Title	Supercoset construction of Yang-Baxter deformed AdS5×S5 backgrounds
Author(s)	Hideki, Kyono
Citation	Journal of Physics: Conference Series (2017), 804
Issue Date	2017-04-04
URL	http://hdl.handle.net/2433/225264
Right	Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Type	Conference Paper
Textversion	publisher

Supercoset construction of Yang-Baxter deformed $AdS_5 \times S^5$ backgrounds

2017 J. Phys.: Conf. Ser. 804 012026

(http://iopscience.iop.org/1742-6596/804/1/012026)

IOP Conf. Series: Journal of Physics: Conf. Series **804** (2017) 012026 doi:10.1088/1742-6596/804/1/012026

Supercoset construction of Yang-Baxter deformed $AdS_5 \times S^5$ backgrounds

Hideki Kyono

Department of Physics, Kyoto University, Kitashirakawa Oiwake-cho, Kyoto 606-8502, Japan

E-mail: h_kyono@gauge.scphys.kyoto-u.ac.jp

Abstract. We consider Yang-Baxter deformations of the $AdS_5 \times S^5$ superstring theory. In previous works, the metric and B-field of some well-known string backgrounds concerned with the AdS/CFT correspondence have been obtained as deformations of $AdS_5 \times S^5$ based on r-matrices satisfying the homogeneous Yang-Baxter equation. Recently, the remaining fields including the Ramond-Ramond fields and the dilaton have been derived completely by performing the supercoset construction for abelian r-matrices. We also discuss the deformation with a non-abelian r-matrix and, in this case, the resulting background is not a solution of the type IIB supergravity. This article is based on the original paper [1].

1. Introduction

The Yang-Baxter sigma model [2] provides a good approach to study integrable deformations of two-dimensional non-linear sigma models. This deformed model was originally invented by Klimcik for principal chiral models. The deformation is characterized by an r-matrix satisfying the classical Yang-Baxter equation (CYBE). One can consider various deformations of the original model by changing the classical r-matrix. It ensures the existence of an associated Lax pair. In this sense, it is an integrable deformation.

Subsequently, this method has been generalized to symmetric (super)coset models, especially, to the type IIB superstring theory on the $AdS_5 \times S^5$ background for the case of r-matrices satisfying the modified classical Yang-Baxter equation (mCYBE) [3] or homogeneous CYBE [4, 5]. In fact, the Green-Schwarz (GS) type action can be constructed with a supercoset [6]. The supercoset has the $\mathbb{Z}_4$-grading structure which ensures the kinematical integrability of this system [7, 8]. Thus some integrable deformed string theories may be provided by Yang-Baxter deformations with some classical r-matrices.

So far, some Yang-Baxter deformed $AdS_5 \times S^5$ backgrounds have been studied. The deformed background with the Drinfel'd-Jimbo type r-matrix satisfying the mCYBE [9] was derived by Arutyunov, Borsato and Frolov, which is often called the η-deformed background [10]. They have obtained the deformed metric and Neveu-Schwarz-Neveu-Schwarz (NS-NS) two-form, and recently succeeded in deriving the Ramond-Ramond (R-R) fields and the dilaton by calculation including the fermionic components [11]. It was found that the resulting full background cannot satisfy the equations of motion of the type IIB supergravity, but it is a solution of the "generalized" type IIB supergravity [12]. On the "generalized" type IIB supergravity background the worldsheet theory does not preserve the Weyl invariance but only the scale invariance.

One can consider the deformations with r-matrices satisfying the homogeneous CYBE. The metrics and NS-NS two forms of some well-known string backgrounds concerned with the AdS/CFT correspondence [13] have been obtained as homogeneous Yang-Baxter deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ background in the series of works [14–22]. Furthermore, we have succeeded in deriving the remaining fields including the R-R fields and the dilaton by performing the supercoset construction [1], for the cases including gravity duals of non-commutative gauge theories [23, 24], γ-deformations of S^5 [25, 26] and Schrödinger spacetimes [27]. These examples of deformed backgrounds satisfy the equations of motion of the usual type IIB supergravity.

As long as considering abelian r-matrices, the resulting backgrounds seem to be solutions of the usual type IIB supergravity. The backgrounds of our abelian examples can be also derived from TsT-transformations, that is, the deformations with abelian r-matrices may be equivalent to TsT-transformations. Recently, this correspondence has been proven generally in [28, 29]. On the other hand, the deformed backgrounds with non-abelian r-matrices seem not to be the usual supergravity solutions. We will give one of these examples which is not a supergravity solution. In [30] it was shown that this background satisfies the equations of motion of the "generalized" supergravity. More generally, the Yang-Baxter deformed backgrounds satisfy the "generalized" equations due to the kappa symmetry of the deformed action [31]. This fact is consistent with the result of the previous work [12]. In the end of this article, we mention briefly recent progress.

This article is organized as follows. Section 2 is a brief review of the Yang-Baxter deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. Section 3 is devoted the discussion of the supercoset construction by following the procedure of [11]. Section 4 shows the concrete examples of classical r-matrices. Section 5 concludes the results and mentions briefly recent progress.

2. A brief review of the Yang-Baxter deformation

Here we introduce the Yang-Baxter deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action. The discussion is based on the previous works for the mCYBE [3] or for the homogeneous CYBE [4, 5].

The Yang-Baxter deformed action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring is given by

$$S = -\frac{\sqrt{\lambda_{\rm c}}}{4}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, (\gamma^{ab} - \epsilon^{ab})\, \mathrm{STr}\Big[A_a\, d\circ \frac{1}{1-\eta R_g\circ d}(A_b)\Big]\,, \tag{1}$$

where the left-invariant one-form A_a is defined as

$$A_a(\tau,\sigma) \equiv -g^{-1}\partial_a g\,, \qquad g = g(\tau,\sigma) \in SU(2,2|4) \tag{2}$$

with the world-sheet index $a = (\tau,\sigma)$ and the deformation parameter η. When $\eta = 0$, this deformed action goes back to the original undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ GS action [6]. Here we have supposed the conformal gauge $\gamma^{ab} = \mathrm{diag}(-1,+1)$. Hence there is no coupling of the dilaton to the world-sheet scalar curvature. The anti-symmetric tensor ϵ^{ab} is normalized as $\epsilon^{\tau\sigma} = +1$ and the constant $\lambda_{\rm c}$ is the 't Hooft coupling.

A key ingredient in Yang-Baxter deformations is the operator R_g defined as

$$R_g(X) \equiv g^{-1}R(gXg^{-1})g\,, \qquad X\in \mathfrak{su}(2,2|4)\,, \tag{3}$$

where R is a linear map: $\mathfrak{su}(2,2|4)\to\mathfrak{su}(2,2|4)$ defined as follows[1]

$$R(X) = \mathrm{STr}_2[r(1\otimes X)] = \sum_i \left(a_i\,\mathrm{STr}[b_iX] - b_i\,\mathrm{STr}[a_iX]\right)\,. \tag{4}$$

[1] STr_2 is defined as $\mathrm{STr}_2[(W\otimes X)(Y\otimes Z)] = WY\,\mathrm{STr}[XZ]$.

Here r is a *skew-symmetric* classical r-matrix denoted as

$$r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i) \qquad \text{with} \quad a_i,\ b_i \in \mathfrak{su}(2,2|4)\,. \tag{5}$$

These generators a_i and b_i characterize the directions of deformation. It is important for the integrability that this R-operator satisfies the following equation,

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = c\,[X, Y] \qquad \text{for}\ {}^{\forall}X\,, Y \in \mathfrak{su}(2,2|4)\,. \tag{6}$$

When $c = 0\,(\neq 0)$, it is called the homogeneous (modified) CYBE. In the following, we concentrate on r-matrices satisfying the homogeneous CYBE.

The projection operator d is defined as

$$d \quad \equiv \quad P_1 + 2P_2 - P_3\,, \tag{7}$$

where $P_\ell\,(\ell = 0,1,2,3)$ are projections to the $\mathbb{Z}_4$-graded components of $\mathfrak{su}(2,2|4)$. In particular, $P_0(\mathfrak{su}(2,2|4))$ is a local symmetry of the classical action, $\mathfrak{so}(1,4) \oplus \mathfrak{so}(5)$. The numerical coefficients in the linear combination (7) are fixed by requiring the kappa symmetry [4, 6].

