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Flow equation for the large N scalar model and induced geometries[†]

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We study the proposal that a $(d + 1)$-dimensional induced metric is constructed from a d-dimensional field theory using gradient flow. Applying the idea to the O(N) φ^4 model and normalizing the flow field, we have shown in the large N limit that the induced metric is finite and universal in the sense that it does not depend on the details of the flow equation and the original field theory except for the renormalized mass, which is the only relevant quantity in this limit. We have found that the induced metric describes Euclidean anti-de Sitter (AdS) space in both ultraviolet (UV) and infrared (IR) limits of the flow direction, where the radius of the AdS is bigger in the IR than in the UV.

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Subject Index   B30, B32, B35, B37

1. Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1] (or, more generally, the gravity/gauge theory correspondence) is a surprising but significant finding in field theories and string theories. The original correspondence claims that a d-dimensional conformal field theory is equivalent to some $(d + 1)$-dimensional supergravity theory on an AdS background. After the first proposal, there appeared many pieces of evidence to indicate that the correspondence is true, and applications to various areas in physics, such as AdS/QCD (quantum chromodynamics) or AdS/condensed matter physics, have been successfully investigated. Even though the AdS/CFT correspondence might be explained by the closed string/open string duality, the claimed equivalence is still mysterious and a deeper understanding of the correspondence is necessary.

In a previous paper [2], two of the present authors and their collaborator tried to understand the gravity/field theory correspondences from a different point of view, proposing an alternative method to define a geometry from a field theory: A quantum field theory in d dimensions is lifted to a $(d + 1)$-dimensional one using gradient flow [3–6], where the flow time t, which represents

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[†]We dedicate this work to the memory of Peter Hasenfratz.

the energy scale of the original d-dimensional theory, is interpreted as an additional coordinate. Then the induced metric is defined from this $(d + 1)$-dimensional field. As the metric is derived from the original d-dimensional theory together with its scale dependence, the method proposed in the paper can be applied in principle to all field theories. As a concrete example, the method was applied to the O(N) non-linear sigma model (NLSM) in two dimensions, and the vacuum expectation value (VEV) of the three-dimensional induced metric was shown to describe an AdS space in the massless limit.[1]

There are two key aspects in this proposal. First of all, the $(d + 1)$-dimensional induced metric becomes classical in the large N limit, and the quantum corrections can be calculated order by order in the large N expansion. Secondly, in the example mentioned above, the three-dimensional induced metric, constructed from a product of three-dimensional flow fields at the same point, is free from UV divergences in the large N limit. Instead, if the original two-dimensional fields were used directly to define the two-dimensional metric, it would badly diverge. These two special properties allow us to infer that (the VEV of) the induced metric describes a geometry.

In this paper, we further investigate this proposal, explicitly considering a more general model, the O(N) invariant φ^4 model in d dimensions, which can describe the free scalar model as well as the NLSM as limiting cases. Furthermore, as a generalization of the proposal in Ref. [2], we consider the case that parameters (mass and coupling) are different in the original d-dimensional theory and the flow equation, in order to see how the behavior of the flow field depends on the choice. We also introduce a normalization of the flow field, to interpret the gradient flow as a renormalization group (RG) transformation. Our normalization condition for the flow field requires a rescaling factor that becomes unity in the NLSM limit, so that the result in the previous paper [2] is unchanged. We then define the induced metric from this normalized flow field. It turns out that the metric defined in this way leads to the same geometry, irrespective of the parameters in the flow equation as well as in the original d-dimensional theory. We show that the $(d + 1)$-dimensional space described by the metric becomes AdS both in UV and IR limits as long as $d > 2$.

This paper is organized as follows. In Sect. 2, we first briefly summarize the proposal in Ref. [2], together with a few modifications introduced in this paper. We explain some properties of the d-dimensional model in Sect. 3, and solve the flow equation in the large N limit in Sect. 4. We calculate the induced metric in Sect. 5 and the Einstein tensor in Sect. 6, showing that the space defined by the metric becomes AdS in both UV and IR limits. The behavior of the Einstein tensor between two limits is also investigated in Sect. 6. We finally give a summary of this paper in Sect. 7. In Appendix A, we show that a divergence of the flow field appears for the φ^4 theory in perturbation theory but it disappears non-perturbatively in the large N limit. Some properties of the incomplete gamma function used in the main text are given in Appendix B.

2. Proposal

In this section, we briefly summarize the proposal of Ref. [2] and explain some modifications introduced in this paper.

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[1] There are also studies using a different method on the relation between O(N) vector models in d dimensions and (generalized) gravity theories in $(d + 1)$ dimensions [7]. See also [8] for recent developments.

In this paper we consider the large N scalar field $\varphi^a(x)$ with the d-dimensional space-time coordinate x and the large N index $a = 1, 2, \ldots, N$, whose action S describes the φ^4 model and will be explicitly given in the next section.

The d-dimensional field $\varphi^a(x)$ is extended to $\phi^a(t, x)$ in $(d + 1)$ dimensions, using the gradient flow equation as [3,4]

$$\frac{\partial}{\partial t}\phi^a(t,x) = -\left.\frac{\delta S'}{\delta \varphi^a(x)}\right|_{\varphi \to \phi}, \tag{1}$$

with the initial condition $\phi^a(0, x) = \varphi^a(x)$, where the flow action S' in this paper can be different from the action S in the original d-dimensional theory. Since the length dimension of t is 2 and $t \geq 0$, a new variable $\tau = 2\sqrt{t}$ is introduced. We then denote the $(d + 1)$-dimensional coordinate as $z = (\tau, x) \in \mathbb{R}^+ (= [0, \infty)) \times \mathbb{R}^d$ and the flow field as $\phi^a(z)$.

In Ref. [2], a $(d + 1)$-dimensional metric was defined as

$$\hat{g}_{\mu\nu}(z) \equiv h \sum_{a=1}^{N} \partial_\mu \phi^a(z) \partial_\nu \phi^a(z), \tag{2}$$

where $h > 0$ is a dimensionful constant. In this paper, however, we modify this definition slightly to

$$\hat{g}_{\mu\nu}(z) \equiv h \sum_{a=1}^{N} \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z), \tag{3}$$

where σ^a is the normalized flow field defined by

$$\sigma^a(z) \equiv \frac{\phi^a(z)}{\sqrt{\langle \phi^2(z) \rangle}}, \tag{4}$$

which is equivalent to a constant renormalization, as we will see later, and h is a constant with the mass dimension -2. The average $\langle \phi^2(z) \rangle$ in the above definition will be defined in the next paragraph. The reason why we use σ^a instead of ϕ^a to define the metric is as follows. The RG transformation consists of two procedures: first UV modes are integrated out and then the field is normalized for the RG flow to have a fixed point. Since the gradient flow itself corresponds to the former procedure only, a field normalization is introduced as in Eq. (4). One may adopt a different field normalization, but Eq. (4) leads to interesting results as will be seen in later sections. We call this condition the NLSM normalization, as $\langle \sigma^2(z) \rangle = 1 \ \forall z$.

