \partial_k \Phi H^k{}_{mn} = \mathcal{K}_{mn}, \quad d * \hat{F}_p - H_3 \wedge * \hat{F}_{p+2} = 0, \end{aligned} \quad (3.50)$$

where p is even/odd for type IIA/IIB supergravity.

3.2. Generalized type IIA/IIB equations

As discussed in Sect. 2.2, the equations of motion for the (m)DFT can be obtained by introducing a linear \tilde{x}_m dependence into the DFT dilaton or the conventional dilaton,

$$d \rightarrow d_* \equiv d + I^m \tilde{x}_m, \quad \Phi \rightarrow \Phi_* \equiv \Phi + I^m \tilde{x}_m. \quad (3.51)$$

According to this replacement, the equations of motion for the NS–NS sector are modified [22]. On the other hand, because there is no dilaton dependence in the equations of motion for the R–R sector, one may deduce that the R–R sector should not be modified. However, as we see in various examples,¹⁰ the R–R fields $|A\rangle$ or $|F\rangle$ already include a non-linear dual-coordinate dependence through the dilaton Φ_* ,

$$|A\rangle = e^{-\Phi_*} |\mathcal{A}\rangle, \quad |F\rangle = e^{-\Phi_*} |\mathcal{F}\rangle, \quad (3.52)$$

where \mathcal{A} and \mathcal{F} , to be called the Φ_* -untwisted fields, are supposed to be independent of the dual coordinates. In fact, these rescaled fields, \mathcal{A} and \mathcal{F} , appear more naturally in the vielbein formulations of the DFT discussed in Refs. [35] and [34,36].

In terms of the Φ_* -untwisted fields, we obtain the relation

$$\begin{aligned} |\mathcal{F}\rangle &= e^{\Phi_*} \partial(e^{-\Phi_*} |\mathcal{A}\rangle) = \partial|\mathcal{A}\rangle - (\partial\Phi_*) |\mathcal{A}\rangle = \gamma^m (\partial_m - \partial_m \Phi) |\mathcal{A}\rangle - I^m \gamma_m |\mathcal{A}\rangle \\ &= |d\mathcal{A} - d\Phi \wedge \mathcal{A} - \iota_I \mathcal{A}\rangle. \end{aligned} \quad (3.53)$$

The Bianchi identity is also modified in a similar manner,

$$0 = e^{\Phi_*} \partial(e^{-\Phi_*} |\mathcal{F}\rangle) = |d\mathcal{F} - d\Phi \wedge \mathcal{F} - \iota_I \mathcal{F}\rangle. \quad (3.54)$$

More explicitly, we obtain

$$\mathcal{F}_{p+1} = d\mathcal{A}_p - d\Phi \wedge \mathcal{A}_p - \iota_I \mathcal{A}_{p+2}, \quad d\mathcal{F}_p - d\Phi \wedge \mathcal{F}_p - \iota_I \mathcal{F}_{p+2} = 0. \quad (3.55)$$

If we further introduce the (Φ_*, B) -untwisted fields¹¹ defined as

$$\hat{\mathcal{F}} \equiv e^{-B_2 \wedge} \mathcal{F}, \quad \hat{\mathcal{A}} \equiv e^{-B_2 \wedge} \mathcal{A}, \quad (3.56)$$

¹⁰ J. Sakamoto, Y. Sakatani, and K. Yoshida, work in progress.

¹¹ The flat components of $\hat{\mathcal{C}}$ correspond to the fundamental fields in Refs. [34,36] by taking the conventional parameterization of the generalized vielbein in terms of the vielbein for G_{mn} and B_{mn} .

the relation between the potential and the field strength is represented by

$$\begin{aligned}\hat{\mathcal{F}} &= e^{-B_2} \wedge [d(e^{B_2} \wedge \mathcal{C}) - d\Phi \wedge (e^{B_2} \wedge \mathcal{C}) - \iota_I(e^{B_2} \wedge \mathcal{C})] \\ &= d\mathcal{C} - Z \wedge \mathcal{C} - \iota_I \mathcal{C} + H_3 \wedge \mathcal{C} \quad (Z \equiv d\Phi + \iota_I B_2).\end{aligned}\quad (3.57)$$