3. Supercoset construction

In this section, we introduce the detail of the supercoset construction of deformed models. The strategy is based on [11]. Our purpose here is to read off the type IIB supergravity fields from the deformed action (1) by comparing it with the canonical form of GS action [32]. Hence we will investigate the deformed action at the quadratic level of fermions.

The canonical form of the Lagrangian at order θ^2 [32] is given by[2],

$$\begin{aligned}
\mathcal{L}^{(\theta^2)} \quad &= \quad -\frac{\sqrt{\lambda_c}}{2}\, i\bar{\Theta}_I(\gamma^{ab}\delta^{IJ} + \epsilon^{ab}\sigma_3^{IJ})\,\tilde{e}_a^m\Gamma_m\widetilde{D}_b^{JK}\Theta_K\,, \\
\widetilde{D}_a^{IJ} \quad &\equiv \quad \delta^{IJ}\left(\partial_a - \frac{1}{4}\tilde{\omega}_a^{mn}\Gamma_{mn}\right) + \frac{1}{8}\sigma_3^{IJ}\tilde{e}_a^m H_{mnp}\Gamma^{np} \\
&\quad -\frac{1}{8}\mathrm{e}^{\Phi}\left[\epsilon^{IJ}\Gamma^p F_p + \frac{1}{3!}\sigma_1^{IJ}\Gamma^{pqr}F_{pqr} + \frac{1}{2\cdot 5!}\epsilon^{IJ}\Gamma^{pqrst}F_{pqrst}\right]\tilde{e}_a^m\Gamma_m\,.
\end{aligned} \tag{8}$$

This Lagrangian contains the dilaton Φ, the three-form field strength $H_3 = dB_2$ (B_2 : NS-NS two-form), the one-form field strength $F_1 = d\chi$ (χ : axion or R-R scalar), the three-form field strength $F_3 = dC_2$ (C_2 : R-R two-form), and the five-form field strength $F_5 = dC_4$ (C_4 : R-R four-form). In the following steps, we rewrite the deformed Lagrangian (1) in order to make it easy to read off the type IIB supergravity fields by using the canonical form (8).

3.1. Left-invariant one-form

First of all, it is needed to determine the expression of the left-invariant one-form A. We parametrize the group element g as $g = g_b g_f$ where g_b includes only bosonic generators and g_f includes only fermionic ones. Then the bosonic left-invariant one-form $A_b \equiv g_b^{-1}\mathrm{d}g_b$ can be expressed as

$$A_b \quad = \quad e^m\mathbf{P}_m + \frac{1}{2}\omega^{mn}\,\mathbf{J}_{mn}\,, \tag{9}$$

[2] For the notation, please see the original paper [1].

where e^m is the vielbein for the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime and ω^{mn} is the spin connection. When the fermionic component g_f is parametrized as $g_f = \exp(\mathbf{Q}^I\theta_I)$, the full left-invariant one-form A can be written as follows,

$$\begin{aligned} A &= (e^m + \frac{i}{2}\bar{\theta}_I\gamma^m D^{IJ}\theta_J)\,\mathbf{P}_m - \mathbf{Q}^I\,D^{IJ}\theta_J + \frac{1}{2}\omega^{mn}\,\mathbf{J}_{mn} \\ &\quad -\frac{1}{4}\epsilon^{IJ}\bar{\theta}_I(\gamma^{\check{m}\check{n}}\,\mathbf{J}_{\check{m}\check{n}} - \gamma^{\hat{m}\hat{n}}\,\mathbf{J}_{\hat{m}\hat{n}})D^{JK}\theta_K\,. \end{aligned} \tag{10}$$

Here the covariant derivative for θ is defined as

$$D^{IJ}\theta_J = \delta^{IJ}\left(d\theta_J - \frac{1}{4}\omega^{mn}\gamma_{mn}\theta_J\right) + \frac{i}{2}\epsilon^{IJ}\,e^m\,\gamma_m\theta_J\,. \tag{11}$$

The last term represents the contribution of the R-R five-form field strength.

For later convenience, it is helpful to rearrange the above expansion of A with respect to the order of θ as follows,

$$A = A_{(0)} + A_{(1)} + A_{(2)}\,.$$

Here $A_{(p)}$ is the p-th order of θ and the explicit expressions of $A_{(p)}$ are given by

$$\begin{aligned} A_{(0)} &= e^m\,\mathbf{P}_m + \frac{1}{2}\omega^{mn}\,\mathbf{J}_{mn}\,, \\ A_{(1)} &= -\mathbf{Q}^I\,D^{IJ}\,\theta_J\,, \\ A_{(2)} &= \frac{i}{2}\bar{\theta}_I\gamma^m D^{IJ}\theta_J\,\mathbf{P}_m - \frac{1}{4}\epsilon^{IJ}\bar{\theta}_I(\gamma^{\check{m}\check{n}}\,\mathbf{J}_{\check{m}\check{n}} - \gamma^{\hat{m}\hat{n}}\,\mathbf{J}_{\hat{m}\hat{n}})D^{JK}\theta_K\,. \end{aligned} \tag{12}$$

3.2. Deformation operator

Furthermore, it is necessary to expand the deformation operator in terms of θ. θ is contained in the operator R_g through the group element g.

Let us define the deformation operator $\mathcal{O}$ and expand it in terms of θ as

$$\begin{aligned} \mathcal{O} &\equiv 1 - \eta R_g \circ d \\ &= \mathcal{O}_{(0)} + \mathcal{O}_{(1)} + \mathcal{O}_{(2)} + \mathcal{O}(\theta^3)\,. \end{aligned} \tag{13}$$

Similarly, for the inverse operator, $\mathcal{O}^{\rm inv}_{(i)}$ is defined as

$$\begin{aligned} \mathcal{O}^{\rm inv} &\equiv \frac{1}{1-\eta R_g \circ d} \\ &= \mathcal{O}^{\rm inv}_{(0)} + \mathcal{O}^{\rm inv}_{(1)} + \mathcal{O}^{\rm inv}_{(2)} + \mathcal{O}(\theta^3)\,. \end{aligned} \tag{14}$$

Here, due to the relation $\mathcal{O}\circ\mathcal{O}^{\rm inv} = 1$, each of the components $\mathcal{O}^{\rm inv}_{(p)}$ $(p = 0, 1, 2)$ can be expressed as follows,

$$\begin{aligned} \mathcal{O}^{\rm inv}_{(0)} &= \frac{1}{1-\eta R_{g_b}\circ d}\,, \\ \mathcal{O}^{\rm inv}_{(1)} &= -\mathcal{O}^{\rm inv}_{(0)}\circ\mathcal{O}_{(1)}\circ\mathcal{O}^{\rm inv}_{(0)}\,, \\ \mathcal{O}^{\rm inv}_{(2)} &= -\mathcal{O}^{\rm inv}_{(0)}\circ\mathcal{O}_{(2)}\circ\mathcal{O}^{\rm inv}_{(0)} - \mathcal{O}^{\rm inv}_{(1)}\circ\mathcal{O}_{(1)}\circ\mathcal{O}^{\rm inv}_{(0)}\,. \end{aligned} \tag{15}$$

In the following, we will concentrate only on the bosonic deformations, that is, the generators a_i and b_i are included in $\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)$. Then the action of $R_{g_\mathrm{b}}\circ d$ for the basis of $\mathfrak{su}(2,2|4)$ can be evaluated as below,

$$\begin{aligned} R_{g_\mathrm{b}}\circ d(\mathbf{P}_m) &= 2\left(\lambda_m{}^n\mathbf{P}_n+\frac{1}{2}\lambda_m{}^{np}\mathbf{J}_{np}\right)\,,\\ R_{g_\mathrm{b}}\circ d(\mathbf{J}_{mn}) &= 0\,,\qquad R_{g_\mathrm{b}}\circ d(\mathbf{Q}^I)=0\,. \end{aligned}\tag{16}$$