The above $\hat{g}_{\mu\nu}(z)$ is an induced metric on a $(d + 1)$-dimensional manifold $\mathbb{R}^+ \times \mathbb{R}^d$ from some manifold in $\mathbb{R}^N$ defined by $\sigma^a(z)$, which becomes an $(N - 1)$-dimensional sphere after the quantum average as $\langle \sigma^2(z) \rangle = 1$. Here the expectation values of $\hat{g}_{\mu\nu}$ (and its correlations) are defined as

$$\langle \hat{g}_{\mu\nu}(z) \rangle \equiv \langle \hat{g}_{\mu\nu}(z) \rangle_S, \tag{5}$$

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle \equiv \langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle_S, \tag{6}$$

where $\langle \mathcal{O} \rangle_S$ is the expectation value of $\mathcal{O}(\varphi)$ in d dimensions with the action S as

$$\langle \mathcal{O} \rangle_S \equiv \frac{1}{Z} \int \mathcal{D}\varphi \, \mathcal{O}(\varphi) \, e^{-S}, \quad Z \equiv \int \mathcal{D}\varphi \, e^{-S}. \tag{7}$$

Even though the composite operator $\hat{g}_{\mu\nu}(z)$ contains a product of two local operators at the same point z, $\langle \hat{g}_{\mu\nu}(z) \rangle$ is finite as long as $\tau \neq 0$ for the two-dimensional NLSM in the large N limit [2]. Although the finiteness of the flow field was proven for gauge theories [9] and some scalar theories [10] (see also the latest extension in [11]), it is not guaranteed in general. Indeed, it was pointed out recently that the flow field of the φ^4 model gives an extra UV divergence in perturbation theory [12]. (The modified flow equation can avoid this divergence [13].) As shown later, however, the induced metric $\hat{g}_{\mu\nu}$ in Eq. (3) is free from such UV divergences in the large N limit. In Appendix A, we explicitly demonstrate that a divergence indeed appears in perturbation theory, but it disappears non-perturbatively in the large N limit. On the other hand, if the d-dimensional induced metric were defined from the d-dimensional field φ, it would diverge badly, and hence a geometry could not be defined from it.

Moreover, thanks to the large N factorization, quantum fluctuations of the metric $\hat{g}_{\mu\nu}$ are suppressed in the large N limit. The n-point correlation function of $\hat{g}_{\mu\nu}$ behaves as

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle = \prod_{i=1}^{n} \langle \hat{g}_{\mu_i\nu_i}(z_i) \rangle + O\left(\frac{1}{N}\right), \tag{8}$$

showing that the induced metric $\hat{g}_{\mu\nu}$ becomes classical in the large N limit, and quantum fluctuations are sub-leading and calculable in the large N expansion. This property allows a geometrical interpretation of the metric $\hat{g}_{\mu\nu}$. In Sect. 6, the VEV of the Einstein tensor is calculated directly from $\langle \hat{g}_{\mu\nu} \rangle$, as in the classical theory.

3. Large N model

3.1. Model

In this paper, we consider the N-component scalar φ^4 model in d dimensions, defined by the action

$$S(\mu^2, u) = N \int d^d x \left[\frac{1}{2} \partial^k \varphi(x) \cdot \partial_k \varphi(x) + \frac{\mu^2}{2} \varphi^2(x) + \frac{u}{4!} \left(\varphi^2(x) \right)^2 \right], \tag{9}$$

where $\varphi^a(x)$ is an N-component scalar field, $(\cdot)$ indicates an inner product of N-component vectors such that $\varphi^2(x) \equiv \varphi(x) \cdot \varphi(x) = \sum_{a=1}^{N} \varphi^a(x)\varphi^a(x)$, μ^2 is the bare scalar mass parameter, and u is the bare coupling constant of the φ^4 interaction, whose canonical dimension is $4 - d$.

This model describes the free massive scalar at $u = 0$, while it is equivalent to the NLSM in the $u \to \infty$ limit, whose action is obtained from Eq. (9) as

$$S(\lambda) = \frac{N}{2\lambda} \int d^d x \, \partial^k \sigma(x) \cdot \partial_k \sigma(x), \qquad \sigma^2(x) = 1, \tag{10}$$

with the replacement

$$\sigma^a(x) = \sqrt{\lambda} \varphi^a(x), \qquad \lambda = \lim_{u \to \infty} -\frac{u}{6\mu^2}. \tag{11}$$

In addition, the flow time also has to be rescaled by λ for the flow equation in this limit.

3.2. Calculation in the large N limit

The partition function is evaluated as [14]

$$Z(J) = \int [\mathcal{D}\varphi]\, e^{-S(\mu^2, u) + (J, \varphi)} = \int [\mathcal{D}\varphi][\mathcal{D}\rho]\, \delta\left(\rho - \varphi^2\right) e^{-S(\mu^2, u) + (J, \varphi)}$$

$$= \int [\mathcal{D}\beta] \exp\left[-N S_{\text{eff}}(\beta, J)\right], \tag{12}$$

where the effective action is given by

$$S_{\text{eff}}(\beta, J) = \int d^d x \left[\frac{6}{u}\beta^2(x) + \frac{6\mu^2}{u} i\beta(x)\right] + \frac{1}{2}\text{Tr}\ln D[\beta] - \frac{1}{2N^2}(J, D^{-1}J), \tag{13}$$

with $D[\beta](x, y) \equiv \{-\nabla^2 + 2i\beta(x)\}\delta^{(d)}(x - y)$, and

$$(F, G) \equiv \int d^d x\, F(x) G(x) \tag{14}$$

for arbitrary functions $F(x)$ and $G(x)$.

The large N limit corresponds to the saddle point, $2i\beta(x) = m^2$, determined by the saddle point equation as

$$\left.\frac{\partial S_{\text{eff}}(\beta, J)}{i\partial\beta(x)}\right|_{2i\beta(x) = m^2} = \frac{6}{u}(\mu^2 - m^2) + Z(m) = 0, \tag{15}$$

where

$$Z(m) = \int dp\, \frac{1}{p^2 + m^2} \geq 0, \qquad dp \equiv \frac{d^d p}{(2\pi)^d}, \tag{16}$$

which is divergent at $d > 1$. We therefore introduce some regularization such as lattice, dimensional, or momentum cut-off regularization, but do not specify it unless necessary. In this paper, sending the cut-off to infinity is called "the continuum limit."