Namely, the $(p+1)$ -form field strength is given by

$$\hat{\mathcal{F}}_{p+1} = d\mathcal{C}_p - Z \wedge \mathcal{C}_p - \iota_I \mathcal{C}_{p+2} + H_3 \wedge \mathcal{C}_{p-2}.\quad (3.58)$$

The Bianchi identity becomes

$$d\hat{\mathcal{F}}_p - Z \wedge \hat{\mathcal{F}}_p - \iota_I \hat{\mathcal{F}}_{p+2} + H_3 \wedge \hat{\mathcal{F}}_{p-2} = 0.\quad (3.59)$$

The gauge transformation of the R–R potential is expressed as

$$\delta_\lambda \mathcal{C} = d\hat{\lambda} - Z \wedge \hat{\lambda} - \iota_I \hat{\lambda} + H_3 \wedge \hat{\lambda},\quad (3.60)$$

and the invariance of the field strength requires the nilpotency

$$0 = (d - Z \wedge - \iota_I + H_3 \wedge)^2 \hat{\lambda} = -\mathcal{L}_I \lambda - (dZ + \iota_I H_3 - \iota_I Z) \wedge \lambda.\quad (3.61)$$

But this is indeed satisfied from the strong constraint (2.23) and $\mathcal{L}_I \lambda = 0$.

The equations of motion become (see Ref. [22] for the modification of the NS–NS sector)

$$\begin{aligned}R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} + 2D_m \partial_n \Phi + D_m U_n + D_n U_m &= T_{mn}, \\ R + 4D^m \partial_m \Phi - 4|\partial\Phi|^2 - \frac{1}{2}|H_3|^2 - 4(I^m I_m + U^m U_m + 2U^m \partial_m \Phi - D_m U^m) &= 0, \\ -\frac{1}{2} D^k H_{kmn} + \partial_k \Phi H^k{}_{mn} + U^k H_{kmn} + D_m I_n - D_n I_m &= \mathcal{K}_{mn}, \\ d * \hat{\mathcal{F}}_p - Z \wedge * \hat{\mathcal{F}}_p - \iota_I * \hat{\mathcal{F}}_{p-2} - H_3 \wedge * \hat{\mathcal{F}}_{p+2} &= 0,\end{aligned}\quad (3.62)$$

where we have introduced the following quantities:

$$\begin{aligned}T_{mn} &= \frac{1}{4} \sum_p \left[\frac{1}{(p-1)!} \hat{\mathcal{F}}_{(m}{}^{k_1 \dots k_{p-1}} \hat{\mathcal{F}}_{n)k_1 \dots k_{p-1}} - \frac{1}{2} G_{mn} |\hat{\mathcal{F}}_p|^2 \right], \\ \mathcal{K}_{mn} &= \frac{1}{4} \sum_p \frac{1}{(p-2)!} \hat{\mathcal{F}}_{k_1 \dots k_{p-2}} \hat{\mathcal{F}}_{mn}{}^{k_1 \dots k_{p-2}}.\end{aligned}\quad (3.63)$$

For the type IIB case, these equations are nothing but the generalized type IIB equations (in the democratic form).

3.2.1. Redefinitions of the Ramond–Ramond fields

Our fundamental fields are (A, F) . However, depending on the situation, it may be more convenient to introduce various untwisted quantities. For completeness, let us introduce the following redefinitions

of the R–R fields:

$$\begin{array}{ccccc}
 (\check{C}, \check{F}) & \xleftarrow[\beta\text{-untwist}]{e^{\beta\vee}} & (A, F) & \xrightarrow[B\text{-untwist}]{e^{-B_2\wedge}} & (\hat{C}, \hat{F}) \\
 & & \downarrow e^{I^m \tilde{x}_m} \text{ } J\text{-untwist} & & \\
 (\check{C}, \check{F}) & \xleftarrow[\beta\text{-untwist}]{e^{\beta\vee}} & (A, F) & \xrightarrow[B\text{-untwist}]{e^{-B_2\wedge}} & (\hat{C}, \hat{F}) \\
 & & \downarrow e^{\Phi} \text{ } \Phi\text{-untwist} & & \\
 (\check{C}, \check{F}) & \xleftarrow[\beta\text{-untwist}]{e^{\beta\vee}} & (A, \mathcal{F}) & \xrightarrow[B\text{-untwist}]{e^{-B_2\wedge}} & (\hat{C}, \hat{\mathcal{F}})
 \end{array} \tag{3.64}$$