Here $\lambda_m{}^n$ and $\lambda_m{}^{np}$ are defined as below from the relation in (4),

$$\begin{aligned} \lambda_m{}^n &\equiv (a_i^{g_\mathrm{b}})^n\,(b_i^{g_\mathrm{b}})_m-(b_i^{g_\mathrm{b}})^n\,(a_i^{g_\mathrm{b}})_m\,,\\ \lambda_m{}^{np} &\equiv (a_i^{g_\mathrm{b}})^{np}\,(b_i^{g_\mathrm{b}})_m-(b_i^{g_\mathrm{b}})^{np}\,(a_i^{g_\mathrm{b}})_m\,, \end{aligned}\tag{17}$$

where $(a_i^{g_\mathrm{b}})^m$, $(a_i^{g_\mathrm{b}})^{mn}$, $(b_i^{g_\mathrm{b}})^m$ and $(b_i^{g_\mathrm{b}})^{mn}$ are defined as

$$\begin{aligned} a_i^{g_\mathrm{b}} &\equiv g_\mathrm{b}^{-1}\,a_i\,g_\mathrm{b}=(a_i^{g_\mathrm{b}})^m\,\mathbf{P}_m+\frac{1}{2}(a_i^{g_\mathrm{b}})^{mn}\,\mathbf{J}_{mn}\,,\\ b_i^{g_\mathrm{b}} &\equiv g_\mathrm{b}^{-1}\,b_i\,g_\mathrm{b}=(a_i^{g_\mathrm{b}})^m\,\mathbf{P}_m+\frac{1}{2}(b_i^{g_\mathrm{b}})^{mn}\,\mathbf{J}_{mn}\,. \end{aligned}\tag{18}$$

Now the action of $\mathcal{O}^{\mathrm{inv}}_{(0)}$, $\mathcal{O}^{\mathrm{inv}}_{(1)}$ and $\mathcal{O}^{\mathrm{inv}}_{(2)}$ can be examined as follows.

The action of $\mathcal{O}^{\mathrm{inv}}_{(0)}$ is given by

$$\begin{aligned} \mathcal{O}^{\mathrm{inv}}_{(0)}(\mathbf{P}_m) &\equiv k_m{}^n\,\mathbf{P}_n+\frac{1}{2}l_m{}^{np}\,\mathbf{J}_{np}\,,\\ \mathcal{O}^{\mathrm{inv}}_{(0)}(\mathbf{J}_{mn}) &= \mathbf{J}_{mn}\,,\qquad \mathcal{O}^{\mathrm{inv}}_{(0)}(\mathbf{Q}^I)=\mathbf{Q}^I\,, \end{aligned}\tag{19}$$

where $k_m{}^n$ is determined from $\lambda_m{}^n$ by the following relation,

$$k_m{}^n=(\delta-2\eta\lambda)^{-1}{}_m{}^n\,.\tag{20}$$

When $\eta=0$, $k_m{}^n$ is reduced to $\delta_m{}^n$. Here we have not displayed the explicit form of $l_m{}^{np}$ because it does not appear in the final expression due to the presence of the projection operators.

Then the action of $\mathcal{O}^{\mathrm{inv}}_{(1)}$ is written as

$$\begin{aligned} \mathcal{O}^{\mathrm{inv}}_{(1)}(\mathbf{P}_m) &= i\epsilon^{IJ}k_m{}^n\eta\lambda_n{}^p\,\mathbf{Q}^J\,\gamma_p\theta_I+\frac{1}{2}\delta^{IJ}k_m{}^n\eta\lambda_n{}^{pq}\,\mathbf{Q}^J\,\gamma_{pq}\theta_I\,,\\ \mathcal{O}^{\mathrm{inv}}_{(1)}(\mathbf{J}_{mn}) &= 0\,,\\ \mathcal{O}^{\mathrm{inv}}_{(1)}(\mathbf{Q}^I) &= i\sigma_3^{IJ}\,k_m{}^p\eta\lambda^{nm}\,\bar\theta_J\gamma_n\,\mathbf{P}_p+\frac{1}{2}\sigma_1^{IJ}\,k_m{}^q\eta\lambda^{m,np}\,\bar\theta_J\gamma_{np}\,\mathbf{P}_q+\text{terms with }\mathbf{J}\,. \end{aligned}\tag{21}$$

Here the terms proportional to $\mathbf{J}_{mn}$ are not explicitly written down because they do not contribute to the final expression.

Finally, the action of $\mathcal{O}^{\mathrm{inv}}_{(2)}$ is evaluated as

$$\begin{aligned} \mathcal{O}^{\mathrm{inv}}_{(2)}(\mathbf{P}_m) &= \bar\theta_I\left[\delta^{IJ}(\mathcal{M}^{P\delta}_{(2)})_m{}^n+\epsilon^{IJ}(\mathcal{M}^{P\epsilon}_{(2)})_m{}^n+\sigma_1^{IJ}(\mathcal{M}^{P\sigma_1}_{(2)})_m{}^n+\sigma_3^{IJ}(\mathcal{M}^{P\sigma_3}_{(2)})_m{}^n\right]\theta_J\,\mathbf{P}_n\\ &\quad+\text{terms with }\mathbf{J}\,,\\ \mathcal{O}^{\mathrm{inv}}_{(2)}(\mathbf{J}_{mn}) &= 0\,,\qquad \mathcal{O}^{\mathrm{inv}}_{(2)}(\mathbf{Q}^I)=\text{irrelevant terms}\,, \end{aligned}\tag{22}$$

where $\mathcal{M}_{(2)}^{P\delta}$, $\mathcal{M}_{(2)}^{P\epsilon}$, $\mathcal{M}_{(2)}^{P\sigma_1}$, and $\mathcal{M}_{(2)}^{P\sigma_3}$ are defined as

$$
\begin{aligned}
(\mathcal{M}_{(2)}^{P\delta})_m{}^n &\equiv -\frac{i}{4}\Big[(k_r{}^n\gamma^r)(k_m{}^s\eta\lambda_s{}^{pq}\gamma_{pq}) - (k_r{}^n\eta\lambda^{r,pq}\gamma_{pq})(k_m{}^s\gamma_s)\Big]\,,\\
(\mathcal{M}_{(2)}^{P\epsilon})_m{}^n &\equiv -\frac{1}{2}\Big[(k_p{}^n\gamma^p)(k_m{}^s\eta\lambda_s{}^q\gamma_q) - (k_p{}^n\eta\lambda^{pq}\gamma_q)(k_m{}^s\gamma_s)\Big]\,,\\
(\mathcal{M}_{(2)}^{P\sigma_1})_m{}^n &\equiv (k_s{}^n\eta\lambda^{rs}\gamma_r)(k_m{}^q\eta\lambda_q{}^p\gamma_p) + \frac{1}{4}(k_s{}^n\eta\lambda^{s,rt}\gamma_{rt})(k_m{}^u\eta\lambda_u{}^{pq}\gamma_{pq})\,,\\
(\mathcal{M}_{(2)}^{P\sigma_3})_m{}^n &\equiv \frac{i}{2}\Big[(k_s{}^n\eta\lambda^{rs}\gamma_r)(k_m{}^t\eta\lambda_t{}^{pq}\gamma_{pq}) + (k_s{}^n\eta\lambda^{s,rt}\gamma_{rt})(k_m{}^q\eta\lambda_q{}^p\gamma_p)\Big]\,.
\end{aligned}
\tag{23}
$$

Here the terms proportional to $\mathbf{J}_{mn}$ have not been written down on the same reasoning. Furthermore, the explicit expression of $\mathcal{O}_{(2)}^{\text{inv}}(\mathbf{Q}^I)$ is not necessary for our argument because it always leads to higher-order contributions with $\mathcal{O}(\theta^4)$ in the resulting Lagrangian.