The two-point function is then evaluated as

$$\langle \varphi^a(x)\varphi^b(y)\rangle = \delta^{ab}\frac{1}{N}\int dp\, \frac{e^{ip(x-y)}}{p^2 + m^2} \tag{17}$$

at $u \neq \infty$, where m is regarded as the renormalized mass. Therefore μ^2 must be tuned to keep m^2 finite in the continuum limit. The saddle point equation,

$$\mu^2 = m^2 - \frac{u}{6}Z(m), \tag{18}$$

says that $\mu^2 \to -\infty$ in the continuum limit at $d > 1$ as long as $u > 0$.

In the NLSM limit $u \to \infty$, we have

$$\langle \sigma^a(x)\sigma^b(y)\rangle = \delta^{ab}\frac{\lambda}{N}\int dp\, \frac{e^{ip(x-y)}}{p^2 + m^2}, \tag{19}$$

where

$$\frac{1}{\lambda} = Z(m) = \int dp\, \frac{1}{p^2 + m^2}. \tag{20}$$

Therefore $\lambda \to 0$ in the continuum limit at $d > 1$.

4. Flow equation and its solution in the large N limit

4.1. Flow equation

In this paper, we consider the flow equation, given by

$$\frac{\partial}{\partial t}\phi^a(t,x) = -\left.\frac{\delta S(\mu_f^2, u_f)}{\delta \varphi^a(x)}\right|_{\varphi \to \phi} = \left(\Box - \mu_f^2\right)\phi^a(t,x) - \frac{u_f}{6}\phi^a(t,x)\phi^2(t,x), \tag{21}$$

$$\phi^a(0,x) = \varphi^a(x),$$

where μ_f^2 and u_f can be different from μ^2 and u in the original d-dimensional theory. The flow with $\mu_f = \mu$ and $u_f = u$ is called gradient flow, as it is given by the gradient of the original action.

4.2. The solution in the large N limit

In the large N limit, the solution has the following form in momentum space [15]:

$$\phi(t,p) = f(t)e^{-p^2 t}\varphi(p). \tag{22}$$

The flow equation (21) in the large N limit leads to the equation for $f(t)$:

$$\dot{f}(t) = -\mu_f^2 f(t) - \frac{u_f}{6}f^3(t)\zeta_0(t), \tag{23}$$

where

$$\zeta_0(t) = \int dp\, \frac{e^{-2p^2 t}}{p^2 + m^2}, \qquad \zeta_0(0) = Z(m). \tag{24}$$

Using $X(t) = f^{-2}(t)$, the equation becomes

$$\dot{X}(t) = 2\mu_f^2 X(t) + \frac{u_f}{3}\zeta_0(t), \tag{25}$$

so that the solution is given by

$$X(t) = e^{2t\mu_f^2}\left[1 + \frac{u_f}{3}\int_0^t dx\, \zeta_0(x)e^{-2x\mu_f^2}\right]. \tag{26}$$

Introducing m_f by the relation

$$\mu_f^2 = m_f^2 - \frac{u_f}{6}Z(m_f), \tag{27}$$

the solution is further reduced as

$$\begin{aligned}
X(t) &= \frac{e^{2t\mu_f^2}}{m_f^2 - \mu_f^2}\left[m_f^2 - \mu_f^2 + \frac{u_f}{3}\int_0^t dx \int dp\, \left(\frac{p^2 + m_f^2}{p^2 + m^2} - \frac{p^2 + \mu_f^2}{p^2 + m^2}\right)e^{-2x(p^2+\mu_f^2)}\right] \\
&= \frac{\zeta(t)}{Z(m_f)}, \qquad \zeta(t) \equiv \zeta_0(t) + \Delta(t),
\end{aligned} \tag{28}$$

where

$$\Delta(t) = \left\{Z(m_f) - Z(m)\right\}e^{2t\mu_f^2} + \int dp\, \left(\frac{p^2 + m_f^2}{p^2 + m^2}\right)\frac{e^{2t\mu_f^2} - e^{-2tp^2}}{p^2 + \mu_f^2}. \tag{29}$$

In the case of the interacting flow with $u_f > 0$, μ_f^2 negatively diverges ($\mu_f^2 \to -\infty$), as $Z(m_f) \to +\infty$ (the continuum limit at $d > 1$) or $u_f \to +\infty$ (the NLSM limit). Therefore $\Delta(t)$ vanishes as

$$\lim_{\mu_f^2 \to -\infty} \Delta(t) \simeq -\frac{m_f^2 \zeta_0(t) - \dot{\zeta}_0(t)/2}{\mu_f^2} + O\left(\mu_f^{-4}\right) \tag{30}$$

for $t > 0$.

For the flow with $u_f = 0$, which we call "the free flow," we simply have $f(t) = e^{-m_f^2 t}$ for all $t \geq 0$.

4.3.   Two-point function

The two-point function of the flow field with $u_f > 0$ is given by

$$\langle \phi^a(t,x)\phi^b(s,y) \rangle = \frac{\delta^{ab}}{N} \frac{Z(m_f)}{\sqrt{\zeta(t)\zeta(s)}} \int dp \, \frac{e^{-(t+s)p^2} e^{ip(x-y)}}{p^2 + m^2}. \tag{31}$$

In the continuum or NLSM limit, we have

$$\langle \phi^a(t,x)\phi^b(s,y) \rangle = \frac{\delta^{ab}}{N} \frac{Z(m_f)}{\sqrt{\zeta_0(t)\zeta_0(s)}} \int dp \, \frac{e^{-(t+s)p^2} e^{ip(x-y)}}{p^2 + m^2}, \tag{32}$$

which only depends on renormalized quantities m_f and m, but does not depend on the bare parameters μ_f^2, u_f, μ^2, and u. Note, however, that $Z(m_f)$ diverges in the continuum limit for $d > 1$.

If we take the NLSM limit for the flow equation ($u_f \to \infty$) but without taking the continuum limit, the two-point function of $\sigma^a(t,x) = \sqrt{\lambda_f}\phi^a(t,x)$ with $\lambda_f = 1/Z(m_f)$ is given by

$$\langle \sigma^a(t,x)\sigma^b(s,y) \rangle = \frac{\delta^{ab}}{N} \frac{1}{\sqrt{\zeta_0(t)\zeta_0(s)}} \int dp \, \frac{e^{-(t+s)p^2} e^{ip(x-y)}}{p^2 + m^2}. \tag{33}$$

This two-point function is finite even in the continuum limit, as shown explicitly at $d = 2$ [15,16].