For example, the (I, B) -untwisted fields are defined as

$$\hat{C} \equiv e^{I^m \tilde{x}_m} e^{-B_2\wedge} A = e^{-\Phi} \mathcal{C}, \quad \hat{F} \equiv e^{I^m \tilde{x}_m} e^{-B_2\wedge} F = e^{-\Phi} \hat{\mathcal{F}}, \tag{3.65}$$

the potentials and the field strengths are related through

$$\hat{F}_{p+1} = d\hat{C}_p - U \wedge \hat{C}_p - \iota_I \hat{C}_{p+2} + H_3 \wedge \hat{C}_{p-2}, \tag{3.66}$$

and the Bianchi identities become

$$d\hat{F}_p - U \wedge \hat{F}_p - \iota_I \hat{F}_{p+2} + H_3 \wedge \hat{F}_{p-2} = 0. \tag{3.67}$$

In particular, when we study non-geometric T -folds, it is more convenient to define the (I, β) -untwisted fields¹²

$$|\check{C}\rangle \equiv e^{I^m \tilde{x}_m} e^{\beta} |A\rangle, \quad |\check{F}\rangle \equiv e^{I^m \tilde{x}_m} e^{\beta} |F\rangle \tag{3.68}$$

or

$$\check{C} \equiv e^{I^m \tilde{x}_m} e^{\beta\vee} A, \quad \check{F} \equiv e^{I^m \tilde{x}_m} e^{\beta\vee} F, \tag{3.69}$$

where we have defined an operation $\beta\vee$ that acts on a p -form α_p as

$$\beta \vee \alpha_p \equiv \frac{1}{2} \beta^{mn} \iota_m \iota_n \alpha_p. \tag{3.70}$$

For the β -untwisted quantities, the natural metric is the metric \tilde{g}_{mn} introduced in Eq. (3.17). For example, the self-duality relation (3.28) takes the form

$$\star \check{F}_p = (-1)^{\frac{p(p+1)}{2}+1} \check{F}_{10-p}, \quad \check{F}_p = (-1)^{\frac{p(p-1)}{2}} \star \check{F}_{10-p}, \tag{3.71}$$

and the same relations hold for $\check{\mathcal{F}}$ and $\check{\mathcal{F}}$, where \star is the Hodge star operator associated with \tilde{g}_{mn} . Also, if there is no modification I^m , the Lagrangian for the R–R sector becomes

$$\mathcal{L}_{\text{R-R}} = -\frac{1}{4} \sqrt{|\tilde{g}|} \sum_p \frac{1}{p!} \tilde{g}^{m_1 n_1} \dots \tilde{g}^{m_p n_p} \check{F}_{m_1 \dots m_p} \check{F}_{n_1 \dots n_p}. \tag{3.72}$$

¹² J. Sakamoto, Y. Sakatani, and K. Yoshida, work in progress. A definition of β -untwisted R–R fields is discussed in Ref. [41].

3.3. T -duality transformation rules

In this subsection, we present the T -duality transformation rule in a coordinate system in which (2.15) is realized. If the fields $(\mathcal{H}_{MN}, d, A_p)$ are independent of a coordinate z , the following T -duality transformation along the z -direction maps a solution of the (m)DFT to another one of the (m)DFT:

$$\begin{aligned} \mathcal{H}_{MN} &\rightarrow \mathcal{H}'_{MN} = (\Lambda^T \mathcal{H} \Lambda)_{MN}, & \Lambda &\equiv (\Lambda^M{}_N) \equiv \begin{pmatrix} \delta_n^m - \delta_z^m \delta_n^z & \delta_z^m \delta_n^z \\ \delta_m^z \delta_n^z & \delta_m^n - \delta_m^z \delta_n^z \end{pmatrix}, \\ |A\rangle_{\text{IIA}} &\rightarrow |\mathcal{A}'\rangle_{\text{IIB}} = (\gamma_z - \gamma^z) |A\rangle_{\text{IIA}}, & |A\rangle_{\text{IIB}} &\rightarrow |\mathcal{A}'\rangle_{\text{IIA}} = (\gamma^z - \gamma_z) |A\rangle_{\text{IIB}}, \\ d &\rightarrow d' = d + I^z z, & I^z &\rightarrow I'^z = 0, & I^i &\rightarrow I'^i = I^i. \end{aligned} \tag{3.73}$$

Here, x^i is an arbitrary coordinate other than z . In the component expression, the above rule is represented by the following map:

$$\begin{aligned} G'_{ij} &= G_{ij} - \frac{G_{iz} G_{jz} - B_{iz} B_{jz}}{G_{zz}}, & G'_{iz} &= \frac{B_{iz}}{G_{zz}}, & G'_{zz} &= \frac{1}{G_{zz}}, \\ B'_{ij} &= B_{ij} - \frac{B_{iz} G_{jz} - G_{iz} B_{jz}}{G_{zz}}, & B'_{iz} &= \frac{G_{iz}}{G_{zz}}, \\ \Phi' &= \Phi + \frac{1}{4} \ln \left| \frac{\det G'_{mn}}{\det G_{mn}} \right| + I^z z, & I'^i &= I^i, & I'^z &= 0, \\ \mathcal{A}'_{i_1 \dots i_{p-1} z} &= \mathcal{A}_{i_1 \dots i_{p-1}}, & \mathcal{A}'_{i_1 \dots i_p} &= \mathcal{A}_{i_1 \dots i_p z}. \end{aligned} \tag{3.74}$$

For the other R–R fields, the transformation rules are given by

$$\begin{aligned} \mathbf{A}'_{i_1 \dots i_{p-1} z} &= e^{-I^z z} \mathbf{A}_{i_1 \dots i_{p-1}}, & \mathbf{A}'_{i_1 \dots i_p} &= e^{-I^z z} \mathbf{A}_{i_1 \dots i_p z}, \\ \hat{\mathcal{C}}'_{i_1 \dots i_{p-1} z} &= \hat{\mathcal{C}}_{i_1 \dots i_{p-1}} - (p-1) \frac{\hat{\mathcal{C}}_{[i_1 \dots i_{p-2} | z] G_{i_{p-1} | z}}}{G_{zz}}, \\ \hat{\mathcal{C}}'_{i_1 \dots i_p} &= \hat{\mathcal{C}}_{i_1 \dots i_p z} + p \hat{\mathcal{C}}_{[i_1 \dots i_{p-1} | z] B_{i_p | z}} + p(p-1) \frac{\hat{\mathcal{C}}_{[i_1 \dots i_{p-2} | z] B_{i_{p-1} | z] G_{i_p | z}}}{G_{zz}}. \end{aligned} \tag{3.75}$$

4. Scale invariance and Weyl invariance

Let us recall the conventional bosonic string sigma model in the critical dimension $D = 26$,¹³

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\gamma} (G_{mn} \gamma^{ab} - B_{mn} \varepsilon^{ab}) \partial_a X^m \partial_b X^n, \tag{4.1}$$

where $a, b = \tau, \sigma$, $\varepsilon^{\tau\sigma} = +1/\sqrt{-\gamma}$, and $\varepsilon_{\tau\sigma} = -\sqrt{-\gamma}$. This system is scale invariant (or UV finite) at the one-loop level [26], if the beta functions

$$\beta_{mn}^G = \alpha' \left(R_{mn} - \frac{1}{4} H_{mpq} H_n{}^{pq} \right), \quad \beta_{mn}^B = \alpha' \left(-\frac{1}{2} D^k H_{kmn} \right), \tag{4.2}$$

take the following forms:¹⁴

$$\beta_{mn}^G = -2\alpha' D_{(m} Z_{n)}, \quad \beta_{mn}^B = -\alpha' (Z^k H_{kmn} + 2D_{[m} I_{n]}), \tag{4.3}$$

¹³ In the following discussion, we will not show the ghost sector explicitly.