3.3. The deformed Lagrangian at quadratic level of θ

Now the Lagrangian in (1) can be rewritten as

$$
\mathcal{L} = -\frac{\sqrt{\lambda_c}}{4}\left(\gamma^{ab} - \epsilon^{ab}\right)\mathrm{STr}\left[\tilde{d}(A_a)\mathcal{O}^{\text{inv}}(A_b)\right]\,,
\tag{24}
$$

where $\tilde{d}$ is the transpose operator of d and it is defined as

$$
\tilde{d} \equiv -P_1 + 2P_2 + P_3\,.
\tag{25}
$$

This Lagrangian can be expanded in terms of θ at quadratic level as

$$
\mathcal{L} = \mathcal{L}_{(0)} + \mathcal{L}_{(2,0,0)} + \mathcal{L}_{(0,0,2)} + \mathcal{L}_{(1,1,0)} + \mathcal{L}_{(0,1,1)} + \mathcal{L}_{(0,2,0)} + \mathcal{L}_{(1,0,1)} + \mathcal{O}(\theta^4)\,.
\tag{26}
$$

Here, $\mathcal{L}_{(0)}$ does not include any θ. The second-order term $\mathcal{L}_{(l,m,n)}$ contains two θs. The set of subscripts (l,m,n) indicates the numbers of θ included in $\tilde{d}(A_a)$, $\mathcal{O}^{\text{inv}}$ and A_b, respectively. For example, in the case of $\mathcal{L}_{(2,0,0)}$, the two θs are included in $\tilde{d}(A_a)$, and there is no θ in $\mathcal{O}^{\text{inv}}$ and A_b. That is, $\mathcal{L}_{(2,0,0)}$ is given by

$$
\mathcal{L}_{(2,0,0)} = -\frac{\sqrt{\lambda_c}}{4}\left(\gamma^{ab} - \epsilon^{ab}\right)\mathrm{STr}\left[\tilde{d}((A_{(2)})_a)\mathcal{O}_{(0)}^{\text{inv}}((A_{(0)})_b)\right]\,.
\tag{27}
$$

In the following, let us see each term of the expansion (26). The first one can be rewritten into the standard form as follows,

$$
\begin{aligned}
\mathcal{L}_{(0)} &= -\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab} - \epsilon^{ab})\,\mathrm{STr}\left[\tilde{d}((A_{(0)})_a)\mathcal{O}_{(0)}^{\text{inv}}((A_{(0)})_b)\right]\\
&= -\frac{\sqrt{\lambda_c}}{2}(\gamma^{ab} - \epsilon^{ab})\,e_a^m e_b^n\,k_{nm}\\
&= -\frac{\sqrt{\lambda_c}}{2}\left[\gamma^{ab}e_\mu^m e_\nu^n\,k_{(mn)}\,\partial_a X^\mu\partial_b X^\nu - \epsilon^{ab}e_\mu^m e_\nu^n\,k_{[nm]}\,\partial_a X^\mu\partial_b X^\nu\right]\,.
\end{aligned}
\tag{28}
$$

Here we have used the relation $e_a^m = e_\mu^m\partial_a X^\mu$, and X^μ are the target-spacetime coordinates. From the last expression of (28), one can obtain the deformed spacetime metric $\widetilde{G}$ and NS-NS two-form B,

$$
\begin{aligned}
\widetilde{G}_{MN} &\equiv e_M^m e_N^n\,k_{(mn)} = \tilde{e}_M^m\tilde{e}_{mN}\,,\\
B_{MN} &\equiv e_M^m e_N^n\,k_{[nm]}\,.
\end{aligned}
\tag{29}
$$

Here, for our later convenience, we have introduced the vielbein $\tilde{e}^m_M$ for the deformed metric. Note that the index M is raised and lowered by $\widetilde{G}^{MN}$ and $\widetilde{G}_{MN}$, respectively.

Then let us evaluate the combination $\mathcal{L}_{(2,0,0)} + \mathcal{L}_{(0,0,2)}$. From the point of view of symmetry, this combination is convenient and can be evaluated as

$$\begin{aligned}
&\mathcal{L}_{(2,0,0)} + \mathcal{L}_{(0,0,2)} \\
= &-\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\big[i\bar{\theta}_I\gamma^m D^{IJ}_a\theta_J\,\mathbf{P}_m\, e^n_b k_n{}^p\,\mathbf{P}_p + ie^m_a\mathbf{P}_m\,\bar{\theta}_I\gamma^n D^{IJ}_b\theta_J\, k_n{}^p\mathbf{P}_p\big] \\
= &-\frac{i\sqrt{\lambda_c}}{4}(\gamma^{ab}-\epsilon^{ab})\,\bar{\theta}_I\left(e^n_b k_{nm}\gamma^m D^{IJ}_a + e^m_a k_{nm}\gamma^n D^{IJ}_b\right)\theta_J\,. \qquad (30)
\end{aligned}$$

By the same reasoning, it is helpful to evaluate the combination $\mathcal{L}_{(1,1,0)} + \mathcal{L}_{(0,1,1)}$. The resulting expression is given by

$$\begin{aligned}
&\mathcal{L}_{(1,1,0)} + \mathcal{L}_{(0,1,1)} \qquad (31) \\
= &-\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\big[\sigma_3^{IJ}\mathbf{Q}^J D^{IK}_a\theta_K\mathcal{O}^{\mathrm{inv}}_{(1)}(e^m_b\mathbf{P}_m) + 2e^m_a\mathbf{P}_m\mathcal{O}^{\mathrm{inv}}_{(1)}(-\mathbf{Q}^I D^{IJ}_b\theta_J)\big] \\
= &-\frac{\sqrt{\lambda_c}}{2}(\gamma^{ab}-\epsilon^{ab})\,\bar{\theta}_I\left[i\eta\lambda_n{}^p\gamma_p\sigma_3^{IJ} - \frac{1}{2}\eta\lambda_n{}^{pq}\gamma_{pq}\sigma_1^{IJ}\right]\left(e^m_b k_m{}^n D^{JK}_a + e^m_a k^n{}_m D^{JK}_b\right)\theta_K\,.
\end{aligned}$$

Finally, $\mathcal{L}_{(0,2,0)}$ and $\mathcal{L}_{(1,0,1)}$ are evaluated as, respectively,

$$\begin{aligned}
\mathcal{L}_{(0,2,0)} &= -\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\left[2e^m_a\mathbf{P}_m\mathcal{O}^{\mathrm{inv}}_{(2)}(e^n_b\mathbf{P}_n)\right] \\
&= -\frac{\sqrt{\lambda_c}}{2}(\gamma^{ab}-\epsilon^{ab})\,e^m_a e^n_b\,\bar{\theta}_I\Big[\epsilon^{IJ}(\mathcal{M}^{P\epsilon}_{(2)})_{nm} + \delta^{IJ}(\mathcal{M}^{P\delta}_{(2)})_{nm} \\
&\qquad + \sigma_1^{IJ}(\mathcal{M}^{P\sigma_1}_{(2)})_{nm} + \sigma_3^{IJ}(\mathcal{M}^{P\sigma_3}_{(2)})_{nm}\Big]\theta_J\,, \qquad (32) \\
\mathcal{L}_{(1,0,1)} &= -\frac{\sqrt{\lambda_c}}{4}(\gamma^{ab}-\epsilon^{ab})\,\mathrm{STr}\left[\sigma_3^{IJ}\mathbf{Q}^J D^{IK}_a\theta_K(-\mathbf{Q}^L D^{LM}_b)\theta_M\right] \\
&= -\frac{i\sqrt{\lambda_c}}{2}\epsilon^{ab}\sigma_3^{IJ}\,\bar{\theta}_I e^m_a\gamma_m D^{JK}_b\theta_K\,. \qquad (33)
\end{aligned}$$

So far, we have derived the deformed Lagrangian at the quadratic level of θ. However, coordinate transformations of X^M and θ_I are still needed to recast the Lagrangian into the canonical form. In the following, we perform a shift of X^M and a rotation of θ_I.