At $u_f = 0$, on the other hand, we obtain

$$\langle \phi^a(t,x)\phi^b(s,y) \rangle = \frac{\delta^{ab}}{N} \int dp \, \frac{e^{-(t+s)(p^2+m_f^2)} e^{ip(x-y)}}{p^2 + m^2}, \tag{34}$$

which is of course manifestly finite.

4.4.   Normalized flow field

As mentioned before, to construct an RG transformation using the flow equation, which is merely a kind of smearing procedure for UV fluctuations, we introduce a normalization condition for the flow field.

There is no unique way to define the normalized flow field. In the standard block spin transformation, for example, the normalization factor for the field is so chosen so that some fixed point can appear for the defined RG transformation. In this paper, we propose the following normalization for the flow field:

$$\sigma^a(t,x) = \frac{\phi^a(t,x)}{\sqrt{\langle \phi^2(t,x) \rangle}}, \tag{35}$$

so that the normalized flow field is dimensionless and satisfies the non-linear constraint $\langle \sigma^2(t,x) \rangle = 1$. Because of this property, we may call it the NLSM normalization.

For the interacting flow ($u_f \neq 0$), since

$$\langle \phi^2(t,x) \rangle = \frac{Z(m_f)}{\zeta(t)} \int dp \, \frac{e^{-2tp^2}}{p^2 + m^2} = Z(m_f) \frac{\zeta_0(t)}{\zeta(t)}, \tag{36}$$

we have

$$\sigma^a(t,p) = \frac{1}{\sqrt{\zeta_0(t)}} e^{-p^2 t} \varphi^a(p), \tag{37}$$

which gives

$$\langle \sigma^a(t,x) \sigma^b(s,y) \rangle = \frac{\delta^{ab}}{N} \frac{1}{\sqrt{\zeta_0(t)\zeta_0(s)}} \int dp \, \frac{e^{-(t+s)p^2} e^{ip(x-y)}}{p^2 + m^2}. \tag{38}$$

This result is not only finite in the continuum limit but also independent of bare parameters (μ_f^2, u_f, μ^2, u) as well as the renormalized flow mass m_f even without taking the continuum or NLSM limit.

For the free flow ($u_f = 0$), we have

$$\langle \phi^2(t,x) \rangle = \int dp \, \frac{e^{-2t(p^2 + m_f^2)}}{p^2 + m^2} = \zeta_0(t) e^{-2tm_f^2}, \tag{39}$$

which also leads to Eq. (38).

We thus conclude that the normalized flow field defined by Eq. (35) leads universally to Eqs. (37) and (38), which do not depend on flow parameters or bare parameters μ^2 and u in the original theory. The final result is universal and depends only on the renormalized mass m in the original theory. Even the free massless flow equation ($u_f = m_f = 0$) gives the same result as long as the normalized flow field is used. Note also that the flow field in the NLSM automatically leads to Eqs. (37) and (38) without introducing the non-trivial normalization factor.

5. Induced metric and geometry

5.1. Induced metric

As proposed in Sect. 2, we define the symmetric second-rank tensor field as

$$\hat{g}_{\mu\nu}(z) \equiv h \sum_{a=1}^{N} \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z), \tag{40}$$

where μ, ν run from 0 to d, and h is a constant with mass dimension -2. We can interpret this field as the induced metric on the $(d+1)$-dimensional manifold $\mathbb{R}^+ \times \mathbb{R}^d$ from some manifold in $\mathbb{R}^N$ defined by $\sigma^a(z)$, which classically becomes the $(N-1)$-dimensional sphere.

Using Eq. (38), the VEV of $\hat{g}_{\mu\nu}$ is given by

$$g_{\mu\nu}(z) \equiv \langle \hat{g}_{\mu\nu}(z) \rangle = \begin{pmatrix} g_{\tau\tau}(\tau) & 0 \\ 0 & g_{ij}(\tau) \end{pmatrix}, \tag{41}$$

where

$$g_{\tau\tau}(\tau) = \frac{h\tau^2}{16} \frac{d^2 \log \zeta_0(t)}{dt^2}, \qquad g_{ij}(\tau) = -\delta_{ij} \frac{h}{2d} \frac{d \log \zeta_0(t)}{dt}. \tag{42}$$

Explicitly, $\zeta_0(t)$ is evaluated as

$$\zeta_0(t) = \frac{m^{d-2}e^{2m^2 t}}{(4\pi)^{d/2}}\Gamma(1-d/2, 2m^2 t). \tag{43}$$

See Appendix B for the definition and properties of the incomplete gamma function $\Gamma(a, x)$.

5.2.   The metric in the IR limit

In the IR limit $m\tau \gg 1$, we have

$$\zeta_0(t) \simeq \frac{1}{(4\pi)^{d/2}m^2(2t)^{d/2}}, \tag{44}$$

so that

$$g_{\tau\tau}(\tau) = \frac{hd}{2\tau^2}, \qquad g_{ij}(\tau) = \frac{h\delta_{ij}}{\tau^2}. \tag{45}$$

This result shows that $g_{\mu\nu}$, the VEV of the induced metric $\hat{g}_{\mu\nu}$, describes a Euclidean AdS space in $(d+1)$ dimensions at $m\tau \gg 1$ for any d. Indeed, with a new variable $u = \sqrt{d/2}\tau$, the world line element is expressed as

$$ds^2 = \frac{hd}{2u^2}(du^2 + dx^2). \tag{46}$$

The condition that $ds^2 > 0$ with the Euclidean signature requires $h > 0$.

5.3.   The metric in the UV limit

In the opposite limit $m\tau \ll 1$, on the other hand, we obtain,

$$\zeta_0(t) \simeq \begin{cases} \dfrac{1}{2m} - \dfrac{(2t)^{1/2}}{\sqrt{\pi}} & d = 1, \\[3ex] -\dfrac{\log(2tm^2)}{4\pi} & d = 2, \\[3ex] \dfrac{2}{d-2}\dfrac{(2t)^{1-d/2}}{(4\pi)^{d/2}} & d \geq 3, \end{cases} \tag{47}$$

which leads to

$$g_{\tau\tau}(\tau) \simeq h \begin{cases} \sqrt{\dfrac{2}{\pi}}\dfrac{m}{4\tau} & d = 1, \\[3ex] -\dfrac{1}{\tau^2 \log(m^2\tau^2)} & d = 2, \\[3ex] \dfrac{d-2}{2}\dfrac{1}{\tau^2} & d \geq 3, \end{cases} \tag{48}$$

$$
g_{ij}(\tau) \simeq h\delta_{ij}
\begin{cases}
\sqrt{\dfrac{2}{\pi}\dfrac{m}{\tau}} & d = 1, \\[2ex]
-\dfrac{1}{\tau^2 \log(m^2\tau^2)} & d = 2, \\[2ex]
\dfrac{d-2}{d}\dfrac{1}{\tau^2} & d \geq 3.
\end{cases}
\tag{49}
$$

The VEV of the metric describes a Euclidean AdS space at $d \geq 3$, while a log correction appears at $d = 2$.