¹⁴ In terms of the DFT, these equations can neatly be summarized as $\mathcal{S}_{MN} = \hat{\xi}_Y \mathcal{H}_{MN}$ [22], where $(Y^M) = (U^m, I_m + B_{mn} U^n)$ is an arbitrary generalized vector and $Z_m = \partial_m \Phi + U_m$.

where I^m and Z_m are arbitrary vectors. The condition for Weyl invariance is more restrictive, and it is satisfied only when I^m and Z_m have the following forms:

$$I^m = 0, \quad Z_m = \partial_m \Phi. \tag{4.4}$$

This is because the trace of the energy–momentum tensor takes the form

$$2\alpha' \langle T \rangle = (\beta_{mn}^G \gamma^{ab} - \beta_{mn}^B \varepsilon^{ab}) \partial_a X^m \partial_b X^n, \tag{4.5}$$

and the non-vanishing β -function can be canceled by adding the Fradkin–Tseytlin term,

$$S_{\text{FT}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{-\gamma} R^{(\gamma)} \Phi, \tag{4.6}$$

only when Eq. (4.4) is satisfied. Indeed, S_{FT} gives a contribution to $\langle T \rangle$ as

$$\frac{4\pi}{\sqrt{-\gamma}} \gamma^{ab} \frac{\delta S_{\text{FT}}}{\delta \gamma^{ab}} = \left(D_m D_n \Phi \gamma^{ab} - \frac{1}{2} \partial_k \Phi H^k{}_{mn} \varepsilon^{ab} \right) \partial_a X^m \partial_b X^n + D^m \Phi \frac{2\pi \alpha'}{\sqrt{-\gamma}} \frac{\delta S}{\delta X^m}, \tag{4.7}$$

and by using the equations of motion for X^m ,

$$0 = \frac{2\pi \alpha'}{\sqrt{-\gamma}} \frac{\delta S}{\delta X^m} = G_{mn} \left[\mathcal{D}^a \partial_a X^n + \left(\gamma_{kl}^n \gamma^{ab} + \frac{1}{2} H^n{}_{kl} \varepsilon^{ab} \right) \partial_a X^k \partial_b X^l \right], \tag{4.8}$$

where the covariant derivative \mathcal{D}_a is associated with the intrinsic metric γ_{ab} and γ_{kl}^n is the Christoffel symbol associated with G_{mn} , this contribution to $\langle T \rangle$ precisely cancels the Weyl anomaly (4.5) for (4.4). For general I^m and Z_m , we cannot find a suitable local counterterm and the Weyl anomaly cannot be canceled.

In the following, by considering the doubled spacetime, we show that the Fradkin–Tseytlin term can completely cancel the Weyl anomaly if I^m and Z_m satisfy the conditions (2.2) and (2.5). When these conditions are satisfied, I^m and Z_m can be replaced by the (m)DFT dilaton d_* or Φ_* , which have dual-coordinate dependence. In order to treat the dual coordinates, we consider the DSM in which the number of the embedding functions is doubled: $(\mathbb{X}^M) = (X^m, \tilde{X}_m)$. For convenience, we choose the coordinates

$$(x^m) = (x^\mu, y^i) \quad (\mu = 0, \dots, D - N - 1; i = 1, \dots, N) \tag{4.9}$$

on the target space such that the background fields (G_{mn}, B_{mn}, Φ) depend only on x^μ , and the modified dilaton Φ_* has the form $\Phi_* = \Phi + I^i \tilde{y}_i$. Then, the essence of our argument is that the contribution of the Fradkin–Tseytlin term (with Φ replaced by Φ_*) to the trace $\langle T \rangle$ can be written as