3.4. Shift of X

Let us see the terms with $\gamma^{ab}\partial_b\theta$ in $\mathcal{L}$. The relevant parts are

$$(a)\quad \mathcal{L}^\gamma_{(2,0,0)} + \mathcal{L}^\gamma_{(0,0,2)} \qquad \text{and} \qquad (b)\quad \mathcal{L}^\gamma_{(1,1,0)} + \mathcal{L}^\gamma_{(0,1,1)}\,.$$

One can realize that the terms should appear with δ^{IJ} from the expression of the canonical form (8). There is no obstacle for (a), however (b) involves terms like

$$\frac{\sqrt{\lambda_c}}{2}\,\bar{\theta}_I\,\gamma^{ab}\,\sigma_1^{IJ}\,e^m_a\,k_{(mn)}\eta\,\lambda^{n\,,pq}\gamma_{pq}\partial_b\theta_J\,. \qquad (34)$$

Such terms proportional to σ_1^{IJ} do not appear in the canonical form (8) and hence must be removed somehow. A possible resolution is to shift X as [11]

$$X^\mu \longrightarrow X^\mu + \bar{\theta}_I\delta X^{\mu\,IJ}\theta_J\,, \qquad \delta X^{\mu\,IJ} \equiv \frac{1}{4}\sigma_1^{IJ}e^{n\,\mu}\,\eta\,\lambda_n{}^{pq}\gamma_{pq}\,. \qquad (35)$$

While this shift removes the problematic terms, it generates additional ones,

$$-\frac{\sqrt{\lambda_c}}{2}\, i\bar{\theta}_I\, \gamma^{ab}\delta^{IJ}\Big[-\frac{i}{2}\,\sigma_1^{JK}\, e_a^m\, e_N^n\, k_{(mn)}\partial_b(e^{n\,N}\,\eta\,\lambda_n{}^{pq}\gamma_{pq}) -\frac{i}{4}\sigma_1^{JK}\,\partial_P\tilde{G}_{MN}\partial_a X^M\,\partial_b X^N\, e^{n\,P}\,\eta\,\lambda_n{}^{pq}\gamma_{pq}\Big]\theta_K\,. \tag{36}$$

Note here that these terms do not involve derivatives of θ.

At this stage, the quadratic Lagrangian including γ^{ab} is written down as

$$\begin{aligned}
\mathcal{L}^\gamma \;=\; & -i\frac{\sqrt{\lambda_c}}{2}\,\gamma^{ab}\delta^{IJ}\bar{\theta}_I\Bigg[e_a^p\, k_{(pn)}\,(\eta^{nm}-(-1)^J 2\eta\lambda^{nm})\gamma_m D_b^{JK} \\
& +\frac{1}{2}\sigma_3^{JK}\, e_a^m k_{(mn)}\eta\lambda^{n,\,pq}\gamma_{pq}\, e_b^r\,\gamma_r - \frac{1}{4}\delta^{JK}\, e_a^m\, k_{nm}\left(\eta^{np}-(-1)^J\, 2\eta\,\lambda^{np}\right)\gamma_p\, e_b^q k_q{}^r\,\eta\,\lambda_r{}^{st}\,\gamma_{st} \\
& +\frac{1}{4}\delta^{JK}\, e_a^m k_{mn}\,\eta\,\lambda^{n,\,pq}\,\gamma_{pq}\, e_b^r\, k_{rs}\left(\eta^{st}-(-1)^J\, 2\eta\,\lambda^{st}\right)\gamma_t \\
& +\frac{i}{2}\epsilon^{JK} e_a^m\, k_{nm}\left(\eta^{np}-(-1)^J\, 2\eta\,\lambda^{np}\right)\gamma_p\, e_b^q\, k_q{}^r\,\eta\,\lambda_r{}^s\,\gamma_s \\
& -\frac{i}{4}\sigma_1^{JK}\, e_a^m k_{(mn)}\,\eta\,\lambda^{n,\,pq}\,\gamma_{pq}\,\omega_b^{rs}\,\gamma_{rs} - \frac{i}{2}\epsilon^{JK}\, e_a^m\, k_{nm}\,\eta\,\lambda^{np}\,\gamma_p\, e_b^q\, k_q{}^r\,\gamma_r \\
& -\frac{i}{4}\sigma_1^{JK}\, e_a^m\, k_{nm}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\, e_b^r\, k_r{}^s\,\eta\,\lambda_s{}^{tu}\,\gamma_{tu} - \frac{i}{2}\sigma_1^{JK}\, e_a^m\, e_M^n\, k_{(mn)}\,\partial_b\left(e^{p,\,M}\,\eta\,\lambda_p{}^{qr}\,\gamma_{qr}\right) \\
& -\frac{i}{4}\sigma_1^{JK}\partial_P\widetilde{G}_{MN}\partial_a\, X^M\,\partial_b\, X^N\, e^{m,\,P}\,\eta\,\lambda_m{}^{np}\,\gamma_{np}\Bigg]\theta_K\,.
\end{aligned} \tag{37}$$

The next step is to see the terms with $\epsilon^{ab}\partial_b\theta$ in $\mathcal{L}$. This part has the terms involving σ_1^{IJ} as well. The sift of X in (35) can eliminate the problematic terms simultaneously, while some additional terms including ϵ^{ab} are again generated. Then the quadratic Lagrangian including ϵ^{ab} is written down as

$$\begin{aligned}
\mathcal{L}^\epsilon \;=\; & -i\,\frac{\sqrt{\lambda_c}}{2}\,\epsilon^{ab}\,\bar{\theta}_I\Bigg[\Big(\delta^{IJ}\, e_a^m\, k_{[mn]}\,\gamma^n + \sigma_3^{IJ}\left[e_a^m\, k_{[mn]}\, 2\,\eta\,\lambda^{np}\,\gamma_p + e_a^m\,\gamma_m)\right]\Big)D_b^{JK} \\
& +i\,\sigma_1^{IJ}\, e_a^m\, k_{[mn]}\,\eta\,\lambda^{n\,,pq}\,\gamma_{pq}\left(-\frac{1}{4}\delta^{JK}\,\omega_b^{rs}\,\gamma_{rs} + \frac{i}{2}\epsilon^{JK} e_b^r\,\gamma_r\right) \\
& +\frac{1}{4}\,\delta^{IK} e_a^m\, k_{nm}\left(\eta^{np}-(-1)^I\, 2\eta\,\lambda^{np}\right)\gamma_p\, e_b^q k_q{}^r\,\eta\,\lambda_r{}^{st}\,\gamma_{st} \\
& -\frac{1}{4}\delta^{IK}\, e_a^m k_{mn}\,\eta\,\lambda^{n,\,pq}\,\gamma_{pq}\, e_b^r\, k_{rs}\left(\eta^{st}-(-1)^I\, 2\eta\,\lambda^{st}\right)\gamma_t \\
& -\frac{i}{2}\epsilon^{IK} e_a^m\, k_{nm}\left(\eta^{np}-(-1)^I\, 2\eta\,\lambda^{np}\right)\gamma_p\, e_b^q\, k_q{}^r\,\eta\,\lambda_r{}^s\,\gamma_s \\
& +\frac{i}{2}\epsilon^{IK}\, e_a^m\, k_{nm}\,\eta\,\lambda^{np}\,\gamma_p\, e_b^q\, k_q{}^r\,\gamma_r + \frac{i}{4}\sigma_1^{IK}\, e_a^m\, k_{nm}\,\eta\,\lambda^{n,pq}\,\gamma_{pq}\, e_b^r\, k_r{}^s\,\eta\,\lambda_s{}^{tu}\,\gamma_{tu} \\
& +\frac{i}{2}\sigma_1^{IK} B_{MN}\partial_a X^M\,\partial_b\left(e^{n\,N}\,\eta\,\lambda_n{}^{pq}\right)\gamma_{pq} \\
& +\frac{i}{4}\sigma_1^{IK}\partial_P B_{MN}\partial_a X^M\partial_b X^N e^{n\,P}\eta\lambda_n{}^{pq}\gamma_{pq}\Bigg]\theta_K\,.
\end{aligned} \tag{38}$$

For the next step, it is convenient to switch from the 16×16 gamma matrices γ to the 32×32 ones Γ, and hence we will work in the 32×32 notation in the following. The lift-up rule is summarized in [1], and it is straightforward to rewrite the Lagrangian.

