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The impact of Brexit on trade patterns and industry location: a NEG analysis

Pasquale Commendatore
Ingrid Kubin
Iryna Sushko

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P. Commendatore[a], I. Kubin[b], I. Sushko[c]

[a]University of Naples 'Federico II', Italy,

[b]Vienna University of Economics and Business Administration, Austria,

[c]Institute of Mathematics, NASU, and Kyiv School of Economics, Ukraine

Abstract

We explore the effects of Brexit on trade patterns and on the spatial distribution of industry between the United Kingdom and the European Union and within the EU. Our study adopts a new economic geography (NEG) perspective developing a linear model with three regions, the UK and two separated regions composing the EU. The 3-region framework and linear demands allow for different trade patterns. Two possible ante-Brexit situations are possible, depending on the interplay between local market size, local competition and trade costs: industrial agglomeration or dispersion. Considering a soft and a hard Brexit scenario, the ante-Brexit situation is altered substantially, depending on which scenario prevails. UK firms could move to the larger EU market, even in the peripheral region, reacting to the higher trade barriers, relocation representing a substitute for trade. Alternatively, some EU firms could move in the more isolated UK market finding shelter from the competition inside the EU. We also consider the post-Brexit scenario of deeper EU integration, leading to a weakening of trade links between the EU and the UK. Our analysis also reveals a highly complex bifurcation sequence leading to many instances of multistability, intricate basins of attraction and cyclical and chaotic dynamics.

1 Introduction

The United Kingdom is one of the oldest members of the European Union. It joined already in 1973 during the first round of EU enlargement, after it applied to join for a first time as early as 1961. This was well before the 5th enlargement round in 2004, which was the biggest and in which ten countries joined in (Bulgaria and Romania joined in 2007 and Croatia, the most recent entrant, in 2013). The United Kingdom comprises several highly active regions in manufacturing and in services; it is highly integrated in the international network of trade and factor mobility (that concerns capital as well as labour). The region of Greater London can rightly be considered as one of the economic cores of Europe. However, on June 23, 2016 the UK voted for leaving the European Union. In the ongoing negotiations on the conditions of this exit, various Brexit scenarios emerge, ranging from a hard Brexit (involving a fall back to World Trade Organization rules) to a soft Brexit, in which many features of the Single Market membership would continue (like Norway, which is member of the European Economic Area, but not of the EU). Irrespective of the negotiations' outcome, Brexit will definitely have considerable repercussions in the economic activity across Europe. Empirical studies predict substantial repercussions in trade patterns (see e.g. [10], [5] and [16], who give also a brief survey of other empirical studies on Brexit; for a survey see also [6]): With a hard Brexit, UK exports to the European Union may fall by up to 40%. UK imports from the European Union are also expected to decline, though by a lesser extent. With a differentiated methodological approach [16] expect the decline in imports to be only half of the reduction in exports. Note that [10] take into consideration a continuing deeper integration within the EU even after Brexit.

Several studies ([10]; see also the surveys by [6], [19] and [18]) point out that welfare effects of Brexit do not only derive from a change in trade patterns, but in particular also from a change in the pattern of factor migration, stressing the effects on FDI flows.

As [11] note, the UK is "one of Europe's most popular destination for FDI flows". It derives its attractiveness as industry location not only from supply side factors (such as local comparative advantages) or from the extent of the local UK market, but in particular from the fact that the UK provided access to the Europe's Single Market. With Brexit, the authors expect a marked decline in FDI flows to the UK and, more generally, a marked decline in the attractiveness of the UK as industry location.

To a large extent, effects on trade and on industry location are discussed separately in those empirical studies. However, the perspective of the New Economic Geography (NEG) highlights that they are intimately intertwined. It combines a trade model à la Krugman ([12]) based upon monopolistic competition, trade costs and productive factors that choose location according to expected factor rewards. In these models, access to a larger home market and the possibility to reach the other markets with low trade costs translates into higher factor rewards, which attracts firms to shift their production to this particular location and to serve the local as well as to export to the international market. This market access effect fosters agglomeration of industry in few regions. It is mitigated by a competition effect: more firms in a location reduce factor rewards. In these models, Brexit is represented by an increase in trade costs. This will change the access to international markets and thus trade patterns; but at the same time, the attractiveness of a region for industry location changes. NEG models focus on studying the long-run effects of a change in trade costs simultaneously on industry location and trade patterns.

Since Krugman's seminal contribution ([13]), a plethora of NEG models emerged differing in particular regarding the productive factor that is considered as internationally mobile and the specification of the demand function. At the core of our study is the mobility of high-qualified labour and capital; accordingly, we choose a footloose entrepreneur (FE) model. We use a FE model with a linear (instead of an iso-elastic) demand function, in which "zero trade" situations are possible allowing therefore to shed a very clear light on changes in trade patterns (linear demand versions of the standard 2-region NEG model have been proposed, among others, by [17], [2], [3] and [15]). Finally, we view the EU not as a homogenous integrated area, but rather as split between central and peripheral regions (indeed, with the UK leaving, part of the centre leaves the Union). Therefore, we setup a 3-region model. Our contribution extends previous works on 3-region NEG linear models (see, for example, [1], [4], [8] and [9]) in three directions: i) we assume a different and more general geography of trade barriers allowing for a much larger set of possible trade network structures; ii) we study the impact on dynamics of two different types of trade costs; iii) we apply our study to evaluate the consequences of a timely historical event.

Indeed, in this model, we study various effects of Brexit. Given the notorious analytic complexity of multiregional NEG models (for a comprehensive review on multi-regional NEG modelling, see [7]), we primarily present simulation results complemented by intuitive explanations of the underlying economic forces. We show that two ante Brexit situations are possible, as a NEG perspective suggests. They depend upon the specific interplay between local market size, local competition and trade costs: First, the economy is well-integrated with small local market sizes and low trade costs; NEG models typically predict agglomeration of economic activity. Second, local market sizes are bigger and NEG models suggest an equal distribution of economic activity. Accordingly, we choose two different values for the local market size.

For both options, we study a soft vs a hard Brexit scenario involving different increases in trade cost between the UK and the regions remaining in the Union. In addition, we study the effects of a deeper integration within the Union as a reaction to Brexit.

We show that Brexit triggers a highly complex bifurcation sequence involving many instances of coexisting attractors (with complicate basins of attraction) and of cyclical and complex dynamics. Among our results, we find the following of particular interest:

- Brexit may induce firms to leave the UK and move to the regions within the Union; the firms in the Union (continue to) export, while the firms remaining in the UK do not (any more) – thus, firm relocation acts as substitute for trade;

- most remarkably, Brexit may induce firm relocation also to a peripheral region within the Union (that did not host industry before Brexit);

- envisaging a post- hard Brexit scenario, the progressing of EU integration may lead to a weakening of the trade links between the EU and the UK and put peripheral regions in danger to lose again their industry;

- in some instances, intense competition within the Union may lead to the opposite result with firm moving to the UK to reduce the competitive pressure.

The paper is structured as follows: Section 2 states its main assumptions, whereas Section 3 analyses the short run equilibrium, paying particular attention to the different possible trade structures. Section 4 studies the long run implications of the model and analyses the various Brexit scenarios. Section 5 concludes.

2 Main assumptions

The economy is composed of three regions, labelled R_r with $r = 1, 2, 3$; two sectors: agriculture A and manufacturing M; two types of agents: workers (L, endowed with unskilled labour) and entrepreneurs (E, endowed with human capital). Workers are mobile across sectors but immobile across regions; entrepreneurs migrate across regions but are specific to manufacturing. Finally, the A-sector does not necessarily represents only Agriculture but it could also include the tertiary sector which is typically less mobile than industry.

The three regions have the same endowment of labour, $L_1 = L_2 = L_3 = \frac{L}{3}$, share the same technology and consumer's preferences.

The A-sector is perfectly competitive, constant returns prevail and production involves one unit of labour to produce one unit of the output. In the monopolistically competitive M-sector, the N varieties of a differentiated commodity are produced by using one entrepreneur as a fixed component and η units of labour for each additional unit of output. The total cost of production (TC) is expressed as $TC = \pi_i + w\eta q_i$, where π_i represents the operating profit and the remuneration of one entrepreneur, w the wage rate, η the labour input requirement and q_i the quantity produced of variety i. There are no economies of scope, thus due to increasing returns each firm produces only a variety. Following from the assumption that one entrepreneur is required to activate the production of a variety, the total number of varieties is equal to to the total number of entrepreneurs, $E = N$. Denoting by λ_r the share of entrepreneurs located in region R_r, the number of varieties produced in this region corresponds to $N_r = \lambda_r N = \lambda_r E$.

The representative consumer's (unskilled worker or entrepreneur) preferences are quasi-linear (see [17]), composed of a quadratic sub-utility defining the choice across the N varieties of the M-good and a linear part for the consumption of the A-good:

$$U = \alpha \sum_{i=1}^{N} c_i - \left(\frac{\beta - \delta}{2}\right) \sum_{i=1}^{N} c_i^2 - \frac{\delta}{2} \left(\sum_{i=1}^{N} c_i\right)^2 + C_A, \tag{1}$$

where c_i is the consumption of variety i and C_A the consumption of the A-good. The parameters are interpreted as follows: $\alpha > 0$ is the intensity of preferences over the M-varieties, $\delta > 0$ the degree of substitutability across those varieties and the difference $\beta - \delta$ measures the taste for variety; where $\beta > \delta > 0$.

The budget constraint is

$$\sum_{i=1}^{N} p_i c_i + p_A \overline{C_A}, \tag{2}$$

where p_i is the price of variety i, p_A the price of the agricultural good, y the consumer's income and $\overline{C_A}$ her endowment of the agricultural good, sufficiently large to allow for positive consumption in equilibrium.

The cost of trading varieties of the M-good between regions, let's say from R_r to R_s (or in the opposite direction from R_s to R_r) is $T_{rs}(= T_{sr})$; with $T_{rs} > 0$ for $r \neq s$, $T_{rr} = 0$ and $r, s = 1, 2, 3$. Trade costs separate the regions introducing the spatial dimension into the economy. Different configurations are possible, for our purposes we assume that the trade distance between R_1 and R_3 and R_2 and R_3 is the same, whereas the distance between R_1 and R_2 could be shorter: $T_{13} = T_{23} = T_E \geq T_{12} = T_U$. These assumptions on trade costs describe a 3-region economy where R_1 and R_2 are part of a more integrated area, whereas R_3 could be less integrated with the rest of the economy. Thus, we provide a stylized set-up that can be used to describe different scenarios following the UK's choice to leave the EU. We first consider the effects of an increase in T_E starting from the initial state $T_E = T_U$. We define it as the first phase of Brexit (the exit of the UK from the EU). We then consider the effects of a reduction of T_U when $T_E > T_U$, in order to study the consequences of a second phase following Brexit (EU deeper integration).

3 Short-run equilibrium

In a short-run equilibrium the distribution of entrepreneurs across the regions is given. All markets are in equilbrium. We choose the A-good as the numeraire. From perfect competition in the A-sector, it follows $p_A = w = 1$.

To determine the short-run equilibrium solutions related to the M-sector, we proceed as follows. Maximizing the utility (1) subject to the constraint (2), we obtain the first order conditions for $i = 1, ..., N$:

$$\frac{\partial U}{\partial c_i} = \alpha - (\beta - \delta)c_i - \delta \sum_{i=1}^{N} c_i - p_i = 0,$$

from which

$$p_i = \alpha - (\beta - \delta)c_i - \delta \sum_{i=1}^{N} c_i.$$

Solving for c_i, we obtain the individual linear demand function for each variety i:

$$c_i = \max[0, a - (b + cN)p_i + cP],$$

where $P = \sum_{i=1}^{N} p_i$ is the price index and

$$a = \frac{\alpha}{(N-1)\delta + \beta}, \; b = \frac{1}{(N-1)\delta + \beta} \text{ and } c = \frac{\delta}{(\beta - \gamma)[(N-1)\delta + \beta]}.$$

Moreover, we define $\widetilde{p_i} = \frac{a+cP}{b+cN}$ the cut-off price only below which the demand for variety i is positive: that is, $c_i > 0$ for $p_i < \widetilde{p_i}$.

The representative consumer's indirect utility corresponds to

$$V = S + y + \overline{C_A},$$

where S is the consumer's surplus:

$$S = U - \sum_{i=1}^{N} p_i c_i - C_A = \frac{a^2 N}{2b} + \frac{b + cN}{2} \sum_{i=1}^{N} p_i^2 - aP - \frac{c}{2} P^2.$$

The consumer's demand originating from R_s $(s = 1, 2, 3)$ for a good produced in R_r $(r = 1, 2, 3)$, dropping the subscript i because of symmetric firm behavior (a typical assumption of NEG models), is:

$$c_{rs} = \max[0, a - (b + cN)p_{rs} + cP_s],$$

where p_{rs} is the price of a good produced in R_r and consumed in R_s and

$$P_s = \sum_{k=1}^{3} n_k p_{ks} = \sum_{k=1}^{3} \lambda_k E p_{ks} \tag{3}$$

is the price index in R_s. As before $c_{rs} > 0$ if and only if $p_{rs} < \widetilde{p_s} = \frac{a+cP_s}{b+cN}$.

Taking into account that workers are equally spread across the regions, $L_1 = L_2 = L_3 = \frac{L}{3}$, with segmented markets, the operating profit of a representative firm located in R_r $(r = 1, 2, 3)$ is:

$$\pi_r = \sum_{s=1}^{3} (p_{rs} - \eta - T_{rs}) q_{rs} \left(\frac{L}{3} + \lambda_s E \right). \tag{4}$$

In a short-run equilibrium, demand is equal to supply in each segmented market (labelled $r, s = 1, 2, 3$): $c_{rs} = q_{rs}$. From profit maximization, recalling that $N = E$ and that firms consider the price index as given, the first order conditions follow

$$p_{rr} = \frac{a + cP_r + \eta(b + cE)}{2(b + cE)} = \frac{1}{2}(\widetilde{p}_r + \eta), \tag{5}$$

which is the price that firms quote in the market where they are located, where $\widetilde{p}_r = \frac{a+cP_r}{b+cE}$; and

$$p_{rs} = \begin{cases} \frac{a+cP_s+(\eta+T_{rs})(b+cE)}{2(b+cE)} = \frac{1}{2}(\widetilde{p}_s + \eta + T_{rs}) & if \quad T_{rs} < \widetilde{p}_s - \eta \\ \widetilde{p}_s & if \quad T_{rs} \geq \widetilde{p}_s - \eta \end{cases}, \tag{6}$$

which is the price that a firm located in R_r quotes in the market s, with $r, s = 1, 2, 3$ and $r \neq s$.