6. Einstein tensor

In this section, we consider the VEV of the Einstein tensor $G_{\mu\nu}$. As mentioned in Sect. 2, quantum corrections can be neglected in the large N limit as

$$
\langle G_{\mu\nu}(\hat{g}_{\mu\nu})\rangle = G_{\mu\nu}(\langle \hat{g}_{\mu\nu}\rangle) + O\left(\frac{1}{N}\right).
\tag{50}
$$

Therefore, the VEV of the Einstein tensor $G_{\mu\nu}$ can be calculated from the VEV of the induced metric, $g_{\mu\nu} = \langle \hat{g}_{\mu\nu}\rangle$, in this limit.

6.1. Einstein tensor from $g_{\mu\nu}$

If the metric has the simple form

$$
g_{\mu\nu}(\tau) =
\begin{pmatrix}
B(\tau) & 0 \\
0 & \delta_{ij}A(\tau)
\end{pmatrix},
\tag{51}
$$

then the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/2$ becomes

$$
G_{\mu\nu}(\tau) =
\begin{pmatrix}
G_{\tau\tau} & 0 \\
0 & G_{ij}
\end{pmatrix},
\tag{52}
$$

where

$$
G_{\tau\tau} = \frac{d(d-1)}{8}\frac{\dot{A}^2}{A^2}, \qquad \dot{f} \equiv \frac{df(\tau)}{d\tau},
\tag{53}
$$

$$
G_{ij} = \delta_{ij}\frac{(d-1)}{4}\left[\frac{(d-4)}{2}\frac{\dot{A}^2}{AB} + 2\frac{\ddot{A}}{B} - \frac{\dot{A}\dot{B}}{B^2}\right].
\tag{54}
$$

From Eq. (42), we have

$$
A(\tau) = -\frac{h}{d}m^2 Y(x), \quad B(\tau) = \frac{h}{4}m^2(m\tau)^2\frac{dY(x)}{dx},
\tag{55}
$$

with $x = m^2\tau^2/2$, where

$$
Y(x) = \frac{d}{dx}\left[\log\left\{\frac{m^{d-2}e^x}{(4\pi)^{d/2}}\Gamma(1-d/2, x)\right\}\right] = 1 + \frac{d}{dx}\log\Gamma(1-d/2, x).
\tag{56}
$$

After a little algebra, we obtain

$$G_{\tau\tau}(\tau) = \frac{d(d-1)m^2}{8}(m\tau)^2 \left[\frac{d\log Y(x)}{dx}\right]^2_{x=m^2\tau^2/2}, \tag{57}$$

$$G_{ij}(\tau) = -\delta_{ij}\frac{(d-1)m^2}{d}\left[\frac{(d-4)}{2}\frac{d\log Y(x)}{dx} + \frac{d}{dx}\left\{\log\left(\frac{dY(x)}{dx}\right)\right\}\right]_{x=m^2\tau^2/2}, \tag{58}$$

so that $G_{\mu\nu} = 0$ at $d = 1$.

Two scalar functions $\Lambda_\tau(m\tau)$ and $\Lambda_d(m\tau)$, defined by

$$G_{\tau\tau}(\tau) = -\Lambda_\tau(m\tau)g_{\tau\tau}(\tau), \qquad G_{ij}(\tau) = -\Lambda_d(m\tau)g_{ij}(\tau), \tag{59}$$

become

$$\Lambda_\tau(m\tau) = -\frac{d(d-1)}{2h}\left.\frac{\frac{d\log Y(x)}{dx}}{Y(x)}\right|_{x=m^2\tau^2/2}, \tag{60}$$

$$\Lambda_d(m\tau) = \Lambda_\tau(m\tau) + \delta\Lambda(m\tau), \qquad \frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} = \frac{2}{d}\left[\frac{\frac{d}{dx}\log\left(\frac{dY(x)}{dx}\right)}{\frac{d\log Y(x)}{dx}} - 2\right]_{x=m^2\tau^2/2}. \tag{61}$$

6.2. IR behaviors

For $m\tau \gg 1$, we have

$$\Lambda_\tau(m\tau) \simeq -\frac{(d-1)}{h}\left[1 + \frac{2(d+2)}{(m^2\tau^2)^2} + \cdots\right], \tag{62}$$

$$\frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} \simeq \frac{(2d+4)}{d}\frac{4}{(m^2\tau^2)^2}. \tag{63}$$

Using the above results, we obtain

$$G_{\mu\nu}(\tau) \simeq -\Lambda g_{\mu\nu}(\tau), \quad m\tau \gg 1, \tag{64}$$

where the cosmological constant is given by $\Lambda = -(d-1)/h$. Therefore the space defined by the metric $g_{\mu\nu}$ is the Euclidean AdS, except for $d = 1$ where the space is flat ($\Lambda = 0$). This is consistent with the result in the previous section.

6.3. UV limit

We next consider the behavior of $G_{\mu\nu}$ in the UV limit ($m\tau \ll 1$).

At $d = 2$, we have

$$\Lambda_\tau(m\tau) \simeq \frac{1}{h}\left(L + 1 - 2TL^2 - T\right), \qquad \frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} \simeq -\frac{L}{(L+1)^2}, \qquad m\tau \ll 1, \tag{65}$$

where $T \equiv m^2\tau^2/2$ and $L = \log(T) + \gamma$. Therefore we obtain

$$G_{\mu\nu}(\tau) \simeq -\Lambda(m\tau)g_{\mu\nu}(\tau), \qquad m\tau \ll 1, \tag{66}$$

where the scalar function $\Lambda(\tau)$ has a logarithmic singularity at $m\tau = 0$:

$$\Lambda(m\tau) \simeq \frac{\log(m^2\tau^2)}{h}. \tag{67}$$

Similarly, we obtain

$$\Lambda_\tau(m\tau) \simeq -\frac{6}{h}\left[1 - \frac{3\sqrt{\pi}}{2}T^{1/2} + 8T - \frac{25\sqrt{\pi}}{2}T^{3/2}\right], \tag{68}$$

$$\frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} \simeq \frac{\sqrt{\pi}}{2}T^{1/2} + \frac{(3\pi - 16)}{3}T + 5\sqrt{\pi}\frac{(9\pi - 20)}{24}T^{3/2} \tag{69}$$

at $d = 3$,

$$\Lambda_\tau(m\tau) \simeq -\frac{6}{h}\left[1 + T(2L + 3) + T^2(9L + 3)\right], \tag{70}$$

$$\frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} \simeq -\frac{1}{2}\left\{T(2L + 5) - T^2(10 + 7L + 6L^2)\right\} \tag{71}$$

at $d = 4$, and

$$\lim_{m\tau \to 0}\Lambda_\tau(m\tau) = -\frac{d(d-1)}{h(d-2)}, \qquad \lim_{m\tau \to 0}\frac{\delta\Lambda(m\tau)}{\Lambda_\tau(m\tau)} = 0 \tag{72}$$

for general $d \geq 3$. Therefore we have

$$G_{\mu\nu}(\tau) \simeq -\Lambda g_{\mu\nu}(\tau), \qquad \Lambda \equiv -\frac{d(d-1)}{h(d-2)}, \qquad m\tau \ll 1, \tag{73}$$

which shows that the space becomes Euclidean AdS in the UV limit ($m\tau \ll 1$) at $d \geq 3$. These results are also consistent with the results in the previous section.