3.5. Rotation of θ

After shifting X, the resulting derivative terms of θ involving γ^{ab} take the following form,

$$-\frac{\sqrt{\lambda_c}}{2}\, i\bar{\Theta}_I\, \gamma^{ab}\delta^{IJ}\, \tilde{e}_{(I)a}^{\ \ m}\, \Gamma_m \partial_b\, \Theta_J\,. \tag{39}$$

Here, the vielbeins[3] $\tilde{e}_{(I)a}^{\ \ m}$ $(I = 1, 2)$ are defined as

$$\tilde{e}_{(I)a}^{\ \ m} \equiv e_a^p\, k_{(pn)} \Big[\eta^{nm} - (-1)^I 2\eta\lambda^{nm}\Big] \tag{40}$$

and depend on the index I. Hence we need to perform a Lorentz transformation for the spinor θ to remove the I dependence.

The first step is to determine the I-independent form of the vielbeins as a reference frame. Hereafter, it is fixed by taking $I = 1$ in (40) as

$$\tilde{e}_a^m = e_a^p\, k_{(pn)} \Big[\eta^{nm} + 2\eta\lambda^{nm}\Big]\,. \tag{41}$$

Then, by performing a Lorentz transformation for θ, this term can be rewritten as

$$\begin{aligned} &\bar{\Theta}_I\, \tilde{e}_{(I)a}^{\ \ m}\, \bar{U}_{(I)}\Gamma_m\, U_{(I)}\partial_b\Theta_I + (\text{the derivative term of } U) \\ =\ &\bar{\Theta}_I\, \tilde{e}_{(I)a}^{\ \ m}\, \Lambda_{(I)m}{}^n\Gamma_n\, \partial_b\Theta_I + (\text{the derivative term of } U)\,. \end{aligned} \tag{42}$$

Note that the Lorentz transformation performed here depends on the index I.

In order to realize the I-independent form (41), the transformation Λ should be taken as

$$\Lambda_{(I)m}{}^n = \big[\delta_m{}^p + (-1)^I 2\eta\lambda_m{}^p\big](\delta - 2\eta\lambda)^{-1}{}_p{}^n\,. \tag{43}$$

Then the spinor transformation $U_{(I)}$ and its inverse $\bar{U}_{(I)}$ have to be determined through the following relation,

$$\bar{U}_{(I)}\Gamma_m U_{(I)} = \Lambda_{(I)m}{}^n\Gamma_n\,. \tag{44}$$

After all this, we have obtained the canonical form of the Lagrangian[4]. In the canonical form of the Lagrangian (8), the dilaton and R-R field strength appear as the product of them. From this reason, we use the following formula to decide the dilaton,

$$\mathrm{e}^\Phi \ =\ \frac{1}{\det_{10}(\delta_m{}^n + 2\,\eta\,\lambda_m{}^n)^{\frac{1}{2}}}\,, \tag{45}$$

where $\det_D$ means the determinant of a $D \times D$ matrix. Note that this is similar to the formula in [33] for λ-deformation. It works well for well-known examples, including the examples discussed in Sec. 4.

In the actual calculation, we used a concrete expression of a classical r-matrix and computation software like Mathematica or Maple to decide the R-R fields and the dilation. The resulting backgrounds for some examples are presented in Sec. 4[5].

[3] Note that $\tilde{e}_{(I)a}^{\ \ m}$ satisfy the relation

$$\tilde{e}_{(I)a}^{\ \ m}\, \tilde{e}_{(I)b\,m} = e_a^m e_b^n\, k_{(mn)} = \widetilde{G}_{MN}\partial_a X^M \partial_b X^N \qquad (\text{for}\quad I = 1\,,2)\,.$$

[4] To read off the R-R fields, we still have to take the trace of gamma matrices in the quadratic Lagrangian.

[5] The deformed R-R fields can also be determined by using the concrete form of the kappa symmetry of the deformed action as another method to determine. The results are consistent with those from the supercoset construction.

4. Examples

In this section, we show the results for some r-matrices satisfying the homogeneous CYBE. For the following argument, let us introduce the terms "abelian" and "non-abelian" classical r-matrices. Suppose that a classical r-matrix is given by $r = a \wedge b$. It is called "abelian" when a and b commute with each other. If not, it is "non-abelian".

4.1. Gravity duals of noncommutative gauge theories

Let us discuss the following classical r-matrix,

$$r = P_2 \wedge P_3 \,. \tag{46}$$

This is an abelian classical r-matrix and satisfies the homogeneous CYBE. The bosonic part of the deformation with this r-matrix has already been studied in [16]. Here P_μ are the translation operators of $\mathfrak{so}(2,2) \in \mathfrak{so}(2,2|4)$. By this r-matrix, the AdS_5 part is deformed.

Through the general argument in Sec. 3, the following deformed background can be obtained,

$$\begin{aligned} ds^2 &= \frac{-(dx^0)^2 + (dx^1)^2}{z^2} + \frac{z^2\left[(dx^2)^2 + (dx^3)^2\right]}{z^4 + 4\eta^2} + \frac{dz^2}{z^2} + ds^2_{\mathrm{S}^5}\,, \\ B_2 &= \frac{2\eta}{z^4 + 4\eta^2}\, dx^2 \wedge dx^3\,, \\ F_3 &= \frac{8\,\eta}{z^5}\, dx^0 \wedge dx^1 \wedge dz\,, \\ F_5 &= 4\left(\mathrm{e}^{2\Phi}\,\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right)\,, \qquad \Phi = \frac{1}{2}\log\left(\frac{z^4}{z^4 + 4\eta^2}\right)\,. \end{aligned} \tag{47}$$

This is nothing but the solution found in [23, 24] as a gravity dual of noncommutative gauge theories.

4.2. γ-deformations of S^5

We shall discuss three-parameter γ-deformations of S^5 with the following classical r-matrix,

$$r = \frac{1}{8}\left(\nu_3\, h_1 \wedge h_2 + \nu_1\, h_2 \wedge h_3 + \nu_2\, h_3 \wedge h_1\right)\,. \tag{48}$$

Here ν_i $(i = 1, 2, 3)$ are real constant parameters, and h_a $(a = 1, 2, 3)$ are the Cartan generators of $\mathfrak{su}(4)$. This is also an abelian classical r-matrix and satisfies the CYBE. The bosonic part has already been studied in [15]. The remaining task is to perform supercoset construction in order to determine the R-R sector and the dilaton.

Then, by following the general discussion, the full solution presented in [25, 26] can be reproduced as

$$ds^2 = ds^2_{\mathrm{AdS}_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\,\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2\left(\sum_{i=1}^{3}\hat{\gamma}_i\, d\phi_i\right)^2\,, \tag{49}$$

$$\begin{aligned} B_2 &= G\left(\hat{\gamma}_3\,\rho_1^2\rho_2^2\, d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1\,\rho_2^2\rho_3^2\, d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2\,\rho_3^2\rho_1^2\, d\phi_3 \wedge d\phi_1\right)\,, \\ F_3 &= -4\sin^3\alpha\,\cos\alpha\,\sin\theta\,\cos\theta\left(\sum_{i=1}^{3}\hat{\gamma}_i\, d\phi_i\right) \wedge d\alpha \wedge d\theta \\ F_5 &= 4\left(\omega_{\mathrm{AdS}_5} + G\,\omega_{\mathrm{S}^5}\right)\,, \qquad \Phi = \frac{1}{2}\log G\,. \end{aligned} \tag{50}$$

Here we have introduced a scalar function G and $\hat{\gamma}_i$ $(i = 1, 2, 3)$ defined as

$$G^{-1} \equiv 1 + \hat{\gamma}_3^2\,\rho_1^2\rho_2^2 + \hat{\gamma}_1^2\,\rho_2^2\rho_3^2 + \hat{\gamma}_2^2\,\rho_3^2\rho_1^2\,, \qquad \hat{\gamma}_i \equiv \eta\nu_i\,. \tag{51}$$

Three coordinates ρ_i satisfying the constraint $\sum_{i=1}^3 \rho_i^2 = 1$ are parametrized by two angle variables α and θ through the relation,

$$\rho_1 \equiv \sin\alpha\cos\theta\,, \quad \rho_2 \equiv \sin\alpha\sin\theta\,, \quad \rho_3 \equiv \cos\alpha\,. \tag{52}$$

It should be remarked that the resulting background is non-supersymmetric other than for exceptional cases like $\nu_1 = \nu_2 = \nu_3$. But the supercoset construction still works well.