Using the demand function and the price solutions, we can write:

$$q_{rr} = (b + cE)(p_{rr} - \eta), \tag{7}$$

which is the quantity sold in the local market; and

$$q_{rs} = \begin{cases} (b + cE)(p_{rs} - \eta - T_{rs}) & if \quad T_{rs} < \widetilde{p}_s - \eta \\ \widetilde{p}_s & if \quad T_{rs} \geq \widetilde{p}_s - \eta \end{cases}, \tag{8}$$

which is the quantity that a firm located in R_r sells in R_s, with $r, s = 1, 2, 3$ and $r \neq s$.

According to (6) and (8), if a firm located in R_r quotes in the market of R_s a price larger or equal than the cut-off price $\widetilde{p}_s$ (i.e. a price which is above the maximum reservation price consumers living in R_s are prepared to pay for a positive quantity of a variety), the export from R_r to R_s is zero. The boundary conditions for trade, as reported in these expressions, are crucial to determine the patterns of trade between the regions, as we shall see in the analysis below.

The indirect utility for a r-entrepreneur is

$$V_r = S_r + \pi_r + \overline{C_A}, \tag{9}$$

where

$$S_r = \frac{a^2 E}{2b} + \frac{b + cE}{2} \sum_{s=1}^{3} \lambda_s E p_{sr}^2 - aP_r - \frac{c}{2} P_r^2 \tag{10}$$

is the surplus enjoyed by a r-entrepreneur as a consumer.

3.1 Trade network structures

From the discussion above, the occurrence of trade between regions depends on trade costs. Above a threshold the price quoted by foreign firms is too high and exports cannot take place. It follows that the trade network structure between the regions is strongly affected by their trade distance; and it also affected by the spatial distribution of industry. In this subsection, we make explicit the conditions for trade between the three regions and verify that not all network structures are possible given the chosen trade costs configuration.

Considering the three regions, R_1, R_2 and R_3, the existence of a trade link from one of them, labelled R_r, to a second one, labeled R_s, depends on trade costs and on competition in the local market originating both from local and foreign firms. The latter is affected by the existence or absence of another link from the third region, labelled R_k, to R_s, with $r, s, k = 1, 2, 3$ and $r \neq s \neq k$. If such a link is absent, r-firms (i.e. those located in R_r) only face competition from the local s-firms; instead, if it is present, r-firms face competition also from k-firms (i.e. those located in R_k) exporting to R_s.

In general, (see above expressions (6) and (8)), the condition for trade (resp. no trade) from R_r to R_s is:

$$T_{rs} < (\geq)\widetilde{p}_s - \eta = 2(p_{ss} - \eta).$$

When trade costs are too high for a link from R_r to R_s, $T_{rs} \geq 2(p_{ss} - \eta)$, and for a link from R_k to R_s as well, $T_{sk} \geq 2(p_{ss} - \eta)$, the price quoted by s-firms in the local market is:

$$p_{ss} = \frac{a + \eta(b + c\lambda_s E)}{2b + c\lambda_s E}.$$

If this equation holds, the condition for no trade from R_r to R_s becomes:

$$T_{rs} \geq \frac{2(a - \eta b)}{2b + c\lambda_s E}. \tag{11}$$

It is useful to rewrite the above condition as

$$\lambda_s \geq \frac{2(a - \eta b - bT_{rs})}{cET_{rs}}. \tag{12}$$

When trade costs are sufficiently low so that $T_{rs} < 2(p_{ss} - \eta)$ and a link from R_r to R_s is allowed, but a link from R_k to R_s is still missing, $T_{ks} \geq 2(p_{ss} - \eta)$, the local price is:

$$p_{ss} = \frac{a + \eta[b + c(\lambda_r + \lambda_s)E] + \frac{T_{rs}}{2} c\lambda_r E}{2b + c(\lambda_r + \lambda_s)E}.$$

If this equation holds, the condition for trade from R_r to R_s is:

$$T_{rs} < \frac{2(a - \eta b)}{2b + c\lambda_s E}. \tag{13}$$

For our purposes we rewrite the above expression as

$$\lambda_s < \frac{2(a - \eta b - bT_{rs})}{cET_{rs}}. \tag{14}$$

Note that since $T_E \geq T_U$, condition (13) can only be applied to $r, s = 1, 2$ and $k = 3$.[1] Finally, looking at conditions (11), (12), (13) and (14), it is easy to verify that the condition for trade (no trade) is less (more) stringent the smaller are λ_s and T_{rs}. That is, trade from R_r and R_s is more likely the less competive is the market in R_s (where the degree of competition is given by the number of local s-firms) and the closer are R_r and R_s; vice versa, a more competitive market in R_s and a long trade distance between R_r and R_s may impede r-firms exporting towards R_s.

When a trade link from R_k to region R_s exists, $T_{sk} < 2(p_{ss} - \eta)$, and trade costs between R_r and R_s are high, $T_{rs} \geq 2(p_{ss} - \eta)$, the price fixed locally by s-firms is:

$$p_{ss} = \frac{a + \eta[b + c(\lambda_s + \lambda_k)E] + \frac{T_{sk}}{2} c\lambda_k E}{2b + c(\lambda_s + \lambda_k)E}.$$

If this equation holds, the condition for the absence of one-way trade from R_r to R_s is:

$$T_{rs} \geq \frac{2(a - \eta b) + cE\lambda_k T_{sk}}{2b + c(\lambda_s + \lambda_k)E}. \tag{15}$$

Note that since $T_E \geq T_U$, this inequality can only be applied for $r = 3$ and $s, k = 1, 2$.[2] It can be alternatively expressed as:

$$\lambda_s \geq \frac{2(a - \eta b - bT_{rs})}{cET_{rs}} + \frac{T_{sk} - T_{rs}}{T_{rs}} \lambda_k. \tag{16}$$

[1] Because of the inequality $T_E \geq T_U$ some of the trade structures are not allowed. Specifically, if a link from R_1 to R_2 is not possible (or from R_2 to R_1) then a link from R_3 to R_2 (or from R_3 to R_1) is not allowed as well (alternatively, if a link from R_3 to R_2 (or from R_3 to R_1) occurs, then a link from R_1 to R_2 (or from R_2 to R_1) must occur as well). Moreover, due to symmetry, if a link from R_1 to R_3 is not possible, then a link from R_2 to R_3 is not possible as well (alternatively, the link from R_1 to R_3 and the link from R_2 to R_3 occur simultaneously).

[2] See the discussion in the previous footnote.

Finally, when $T_{sk} < 2(p_{ss} - \eta)$ and $T_{rs} < 2(p_{ss} - \eta)$, we have that

$$p_{ss} = \frac{a + \eta(b + cE) + \left(\frac{T_{rs}}{2}\lambda_r + \frac{T_{sk}}{2}\lambda_k\right) cE}{2b + cE}.$$

If this equation holds, the condition for the presence of one-way trade between R_r and R_s is:

$$T_{rs} < \frac{2(a - \eta b) + cE\lambda_k T_{sk}}{2b + c(\lambda_s + \lambda_k)E}, \tag{17}$$

that can be alternatively expressed as:

$$\lambda_s < \frac{2(a - \eta b - bT_{rs})}{cET_{rs}} + \frac{T_{sk} - T_{rs}}{T_{rs}}\lambda_k. \tag{18}$$

Finally, looking at conditions (15), (16), (17) and (18), the condition for trade (no trade) from R_r and R_s is less (more) stringent the smaller are λ_s, λ_k (therefore, the larger is λ_r) and T_{rs} and the larger is T_{sk}. That is, trade from R_r and R_s is more (less) likely the less competitive is the market is R_s (where now the degree of competition is also determined by the number of k-firms selling in market s), the closer are Regions R_r and R_s and the farther away are regions R_s and R_k.

Combining these conditions (as shown in the Appendix), the possible trade network structures (NS) are eighteen (see Fig.1), grouped into ten types (which are isomorphic) and characterised as follows:[3]

$NS1$: the three-region economy is fully autarkic, no trade link is present (this is called empty graph in the language of social network analysis (SNA)). Only one network structure of this type exists;

$NS2$: one-way (or unilateral) trade from R_r to R_s, where $r, s = 1, 2$ and $r \neq s$ (single edge in the language of SNA). Two structure of this type exists ($NS21$, $NS22$);

$NS3$: two-way (or bilateral) trade between R_r and R_s, where $r, s = 1, 2$ and $r \neq s$ (mutual edge). Only one structure of this type exists;

$NS4$: one-way trade from R_r to R_s and from R_k to R_s, where $r, s, k = 1, 2, 3$ and $r \neq s \neq k$ (in star). Three structures of this type exists ($NS41$, $NS42$, $NS43$);

$NS5$: two-way trade between R_r and R_s and one-way trade from R_k to R_r, where $r, s = 1, 2$, $k = 3$ and $r \neq s$ (mutual edge + in). Two structures of this type exist ($NS51$, $NS52$);

$NS6$: one-way trade from R_r to R_s, R_r to R_k and R_s to R_k, where $r, s = 1, 2$, $k = 3$ and $r \neq s$ (transitive). Two structures of this type exist ($NS61$, $NS62$).

$NS7$: one-way trade from R_r to R_s and from R_r to R_k and bilateral trade between R_s and R_k, where $r, s, k = 1, 2, 3$ and $r \neq s \neq k$ (mutual edge + double in). Three structures of this type exist ($NS71$, $NS72$, $NS73$);

$NS8$: two-way trade between R_r and R_s, one way trade from R_r to R_k and from R_s to R_k, where $r, s = 1, 2$, $k = 3$ and $r \neq s$ (mutual edge + double out). Only one structure of this type exists;

$NS9$: two-way trade between R_r and R_s and R_s and R_k and one-way trade from R_r to R_k, where $r, s = 1, 2$, $k = 3$ and $r \neq s$ (almost complete graph). Two structure of this type exist ($NS91$, $NS92$);

$NS10$: All regions are engaged in mutual trade (complete graph). Only one structure of this type exists.

Note that in the special case $T_U = T_E$, only eight structures are possible grouped into four isomorphic cases ($NS1$, $NS41$, $NS42$, $NS43$, $NS71$, $NS72$, $NS73$, $NS10$).

3.2 Short-run solutions

To each trade network configuration – which depends on trade costs and on the spatial distribution of entrepreneurs – corresponds a different set of short-run solutions. We cannot present here the whole set of solutions (but see the Appendix). Figs 2a-d present examples of possible configurations of trade costs giving rise to different trade network configurations. All figures are plotted for $a = b = c = \frac{1}{3}$, $\eta = 0$, $E = 10$ and for a) $T_U = T_E = 0.325$, b) $T_U = 0.325$ and $T_E = 0.37$, c) $T_U = 0.325$ and $T_E = 0.45$ and d) $T_U = 0.25$ and $T_E = 0.45$.

[3] These network structures are known in the language of social network analysis as triads.

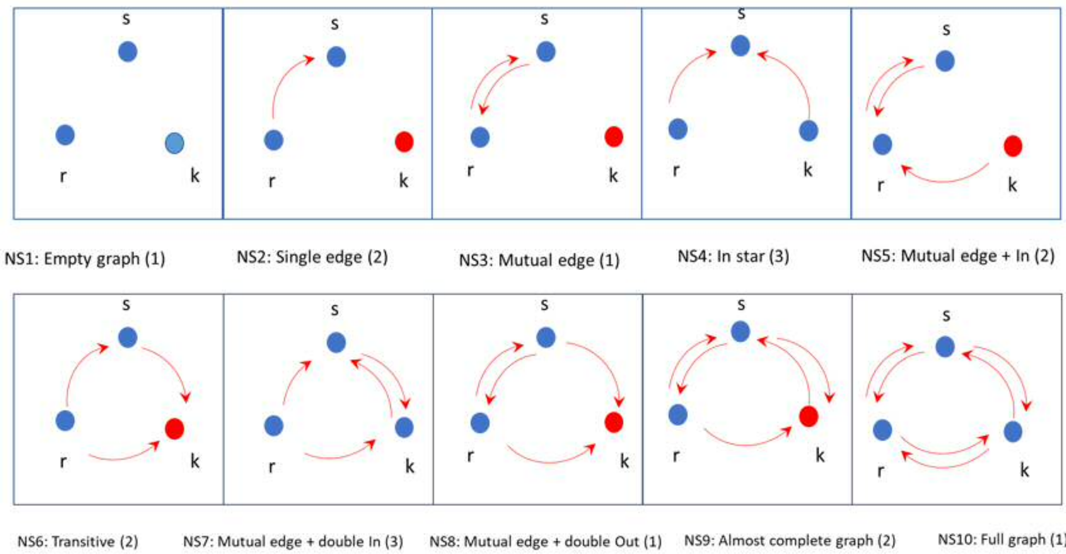

Figure 1: Trade network structures.

In these figures, the combinations of λ_1 and λ_2 (after taking into account that $\lambda_3 = 1 - \lambda_1 - \lambda_2$) that allow for a specific network configuration are represented by areas with different colors. A dotted line, as the lines A_1, A_2 and C, corresponds to the conditions (12) and (14), that is, there is not an incoming trade link from a third region affecting the existence of a link between two regions; and a solid line, as the lines B_1 and B_2, to the conditions (16) and (18), that is, such an incoming link exists (however, notice that when $T_U = T_E$ the two sets of conditions are identical). A line is red when only R_1 and R_2 are involved, as the lines A_1 and A_2, and it is blue when also R_3 is involved, as the lines B_1, B_2 and C. Moreover, trade is allowed (not allowed) on the left (right) of A_1 and B_1, below (above) A_2 and B_2 and above (below) C (see also next Section). Finally, consider that borders and vertices of the triangles in Fig. 2 represent special cases. On a single border firms are located in only two regions, whereas the third region is empty and on a vertex (the crossing of two borders) all industry is agglomerated in one region. Therefore, some of the outward links (involving exporting firms), that may occur in a neighborhood of a point on a border or of a vertex (where all the industry shares are positive), are necessarily absent in those points without industry.