$$\begin{aligned} \frac{4\pi}{\sqrt{-\gamma}} \gamma^{ab} \frac{\delta S_{\text{FT}}}{\delta \gamma^{ab}} &= \mathcal{D}^a [\partial_M \Phi_*(\mathbb{X}) \partial_a \mathbb{X}^M] = \mathcal{D}^a [\partial_\mu \Phi(X) \partial_a X^\mu + I^i \partial_a \tilde{y}_i] \\ &= \mathcal{D}^a (Z_m \partial_a X^m) - \partial_{[m} I_{n]} \varepsilon^{ab} \partial_a X^m \partial_b X^n + \mathcal{D}^a [I^i (\partial_a \tilde{y}_i - G_{in} \varepsilon^b{}_a \partial_b X^n - B_{in} \partial_a X^n)] \\ &= \frac{1}{2} [2 D_{(m} Z_{n)} \gamma^{ab} - (Z^k H_{kmn} + 2 \partial_{[m} I_{n]}) \varepsilon^{ab}] \partial_a X^m \partial_b X^n \\ &\quad + Z^m \frac{2\pi \alpha'}{\sqrt{-\gamma}} \frac{\delta S}{\delta X^m} + I^i \mathcal{D}^a (\partial_a \tilde{y}_i - G_{in} \varepsilon^b{}_a \partial_b X^n - B_{in} \partial_a X^n). \end{aligned} \tag{4.10}$$

By using the equations of motion for X^m and the self-duality relations

$$\partial_a \tilde{y}_i - G_{in} \varepsilon^b{}_a \partial_b X^n - B_{in} \partial_a X^n = 0, \tag{4.11}$$

which are also obtained as the equations of motion of the DSM, as we will explain below, the contribution from the Fradkin–Tseytlin term can completely cancel the Weyl anomaly (4.5).

In order to explain the self-duality relation, let us consider Hull’s double sigma model,

$$S = \frac{1}{4\pi\alpha'} \int \left[\frac{1}{2} \mathcal{H}_{MN}(X) \mathcal{P}^M \wedge *_\gamma \mathcal{P}^N - (d\tilde{X}_m + C_m) \wedge dX^m \right], \quad (4.12)$$

where we have introduced the quantities

$$\mathcal{P}^M(\sigma) \equiv d\mathbb{X}^M(\sigma) + C^M(\sigma), \quad (C^M) = \begin{pmatrix} 0 \\ C_m \end{pmatrix}, \quad (4.13)$$

and the generalized metric $\mathcal{H}_{MN}(X)$ are supposed to be independent of $(Y^I) \equiv (Y^i, \tilde{Y}_i)$. The equations of motion for C_m give rise to

$$d\tilde{X}_m + C_m = G_{mn} *_\gamma dX^n + B_{mn} dX^n, \quad (4.14)$$

which is equivalent to the well-known self-duality relation

$$\mathcal{P}_M = *_\gamma \mathcal{H}_M{}^N \mathcal{P}_N. \quad (4.15)$$

Using the above equations of motion for C_m and eliminating the combination $d\tilde{X}_m + C_m$ from the above action, we obtain the conventional string sigma model action for X^m ,

$$S = \frac{1}{4\pi\alpha'} \int (G_{mn} dX^m \wedge *_\gamma dX^n + B_{mn} dX^m \wedge dX^n). \quad (4.16)$$

Thus, the DSM is classically equivalent to the conventional sigma model. By combining the equations of motion for Y^i and C_m , we can show that

$$dC_i = d(G_{in} *_\gamma dX^n + B_{in} dX^n) = 0, \quad (4.17)$$

and C_i is a closed form. Thus, using the local symmetry

$$\tilde{Y}_i(\sigma) \rightarrow \tilde{Y}_i(\sigma) + \alpha_i(\sigma), \quad C_i(\sigma) \rightarrow C_i - d\alpha_i(\sigma), \quad (4.18)$$

we can (at least locally) set $C_i = 0$, and the equations of motion for C_i become

$$d\tilde{Y}_i = G_{in} *_\gamma dX^n + B_{in} dX^n. \quad (4.19)$$

This is nothing but the key relation (4.11).