4.3. Schrödinger spacetimes

Let us consider Schrödinger spacetimes by employing the following classical r-matrix,

$$r = \frac{i}{4}\,P_- \wedge (h_1 + h_2 + h_3)\,. \tag{53}$$

Here $P_- \equiv (P_0 - P_3)/\sqrt{2}$ is a light-cone generator in $\mathfrak{su}(2,2)$, and h_1, h_2, h_3 are the Cartan generators in $\mathfrak{su}(4)$. This is also an abelian classical r-matrix and satisfies the CYBE. The bosonic part has already been studied in [18].

After all, the full solution [27] has been reproduced as

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \eta^2\frac{(dx^+)^2}{z^4} + ds^2_{\mathrm{S}^5}\,, \\ B_2 &= \frac{\eta}{z^2}\,dx^+ \wedge (d\chi + \omega)\,, \\ F_5 &= 4\,(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5})\,, \qquad \Phi = \text{const.}, \end{aligned} \tag{54}$$

and the other fields are zero. Here, the light-cone coordinates are defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^3)\,.$$

Note that the R-R sector has not been deformed and the dilaton remains constant, though the expression of the fermionic sector is very complicated in the middle of the computation.

4.4. A non-abelian classical r-matrix

So far, we have considered abelian classical r-matrices, for which it seems likely that the supercoset construction works well. Here we study a non-abelian classical r-matrix.

As for non-abelian classical r-matrices, there is no well-known example of the associated background. A nice candidate for non-abelian classical r-matrices [17] is given by

$$\begin{aligned} r &= \frac{1}{\sqrt{2}}E_{24} \wedge (c_1E_{22} - c_2E_{44}) \qquad \Big[(E_{ij})_{kl} \equiv \delta_{ik}\,\delta_{jl}\Big] \\ &= -\frac{1}{2}P_- \wedge \left[\frac{c_1 + c_2}{2}\,(D - L_{03}) + i\,\frac{c_1 - c_2}{2}\,\left(L_{12} - \frac{i}{2}\,\mathbf{1}_4\right)\right]. \end{aligned} \tag{55}$$

Note here that $\mathbf{1}_4$ is included in the expression and hence the image is extended from $\mathfrak{su}(2,2|4)$ to $\mathfrak{gl}(4|4)$. However, it can be ignored due to the presence of the projection operator in the classical action as pointed out in [19].

To ensure that the resulting metric and NS-NS two-form are real, it is necessary to impose the reality condition [17]

$$c_2 = c_1^* \,. \tag{56}$$

It is now convenient to introduce a_1 and a_2 as follows,

$$a_1 \equiv \frac{c_1 + c_2}{2} = \mathrm{Re}(c_1)\,, \quad a_2 \equiv i\frac{c_1 - c_2}{2} = -\mathrm{Im}(c_1)\,. \tag{57}$$

Note here that the classical r-matrix (55) is non-abelian in general. The case that c_1 is pure imaginary (i.e., $a_1 = 0$) is exceptional and it becomes abelian.

The bosonic part has already been studied well [14,17,19], and the R-R sector and the dilaton can be obtained by performing supercoset construction,

$$\begin{aligned}
ds^2 &= \frac{-2dx^+dx^- + d\rho^2 + \rho^2 d\phi^2 + dz^2}{z^2} - \eta^2\left[(a_1^2 + a_2^2)\frac{\rho^2}{z^6} + \frac{a_1^2}{z^4}\right](dx^+)^2 + ds^2_{\mathrm{S}^5}\,,\\
B_2 &= \eta\left[\frac{a_1x^1 + a_2x^2}{z^4}dx^+\wedge dx^1 + \frac{a_1x^2 - a_2x^1}{z^4}dx^+\wedge dx^2 + a_1\frac{1}{z^3}dx^+\wedge dz\right]\,,\\
F_3 &= 4\eta\left[\frac{a_2x^1 - a_1x^2}{z^5}dx^+\wedge dx^1\wedge dz + \frac{a_1x^1 + a_2x^2}{z^5}dx^+\wedge dx^2\wedge dz + \frac{a_1}{z^4}dx^+\wedge dx^1\wedge dx^2\right]\,,\\
F_5 &= 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right)\,, \qquad \Phi = \mathrm{const}\,.,
\end{aligned} \tag{58}$$

and the other components are zero. Here the light-cone coordinates are defined in the same way as the previous example. Notice that the background (58) does not satisfy the equation of motion of B_2 because the Bianchi identity for F_3 is broken. Thus the classical r-matrix (55) does not lead to a solution of the usual type IIB supergravity.

It is worth noting that the pathology vanishes when $a_1 = 0$. This is an exceptional case in which the classical r-matrix becomes abelian and the background (58) is reduced to the Hubeny-Rangamani-Ross solution [34]. This correspondence was originally argued in [17] and elaborated in [19].

5. Conclusion and recent progress

In this article, we have discussed the supercoset construction in Yang-Baxter deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring theories based on the homogeneous CYBE. For abelian classical r-matrices, perfect agreements have been shown for well-known examples including gravity duals of non-commutative gauge theories, γ-deformations of S^5 and Schrödinger spacetimes. For non-abelian classical r-matrices, we have concentrated on a certain example. The resulting background does not satisfy the equation of motion of the NS-NS two-form. Thus, at this stage, it seems that there would be no problem for abelian classical r-matrices, while there are some potential problems in the non-abelian cases.

Recently, some essential progress has been made. Firstly, it is found that the deformed background (58) is a solution of the "generalized" supergravity [30,31] like as the η-deformation [12]. In general, the GS action on a type II supergravity background has the kappa-symmetry [35]. The converse, however, is not true. The presence of the kappa-symmetry ensures only that the background field is a solution of the "generalized" supergravity and, in general, it is not a usual solution [31]. Because the Yang-Baxter deformed action preserves the kappa-symmetry, the Yang-Baxter deformed backgrounds are solutions of the "generalized" supergravity [36].

In our result, the examples of abelian r-matrices lead supergravity solutions. It is known that these backgrounds can also be derived from TsT-transformations. In general, it have been shown that the Yang-Baxter deformation with an abelian r-matrix corresponds to a certain TsT-transformation [28].

Recently, Borsato and Wulff have discovered the condition for r-matrices to lead solutions of the usual supergravity and it is called the unimodularity condition [36]. Furthermore, they have proven that in the case of the S^5 deformations only abelian classical r-matrices lead the usual supergravity solutions. They have also conjectured that in the case of the deformations of the AdS_5 part, the deforamations are equivalent to sequences of TsT-transformations. Recently, Hoare and Tseytlin have conjectured the equivalence between the homogeneous Yang-Baxter deformations and non-abelian T-duals [29]. They have pointed out that a TsT-transformation can be interpreted as a special case of non-abelian duality and proven it for some examples.

By these works, a relation between the Yang-Baxter deformations with the homogeneous CYBE and (non-abelian) T-dual transformations has been revealed. It is interesting to search such a relation in the case of deformations based on the mCYBE. In [30], it is pointed out that some "generalized" supergravity backgrounds form the the YB deformation can be returned the original undeformed backgrounds by a formal T-duality. It seems also interesting to consider a string interpretation of the "generalized" supergravity background.

Acknowledgments
We are grateful to Andrzej Borowiec, Jerzy Lukierski, Takuya Matsumoto and Jun-ichi Sakamoto for the previous works. We are also very grateful Kentaroh Yoshida for the collaboration of the original work and for helpful comments on the draft. The work of H.K. is supported by the Japan Society for the Promotion of Science (JSPS).