Differences in indirect utilities drives the dynamic process as discussed in the following Section. Before that it could be of some interest to see what happens when a boundary line is crossed starting from a given distribution of firms. We consider a specific example: suppose that the initial distribution of firms corresponds to a point in $NS51$ in Fig. 2b, corresponding to the horizontal stripe part of $NS5$, somewhere in the middle of that area. In $NS51$, R_1 and R_2 are undertaking bilateral trade and R_3 is exporting towards R_1. The given distribution of firms leads most probably to the long-run equilibrium BA_2 where firms are located only in R_2 and R_3 (see Fig.4b, where T_E is slightly higher and where points in $RS51$ mostly converge to that equilibrium). Now let T_E increase, so that the point enters in $NS3$. We have that, the link from R_3 to R_1 is cut. For the given distribution of firms, profits for R_1 and R_2 increase and decrease for R_3, moreover the surplus of R_1 decreases while for R_2 and R_3 it does not change (see the Appendix). Therefore, V_2 increases, V_3 decreases while V_1 could increase or decrease. These changes lead to a different long-run outcome, that is, IA', where firms are located in all three regions (see Fig.4d).

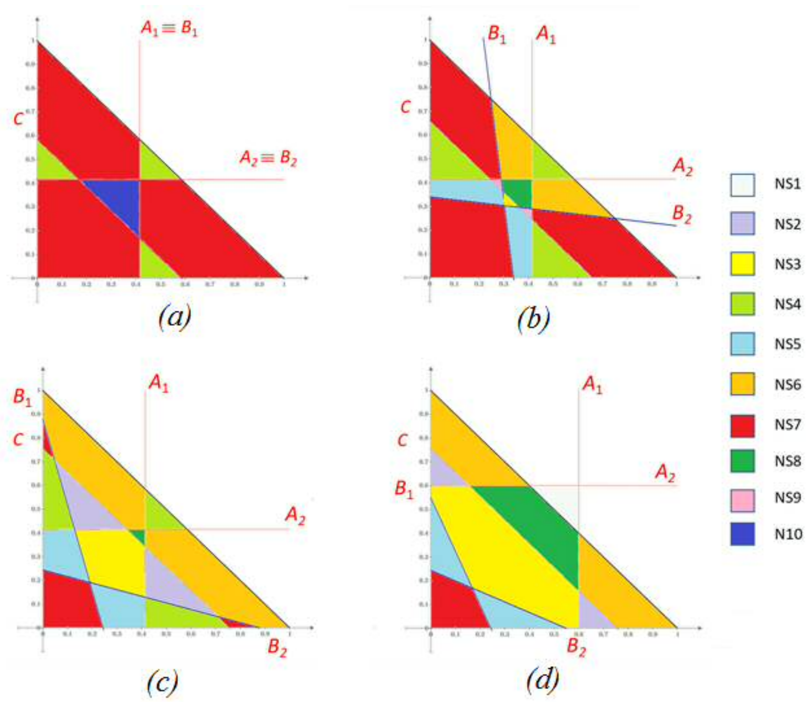

Figure 2: Examples of possible configurations of trade costs giving rise to different trade network configurations. Here $a = b = c = \frac{1}{3}$, $\eta = 0$, $E = 10$ and $T_U = T_E = 0.325$ in (a), $T_U = 0.325$, $T_E = 0.37$ in (b), $T_U = 0.325$, $T_E = 0.45$ in (c) and $T_U = 0.25$, $T_E = 0.45$ in (d).

4 Dynamics

The dynamics of the considered NEG model is described by a two-dimensional (2D) piecewise smooth map $Z : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows:

$$Z : \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mapsto \begin{pmatrix} Z_1(\lambda_1, \lambda_2) \\ Z_2(\lambda_1, \lambda_2) \end{pmatrix}, \tag{19}$$

where

$$Z_i(\lambda_1, \lambda_2) = \left\{ \begin{array}{cll} 0 & if & F_i \leq 0, \\ F_i & if & F_i > 0, \quad F_j > 0, \quad F_i + F_j < 1, \\ \frac{F_i}{F_i + F_j} & if & F_i > 0, \quad F_j > 0, \quad F_i + F_j \geq 1, \\ \frac{F_i}{1 - F_j} & if & F_i > 0, \quad F_j \leq 0, \quad F_i + F_j < 1, \\ 1 & if & F_i > 0, \quad F_j \leq 0, \quad F_i + F_j \geq 1, \end{array} \right.$$

$$\text{with } i = 1,\ j = 2 \text{ for } Z_1(\lambda_1, \lambda_2) \text{ and } i = 2,\ j = 1 \text{ for } Z_2(\lambda_1, \lambda_2)\ ,$$

$$F_r(\lambda_1, \lambda_2) = \lambda_r(1 + \gamma\Omega_r(\lambda_1, \lambda_2)),\ r = 1, 2\ ,$$

$$\Omega_r(\lambda_1, \lambda_2) = \frac{V_r(\lambda_1, \lambda_2)}{\lambda_1 V_1(\lambda_1, \lambda_2) + \lambda_2 V_2(\lambda_1, \lambda_2) + (1 - \lambda_1 - \lambda_2) V_3(\lambda_1, \lambda_2)} - 1.$$

The indirect utilities $V_i(\lambda_1, \lambda_2)$, $i = 1, 2, 3$, of an entrepreneur in regions 1, 2 and 3, respectively, are defined in the Appendix.

The following properties of map Z follow immediately from its definition:

Property 1. In the (λ_1, λ_2)-phase plane any trajectory of map Z is trapped in a triangle denoted S, whose *sides* or *borders*

$$I_{b1} = \{(\lambda_1, \lambda_2) : \lambda_2 = 0\},\ I_{b2} = \{(\lambda_1, \lambda_2) : \lambda_1 = 0\},\ I_{b3} = \{(\lambda_1, \lambda_2) : \lambda_2 = 1 - \lambda_1\} \tag{20}$$

are *invariant lines* of map Z.

Property 2. Map Z is symmetric with respect to the diagonal $D = \{(\lambda_1, \lambda_2) : \lambda_1 = \lambda_2\}$, which is one more invariant line of Z.

Property 2 implies that the phase portrait of map Z is symmetric with respect to D, i.e., any invariant set $\mathcal{A}$ is either itself symmetric with respect to D or there exists one more invariant set $\mathcal{A}'$ symmetric to $\mathcal{A}$.

Property 3. The vertices of S are *Core-periphery* (CP) fixed points

$$CP_0 : (\lambda_1, \lambda_2) = (0, 0),\ CP_1 : (\lambda_1, \lambda_2) = (1, 0),\ CP_2 : (\lambda_1, \lambda_2) = (0, 1), \tag{21}$$

characterised by full spatial agglomeration of the industrial activity, with all the entrepreneurs located in only one region.

Property 4. Any *interior fixed point* of Z, if it exists, is given by intersection of the curves

$$\Omega_1 = \{(\lambda_1, \lambda_2) : \Omega_1(\lambda_1, \lambda_2) = 0\} \text{ and } \Omega_2 = \{(\lambda_1, \lambda_2) : \Omega_2(\lambda_1, \lambda_2) = 0\}. \tag{22}$$

An interior equilibrium is characterised by positive shares of entrepreneurs in all regions.

Property 5. Any *border fixed point* belonging to I_{bi}, $i = 1, 2$, if it exists, is an intersection point of Ω_i and I_{bi}, while any border fixed point belonging to I_{b3} is an intersection point of Ω_1, Ω_2 and I_{b3}. A border equilibrium is characterised by positive shares of entrepreneurs in two regions and no entrepreneurs in the third one.

We denote an interior symmetric fixed point as $IS : (\lambda_1, \lambda_2) = \left(\frac{1}{3}, \frac{1}{3}\right) \in D$, and an interior asymmetric fixed point as IA. The border symmetric / asymmetric fixed points are denoted as BS_i / $BA_i \in I_{bi}$, $i = 1, 2, 3$. In case of coexisting fixed points of the same type we use additional labels. Note that a border symmetric equilibrium is such that, when positive, the two shares are equal to $\frac{1}{2}$. For $T_E \neq T_U$ map Z has only one symmetric equilibrium, $BS_3 : (\lambda_1, \lambda_2) = \left(\frac{1}{2}, \frac{1}{2}\right) \in I_{b3}$.

Besides the borders I_{bi} of the triangle S, map Z changes its definition along five more borders (which depending on the parameters may intersect or not the triangle S):

$$\begin{array}{ll} A_1 : \left\{(\lambda_1, \lambda_2) : \lambda_1 = \widetilde{\lambda}\right\}; & A_2 : \left\{(\lambda_1, \lambda_2) : \lambda_2 = \widetilde{\lambda}\right\}; \\ B_1 : \left\{(\lambda_1, \lambda_2) : \lambda_1 = \overline{\lambda} - k\lambda_2\right\}; & B_2 : \left\{(\lambda_1, \lambda_2) : \lambda_2 = \overline{\lambda} - k\lambda_1\right\}; \\ C : \left\{(\lambda_1, \lambda_2) : \lambda_1 = 1 - \overline{\lambda} - \lambda_2\right\}; & \end{array} \tag{23}$$

where

$$\overline{\lambda} = \frac{2(a - \eta b - bT_E)}{cET_E}, \quad \widetilde{\lambda} = \frac{2(a - \eta b - bT_U)}{cET_U}, \quad k = \frac{T_E - T_U}{T_E}.$$

These borders refer to the conditions for trade between the regions R_1, R_2 and R_3, namely,

- A_1 : If $\lambda_1 \geq \widetilde{\lambda}$, trade from R_2 to R_1 cannot occur, while if $\lambda_1 < \widetilde{\lambda}$ it is possible;
- A_2 : If $\lambda_2 \geq \widetilde{\lambda}$, trade from R_1 to R_2 cannot occur, while if $\lambda_2 < \widetilde{\lambda}$ it is allowed;
- B_1 : If $\lambda_1 \geq \overline{\lambda} - k\lambda_2$, trade from R_3 to R_1 cannot occur, otherwise, if $\lambda_1 < \overline{\lambda} - k\lambda_2$, trade is possible;
- B_2 : If $\lambda_2 \geq \overline{\lambda} - k\lambda_1$, trade from R_3 to R_2 cannot occur, while if $\lambda_2 < \overline{\lambda} - k\lambda_1$ it is allowed;
- C : If $\lambda_1 \leq 1 - \overline{\lambda} - \lambda_2$, trade from R_1 to R_3 and from R_2 to R_3 is not possible. Instead, when $\lambda_1 > 1 - \overline{\lambda} - \lambda_2$, trade from R_1 to R_3 and from R_2 to R_3 is allowed.

To investigate dependence of the dynamics of map Z on the parameters T_E, T_U and L in our simulations we fix

$$a = b = c = \frac{1}{3},\ \overline{C_A} = 1,\ \eta = 0,\ \gamma = 10,\ E = 10, \tag{24}$$

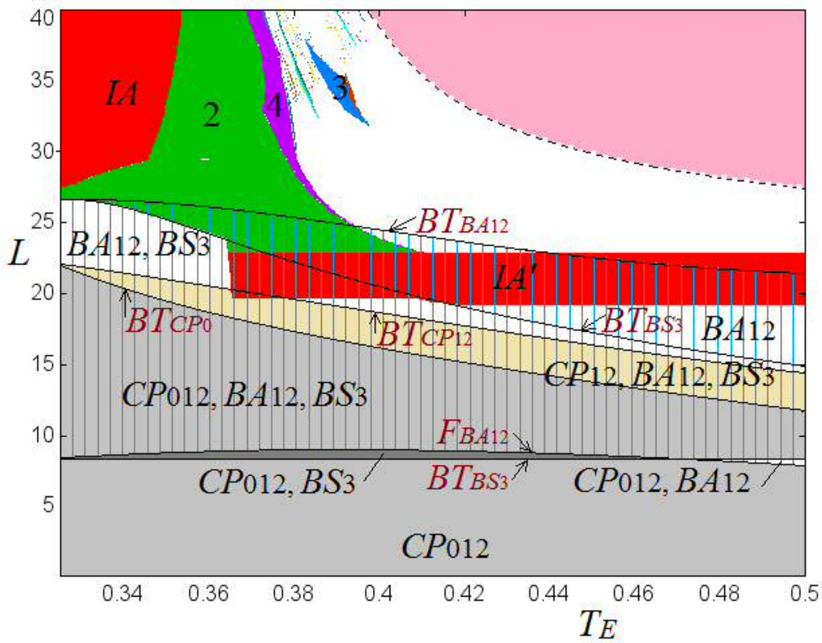

Figure 3: Bifurcation structure of the (T_E, L)-parameter plane of map Z at $T_U = 0.325$ and other parameters fixed as in (24).

and consider first the bifurcation structure of the (T_E, L)-parameter plane for fixed $T_U = 0.325$, then of the (T_E, T_U)-parameter plane for fixed $L = 20$.

In Fig.3 we present 2D bifurcation diagram in the (T_E, L)-parameter plane for $T_U = 0.325$, where the regions of qualitatively similar dynamics are marked. In particular, the region denoted CP_{012} is related to coexisting attracting Core-periphery fixed points CP_0, CP_1, CP_2; the regions BA_{12} and BS_3 to the border fixed points BA_1, BA_2 and BS_3, respectively; the region shown in red is related to an attracting interior fixed point(s); the regions shown in green, light blue and magenta are associated with attracting 2-, 3- and 4-cycles, respectively; the pink region corresponds either to a Milnor attractor[4] on the border I_{b3} or to the M-attracting[5] fixed points CP_1 and CP_2. All the regions are separated by the boundaries related to various bifurcations, among which the boundaries denoted by BT and F correspond to the border-transcritical and fold bifurcations, respectively, of fixed points indicated in the lower index.

Given that only one initial point is used to produce the bifurcation diagram shown in Fig.3, coexisting attractors can be discovered considering their basins of attraction. Below we present examples of attractors and their basins in Figs 4, 6 and 9, where attracting, repelling and saddle fixed points are marked by black, white and gray circles, respectively; the curves Ω_i, $i = 1, 2$, given in (22), as well as the border lines A_i, B_i and C given in (23) are also shown.

4.1 The consequences of Brexit: phase 1

In this subsection, we study the effects of Brexit, following an increase in T_E for a given T_U. As a NEG perspective suggests, we will consider two different ante-Brexit scenarios depending on the interplay between the value of the immobile component of consumers' demand, represented by the parameter L, local competition and trade costs. In the first scenario, this component of consumers' demand is relatively small, $L = 20$, that

[4] An attractor according to the topological definition is a closed invariant set with a dense orbit, which has a neighborhood each point of which is attracted to the attractor. An attractor in Milnor sence (see [14]) does not require existence of such a neighborhood, but only a set of points of positive measure, attracted to the attractor.