6.4. *Behavior of the Einstein tensor from UV to IR*

If we define the radius of the AdS space R as $R^2 = -1/\Lambda$, we have

$$R_{\text{UV}}^2 = \frac{d-2}{d}R_{\text{IR}}^2 < R_{\text{IR}}^2 \tag{74}$$

at $d \geq 3$. The radius increases toward the infrared.

We show this behavior more explicitly at $d = 3$ and 4, using the formula given in Eqs. (60) and (61). In Fig. 1, $h\Lambda_\tau(m\tau)$ is plotted as a function of $m\tau$ at $d = 3$ (blue solid line) and $d = 4$ (red dashed line), where the horizontal axis is $x = \arctan(m\tau)$, together with asymptotic behaviors at UV ($x \simeq 0$) given in Eqs. (68) and (70), and those at IR ($x \simeq \pi/2$) given in Eq. (62). Starting from $h\Lambda_\tau(0) = -6$ in the UV limit, $h\Lambda_\tau(m\tau)$ increases toward the IR limit with $h\Lambda_\tau(\infty) = -2$ at $d = 3$ and $h\Lambda_\tau(\infty) = -3$ at $d = 4$.

Figure 2, on the other hand, shows $\delta\Lambda(m\tau)/\Lambda_\tau(m\tau)$, which represents the violation of $\Lambda_\tau(m\tau) = \Lambda_d(m\tau)$, as a function of $m\tau$ at $d = 3$ (blue solid line) and $d = 4$ (red dashed line), where the horizontal axis is $x = \arctan(m\tau)$, together with asymptotic behaviors at UV ($x \simeq 0$) given in Eqs. (69) and (71), and those at IR ($x \simeq \pi/2$) given in Eq. (63). As expected, $\delta\Lambda(\tau)/\Lambda_\tau(m\tau)$ becomes zero in both UV and IR limits, and the violation becomes maximum at $m\tau \simeq 0.66$ ($x \simeq 0.58$) or at $m\tau \simeq 0.90$ ($x \simeq 0.73$), which is about 14% at $d = 3$ or 8% at $d = 4$, respectively.
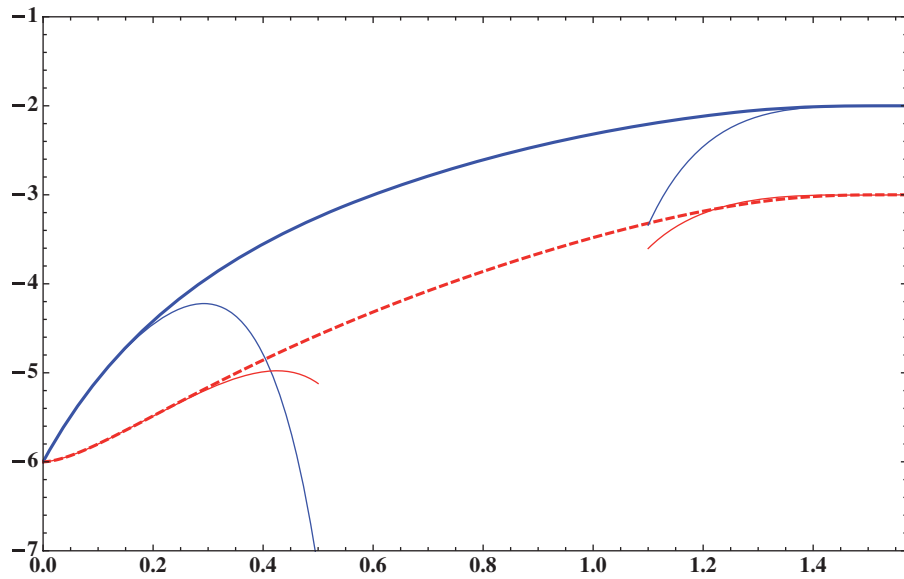
Fig. 1. $h\Lambda_\tau(m\tau)$ as a function of $m\tau$ at $d = 3$ (blue solid line) and $d = 4$ (red dashed line), where the horizontal axis is $x = \arctan(m\tau)$, together with asymptotic behaviors at UV ($x \simeq 0$) and IR ($x \simeq \pi/2$).
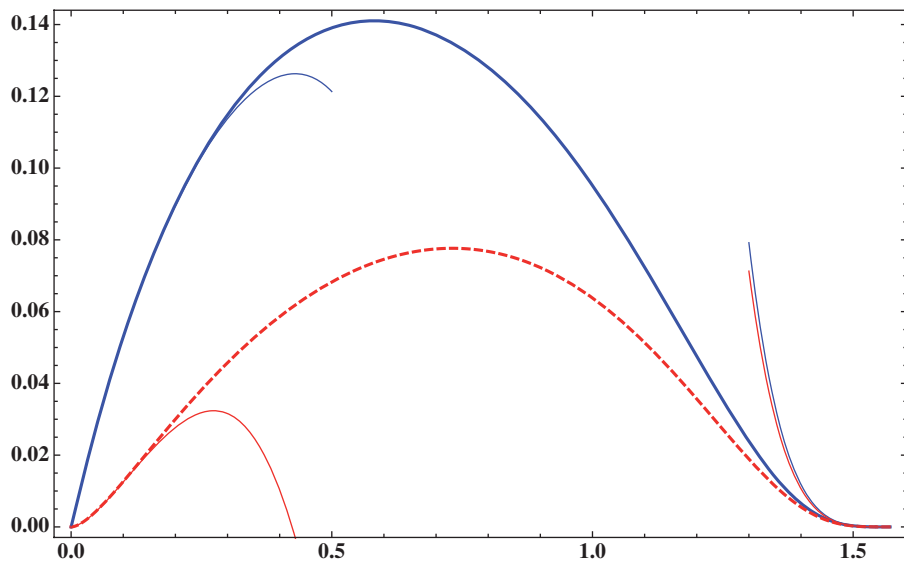
Fig. 2. $\delta\Lambda(m\tau)/\Lambda_\tau(m\tau)$ as a function of $m\tau$ at $d = 3$ (blue solid line) and $d = 4$ (red dashed line), where the horizontal axis is $x = \arctan(m\tau)$, together with asymptotic behaviors at UV ($x \simeq 0$) and IR ($x \simeq \pi/2$).