The one-loop β -function of the DSM is computed in Refs. [42–44] by using the background field method, and the result is consistent with the conventional string sigma model. There, in order to cancel the Weyl anomaly, the T -duality-invariant Fradkin–Tseytlin term [19],¹⁵

$$S_{\text{FT}} = \frac{1}{8\pi} \int d^2\sigma \sqrt{-\gamma} R^{(\nu)} d(X), \quad (4.20)$$

has been introduced, though the dilaton is supposed to be independent of the dual coordinates and the equations of motion of the (m)DFT have not been reproduced. By replacing $d(X)$ with $d_*(X, \tilde{Y})$,

¹⁵ This reduces to the conventional one after integrating over the auxiliary fields C_m [19].

Weyl invariance follows from the same calculation as (4.10). Therefore, the bosonic string sigma model is Weyl invariant, as long as the background fields satisfy the GSE. When we consider the type II Green–Schwarz superstring theories, the β -functions receive additional corrections coming from the R–R field strength [i.e. T_{mn} and \mathcal{K}_{mn} in Eq. (3.62)] [1]. However, there is no explicit I^m and Z_m dependence in T_{mn} and \mathcal{K}_{mn} , hence the (modified) Fradkin–Tseytlin term is enough for the cancelation of the Weyl anomaly for the NS–NS sector of the general backgrounds of the GSE. As for the β -functions for the R–R fields, we still have much to do.

In the literature [26–29], the difference between scale invariance (or finiteness) and Weyl invariance has been studied, but our result has alleviated the gap. The only thing remaining is that the scale invariance does not require any condition for I^m and Z_m , while the Weyl invariance is realized only when the conditions in Eqs. (2.2) and (2.5) are satisfied. We expect that this small difference is intrinsic to the bosonic string, and when we consider the type II Green–Schwarz superstring theory the scale invariance may require the same conditions (2.2) and (2.5), which are also required from the kappa symmetry of the superstring theory.

Before closing this section, let us comment on the central charge identity [24,45], namely the constancy of \mathcal{S} . As discussed in Ref. [22], one can show the identity only from the differential Bianchi identity and the equations of motion for the generalized metric, $\mathcal{S}_{MN} = 0$,

$$\partial_M \mathcal{S} = 2 \mathcal{H}_{MN} \nabla_K \mathcal{S}^{KN} = 0. \quad (4.21)$$

In Ref. [22], the differential Bianchi identity has not been proven in the presence of X^M , and it has not been clear whether $\mathring{\mathcal{S}}_{MN} = 0$ can generally show the central charge identity. However, as we have found that the mDFT is just the conventional DFT, the differential Bianchi identity and $\mathring{\mathcal{S}}_{MN} = 0$ indicate that $\mathring{\mathcal{S}}$ is constant.

5. Conclusion and discussion

In this paper, the bosonic part of the GSE is completely reproduced from the (m)DFT. When all of the fields are independent of the dual coordinates, the equations of motion of the DFT lead to the conventional supergravity equations, while when the dilaton has a linear dual-coordinate dependence, the GSE are reproduced. The type IIA supergravity equations of motion have been presented in the same manner as the type IIB equations, and the T -duality transformation rules between the type IIA and the type IIB have been provided. The discrepancy between the scale invariance and the Weyl invariance of the bosonic string theory has been resolved by introducing dual-coordinate dependence into the Fradkin–Tseytlin term. The existence of the doubled space is indispensable to our discussion.

In this paper, we have considered a T -duality covariant language of the DFT for simplicity, but the GSE can also be derived from a U -duality covariant generalization, the exceptional field theory (EFT) [23]. When we describe the type IIB theory in the $E_{d(d)}$ EFT, following the convention of Ref. [46], the generalized coordinates are introduced as

$$(x^\mu, \mathbf{x}^M) = (x^\mu, \mathbf{x}^m, \mathbf{y}_m^\alpha, \mathbf{y}_{m_1 m_2 m_3}^\alpha, \mathbf{y}_{m_1 \dots m_5}^\alpha, \mathbf{y}_{m_1 \dots m_6, m}, \dots), \quad (5.1)$$