[5] For short we say that an invariant set is M-attractor if it is attracting in Milnor sense, but not in a sense of the topological definition.

is, twice the size of the overall number of entrepreneurs, E, representing the mobile component of demand; the economy is well-integrated with a comparatively small immobile component of demand; for low trade costs, NEG models typically predict agglomeration of economic activity. In the second scenario, local market sizes are bigger, $L = 30$, that is, three times the size of E. For this scenario, NEG models suggest an equal distribution of economic activity, even when trade costs are not high. In our discussion, we proceed as follow: we describe how stability properties of the equilibria and the dynamics are affected by changing the relevant parameters, followed by the economic interpretation of the results.

Let us investigate how the dynamics changes for increasing value of T_E at fixed $L = 20$. We begin from the value $T_E = T_U = 0.325$. As one can see in Fig.4a, for such parameter values there are six coexisting attracting fixed points: Core-periphery fixed points CP_0, CP_1, CP_2 and symmetric border points BS_1, BS_2 and BS_3. The basins of attraction of the Core-periphery fixed points are relatively small, and for increasing T_E at first CP_0 becomes unstable via a border-transcritical bifurcation (when the boundary BT_{CP0} is crossed, see Fig.3) leading to five coexisting attracting fixed points (see Fig.4b where $T_E = 0.35$). If we continue to increase T_E the interior fixed point denoted IA' (see Fig.4b) collides with the border B. As a result a border collision bifurcation (BCB for short) occurs due to which IA' becomes stable and a couple of new interior saddle fixed points are born. Then CP_1 and CP_2 also lose their stability via a border-transcritical bifurcation (when the boundary BT_{CP12} is crossed). After this bifurcation map Z has four coexisting attracting fixed points: BA_1, BA_2, BS_3 and IA' (see Fig.4c where $T_E = 0.38$). For further increasing T_E, the fixed point BS_3 loses its stability due to a border-transcritical bifurcation (when the boundary BT_{BS3} is crossed) merging with the fixed point IA, so that only three attractors are left: the fixed points BA_1, BA_2 and IA' (see Fig.4d where $T_E = 0.45$).

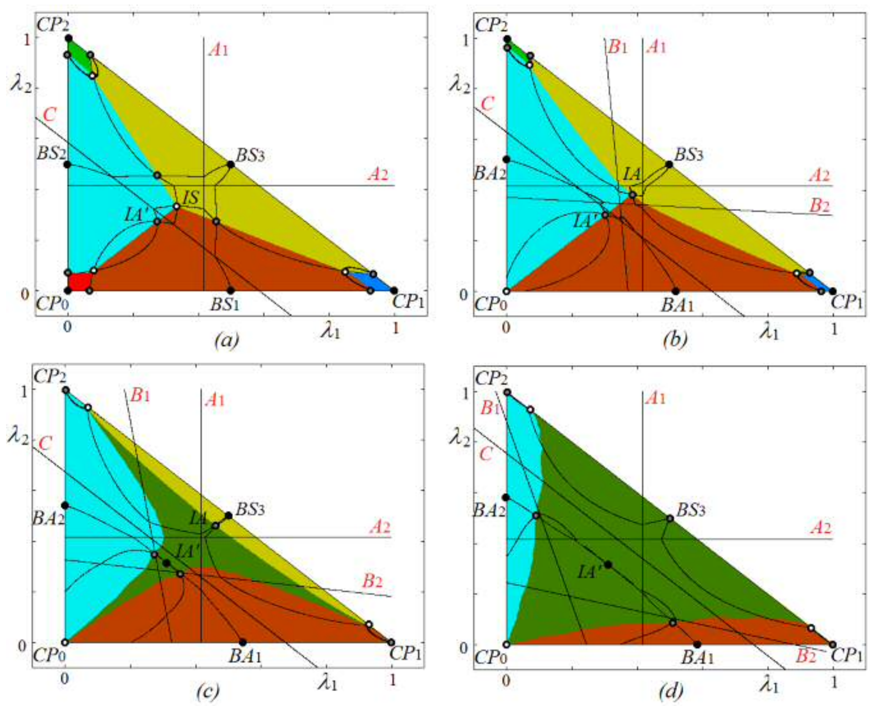

Figure 4: Basins of coexisting attracting fixed points of map Z for $T_U = 0.325$, $L = 20$ and $T_E = 0.325$ in (a), $T_E = 0.35$ in (b), $T_E = 0.38$ in (c) and $T_E = 0.45$ in (d). The other parameters are fixed as in (24).

In Fig.4a we have chosen the parameter values according to the first scenario mentioned above and the interior symmetric fixed point, IS, is unstable. Four ante-Brexit constellations, which involve full or partial agglomeration and which are different from an economic point of view (though some of them are analytically isomorphic), are possible stable long-run positions:

- Agglomeration of industrial activity in the leaving region (R_3, the UK), represented by CP_0;

- agglomeration of industrial activity in one of the Union's regions, ie CP_1 or CP_2;
- equal distribution of industrial activity between the leaving region R_3 and one of the regions remaining in the Union, whereas one of the Union's regions is peripheral, insofar it does not host any industrial production, ie BS_1 or BS_2;
- equal distribution of industrial activity between the two regions that remain in the Union, while R_3 had no industry, i.e. BS_3.

Note that those constellations exhibit different trade patterns: A CP equilibrium only involves outwards trade, the Core is exporting to both the other two regions (points in the neighborhood of such equilibria with positive λs are characterised by network structure $NS7$). Instead, in a BS equilibrium the two regions with industry only export towards the empty region and are not trading with each other (points in the neighborhood of such equilibria with positive λs are characterised by network structure $NS4$). In Fig. 5 the panels in the top line represent the four possible ante- Brexit constellations; a square around the number of a region indicates that it is not empty, ie that industry is located in this region; a solide line indicates a trade link.

The consequences of a soft Brexit can be seen by comparing Fig 4a with Fig.4b, where T_E is slightly increased above T_U. We differentiate according to the ante-Brexit situation:

- The CP_0 equilibrium (in which industry was agglomerated in the UK) loses stability. With small shocks, some of the firms move to one of the Union's regions in order to gain market access (while the other Union's region continues to be peripheral, i.e. without industry) leading to BA_1 or BA_2. Trade patterns will also change and are given by $NS4 - R_3$ will no longer export to the region to which firms have moved, but only to the peripheral region; instead, the newly industrialised region within the Union starts to export also to the peripheral region; this situation is depicted in Fig. 5 middle line, first panel (where a dashed line indicates firm movement);
- agglomeration of industrial activity in one of the Union's regions, i.e. CP_1 or CP_2, is still possible; however, the basin of attraction shrunk and small shocks will again push the economy to either BA_1 or BA_2, in which case some of the firms move from the Union to R_3. Trade patterns change to $NS4$: The industrialised region within the Union will continue to export to the other region within the Union, but does no longer export to R_3; instead, R_3 will start exporting to the peripheral region within the Union; this situation is depicted in Fig. 5, middle line, second panel;
- border equilibria, in which industry is found in one of the Union's regions and in the leaving region R_3, will exhibit asymmetries; firms move from the leaving region R_3 to one of the Union's regions; BA_1 and BA_2 involve a higher share of industry for the Union region, and a lower share for the leaving region; in addition, their basin of attraction has increased. Trade patterns continue to be of the $NS4$ type; this situtaion is depicted in Fig. 5, middle line, third panel;
- the border equilibrium BS_3 is still possible, but with a smaller basin of attraction making this constellation less robust to shocks. Trade patterns continue to be of the $NS4$ type (Fig. 5; middle line, fourth panel).

Interestingly, imposing a harder Brexit makes an interior solution, in which industry is distributed over all three regions, more likely: IA' gains stability and its basin of attraction expands. Note that this equilibrium involves a larger share of firms in region R_3 compared to IS. Note in addition, that IA' is located in area $NS3$, thus trade is only between the regions within the Union; and no trade occurs with R_3 (which becomes autarkic). Comparing Fig. 4a with Fig. 4d reveals the following consequences of a hard Brexit:

- The CP_0 equilibrium (in which industry was agglomerated in the UK) loses stability and IA' is the most likely outcome. Firms from the UK move to the two regions within the Union in order to get market access. Trade pattern change: The two regions within the Union start trading with each other, but not with R_3; firms remaining in R_3 will no longer export (because of the higher trade costs and the higher competition within the Union). Interestingly, R_3 changed from being the only exporting region to an autarkic region; this situation is depicted in Fig. 5, bottom line, first panel;

- agglomeration of industrial activity in one of the Union's regions, i.e. CP_1 or CP_2, is no longer a likely outcome. Similar to a soft Brexit, small shocks will lead to BA_1 or BA_2, i.e. to a situation in which some of the firms leave the Union to move to R_3 (the UK); a larger shock will push the economy to IA', in which firms also move to the peripheral region within the Union (see Fig. 5, bottom line, second panel);
- border equilibria, in which industry is found in one of the Union's regions and in the leaving region R_3, are still possible, but will exhibit asymmetries (larger share of industry for the region within the Union) and have a reduced basin of attraction. Again, larger shocks will push the economy to IA'. Again, the peripheral region within the Union attracts industry from the leaving region R_3, because it provides access to the Union market, and from the Union's industrialised region because the overall market is now smaller (since R_3 is more difficult to be reached and has less firms) and it is no longer sufficient to sustain agglomeration within the Union. At the same time, firms in the once peripheral region start to export to the Union's other region since now that market is comparatively more accessible and less competitive (see Fig. 5, bottom line, third panel);
- the border equilibrium BS_3, in which industry is only located within the Union, loses stability and the economy will move to IA', some firms leave the Union in order to locate in R_3, where they find less competition than in the Union. The larger T_E the stronger this motivation becomes (Fig. 5, bottom line, fourth panel).

To sum up: Before Brexit, the Union was well integrated and – corresponding to a NEG logic – agglomeration in one or two regions was a very likely outcome. In a soft Brexit scenario, core-periphery outcomes become less likely and agglomeration in the Union is possible but less likely as T_E increases. With a hard Brexit, the equilibrium IA', in which industry is located in all three regions, becomes more likely. Thus, with a hard Brexit, peripheral regions that did not have industry can attract firms. Note that trade will be only between the Union's regions, whereas R_3 becomes autarkic.

As can be seen in Fig. 3, for larger values of L the bifurcation structure becomes more complicated, involving attracting cycles of different periods (this occurs because the value of γ is sufficiently large). As a second scenario in Fig. 6, we assume $L = 30$ to be fixed and the value of T_E be increasing. As before we begin with $T_E = T_U = 0.325$. As illustrated in Fig.6a, for such parameter values map Z has a unique attractor, which is the interior symmetric fixed point IS. Note that border fixed points have already lost their stability via a flip bifurcation, so that on the borders of the triangle S there are saddle cycles of period 2. Bifurcations occurring for increasing T_E in the 1D map, which is a restriction of map Z to the diagonal, are illustrated in Fig.7 by means of a 1D bifurcation diagram λ_1 versus T_E for $\lambda_1 = \lambda_2$ and $T_U < T_E < 0.4$.

As one can see in Fig.7, for increasing T_E the interior fixed point IA remains stable up to a flip BCB which occurs when IA collides with the border C. After this bifurcation the unique attractor of Z is a 2-cycle belonging to the diagonal (see Fig.6b where $T_E = 0.37$). Then the 2-cycle collides with the border B. This collision does not lead to a qualitative change (i.e., a persistence border collision occurs), after which a flip bifurcation of the 2-cycle leads to an attracting 4-cycle (see Fig.6c where $T_E = 0.38$). For further increasing T_E the 4-cycle collides with the border C. Such a BCB leads to a 4-cyclic or 4-piece chaotic attractor, which then undergoes a merging bifurcation giving rise to a 2-cyclic or 2-piece chaotic attractor. This attractor in its turn also undergoes a merging bifurcation and is transformed into one-piece chaotic attractor (see Fig.6d where a chaotic attractor is shown for $T_E = 0.4$). For further increasing T_E the attractor on the diagonal disappears (more precisely, it is transformed into a chaotic repellor) due a contact with the border I_{b3} (when the parameter point enters the pink region in Fig.3) after which almost all the initial points of S are attracted to an M-attractor belonging to I_{b3}. This attractor eventually disappears due to a contact with the fixed points CP_1 and CP_2, so that they become M-attracting.

From an economic point of view, the case of $L = 30$ represents a different situation from the one discussed regarding Fig. 4: Markets are larger, and trade costs are not sufficiently low to make the Union a well-integrated area. As NEG models would predict, before Brexit every region hosts the same number of firms and trades with all other regions, i.e. the initial equilibrium is the interior symmetric equilibrium IS located in $NS10$.

From the analytic discussion above (see also Fig. 7), note that Brexit does not destroy the symmetry between the two regions remaining in the Union ($\lambda_1 = \lambda_2 = \lambda_D$ prevails); a moderate increase in T_E, i.e. a soft Brexit,

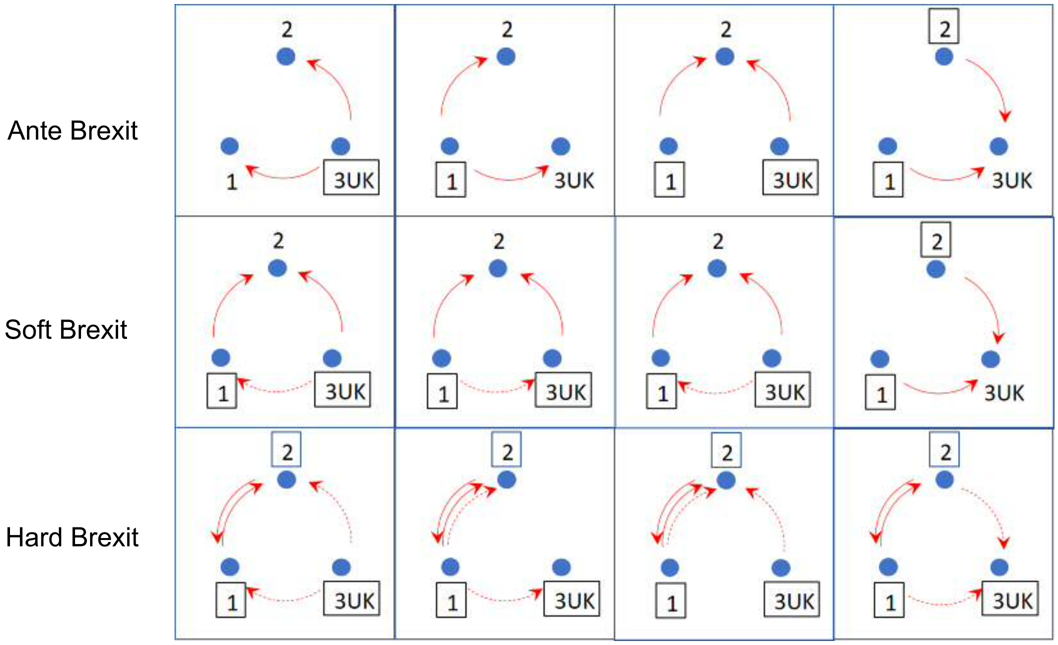

Figure 5: Industry location and network structure, top line: Ante Brexit; middle line: Soft Brexit, bottom line: Hard Brexit; a box around a region's number indicates industry loction in that region; a solid arrow indicates a trade link; a dashed line indicates firm movement

increases the incentive for firms to move to region R_3 (where they are sheltered from competition) and the regions within the Union R_1 and R_2 lose firms. In contrast, a hard Brexit involving a sharper increase in T_E may have a destabilizing effect leading to cyclical and chaotic patterns. Most remarkable, also these complex dynamic patterns retain the symmetry between the regions within the Union R_1 and R_2. Finally, note that we do not observe – full or partial – agglomeration of industry; industry stays active in all regions.