References

[1] H. Kyono and K. Yoshida, "Supercoset construction of Yang-Baxter deformed $AdS_5 \times S^5$ backgrounds," 10.1093/ptep/ptw111 [arXiv:1605.02519 [hep-th]].
[2] C. Klimcik, "Yang-Baxter sigma models and dS/AdS T duality," JHEP **0212** (2002) 051 [hep-th/0210095]; "On integrability of the Yang-Baxter sigma-model," J. Math. Phys. **50** (2009) 043508 [arXiv:0802.3518 [hep-th]]; "Integrability of the bi-Yang-Baxter sigma model," Lett. Math. Phys. **104** (2014) 1095 [arXiv:1402.2105 [math-ph]].
[3] F. Delduc, M. Magro and B. Vicedo, "On classical q-deformations of integrable sigma-models," JHEP **1311** (2013) 192 [arXiv:1308.3581 [hep-th]]; "An integrable deformation of the $AdS_5 \times S^5$ superstring action," Phys. Rev. Lett. **112** (2014) 051601 [arXiv:1309.5850 [hep-th]].
[4] I. Kawaguchi, T. Matsumoto and K. Yoshida, "Jordanian deformations of the AdS_5xS^5 superstring," JHEP **1404**, 153 (2014) [arXiv:1401.4855 [hep-th]].
[5] T. Matsumoto and K. Yoshida, "Yang-Baxter sigma models based on the CYBE," Nucl. Phys. B **893** (2015) 287 [arXiv:1501.03665 [hep-th]].
[6] R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in $AdS_5 \times S^5$ background," Nucl. Phys. B **533** (1998) 109 [hep-th/9805028].
[7] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the $AdS_5 \times S^5$ superstring," Phys. Rev. D **69** (2004) 046002 [hep-th/0305116].
[8] G. Arutyunov and S. Frolov, "Foundations of the $AdS_5 \times S^5$ Superstring. Part I," J. Phys. A **42** (2009) 254003 [arXiv:0901.4937 [hep-th]].
[9] V. G. Drinfel'd, "Hopf algebras and the quantum Yang-Baxter equation," Sov. Math. Dokl. **32** (1985) 254;"Quantum groups," J. Sov. Math. **41** (1988) 898 [Zap. Nauchn. Semin. **155**, 18 (1986)]. M. Jimbo, "A q difference analog of $U(g)$ and the Yang-Baxter equation," Lett. Math. Phys. **10** (1985) 63.
[10] G. Arutyunov, R. Borsato and S. Frolov, "S-matrix for strings on η-deformed $AdS_5 \times S^5$ s," JHEP **1404** (2014) 002 [arXiv:1312.3542 [hep-th]].
[11] G. Arutyunov, R. Borsato and S. Frolov, "Puzzles of eta-deformed $AdS_5 \times S^5$," arXiv:1507.04239 [hep-th].
[12] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A. A. Tseytlin, "Scale invariance of the eta-deformed $AdS_5 \times S^5$ superstring, T-duality and modified type II equations," arXiv:1511.05795 [hep-th].

[13] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999) 1113 [Adv. Theor. Math. Phys. **2** (1998) 231] [hep-th/9711200].

[14] I. Kawaguchi, T. Matsumoto and K. Yoshida, "A Jordanian deformation of AdS space in type IIB supergravity," JHEP **1406**, 146 (2014) [arXiv:1402.6147 [hep-th]].

[15] T. Matsumoto and K. Yoshida, "Lunin-Maldacena backgrounds from the classical Yang-Baxter equation - towards the gravity/CYBE correspondence," JHEP **1406** (2014) 135 [arXiv:1404.1838 [hep-th]].

[16] T. Matsumoto and K. Yoshida, "Integrability of classical strings dual for noncommutative gauge theories," JHEP **1406** (2014) 163 [arXiv:1404.3657 [hep-th]].

[17] T. Matsumoto and K. Yoshida, "Yang-Baxter deformations and string dualities," JHEP **1503** (2015) 137 [arXiv:1412.3658 [hep-th]].

[18] T. Matsumoto and K. Yoshida, "Schrödinger geometries arising from Yang-Baxter deformations," JHEP **1504** (2015) 180 [arXiv:1502.00740 [hep-th]].

[19] S. J. van Tongeren, "On classical Yang-Baxter based deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring," JHEP **1506** (2015) 048 [arXiv:1504.05516 [hep-th]].

[20] S. J. van Tongeren, "Yang-Baxter deformations, AdS/CFT, and twist-noncommutative gauge theory," Nucl. Phys. B **904** (2016) 148 [arXiv:1506.01023 [hep-th]].

[21] T. Matsumoto and K. Yoshida, "Integrable deformations of the $\mathrm{AdS}_5 \times S^5$ superstring and the classical Yang-Baxter equation -*Towards the gravity/CYBE correspondence*-," J. Phys. Conf. Ser. **563** (2014) 1, 012020 [arXiv:1410.0575 [hep-th]]; "Towards the gravity/CYBE correspondence – current status –," J. Phys. Conf. Ser. **670** (2016) no.1, 012033.

[22] T. Kameyama, H. Kyono, J. Sakamoto and K. Yoshida, "Lax pairs on Yang-Baxter deformed backgrounds," JHEP **1511** (2015) 043 [arXiv:1509.00173 [hep-th]].

[23] A. Hashimoto and N. Itzhaki, "Noncommutative Yang-Mills and the AdS / CFT correspondence," Phys. Lett. B **465** (1999) 142 [hep-th/9907166].

[24] J. M. Maldacena and J. G. Russo, "Large N limit of noncommutative gauge theories," JHEP **9909** (1999) 025 [hep-th/9908134].

[25] O. Lunin and J. M. Maldacena, "Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals," JHEP **0505** (2005) 033 [hep-th/0502086].

[26] S. Frolov, "Lax pair for strings in Lunin-Maldacena background," JHEP **0505** (2005) 069 [hep-th/0503201].

[27] C. P. Herzog, M. Rangamani and S. F. Ross, "Heating up Galilean holography," JHEP **0811** (2008) 080 [arXiv:0807.1099 [hep-th]]; J. Maldacena, D. Martelli and Y. Tachikawa, "Comments on string theory backgrounds with non-relativistic conformal symmetry," JHEP **0810** (2008) 072 [arXiv:0807.1100 [hep-th]]; A. Adams, K. Balasubramanian and J. McGreevy, "Hot Spacetimes for Cold Atoms," JHEP **0811** (2008) 059 [arXiv:0807.1111 [hep-th]].

[28] D. Osten and S. J. van Tongeren, "Abelian Yang-Baxter Deformations and TsT transformations," arXiv:1608.08504 [hep-th].

[29] B. Hoare and A. A. Tseytlin, "Homogeneous Yang-Baxter deformations as non-abelian duals of the AdS_5 sigma-model," arXiv:1609.02550 [hep-th].

[30] D. Orlando, S. Reffert, J. i. Sakamoto and K. Yoshida, "Generalized type IIB supergravity equations and non-Abelian classical r-matrices," arXiv:1607.00795 [hep-th].

[31] L. Wulff and A. A. Tseytlin, "Kappa-symmetry of superstring sigma model and generalized 10d supergravity equations," JHEP **1606**, 174 (2016) [arXiv:1605.04884 [hep-th]].

[32] M. Cvetic, H. Lu, C. N. Pope and K. S. Stelle, "T duality in the Green-Schwarz formalism, and the massless / massive IIA duality map," Nucl. Phys. B **573**, 149 (2000) [hep-th/9907202].

[33] T. J. Hollowood, J. L. Miramontes and D. M. Schmidtt, "An Integrable Deformation of the $AdS_5 \times S^5$ Superstring," J. Phys. A **47**, no. 49, 495402 (2014) [arXiv:1409.1538 [hep-th]].

[34] V. E. Hubeny, M. Rangamani and S. F. Ross, "Causal structures and holography," JHEP **0507** (2005) 037 [hep-th/0504034].

[35] M. T. Grisaru, P. S. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, "$\mathcal{N} = 2$ Super-strings in a Supergravity Background," Phys. Lett. B 162 (1985) 116.

[36] R. Borsato and L. Wulff, "Target space supergeometry of η and λ-deformed strings," arXiv:1608.03570 [hep-th].