where the ellipsis is not necessary for $d \leq 7$. Here, x^μ ($\mu = 0, 10 - d$) are the coordinates on the non-compact directions, \mathbf{x}^m ($m = 1, \dots, d - 1$) are the conventional coordinates on the torus T^{d-1} , and the other coordinates are winding coordinates associated with various type IIB branes (see Ref. [46] for more detail). The index $\alpha = 1, 2$ is for the fundamental representation of the $SL(2)$ S -duality symmetry. In the exceptional space, a doubled torus can be seen as a $2(d - 1)$ -dimensional space

with coordinates $(\mathbf{x}^m, \mathbf{y}_m^1)$. For convenience, let us consider the decomposition

$$(\mathbf{x}^m) = (\mathbf{x}^a, \mathbf{x}^i) \quad (a = 1, \dots, d - N - 1; i = d - N, \dots, d - 1). \quad (5.2)$$

Then, if the dilaton depends on (x^μ, \mathbf{x}^a) and also has a linear coordinate dependence on \mathbf{y}_i^1 , and all of the other fields are functions of (x^μ, \mathbf{x}^a) , the GSE can be reproduced from the equations of motion of the EFT. In the case of $d = 6$ and $N = 1$, where $\mathbf{y}_{*+} \equiv \mathbf{y}_i^1$, the coordinates $(\mathbf{x}^a, \mathbf{y}_{*+})$ are precisely the coordinates used in Ref. [23], which can be identified with the conventional coordinates in the type IIA theory (from the linear map of Ref. [46]). In this sense, the result of this paper is consistent with that of Ref. [23]. However, the Scherk–Schwarz ansatz employed in Ref. [23] was not necessary in our approach. The dual derivatives of the dilaton generate the extra vector fields, and the GSE follows automatically from the equations of motion of the EFT. On the other hand, when we consider the generalized type IIA supergravity, the generalized coordinates are parameterized as

$$(x^\mu, x^I) = (x^\mu, x^i, y_{i_1 i_2}, y_{i_1 \dots i_5}, y_{i_1 \dots i_7, i}, \dots), \quad (5.3)$$

and the \tilde{x}_i coordinates of the DFT can be identified with y_{iz} [46], where x^z is the coordinate on the M -theory circle. Thus, if the dilaton has linear dependence on y_{iz} , the equations of motion of the EFT lead to the generalized type IIA supergravity equations.

Note that the dual coordinate \mathbf{y}_i^1 in the exceptional space is not invariant under the S -duality symmetry. The dilaton also is not a singlet under the S -duality. Thus, the generalized type IIB supergravity is not covariant under the S -duality. Namely, under the S -duality transformation, a solution of the GSE is mapped to a configuration which does not satisfy the GSE, although it is still a solution of the EFT. It may be interesting to see whether there exists a U -duality-covariant generalization of the GSE. Of course, we already know that the EFT can describe all such solutions, and the EFT may be enough. In addition, from the point of view of the eleven-dimensional supergravity, the dilaton is the eleventh component of the metric, and the introduction of dual-coordinate dependence into the dilaton breaks the eleven-dimensional general covariance. That would be the reason why the “generalized eleven-dimensional supergravity” has not been found. Again, it is not necessary to look for such a generalized eleven-dimensional theory and the EFT is enough, which has a huge gauge symmetry that includes the eleven-dimensional diffeomorphisms.

Utilizing the result of this paper, we can straightforwardly study a supersymmetric generalization of the GSE in detail. The type II supersymmetric DFT has been constructed in Ref. [37]. By taking a non-standard section of the (m)DFT, one can identify how the extra vector fields modify the equations of motion and supersymmetry transformations. It will be important to complete this analysis and the obtained result should be compared with that of Ref. [7].

It is also important to study a supersymmetric generalization of the DSM analysis. The GSE can be derived from the requirement of kappa symmetry, while Weyl invariance (at quantum level) has not followed directly from kappa symmetry (at a classical level). The result obtained here implies the equivalence of kappa symmetry and Weyl invariance, and detailed analysis based on a supersymmetric DSM, such as Refs. [47,48], would be important future work.

The present work has provided positive evidence that superstring theories defined on solutions of the GSE are Weyl invariant. We further expect that superstring theories can be consistently defined on arbitrary solutions of the DFT/EFT. It will be an interesting future direction to be addressed.

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