In Figure 7, the lines A, B and C allow an easy determination of the trade patterns and how they change with increasing T_E.

Initially, in the ante-Brexit situation and for a moderated increase in T_E, λ_D lies between the B and C lines; this corresponds to $NS10$ and we observe a full trade network. Interestingly, the period-2 cycle (λ_{D0} and λ_{D1}, with $\lambda_{D0} > \lambda_{D0}$), that is born as λ_{D0} crosses the C line, involves several distinct trade patterns:

- Initially, λ_{D0} continues to lie in $NS10$, which corresponds to the full trade network. However, λ_{D1} lies below the C line in an area which corresponds to $NS7$: The Union's regions R_1 and R_2 trade with each other and R_3 exports to them; however, the Union's regions do not export to R_3, since a low λ_{D1} translates into a high number of firms in R_3 and to an intense competition;

- the trade patterns over the cycle changes, when λ_{D0} further increases, crosses the B line and enters the area corresponding to $NS8$: The Union's regions R_1 and R_2 are still trading with each other. Given the high value of λ_{D0} and the corresponding high (low) number of firms in the Union (R_3), competition is high in the Union's regions, but low in R_3: so exporting to R_3 is attractive for firms located in the Union's regions, but exporting to the Union is not attractive for firms in R_3. λ_{D1} continues to involve $NS7$;

- the next change in the trade pattern occurs when λ_{D1} crosses the B line and enters the area corresponding to $NS3$, implying that region R_3 is autarkic and does not trade at all – given T_E, λ_{D1} is in an intermediate range where there is no incentive for trade between R_3 and the Union's regions. The latter continue to trade with each other and λ_{D0} remains in the area associated with $NS8$.

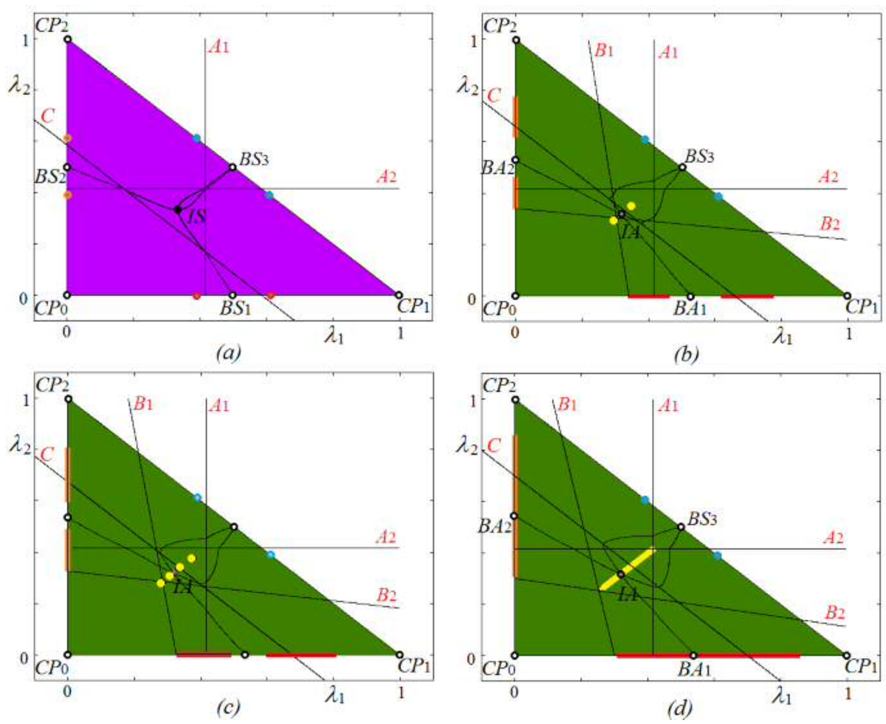

Figure 6: Attractors of map Z for $T_U = 0.325$, $L = 30$ and $T_E = 0.325$ in (a), $T_E = 0.37$ in (b), $T_E = 0.38$ in (c) and $T_E = 0.4$ in (d). The other parameters are fixed as in (24).

Thus, over the period-2 cycle, which involves a switch between a high and a low number of firms within the Union, R_1 and R_2 always trade with each other; they export to R_3 only in every other period, i.e. in the period in which the number of firms is low in R_3. However, the pattern of exports from R_3 to the Union's regions changes markedly. Initially, R_3 always exports to the Union's regions; then, only in every other period, i.e. in periods in which the number of firms in the Union's regions is low (and in R_3 is high); and finally, they never export.

When further increasing T_E, the 4-period cycle is born with now two points in area $NS8$ and two in $NS3$. So the trade pattern continues: R_1 and R_2 trade with each other; and they export to R_3 only in two of the four periods. As the amplitude of the 4-period cycle increases, the fourth point enters in region $NS7$: Over the cycle, the Union's regions always trade with each other; export to R_3 in two periods, import from R_3 in one period and do not trade with R_3 in the last period

Finally, turn to the chaotic attractor (composed first of four pieces, then of two pieces and finally of only one piece). Note that for most of the T_E values it lies in the areas $NS3$ and $NS8$, thus continuing the trade pattern found for lower T_E values.

When the amplitude of the chaotic cycle is sufficiently large (T_E is around 0.4), some of the points enter in area $NS4$ (above line A), which involves no (bilateral) trade between the Union's regions R_1 and R_2 and unilateral exports from the Union's regions to R_3. Thus, that cycle involves $NS3$, $NS4$ and $NS8$, implying that not only the export links between the Union's regions and R_3 are turned on and off, but also the bilateral trade links within the Union.

4.2 The consequences of Brexit: phase 2

We now consider the hypothesis that after Brexit the EU integrates even more and R_1 and R_2 become closer. This corresponds to a reduction in T_U for a given T_E, therefore we assume $T_E > T_U$. As before, we study the dynamic properties of equilibria of different periodicity; then we discuss the economic meaning of the results.

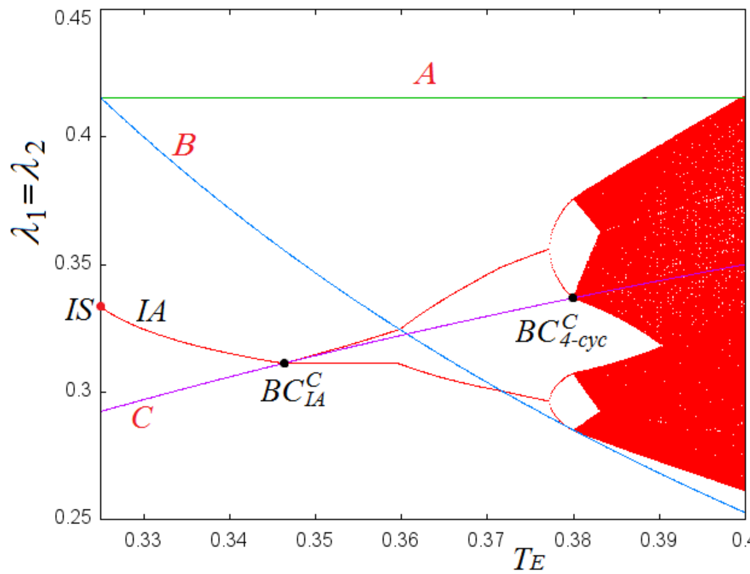

Figure 7: 1D bifurcation diagram λ_1 versus T_E for $\lambda_1 = \lambda_2$ (i.e., for a 1D map which is a restriction of map Z to the diagonal), $L = 30$, $T_U = 0.325$ and $T_U < T_E < 0.4$. All the other parameters are fixed as in (24).

In Fig.8 we present bifurcation structure of the (T_E, T_U)-parameter plane for $L = 20$. In this figure the regions associated with various coexisting attractors of map Z are indicated as well as some boundaries confining these regions. In particular, the boundaries BC^B_{BA12} and BC^C_{BA12} are related to BCBs of the fixed points BA_1 and BA_2 colliding with the borders B and C (see, for example, the first 1D bifurcation diagram in the right panel of Fig.8, related to the cross-section marked by the arrow I); the boundary Fl_{BA12} corresponds to a flip (sub- or supercritical) bifurcation of BA_1 and BA_2 (this bifurcation is illustrated in Fig.10); the boundary F_{BA3} is related to a fold bifurcation leading to two couples of asymmetric fixed points belonging to the border I_{b3} (the attracting fixed points in these couples are denoted BA'_3 and BA''_3 : see the second 1D bifurcation diagram in the right panel of Fig.8, related to the cross-section marked by the arrow II); $F_{IA''}$ denotes the fold bifurcation curve crossing which two couples of interior fixed points are born (the attracting fixed points in these couples are denoted IA''_1 and IA''_2 : see the third 1D diagram in the right panel of Fig.8, related to the cross-section marked by the arrow III); the BCB boundaries are denoted by BC with lower index indicating the colliding point and the upper index indicating the border line involved in the collision. For example, the curve $BC^B_{IA'}$ is associated with the BCB of the fixed point $IA' \in D$ colliding with the border $B = B_1 \cap B_2$. Below we comment a bifurcation sequence observed for fixed $T_E = 0.45$ and decreasing T_U, noting that the bifurcation scenario commented in the previous section and illustrated in Fig.4 is related to the fixed value $T_U = 0.325$ and the value T_E increasing along the dashed line. We leave an exhaustive analysis of the bifurcation structure of the (T_E, T_U)-parameter plane for future study. Indeed, in our interpretation, we focussed on two scenarios: a soft Brexit ($T_E = 0.35$) and a hard Brexit ($T_E = 0.45$). The former did not destroy agglomeration patters and looking at Fig. 8 confirms that a deeper integration within the Union may not bring major qualitative effects. Instead, a hard Brexit led to dispersion of industrial activity. Further integration with the Union will transform significantly this pattern and we analyse this case in more detail.

If we begin from the case shown in Fig.4d (where $L = 20$, $T_E = 0.45$, $T_U = 0.325$) and will decrease T_U, the observed sequence of bifurcations is the following (see the blue circles indicated in Fig.8). At first a fold BCB gives rise to two couples of interior fixed points (see the label $BC^A_{IA''}$ in the diagram III of Fig.8) leading to two new interior attracting fixed points, IA''_1 and IA''_2, and two saddles; these saddles quite soon merge with the fixed point IA' and this fixed point loses stability, i.e., a reverse pitchfork bifurcation occurs, after which map Z has four attractors: the border fixed points BA_1 and BA_2 as well as interior fixed points IA''_1 and IA''_2 (see Fig.9a where $T_U = 0.32$). Then attracting and saddle interior fixed points merge in pairs and disappear

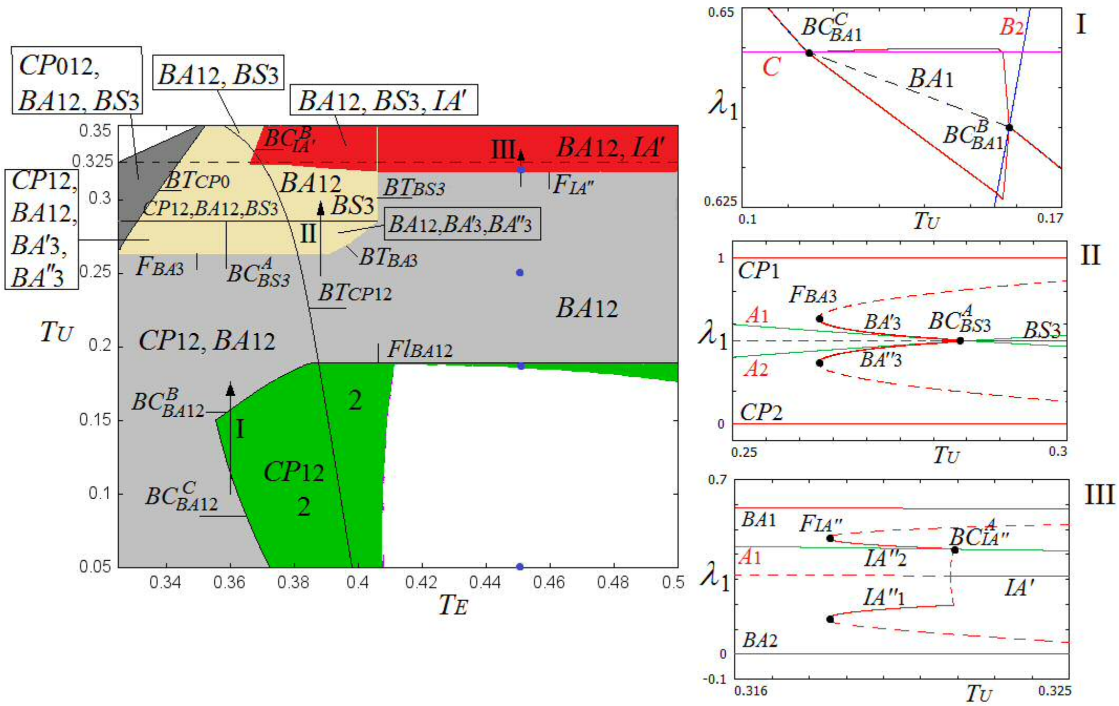

Figure 8: Bifurcation structure of the (T_E, T_U)-parameter plane for $L = 20$ and other parameters fixed as in (24). The attractors and their basins associated with parameter points marked by blue circles are shown in Fig.9. The right pannel shows 1D bifurcation diagrams associated with the crossections marked by the arrows I ($T_E = 0.36$, $0.1 < T_U < 0.17$), II ($T_E = 0.39$, $0.25 < T_U < 0.3$) and III ($T_E = 0.45$, $0.316 < T_U < 0.315$). In particular, in I and II the diagrams λ_1 versus T_U are shown, related to 1D restrictions of map Z to the borders I_{b1} (I_{b2}) and I_{b3}, respectively; In III: 1D diagram λ_1 versus T_U of map Z.

via a reverse fold bifurcation (when the curve $F_{IA''}$ is crossed, see also the label $F_{IA''}$ in the diagram III of Fig.8), leaving only two attractors, BA_1 and BA_2 (see Fig.9b where $T_U = 0.25$). For further increasing T_U the fixed points BA_1 and BA_2 undergo a flip bifurcation, so that attractors of Z are two 2-cycles on the borders I_{b1} and I_{b2} (see Fig.9c where $T_U = 0.186$). Then these 2-cycles collide with the border C, and this BCB leads directly to chaos, namely, to the 2-cyclic chaotic attractors on the borders BA_1 and BA_2. In the mean time the fixed points CP_1 and CP_2 become M-attracting because the flat branches of the functions defining map Z 'enter' the triangle S (see Fig.9d where $T_U = 0.05$; here the basins of CP_1 and CP_2 are shown in green and dark blue, respectively). Evolution of the attractors on the borders I_{b1} and I_{b2} can be clarified by means of the 1D bifurcation diagram λ_1 versus T_U shown in Fig.10 (recall that due to the symmetry of the map the same dynamics is observed on the border I_{b2}). On this diagram one can see that a contact of the 2-cycle with the border C indeed leads to the 2-cyclic chaotic attractor.