7.   Summary

In this paper, we apply the method in Ref. [2] to the O(N) invariant φ^4 model, where the ($d + 1$)-dimensional metric is defined from the d-dimensional field theory through gradient flow in the large N limit. As generalizations of the proposal of Ref. [2], we consider the case where the action for the flow equation is different from the action of the original theory. In addition, we have introduced the NLSM normalization for the flow field, with which the normalized flow field only depends on the renormalized mass of the original d-dimensional theory. Using this normalized flow field, we define the ($d + 1$)-dimensional induced metric, which is shown to describe a Euclidean AdS space in both UV and IR limits at $d \geq 3$.

The induced metric, and thus the geometry, from the flow field with the NLSM normalization, depends only on the renormalized mass m in the original d-dimensional φ^4 theory, but neither bare parameters (μ^2, u) separately nor flow parameters (μ_f^2, u_f) at all. This uniqueness of the induced geometry from the NLSM flow may be natural, since the $O(N)$ φ^4 theory becomes a free massive field theory in the large N limit, which does not depend explicitly on the bare coupling constant u including the free and the NLSM limits. In this sense, the large N scalar field theory in d dimensions corresponds to a $(d + 1)$-dimensional classical geometry in the large N limit, which becomes the Euclidean AdS in UV and IR limits at $d > 2$. A posteriori, the NLSM normalization turns out to be an interesting choice to define the RG transformation, as an AdS space emerges in both UV and IR limits.

Since the information about the renormalized coupling constant appears at the next-to-leading order (NLO) in the large N expansion of the d-dimensional φ^4 model, our next task is to obtain the solution of the flow equation at NLO. Using the solution, we then evaluate quantum corrections to the classical geometry such as the propagation of the induced metric $\hat{g}_{\mu\nu}$ on the classical background. It is interesting to see how the information about interactions in the large N field theory appears in quantum corrections to the induced metric. Although it is rather difficult to solve the flow equation at NLO [15], we are currently working on this problem.

Appendix A. Divergence of the flow field in perturbation theory

In this appendix, we show that an extra divergence appears in the flow field of the $\lambda \varphi^4$ theory, if the bare action is employed for the flow equation, together with perturbation theory in the coupling [12].[2] Although this kind of divergence can be avoided by using the renormalized flow equation such as the free flow equation in [12,17], it is relevant to our flow action, which contains bare parameters of the original theory. We then discuss how this divergence disappears non-perturbatively in the large N limit.

A.1. Renormalization in the original theory

For simplicity, we consider the model at $d = 2, 3$ defined by

$$S = \int d^d x \left[\frac{1}{2} \partial^k \varphi(x) \cdot \partial_k \varphi(x) + \frac{\mu^2}{2} \varphi^2(x) + \frac{\lambda}{4!} \varphi^2(x)^2 \right], \tag{A1}$$

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[2] H. Suzuki, private communication.

where the renormalization is made by

$$\mu^2 = Z_m m^2 = m^2 + \delta m^2, \quad \varphi = Z_\varphi \varphi_R, \quad \lambda = Z_\lambda \lambda_R. \tag{A2}$$

At $d = 2, 3$, we have

$$\delta m^2 = -\frac{N+2}{6}\lambda_R Z(m) + O(\lambda_R^2), \quad Z_\varphi = 1 + O(\lambda_R^2), \quad Z_\lambda = 1 + O(\lambda_R^2). \tag{A3}$$

Hereafter we neglect $O(\lambda_R^2)$ contributions. We then have

$$\langle \varphi^a(x)\varphi^b(y)\rangle = \delta^{ab}\int dp \, \frac{e^{ip(x-y)}}{p^2 + m^2}. \tag{A4}$$

A.2.   Divergence in the flowed field

We now consider the flow equation, given by

$$\dot{\phi}^a(t,x) = (\Box - \mu^2)\phi^a(t,x) - \frac{\lambda}{6}\phi^a(t,x)\phi^2(t,x), \quad \phi^a(0,x) = \varphi^a(x) \tag{A5}$$

at $O(\lambda)$, where we take $\mu_f = \mu$ and $\lambda_f = \lambda$ for simplicity.

Setting $\phi = \phi_0 + \lambda\phi_1$, we obtain

$$\phi_0^a(t,x) = \int dy \, K_t(x-y)\varphi^a(y), \tag{A6}$$

$$K_t(x) = \int dp \, e^{ipx}e^{-t(p^2+m^2)}, \quad K_0(x) = \delta^{(d)}(x), \tag{A7}$$

and

$$\lambda\phi_1^a(t,x) = -\int_0^t ds \int dy \, K_{t-s}(x-y)\phi_0^a(s,y)\left[\delta m^2 + \frac{\lambda}{6}\phi_0^2(s,y)\right]. \tag{A8}$$

The two-point function for the flow field is given by

$$\begin{aligned}
\langle\phi^a(t,x)\phi^b(s,y)\rangle &= \langle\phi_0^a(t,x)\phi_0^b(s,y)\rangle \\
&\quad + \langle\lambda\phi_1^a(t,x)\phi_0^b(s,y)\rangle + \langle\phi_0^a(t,x)\lambda\phi_1^b(s,y)\rangle + O(\lambda^2),
\end{aligned} \tag{A9}$$

where we have

$$\langle\phi_0^a(t,x)\phi_0^b(s,y)\rangle = \delta^{ab}\int dp \, \frac{e^{-(s+t)(p^2+m^2)+ip(x-y)}}{p^2 + m^2}, \tag{A10}$$

$$\begin{aligned}
\langle\lambda\phi_1^a(t,x)\phi_0^b(s,y)\rangle = &-\delta m^2 \int_0^t dt_1 \int dx_1 \, K_{t-t_1}(x-x_1)\langle\phi_0^a(t_1,x_1)\phi_0^b(s,y)\rangle \\
&- \frac{\lambda}{6}\int_0^t dt_1 \int dx_1 \, K_{t-t_1}(x-x_1)\langle\phi_0^a(t_1,x_1)\phi_0^2(t_1,x_1)\phi_0^b(s,y)\rangle.
\end{aligned} \tag{A11}$$

The first term is evaluated as

$$= -\delta^{ab}\delta m^2 \int_0^t dt_1 \int dk\, e^{-(t-t_1)(k^2+m^2)+ik(x-x_1)} \int dp\, \frac{e^{-(t_1+s)(p^2+m^2)+ip(x_1-y)}}{p^2+m^2}$$

$$= -\delta m^2 t \langle \phi_0^a(t,x)\phi_0^b(s,y)\rangle, \tag{A12}$$

while the second one is

$$= -\frac{(N+2)\lambda}{6} \int_0^t dt_1 \int dp\, \frac{e^{-2t_1(p^2+m^2)}}{p^2+m^2} \langle \phi_0^a(t,x)\phi_0^b(s,y)\rangle$$

$$= \frac{(N+2)\lambda}{12} J_1(t)\langle \phi_0^a(t,x)\phi_0^b(s,y)\rangle, \tag{A13}$$

where

$$J_1(t) \equiv \int dp\, \frac{e^{-2t(p^2+m^2)}-1}{(p^2+m^2)^2} \tag{A14}$$

is UV finite at $d = 2, 3$.

