TOPOLOGICAL EQUIVALENCE OF HOLOMORPHIC FOLIATION GERMS OF RANK 1 WITH ISOLATED SINGULARITY IN THE POINCARÉ DOMAIN

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Abstract. We show that the topological equivalence class of holomorphic foliation germs with an isolated singularity of Poincaré type is determined by the topological equivalence class of the real intersection foliation of the (suitably normalized) foliation germ with a sphere centered in the singularity. We use this Reconstruction Theorem to completely classify topological equivalence classes of plane holomorphic foliation germs of Poincaré type and discuss a conjecture on the classification in dimension ≥ 3.

0. Introduction

As for isolated singularities of analytic set germs (see [BK86] in the case of plane curve germs) a standard technique to study the topology of holomorphic foliation germs with isolated singularity looks at the intersection of their integral manifolds with spheres centered in the origin (see [LS11] for a more general Morse-theoretic approach). The technique was particularly successful when analyzing holomorphic foliation germs represented by vector fields, that is, foliation germs with 1-dimensional leaves: Guckenheimer [Guc72] and Camacho, Kuiper and Palis [CKP78] (who use polycylinders instead of spheres) classify foliation germs represented by generic linearizable vector fields, whereas Camacho and Sad [CS82] treat resonant cases of plane foliation germs represented by holomorphic vector fields of Siegel type.

Note that Ito [Ito94] and Ito and Scardua [IS05] investigate a kind of reverse situation: They show that a holomorphic vector field everywhere transversal to a sphere has exactly one singularity in the ball bounded by the sphere, but they do not further investigate the intersection foliation and its relation to the original holomorphic vector field.

In this paper we first prove a reconstruction theorem for holomorphic foliation germs represented by a vector field of Poincaré type, that is, the linear part of the vector field has eigenvalues whose convex hull in $\mathbb{C}$ does not contain $0 \in \mathbb{C}$: The topological equivalence class of such a holomorphic foliation germ is uniquely determined by the real-analytic foliation obtained on a sphere around the singularity when intersecting it with all the leaves of a holomorphically equivalent, normalized foliation germ. One direction of this topological equivalence was already observed by Arnol′d [Arn69]; he showed that the original foliation is topologically equivalent

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to the cone foliation over the intersection foliation. For more details on the other direction see Theorem 2.2 and the preceding discussion in Sections 1 and 2.

A similar reconstruction theorem for holomorphic foliation germs represented by vector fields of Siegel type (that is, not of Poincaré type and the linear part has only non-zero eigenvalues) seems possible. The main obstacles to prove such a theorem are missing normal forms and the fact that leaves of such foliation germs may not intersect spheres around the singularity transversally, but tangentially. In fact, results of Brunella and Sad [BS95] suggest that at least in $\mathbb{C}^2$ transversality of the leaves to spheres implies that the holomorphic foliation is of Poincaré type. However, in sufficiently normal situations also in the Siegel case the intersection of leaves and sphere still combine to a real-analytic foliation on the sphere, and the tangential locus is the polar variety of Limón and Seade [LS11] with useful properties.

In Sections 3 to 6 we use the Reconstruction Theorem 2.2 to completely classify topological equivalence classes of plane holomorphic foliation germs represented by vector fields of Poincaré type. Some of the cases in this classification are well-known, for example by Guckenheimer's Stability Theorem [Guc72] on foliation germs given by vector fields whose linear part has $\mathbb{R}$-linearly independent eigenvalues. Nevertheless we describe the topology of the associated intersection foliations in these cases in full detail because this description is missing in the literature and may be useful for classification in higher dimensions. In Section 7 we speculate how to extend the 2-dimensional picture and Guckenheimer's Stability to higher-dimensional foliation germs represented by vector fields of Poincaré type.

Recently, Marín and Mattei presented a classification of topological equivalence classes of plane holomorphic foliation germs satisfying weak genericity assumptions [MM12], by exhibiting an invariant based on the reduction of plane holomorphic foliation singularities and the holonomy around irreducible exceptional components of the reduction. However this classification does not cover the resonant cases discussed in Section 4 because these are not of generic general type, in the terminology of [MM12] (see Remark 4.5). In any case, Marín and Mattei give no explicit lists of topological equivalent foliation germs.

Similarly, Ilyashenko and Yakovenko do not use their calculation of holonomy along the unique closed leaf of resonant foliation germs [IY08, 27.C] to classify these germs.

1. Preliminaries on holomorphic foliation germs of rank 1

In this section we collect well-known definitions and and results on holomorphic foliations with as much details as necessary to fix notations and certain choices simplifying the later arguments. In essence, all of the following can be found in the monograph [IY08].