Recall (going back to Fig. 4d) that a hard Brexit may lead to two different outcomes. The first possibility is the equilibrium IA', industry is located in all regions and trade network structure is of the $NS3$ type, i.e. bilateral between the Union's regions, while R_3 is autarkic. Alternatively, a hard Brexit may lead to BA_1 or BA_2, in which industry is located in R_3 and in only one of the Union's regions (while the other is left peripheral and without industry) and trade involves only exports from the two industrialised regions towards the peripheral region within the Union ($NS4$).

All panels in Fig. 9 start from a hard Brexit scenario (i.e. $T_E = 0.45$); the panels depict an increasing internal integration (T_U reduces from 0.32 to 0.05).

Notice that in all panels $IA\prime$ has lost stability; thus, EU's deeper integration after a hard Brexit destabilises a symmetric location of industry. Looking at Fig.9a, two additional results emerge. First, two new interior fixed

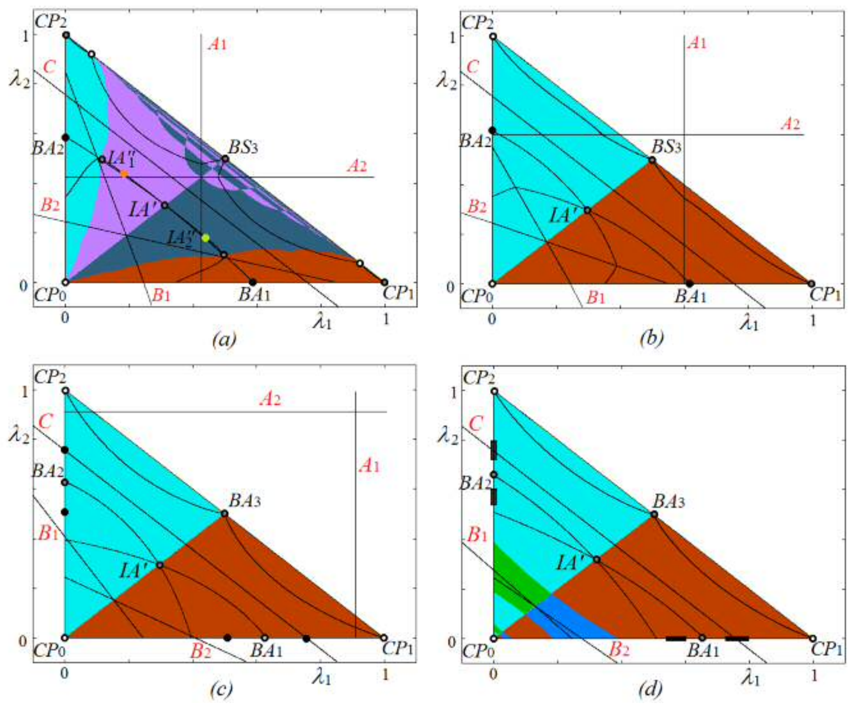

Figure 9: Coexisting attractors of map Z and their basins for $T_E = 0.45$, $L = 20$ and $T_U = 0.32$ in (a), $T_U = 0.25$ in (b), $T_U = 0.186$ in (c) and $T_U = 0.05$ in (d) (see blue circles in Fig.8). The other parameters are fixed as in (24).

points off the diagonal appear that introduce an asymmetry between the Union's regions. They are located in area $NS2$, where only one way trade occurs within the Union, from the more to the less industrialised region; R_3 remains autarkic (as in IA'). Second, the basins of attraction of these two equilibria show an intermingled structure, implying that it is difficult to predict in which of the two equilibria the dynamic process will settle (this holds in particular for initial conditions close to BS_3, in which the Union's regions are almost symmetric). These additional new fixed points (that involve asymmetry between the Union's regions) disappear for a deeper integration within the Union (i.e. for lower T_U values, see Fig. 9b-d).

The two other possible equilibria, BA_1 and BA_2, persist – first as fixed points coexisting with the new interior fixed points (Fig. 9a); then as the only fixed points (Fig. 9b); then as period-2 cycles (Fig. 9c) and finally as two-piece chaotic attractors that coexist with stable CP_1 and CP_2. Fig. 10 allows to analyse these equilibria in more detail (note that for the hard Brexit scenario depicted in Fig. 4d $T_E = 0.45$ was assumed). Fig.10 focusses on BA_1 (BA_2 is symmetric).

First, note that deeper integration within the Union will attract firms from the leaving region R_3 to the industrialised region within the Union, its share in industry increases.

Second, and most interestingly, the trade pattern changes as well, as can easily be seen from Fig. 10 (note that the A_1 line is not relevant, since the equilibrium BA_1 does not involve industry in R_2):

Initially, for $T_U = 0.325$ (Fig. 4d) and 0.32 (Fig 9a) λ_1 was above the B_1 line and below the B_2 line, corresponding to $NS4$ (involving one-way trade from the two industrialised regions to the peripheral region within the Union). Reducing T_U, the trade pattern changes once λ_1 has crossed the B_2 line and enters the area $NS2$ ($T_U = 0.25$, Fig. 9b): still one-way trade within the Union, from the industrialised to the peripheral region, but R_3 is autarkic and does not export anymore to the peripheral region.

BA_1 and BA_2 lose stability and cyclical behavior takes place along the sides I_{b1} and I_{b2}. In Fig. 9c, the 2-cycles do not collide yet with the border C and the trade pattern does not change: These cycles still involve only trade from R_1 to R_2 (in BA_1) or from R_2 to R_1 (in BA_2).

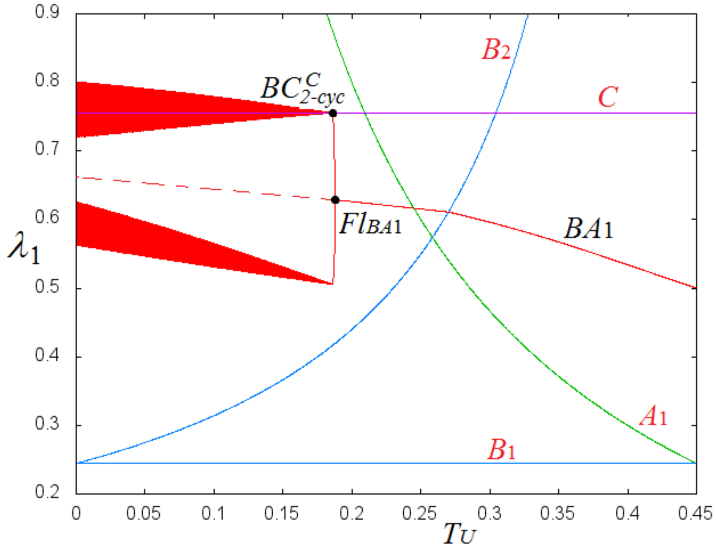

Figure 10: 1D bifurcation diagram λ_1 versus T_U for $\lambda_2 = 0$ (i.e., for a 1D map which is the restriction of map Z to the border I_{b1}), $L = 20$, $T_E = 0.45$ and $0 < T_U < T_E$. Due to Property 2, the same dynamics is observed for λ_2 versus T_U for $\lambda_1 = 0$. All the other parameters are fixed as in (24).

As T_U is further reduced, the period-2 cycles hit the border C. Some of the points of the ensuing 2-piece chaotic attractor lie above the C border, thus in area $NS8$. In these points, the share of firms located in R_3 is sufficiently small that firms located in R_1 find profitable to export towards R_3 as well.

Thus, if one of the Union's regions is peripheral without industry, it will always import from the other region in the Union. R_3 does not export to the industrialised region in the Union. With deeper integration within the Union, R_3 will stop exporting to the Union's peripheral region; it is autarkic for some values of T_U, before it starts importing from the Union's industrialised region. As shown in Fig. 9d, other possible outcomes are CP_1 and CP_2, whose basins of attraction are intermingled, making it difficult to predict the long-run outcome the closer the initial condition is to CP_0. In CP_1 and CP_2 the core, which is in the Union, exports to the other two regions.

In summary, deeper integration of the Union after a hard Brexit involves a loss of stability for the interior equilibrium $IA\prime$. One interesting result is that the reduction in T_U may destroy reciprocal trade within the Union. It also reduces the likelihood of R_3 exporting towards the Union and increases that of non-trading or importing. Other interesting phenomena can emerge like cyclical or even chaotic behaviour, intermingled basins of attraction and unpredictability of long-run outcomes concerning the location of industry and the patterns of trade.

5 Conclusions

As many empirical studies suggest, Brexit will deeply affect Europe's economic landscape, in particular firm location and trade patterns will change substantially with marked differences between the regions. Empirical studies treat these two dimensions as rather unrelated, whereas a NEG perspective suggests that they are intimately related. In this paper, we therefore developed a 3-region footloose entrepreneur (FE) model with linear demand functions that allows an explicit analysis of changes in trade patterns. Given the notorious analytic complexity of multiregional NEG models, we primarily present simulation results.

In order to structure our analysis, we differentiated the two ante-Brexit situations that are quintessential from a NEG perspective.

The relation between market size and trade costs was initially such that the Union was a well-integrated economic area. In that case, NEG models predict (partial or full) agglomeration of economic activity; indeed, we found four agglomeration patterns that are different from an economic point of view. We introduced Brexit as an increase in trade cost towards the exiting region (whereas the trade costs remain fixed within the Union); and we differentiated between a soft and a hard Brexit, the latter involving a more pronounced increase in trade cots. Our analysis suggests that this will lead to a reduction of trade between the Union and the UK, and an intensification of trade within the Union; in many cases firms relocate from the exiting region towards the Union in order to gain market access – in these cases, firm relocation replaces an export link. Remarkably, even a region that was peripheral before Brexit with no industry may gain industry after Brexit (being now a region offering access to the Union's market as well as offering low local competition). In some cases, we also found firm relocation from within the Union to the exiting region, seeking shelter from the intensive competition within the Union.

Alternatively, the ante Brexit relation between market size and trade costs was initially such that the Union has been less integrated. In that case, a NEG perspective suggest dispersion of economic activity and a full trade network, which we represented by our second parameter set. In that case, Brexit does only gradually affect industry location; all regions maintain industry, though asymmetries between the leaving and the remaining regions will develop (the latter maintain their symmetry). With a soft Brexit, the leaving region actually gains industry (firms seeking shelter from competition), the full trade network continues to exist. A harder Brexit involving a more pronounced increase in trade cost will destabilise the equilibrium and cyclical or chaotic patterns of industry location emerge. Most interesting, these changes in the number of firms and thus in the degree of local competition will also affect trade patterns: Bilateral trade within the Union will persist; (unilateral) trade between the Union's regions and the exiting region will only happen with low competition in the destination region (i.e. low number of firms). With very high trade costs, firms in the exiting region will stop to export.

Finally, we studied the effects of a deeper integration within the Union after Brexit, starting from an ante-Brexit well integrated economic area displaying agglomeration features. We argued that in that case a hard Brexit may lead to a more dispersed industry location (with even peripheral regions gaining industry) and to less trade with the exiting region. A deeper integration within the Union may actually reverse the effect on industry location, the peripheral region may again (partly or fully) lose their industry. Trade patterns, however, will continue to show a rather isolated position of the exiting region.

Brexit will change trade costs implying corresponding changes in trade patterns and industry location. As a consequence, economic agents' welfare will change accordingly, since the range of available commodities will vary togheter with their price (due to transport costs and more / less intense local competition), for the mobile factor – entrepreneurs – profit income changes as well. The overall effect is difficult to ascertain and we leave this to further research. However, in many instances we found for the leaving country a reduction in trade, in the number of firms and thus in local competition. These factors – taken in isolation – reduce welfare in the leaving region, an aspect that deserves more attention in any discussions on Brexit.

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References

[1] T. Ago, I. Isono, T. Tabuchi, Locational disadvantage of the hub, *The Annals of Regional Science* 40.4 (2006) 819–848.

[2] K. Behrens, Agglomeration without trade: how non-traded goods shape the space economy, *Journal of Urban Economics* 55 (2004) 68–92.

[3] K. Behrens, How endogenous asymmetries in interregional market access trigger regional divergence, *Regional Science and Urban Economics* 35 (2005) 471–492.

[4] K. Behrens, International integration and regional inequalities: how important is national infrastructure? *The Manchester School* 79.5 (2011) 952–971.