We finally obtain

$$\langle \phi^a(t,x)\phi^b(s,y)\rangle = \langle \phi_0^a(t,x)\phi_0^b(s,y)\rangle$$

$$\times \left[1 - (t+s)\delta m^2 + \frac{(N+2)\lambda}{12}\{J_1(t)+J_1(s)\} \right], \tag{A15}$$

which shows that t, s-dependent renormalization is needed to make the two-point function for the flow field UV finite.

A.3.   *Relation to the large N result*

We now consider this divergence from the result in the large N limit:

$$\langle \phi^a(t,x)\phi^b(s,y)\rangle = \frac{\delta^{ab}}{N\sqrt{X(t)X(s)}} \int dp\, \frac{e^{-(t+s)p^2+ip(x-y)}}{p^2+m^2}, \tag{A16}$$

where

$$X(t) = e^{2t\mu^2}\left(1 + \frac{u}{3}\int_0^t dx \int dp\, \frac{e^{-2(p^2+\mu^2)x}}{p^2+m^2}\right)$$

$$= e^{2tm^2}\left[1 + 2t\delta m^2 - \frac{u}{6}\int dp\, \frac{e^{-2t(p^2+m^2)}-1}{(p^2+m^2)^2}\right] + O(\lambda^2). \tag{A17}$$

Therefore, if the perturbative expansion is employed, we have

$$\langle \phi^a(t,x)\phi^b(s,y)\rangle \simeq \frac{\delta^{ab}e^{-(t+s)m^2}}{N(1+(t+s)\delta m^2 - u\{J_1(t)+J_1(s)\}/12)} \int dp\, \frac{e^{-(t+s)p^2+ip(x-y)}}{p^2+m^2}$$

$$\simeq \frac{\delta^{ab}}{N}\left[1 - (t+s)\delta m^2 + \frac{u}{12}\{J_1(t)+J_1(t)\}\right]\int dp\, \frac{e^{-(t+s)(p^2+m^2)+ip(x-y)}}{p^2+m^2}, \tag{A18}$$

which agrees with the result in Eq. (A15) in the large N limit, after replacing $\lambda = u/N$ and $\phi \rightarrow \phi/\sqrt{N}$ in Eq. (A15). This shows that the divergence that appeared in the perturbation expansion disappears in the large N expansion where potentially divergent contributions are summed up to an exponential form.

Appendix B.   Incomplete gamma function

The incomplete gamma function of the second kind, $\Gamma(a, x)$, is defined by

$$\Gamma(a, x) = \int_x^\infty dy\, e^{-y} y^{a-1}, \tag{B1}$$

whose asymptotic behaviors are given by

$$\Gamma(a, x) = \Gamma(a) e^{-x} \left[e^x - x^a \sum_{k=0}^\infty \frac{x^k}{\Gamma(k+1+a)} \right], \quad x \rightarrow 0, \tag{B2}$$

$$\Gamma(a, x) = x^{a-1} e^{-x} \sum_{k=0}^\infty \frac{\Gamma(a)}{\Gamma(a-k)} x^{-k}, \quad x \rightarrow \infty. \tag{B3}$$

Other useful properties of Γ are

$$\Gamma(a+1, x) = a\Gamma(a, x) + x^a e^{-x}, \quad \Gamma(0, x) = -\mathrm{Ei}(-x), \, x > 0, \tag{B4}$$

$$\Gamma(1/2, x) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{x}), \quad \Gamma(1, x) = e^{-x}, \tag{B5}$$

where erfc is a complementary error function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dy\, e^{-y^2}, \qquad \mathrm{erfc}(0) = 1, \tag{B6}$$

and $\mathrm{Ei}(-x)$ is the exponential integral function defined by

$$\mathrm{Ei}(-x) = -\int_x^\infty dy\, \frac{e^{-y}}{y}. \tag{B7}$$

As $x \rightarrow 0^+$, we have

$$\mathrm{Ei}(-x) = \gamma + \log(x) + \sum_{n=1}^\infty \frac{(-x)^n}{n\, n!}, \quad \mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!}, \tag{B8}$$

while as $x \rightarrow \infty$

$$\mathrm{Ei}(-x) = e^{-x} \sum_{n=1}^\infty \frac{(n-1)!}{(-x)^n}, \quad \mathrm{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}x} \left(1 + \sum_{n=1}^\infty \frac{(-1)^n (2n-1)!!}{(2x^2)^n} \right). \tag{B9}$$

Using the above formulae, we obtain the asymptotic behavior of $\Gamma(1 - d/2, 2m^2 t)$. As $t \rightarrow \infty$, we have

$$\Gamma\left(1 - \frac{d}{2}, 2m^2 t\right) = \frac{1}{(2t)^{d/2} m^d} e^{-2m^2 t} \sum_{k=0}^\infty \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1 - \frac{d}{2} - k)} \frac{1}{(2m^2 t)^k}, \tag{B10}$$

while as $t \to 0$,

$$
\Gamma\left(1 - \frac{d}{2}, 2m^2 t\right) =
\begin{cases}
\sqrt{\pi} - 2 \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k (2m^2 t)^{k+1/2}}{(2k+1)k!} & d = 1, \\[3ex]
-\gamma - \log(2m^2 t) - \sum\limits_{k=1}^{\infty} \dfrac{(-2m^2 t)^k}{k k!} & d = 2, \\[3ex]
\dfrac{2e^{-2m^2 t}}{(2m^2 t)^{1/2}} - 2\sqrt{\pi}\,\mathrm{erfc}(\sqrt{2m^2 t}) & d = 3, \\[3ex]
\dfrac{e^{-2m^2 t}}{2m^2 t} + \mathrm{Ei}(-2m^2 t) & d = 4.
\end{cases}
\tag{B11}
$$

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