Definition 1.1. *A germ of a holomorphic foliation of rank* 1 *in* $\mathbb{C}^n$ *with an isolated singularity in* $0 \in \mathbb{C}^n$ *is an equivalence class of pairs* $[U, \theta]$ *where* $U \subset \mathbb{C}^n$ *is an open neighborhood of* 0 *with holomorphic coordinates* $z_1, \ldots, z_n$ *and*

$$\theta = f_1 \frac{\partial}{\partial z_1} + \ldots + f_n \frac{\partial}{\partial z_n}$$

is a holomorphic vector field such that $f_1, \ldots, f_n \in \mathcal{O}(U)$ *vanish simultaneously only in* 0.

Two such pairs $[U,\theta]$ *and* $[U',\theta']$ *are equivalent if there exists an open neighborhood* $V\subset U\cap U'$ *of* $0\in\mathbb{C}^n$ *and a function* $h\in\mathcal{O}^*(V)$ *such that*

$$h\cdot\theta_{|V}=\theta'_{|V}.$$

We denote such holomorphic foliation germs by $\mathcal{F}$.

This definition is equivalent to other definitions of holomorphic foliation germs, see the discussion in [IY08, I.2]. In particular, if $[U,\theta]$ represents a holomorphic foliation germ $\mathcal{F}$ of rank 1 in $\mathbb{C}^n$ with an isolated singularity in $0\in\mathbb{C}^n$ then for all $p\in U-\{0\}$ there exists an open neighborhood $V\subset U$ of p such that the foliation restricted to V is the *standard holomorphic foliation* in suitable holomorphic coordinates $w_1,\dots,w_n$ on V centred in p, that is,

$$\theta(w_1)=\ldots=\theta(w_{n-1})=0.$$

Furthermore, the integral curves of θ in U are covered by the local leaves or *plaques* of these local standard holomorphic foliations given by the intersection of the level sets of $w_1,\dots,w_{n-1}$. These integral curves are called the *leaves* of $\mathcal{F}$.

We will consider the following topological equivalence relation on holomorphic foliation germs:

Definition 1.2. *Two holomorphic foliation germs* $\mathcal{F}$*,* $\mathcal{F}'$ *of rank* 1 *with an isolated singularity in* $0\in\mathbb{C}^n$ *and represented by* $[U,\theta]$*,* $[U',\theta']$ *are called topologically equivalent if there exists a homeomorphism* $\phi:V\to V'$ *of open neighborhoods* $V\subset U$*,* $V'\subset U'$ *of* 0 *such that* $\phi(0)=0$ *and the leaves of* $\mathcal{F}$ *in* V *are mapped onto the leaves of* $\mathcal{F}'$ *in* V' *by* ϕ.

If ϕ is biholomorphic we say that $\mathcal{F}$ and $\mathcal{F}'$ are *holomorphically equivalent.*

We will focus on a special type of holomorphic foliation germs:

Definition 1.3. *A holomorphic foliation germ of rank* 1 *with an isolated singularity in* $0\in\mathbb{C}^n$ *represented by* $[U,\theta]$ *is said to be of Poincaré type if the eigenvalues* $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ *of the linear part*

$$A=\left(\frac{\partial f_j}{\partial z_i}(0)\right)_{i,j=1,\dots,n}$$

of $\theta=\sum_{j=1}^n f_j\frac{\partial}{\partial z_j}$ *generate a convex hull not containing* $0\in\mathbb{C}$*. Then the tuple of eigenvalues* $(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$ *is said to be in the Poincaré domain.*

The classical theorems of Poincaré and Poincaré-Dulac (see [Arn83, §24.D and E]) state that all holomorphic foliation germs of rank 1 with an isolated singularity in $0\in\mathbb{C}^n$ of Poincaré type are even holomorphically equivalent to such germs $\mathcal{F}$ of Poincaré type represented by an open subset $U\subset\mathbb{C}^n$ containing 0 and a holomorphic vector field $\theta=\sum_{i=1}^n f_i(z)\frac{\partial}{\partial z_i}$ for which the following hold:

(i) In U, the $f_i(z)$ can be developed into powers series in the variables $z_1,\dots,z_n$.

(ii) The linear part $A=\left(\frac{\partial f_j}{\partial z_i}(0)\right)_{i,j=1,\dots,n}$ of θ is in Jordan normal form.