[5] S. Brakman, H. Garretsen, T. Kohl, Consequences of Brexit and options for a 'Global Britain', *Papers in Regional Science* 97 (2018) 55–72.

[6] B. Busch, J. Matthes, Brexit - the economic impact: A meta-analysis, IW-Report No. 10 (2016).

[7] P. Commendatore, V. Filoso, I. Kubin, T. Grafeneder-Weissteiner, Towards a multiregional NEG framework: comparing alternative modelling strategies, in P. Commendatore, S.S. Kayam, I. Kubin (Eds), Complexity and Geographical Economics: Topics and Tools, Springer-Verlag, Heidelberg, 2018, 13-50.

[8] P. Commendatore, I. Kubin, I. Sushko, Dynamics of a developing economy with a remote region: Agglomeration, trade integration and trade patterns, *Communications in Nonlinear Science and Numerical Simulation* 58 (2018) 303-327.

[9] P. Commendatore, I. Kubin, I. Sushko, Emerging trade patterns in a 3-Region linear NEG model: three examples, In P. Commendatore, I. Kubin , S. Bougheas, A. Kirman, M. Kopel, G.I. Bischi (Eds), The Economy as a Complex Spatial System: Macro, Meso and Micro Perspectives, Springer, Cham, 2018, 38-80.

[10] S. Dhingra, H. Huang, G. Ottaviano, J.P. Pessoa, T. Sampson, J. Van Reenen, The costs and benefits of leaving the EU: trade effects, *Economic Policy* (October 2017) 651–705.

[11] S. Dhingra, G. Ottaviano, V. Rappoport, T. Sampson, C. Thomas, UK trade and FDI: A post-Brexit perspective, *Papers in Regional Science* 97 (2018) 9–24.

[12] P. R. Krugman, Scale Economies, product differentiation, and the pattern of trade, *American Economic Review* 70 (1980) 950-959.

[13] P. R. Krugman, Increasing returns and economic geography, *Journal of Political Economy* 99 (1991) 483–499.

[14] J. Milnor, On the concept of attractor, *Commun. Math. Phys.* 99 (1985) 177–195

[15] T. Okubo, P.M. Picard, and J.F. Thisse, On the impact of competition on trade and firm location, *Journal of Regional Science* 54.5 (2014) 731-754.

[16] H. Oberhofer, M. Pfaffermayr, Estimating the Trade and Welfare Effects of Brexit: A Panel Data Structural Gravity Model, WU Department of Economics Working Paper No. 259 (2018).

[17] G.I. Ottaviano, T. Tabuchi, J.F. Thisse, Agglomeration and trade revisited, *International Economic Review* 43 (2002) 409–436.

[18] T. Sampson, Brexit: The Economics of International Disintegration, Journal of Economic Perspectives 31.4 (2017) 163–184.

[19] J. Van Reenen, Brexit's Long-Run Effects on the U.K. Economy, *Brookings Papers on Economic Activity* (Fall 2016) 367-383.

Appendix

The indirect utilities $V_i(\lambda_1, \lambda_2) =: V_i$, $i = 1, 2, 3$, of an entrepreneur in R_1, R_2 and R_3, respectively, are determined as follows, according to the bonduary conditions for trade fixing the possible trade network structures:

$$
\begin{array}{lcll}
\textit{if} & \lambda_1 \geq \widetilde{\lambda},\ \widetilde{\lambda} \leq \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS1},\ V_2 = V_2^{NS1},\ V_3 = V_3^{NS1},\\
\textit{if} & \widetilde{\lambda} \leq \lambda_1 \leq 1-\overline{\lambda}-\lambda_2,\ \overline{\lambda}-\kappa\lambda_1 \leq \lambda_2 < \widetilde{\lambda} & \textit{then} & V_1 = V_1^{NS21},\ V_2 = V_2^{NS21},\ V_3 = V_3^{NS21},\\
\textit{if} & \overline{\lambda}-\kappa\lambda_2 \leq \lambda_1 < \widetilde{\lambda},\ \widetilde{\lambda} \leq \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS22},\ V_2 = V_2^{NS22},\ V_3 = V_3^{NS22},\\
\textit{if} & \overline{\lambda}-\kappa\lambda_2 \leq \lambda_1 < \widetilde{\lambda},\ \overline{\lambda}-\kappa\lambda_1 \leq \lambda_2 < \widetilde{\lambda},\ \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS3},\ V_2 = V_2^{NS3},\ V_3 = V_3^{NS3},\\
\textit{if} & \lambda_1 < \overline{\lambda}-\kappa\lambda_2,\ \widetilde{\lambda} \leq \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS41},\ V_2 = V_2^{NS41},\ V_3 = V_3^{NS41},\\
\textit{if} & \widetilde{\lambda} \leq \lambda_1 \leq 1-\overline{\lambda}-\lambda_2,\ \lambda_2 < \overline{\lambda}-\kappa\lambda_1 & \textit{then} & V_1 = V_1^{NS42},\ V_2 = V_2^{NS42},\ V_3 = V_3^{NS42},\\
\textit{if} & \lambda_1 \geq \widetilde{\lambda},\ \lambda_2 \geq \widetilde{\lambda},\ \lambda_2 > 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS43},\ V_2 = V_2^{NS43},\ V_3 = V_3^{NS43},\\
\textit{if} & \left\{\begin{array}{c} \lambda_1 < \overline{\lambda}-\kappa\lambda_2,\\ \overline{\lambda}-\kappa\lambda_1 \leq \lambda_2 < \widetilde{\lambda},\ \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 \end{array}\right. & \textit{then} & V_1 = V_1^{NS51},\ V_2 = V_2^{NS51},\ V_3 = V_3^{NS51},\\
\textit{if} & \left\{\begin{array}{c} \overline{\lambda}-\kappa\lambda_2 \leq \lambda_1 < \widetilde{\lambda},\\ \lambda_2 < \overline{\lambda}-\kappa\lambda_1,\ \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 \end{array}\right. & \textit{then} & V_1 = V_1^{NS52},\ V_2 = V_2^{NS52},\ V_3 = V_3^{NS52},\\
\textit{if} & \lambda_1 \geq \widetilde{\lambda},\ \overline{\lambda}-\kappa\lambda_1 \leq \lambda_2 < \widetilde{\lambda},\ \lambda_2 > 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS61},\ V_2 = V_2^{NS61},\ V_3 = V_3^{NS61},\\
\textit{if} & \overline{\lambda}-\kappa\lambda_2 \leq \lambda_1 < \widetilde{\lambda},\ \lambda_2 \geq \widetilde{\lambda},\ \lambda_2 > 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS62},\ V_2 = V_2^{NS62},\ V_3 = V_3^{NS62},\\
\textit{if} & \lambda_1 \geq \widetilde{\lambda},\ 1-\overline{\lambda}-\lambda_1 < \lambda_2 < \overline{\lambda}-\kappa\lambda_1 & \textit{then} & V_1 = V_1^{NS71},\ V_2 = V_2^{NS71},\ V_3 = V_3^{NS71},\\
\textit{if} & 1-\overline{\lambda}-\lambda_2 < \lambda_1 < \overline{\lambda}-\kappa\lambda_2,\ \lambda_2 \geq \widetilde{\lambda} & \textit{then} & V_1 = V_1^{NS72},\ V_2 = V_2^{NS72},\ V_3 = V_3^{NS72},\\
\textit{if} & \lambda_1 < \overline{\lambda}-\kappa\lambda_2,\ \lambda_2 < \overline{\lambda}-\kappa\lambda_1,\ \lambda_2 \leq 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS73},\ V_2 = V_2^{NS73},\ V_3 = V_3^{NS73},\\
\textit{if} & \overline{\lambda}-\kappa\lambda_2 \leq \lambda_1 < \widetilde{\lambda},\ \overline{\lambda}-\kappa\lambda_1 \leq \lambda_2 < \widetilde{\lambda},\ \lambda_2 > 1-\overline{\lambda}-\lambda_1 & \textit{then} & V_1 = V_1^{NS8},\ V_2 = V_2^{NS8},\ V_3 = V_3^{NS8},\\
\textit{if} & \left\{\begin{array}{c} \overline{\lambda}-\kappa\lambda_2 \leq \lambda_1 < \widetilde{\lambda},\\ 1-\overline{\lambda}-\lambda_1 < \lambda_2 < \overline{\lambda}-\kappa\lambda_1 \end{array}\right. & \textit{then} & V_1 = V_1^{NS91},\ V_2 = V_2^{NS91},\ V_3 = V_3^{NS91},\\
\textit{if} & \left\{\begin{array}{c} 1-\overline{\lambda}-\lambda_2 < \lambda_1 < \overline{\lambda}-\kappa\lambda_2\\ \overline{\lambda}-\kappa\lambda_1 \leq \lambda_2 < \widetilde{\lambda} \end{array}\right. & \textit{then} & V_1 = V_1^{NS92},\ V_2 = V_2^{NS92},\ V_3 = V_3^{NS92},\\
\textit{if} & \left\{\begin{array}{c} \lambda_1 < \overline{\lambda}-\kappa\lambda_2,\\ 1-\overline{\lambda}-\lambda_1 < \lambda_2 < \overline{\lambda}-\kappa\lambda_1 \end{array}\right. & \textit{then} & V_1 = V_1^{NS10},\ V_2 = V_2^{NS10},\ V_3 = V_3^{NS10},
\end{array}
$$

where $\widetilde{\lambda} = \frac{2(a-\eta b-bT_U)}{cET_U}$, $\overline{\lambda} = \frac{2(a-\eta b-bT_E)}{cET_E}$ and $\kappa = \frac{T_E-T_U}{T_E}$ and where $\overline{\lambda} \leq \widetilde{\lambda}$ since $T_E \geq T_U$.
The expressions for the indirect utilities are given below:

Empty graph (NS1)

$$V_1^{NS1} = S_r^{NS1} + \pi_r^{NS1} + \overline{C_A},\ V_2^{NS1} = S_2^{NS1} + \pi_2^{NS1} + \overline{C_A},\ V_3^{NS1} = S_3^{NS1} + \pi_3^{NS1} + \overline{C_A},$$

$$
\begin{aligned}
\pi_1^{NS1} &= \left(\frac{a-\eta b}{2b+c\lambda_1 E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right),\\
\pi_2^{NS1} &= \left(\frac{a-\eta b}{2b+c\lambda_2 E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right),\\
\pi_3^{NS1} &= \left(\frac{a-\eta b}{2b+c(1-\lambda_1-\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),
\end{aligned}
$$

$$
\begin{aligned}
S_1^{NS1} &= \frac{\lambda_1 E(b+cE)(a-\eta b)^2(b+c\lambda_1 E)}{2b(2b+c\lambda_1 E)^2},\ S_2^{NS1} = \frac{\lambda_2 E(b+cE)(a-\eta b)^2(b+c\lambda_2 E)}{2b(2b+c\lambda_2 E)^2},\\
S_3^{NS1} &= \frac{(1-\lambda_1-\lambda_2)E(b+cE)(a-\eta b)^2(b+c(1-\lambda_1-\lambda_2)E)}{2b(2b+c(1-\lambda_1-\lambda_2)E)^2}.
\end{aligned}
$$

Single edge (NS21, NS22)

$$V_1^{NS21} = S_1^{NS21} + \pi_1^{NS21} + \overline{C_A},\ V_2^{NS21} = S_2^{NS21} + \pi_2^{NS21} + \overline{C_A},\ V_3^{NS21} = V_3^{NS1},$$

$$\pi_1^{NS21} = \left(\frac{a-\eta b}{2b+c\lambda_1 E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) + \left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_2 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_2^{NS21} = \left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_1 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_3^{NS21} = \pi_{33}^{NS1},$$
$$S_1^{NS21} = S_1^{NS1},$$
$$S_2^{NS21} = \frac{aE(b+cE)[b+c(\lambda_1+\lambda_2)E][(a-2b\eta)(\lambda_1+\lambda_2)-2bT_U\lambda_1]}{2b[2b+c(\lambda_1+\lambda_2)E]^2} + \frac{E(b+cE)\left\{4\eta^2 b(\lambda_1+\lambda_2)[b+c(\lambda_1+\lambda_2)E]+T_U\lambda_1\left[8\eta b[b+c(\lambda_1+\lambda_2)E]+T_U\left[4b^2+cE(\lambda_1+\lambda_2)(4b+c\lambda_2 E)\right]\right]\right\}}{8[2b+c(\lambda_1+\lambda_2)E]^2},$$
$$S_3^{NS21} = S_3^{NS1},$$

$$V_1^{NS22} = S_1^{NS22} + \pi_1^{NS22} + \overline{C_A},\ V_2^{NS22} = S_2^{NS22} + \pi_2^{NS22} + \overline{C_A},\ V_3^{NS22} = V_3^{NS1}$$

$$\pi_1^{NS22} = \left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_2 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right)$$
$$\pi_2^{NS22} = \left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_1 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) + \left(\frac{a-\eta b}{2b+c\lambda_2 E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right)$$
$$\pi_3^{NS22} = \pi_{33}^{NS1}$$
$$S_1^{NS22} = \frac{aE(b+cE)[b+c(\lambda_1+\lambda_2)E][(a-2b\eta)(\lambda_1+\lambda_2)-2bT_U\lambda_2]}{2b[2b+c(\lambda_1+\lambda_2)E]^2} + \frac{E(b+cE)\left\{4\eta^2 b(\lambda_1+\lambda_2)[b+c(\lambda_1+\lambda_2)E]+T_U\lambda_2\left[8\eta b[b+c(\lambda_1+\lambda_2)E]+T_U\left[4b^2+cE(\lambda_1+\lambda_2)(4b+c\lambda_1 E)\right]\right]\right\}}{8[2b+c(\lambda_1+\lambda_2)E]^2},$$
$$S_2^{NS22} = S_2^{NS1},\ S_3^{NS22} = S_3^{NS1}.$$

Mutual edge (NS3)

$$V_1^{NS3} = S_1^{NS3} + \pi_1^{NS3} + \overline{C_A},\ V_2^{NS3} = S_2^{NS3} + \pi_2^{NS3} + \overline{C_A},\ V_3^{NS3} = V_3^{NS1},$$

$$\pi_1^{NS3} = \left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_2 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) + \left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_2 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_2^{NS3} = \left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_1 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) + \left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_1 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_3^{NS3} = \pi_{33}^{NS1},$$

$$S_1^{NS3} = S_1^{NS22},\ S_2^{NS3} = S_2^{NS21},\ S_3^{NS3} = S_3^{NS1}.$$

In star (NS41, NS42, NS43)

$$V_1^{NS41} = S_1^{NS41} + \pi_1^{NS41} + \overline{C_A},\ V_2^{NS41} = S_2^{NS41} + \pi_2^{NS41} + \overline{C_A},\ V_3^{NS41} = S_3^{NS41} + \pi_3^{NS1} + \overline{C_A},$$

$$\pi_1^{NS41} = \left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right),$$
$$\pi_2^{NS41} = \left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_2)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) + \left(\frac{a-\eta b}{2b+c\lambda_2 E}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_3^{NS41} = \left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_2 E}{2b+cE}\right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) + \left(\frac{a-\eta b}{2b+c(1-\lambda_1-\lambda_2)E}\right)^2 (b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$

$$S_1^{NS41} = \frac{aE(b+cE)^2\{(a-2b\eta)-2b[\lambda_2 T_U+(1-\lambda_1-\lambda_2)T_E]\}}{2b(2b+cE)^2} + \frac{E(b+cE)\left\{4\eta^2 b(b+cE)+T_U\lambda_2\left[8\eta b(b+cE)+T_U\left[4b^2+cE[4b+c(1-\lambda_2)E]\right]\right]+T_E(1-\lambda_1-\lambda_2)\left[8\eta b(b+cE)+T_E\left[4b^2+cE[4b+c(\lambda_1+\lambda_2)E]-2c^2\lambda_2 E^2 T_U\right]\right]\right\}}{8(2b+cE)^2},$$
$$S_2^{NS41} = S_2^{NS1},$$
$$S_3^{NS41} = S_3^{NS1},$$

$$V_1^{NS42} = S_1^{NS42} + \pi_1^{NS42} + \overline{C_A},\ V_2^{NS42} = S_2^{NS42} + \pi_2^{NS42} + \overline{C_A},\ V_3^{NS42} = S_3^{NS42} + \pi_3^{NS42} + \overline{C_A},$$

$$\pi_1^{NS42} = \left(\tfrac{a-\eta b}{2b+c\lambda_1 E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right) + \left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_1)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2 \left(\tfrac{L}{3}+\lambda_2 E\right),$$
$$\pi_2^{NS42} = \left(\tfrac{a-\eta b+\left(\frac{T_U}{2}\lambda_1+\frac{T_E}{2}\lambda_3\right)cE}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right),$$
$$\pi_3^{NS42} = \left(\tfrac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_1 E}{2b+cE}\right)^2 \left(\tfrac{L}{3}+\lambda_2 E\right) + \left(\tfrac{a-\eta b}{2b+c\lambda_3 E}\right)^2 (b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$

$$S_1^{NS42} = S_1^{NS1},$$
$$S_2^{NS42} = \tfrac{aE(b+cE)^2\{(a-2b\eta)-2b[\lambda_1 T_U+(1-\lambda_1-\lambda_2)T_E]\}}{2b(2b+cE)^2} + \tfrac{E(b+cE)\left\{4\eta^2 b(b+cE)+T_U\lambda_1\left[8\eta b(b+cE)\right]+T_U\left[4b^2+cE[4b+c(1-\lambda_1)E]\right]\right]+T_E(1-\lambda_1-\lambda_2)\left[8\eta b(b+cE)+T_E\left[4b^2+cE[4b+c(\lambda_1+\lambda_2)E]-2c^2\lambda_1 E^2 T_U\right]\right]\right\}}{8(2b+cE)^2},$$
$$S_3^{NS42} = S_3^{NS1},$$

$$V_1^{NS43} = S_1^{NS43} + \pi_1^{NS43} + \overline{C_A},\ V_2^{NS43} = S_2^{NS43} + \pi_2^{NS43} + \overline{C_A},\ V_3^{NS43} = S_3^{NS43} + \pi_3^{NS43} + \overline{C_A},$$

$$\pi_1^{NS43} = \left(\tfrac{a-\eta b}{2b+c\lambda_1 E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right) + \left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_2^{NS43} = \left(\tfrac{a-\eta b}{2b+c\lambda_2 E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right) + \left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_3^{NS43} = \left(\tfrac{a-\eta b+\frac{T_E}{2}c(\lambda_1+\lambda_2)E}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_3 E\right),$$

$$S_1^{NS43} = S_1^{NS1},\ S_2^{NS43} = S_2^{NS1},$$
$$S_3^{NS43} = \tfrac{aE(b+cE)^2\{(a-2b[\eta+(\lambda_1+\lambda_2)T_E]\}}{2b(2b+cE)^2} + \tfrac{E(b+cE)\left\{4\eta^2 b(b+cE)+T_E(\lambda_1+\lambda_2)\left[8\eta b(b+cE)+T_E[4b^2+cE(4b+c(1-\lambda_1-\lambda_2)E)]\right]\right\}}{8(2b+cE)^2}.$$

Mutual edge + In (NS51, NS52)

$$V_1^{NS51} = S_1^{NS51} + \pi_1^{NS51} + \overline{C_A},\ V_2^{NS51} = S_2^{NS51} + \pi_2^{NS51} + \overline{C_A},\ V_3^{NS51} = V_3^{NS41},$$

$$\pi_1^{NS51} = \left(\tfrac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right) + \left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_2 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right),$$
$$\pi_2^{NS51} = \left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_2)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right) + \left(\tfrac{a-\eta b+\frac{T_U}{2}c\lambda_1 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right),$$

$$S_1^{NS51} = S_1^{NS41},\ S_2^{NS51} = S_2^{NS21},$$

$$V_2^{NS52} = S_2^{NS52} + \pi_2^{NS52} + \overline{C_A},\ V_1^{NS52} = S_1^{NS52} + \pi_1^{NS52} + \overline{C_A},\ V_3^{NS52} = V_3^{NS42},$$

$$\pi_1^{NS52} = \left(\tfrac{a-\eta b+\frac{T_U}{2}c\lambda_2 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right) + \left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_1)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right),$$
$$\pi_2^{NS52} = \left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_1 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right) + \left(\tfrac{a-\eta b+\left(\frac{T_U}{2}\lambda_1+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2 (b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right),$$

$$S_1^{NS52}=S_1^{NS22},\ S_2^{NS52}=S_2^{NS42}.$$

Transitive (NS61, NS62)

$$V_1^{NS61}=S_1^{NS61}+\pi_1^{NS61}+\overline{C_A},\ V_2^{NS61}=S_2^{NS61}+\pi_2^{NS61}+\overline{C_A},\ V_3^{NS61}=V_3^{NS43},$$

$$\begin{gathered}\pi_1^{NS61}=\left(\tfrac{a-\eta b}{2b+c\lambda_1 E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right)+\left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_2 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right)+\\+\left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\\ \pi_2^{NS61}=\left(\tfrac{a-\eta b+\frac{T_U}{2}c\lambda_1 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right)+\\+\left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\end{gathered}$$

$$S_1^{NS61}=S_1^{NS1},\ S_2^{NS61}=S_2^{NS21},$$

$$V_1^{NS62}=S_1^{NS62}+\pi_1^{NS62}+\overline{C_A},\ V_2^{NS62}=S_2^{NS62}+\pi_2^{NS62}+\overline{C_A},\ V_3^{NS62}=V_3^{NS43},$$

$$\begin{gathered}\pi_1^{NS62}=\left(\tfrac{a-\eta b+\frac{T_U}{2}c\lambda_2 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right)+\\+\left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\\ \pi_2^{NS62}=\left(\tfrac{a-\eta b}{2b+c\lambda_2 E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right)+\left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_1 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right)+\\+\left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\end{gathered}$$

$$S_1^{NS62}=S_1^{NS22},\ S_2^{NS62}=S_2^{NS1}.$$

Mutual edge + double In (NS71, NS72, NS73)

$$V_1^{NS71}=S_1^{NS71}+\pi_1^{NS71}+\overline{C_A},\ V_2^{NS71}=S_2^{NS71}+\pi_2^{NS71}+\overline{C_A},\ V_3^{NS71}=S_3^{NS71}+\pi_3^{NS71}+\overline{C_A},$$

$$\begin{gathered}\pi_1^{NS71}=\left(\tfrac{a-\eta b}{2b+c\lambda_1 E}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_1 E\right)+\left(\tfrac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_1)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right)+\\+\left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\\ \pi_2^{NS71}=\left(\tfrac{a-\eta b+\left(\frac{T_U}{2}\lambda_1+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right)+\\+\left(\tfrac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\\ \pi_3^{NS71}=\left(\tfrac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_1 E}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+\lambda_2 E\right)+\\+\left(\tfrac{a-\eta b+\frac{T_E}{2}c(\lambda_1+\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\tfrac{L}{3}+(1-\lambda_1-\lambda_2)E\right),\end{gathered}$$

$$S_1^{NS71}=S_1^{NS1},\ S_2^{NS71}=S_2^{NS42},\ S_3^{NS71}=S_3^{NS43},$$

$$V_1^{NS72}=S_1^{NS72}+\pi_1^{NS72}+\overline{C_A},\ V_2^{NS72}=S_2^{NS72}+\pi_2^{NS72}+\overline{C_A},\ V_3^{NS72}=S_3^{NS72}+\pi_3^{NS72}+\overline{C_A},$$

$$\pi_1^{NS72}=\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_2^{NS72}=\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_2)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b}{2b+c\lambda_2 E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_3^{NS72}=\left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_2 E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b+\frac{T_E}{2}c(\lambda_1+\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$S_1^{NS72}=S_1^{NS41},\ S_2^{NS72}=S_2^{NS1},\ S_3^{NS72}=S_3^{NS43},$$

$$V_1^{NS73}=S_1^{NS73}+\pi_1^{NS73}+\overline{C_A},\ V_2^{NS73}=S_2^{NS73}+\pi_2^{NS73}+\overline{C_A},\ V_3^{NS73}=S_3^{NS73}+\pi_3^{NS73}+\overline{C_A},$$

$$\pi_1^{NS73}=\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_1)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_2^{NS73}=\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_2)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_1+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right),$$
$$\pi_3^{NS73}=\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_1 E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b}{2b+c(1-\lambda_1-\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$S_1^{NS73}=S_1^{NS41},\ S_2^{NS73}=S_2^{NS42},\ S_3^{NS73}=S_3^{NS1}.$$

Mutual edge + double Out (NS8)

$$V_1^{NS8}=S_1^{NS8}+\pi_1^{NS8}+\overline{C_A},\ V_2^{NS8}=S_2^{NS8}+\pi_2^{NS8}+\overline{C_A},\ V_3^{NS8}=V_3^{NS43},$$

$$\pi_1^{NS8}=\left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_2 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_2 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_2^{NS8}=\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_1 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+\left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_1 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$

$$S_1^{NS8}=S_1^{NS22},\ S_2^{NS8}=S_2^{NS21}.$$

Almost complete graph (NS91, NS92)

$$V_1^{NS91}=S_1^{NS91}+\pi_1^{NS91}+\overline{C_A},\ V_2^{NS91}=S_2^{NS91}+\pi_2^{NS91}+\overline{C_A},\ V_3^{NS91}=S_3^{NS91}+\pi_3^{NS91}+\overline{C_A},$$

$$\pi_1^{NS91}=\left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_2 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_1)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_2^{NS91}=\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_1 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_1+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_3^{NS91}=\left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_1 E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b+\frac{T_E}{2}c(\lambda_1+\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$

$$S_1^{NS91}=S_1^{NS22},\ S_2^{NS91}=S_2^{NS42},\ S_3^{NS91}=S_3^{NS43},$$

$$V_1^{NS92}=S_1^{NS92}+\pi_1^{NS92}+\overline{C_A},\ V_2^{NS92}=S_2^{NS92}+\pi_2^{NS92}+\overline{C_A},V_3^{NS92}=S_3^{NS92}+\pi_3^{NS92}+\overline{C_A},$$

$$\pi_1^{NS92}=\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c\lambda_2 E)}{2b+c(\lambda_1+\lambda_2)E}\right)^2\left(\frac{L}{3}+\lambda_2 E\right)+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_2^{NS92}=\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_2)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE}\right)^2\left(\frac{L}{3}+\lambda_1 E\right)+\left(\frac{a-\eta b+\frac{T_U}{2}c\lambda_1 E}{2b+c(\lambda_1+\lambda_2)E}\right)^2(b+cE)\left(\frac{L}{3}+\lambda_2 E\right)+$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE}\right)^2\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_3^{NS92}=\left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_2 E}{2b+cE}\right)^2\left(\frac{L}{3}+\lambda_1 E\right)+$$
$$+\left(\frac{a-\eta b+\frac{T_E}{2}c(\lambda_1+\lambda_2)E}{2b+cE}\right)^2(b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$

$$S_1^{NS92}=S_1^{NS41},\ S_2^{NS92}=S_2^{NS21},\ S_3^{NS92}=S_3^{NS43}.$$

Complete graph (NS10)

$$V_1^{NS10}=S_1^{NS10}+\pi_1^{NS10}+\overline{C_A},\ V_2^{NS10}=S_2^{NS10}+\pi_2^{NS10}+\overline{C_A},\ V_3^{NS10}=S_3^{NS10}+\pi_3^{NS10}+\overline{C_A},$$

$$\pi_1^{NS10} = \left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_2+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) +$$
$$+\left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_1)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right) +$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_2^{NS10} = \left(\frac{a-\eta b-\frac{T_U}{2}(2b+c(1-\lambda_2)E)+\frac{T_E}{2}c(1-\lambda_1-\lambda_2)E}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) +$$
$$+\left(\frac{a-\eta b+\left(\frac{T_U}{2}\lambda_1+\frac{T_E}{2}(1-\lambda_1-\lambda_2)\right)cE}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right) +$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}[2b+c(1-\lambda_1-\lambda_2)E]}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$\pi_3^{NS10} = \left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_2 E}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+\lambda_1 E\right) +$$
$$+\left(\frac{a-\eta b-\frac{T_E}{2}(2b+c(\lambda_1+\lambda_2)E)+\frac{T_U}{2}c\lambda_1 E}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+\lambda_2 E\right) +$$
$$+\left(\frac{a-\eta b+\frac{T_E}{2}c(\lambda_1+\lambda_2)E}{2b+cE} \right)^2 (b+cE)\left(\frac{L}{3}+(1-\lambda_1-\lambda_2)E\right),$$
$$S_1^{NS10} = S_1^{NS41},\ S_2^{NS10} = S_2^{NS42},\ S_3^{NS10} = S_3^{NS43}.$$