(iii) If $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A appearing with their algebraic multiplicity, then the non-vanishing monomials $z_1^{m_1}\cdots z_n^{m_n}$ in $f_i(z)$, with $m_i\in\mathbb{N}_0$, satisfy

$$\lambda_i=\sum_{j=1}^n m_j\lambda_j.$$

Note that condition (ii) implies that the linear term in $f_i(z)$ has the form $\lambda_i z_i$ or $\lambda_i z_i + z_{i+1}$. In the latter case $\lambda_i = \lambda_{i+1}$ by the properties of the Jordan normal form, hence all the non-vanishing monomials in these linear terms satisfy condition (iii). For $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ a relation $\lambda_i = \sum_{j=1}^n m_j \lambda_j$ is called a *resonance* of the eigenvalues $\lambda_1, \ldots, \lambda_n$, and the monomial $z^m := z_1^{m_1} \cdots z_n^{m_n}$ is called a *resonant monomial* if it appears in $f_i(z)$.

Remark 1.4. If $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is in the Poincaré domain then there are only finitely many resonances $\lambda_i = \sum_{j=1}^n m_j \lambda_j$. Furthermore, a resonance relation with λ_i on the left hand side is either the *trivial resonance relation* $\lambda_i = \lambda_i$, or λ_i does not appear on the right hand side at all, that is $m_i = 0$. For proofs, see [Arn83, §24.B]. Note finally that we do not require $\sum_i m_i \geq 2$ as in [Arn83] but only distinguish between the trivial resonant monomial z_i in $f_i(z)$ and non-trivial resonant monomials.

Remark 1.5. Let $\mathcal{F}$ be a holomorphic foliation germ satisfying (i), (ii) and (iii). For the tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ in the Poincaré domain there exists a maximal real constant $c > 0$ such that

$$|\sum_{i=1}^n \lambda_i t_i| \geq c \cdot \sum_{i=1}^n t_i,$$

for all real numbers $t_1, \ldots, t_n \geq 0$. The number c can be interpreted as the distance of the convex hull of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$ from 0. By separately rescaling the coordinates we can achieve that the entries of the matrix A on the superdiagonal are arbitrarily small. If the entries are $\frac{c}{2n}$ we will call $\mathcal{F}$ *normalized* (see the next section).

Remark 1.6. If $n = 2$ then every normalized holomorphic foliation germ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type is represented by a vector field of one of the following types:

(1) $\theta = \lambda z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$, where $\lambda \in \mathbb{C} - \mathbb{R}$,
(2) $\theta = \lambda z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$, where $\lambda \in \mathbb{R}_{>0}$,
(3) $\theta = (m z_1 + z_2^m) \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$, where $m \geq 2$, or
(4) $\theta = (z_1 + \frac{1}{4} z_2) \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$ because the constant c of Remark 1.5 is 1 in this case.

2. The intersection foliation

In this section S_ϵ^{2n-1} denotes the (real) $2n-1$-dimensional sphere in $\mathbb{C}^n$ centered in $0 \in \mathbb{C}^n$ with radius ϵ, and B_ϵ^{2n} denotes the (real) $2n$-dimensional ball in $\mathbb{C}^n$ centered in $0 \in \mathbb{C}^n$ with radius ϵ.
$\mathcal{F}$ is always a normalized holomorphic foliation germ of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type, and $[U, \theta]$ represents $\mathcal{F}$, with $\theta = \sum_{i=1}^n f_i(z) \frac{\partial}{\partial z_i}$. Furthermore, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the linear part of θ appearing with their algebraic multiplicity, and $c > 0$ is the real constant for the tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ in the Poincaré domain introduced in Remark 1.5.
Arnol′d [Arn69, Thm. 5] observed that the leaves of holomorphic foliation germs $\mathcal{F}$ as above intersect spheres S_ϵ^{2n-1} with small enough radius ϵ everywhere transversally. In particular, in each point $p \in S_\epsilon^{2n-1}$ the (real) tangent spaces of the leaf of $\mathcal{F}$ through p and of S_ϵ^{2n-1} intersect in a (real) 1-dimensional subspace. This yields a 1-dimensional distribution on the real $\mathcal{C}^\infty$-manifold S_ϵ^{2n-1} denoted by $\mathcal{F} \cap S_\epsilon^{2n-1}$. This distribution is integrable because it is 1-dimensional, see [War83]. Therefore

we obtain a real foliation on S_ϵ^{2n-1} with 1-dimensional leaves , also denoted by $\mathcal{F} \cap S_\epsilon^{2n-1}$ and called the *real intersection foliation* or *trace foliation* of $\mathcal{F}$ with S_ϵ^{2n-1}.

Its leaves can be canonically oriented: In each point $p \in S_\epsilon^{2n-1}$ choose those vectors in the tangent subspace given by the distribution $\mathcal{F} \cap S_\epsilon^{2n-1}$ in p which together with the tangent vectors of the leaf of $\mathcal{F}$ through p pointing away from $0 \in \mathbb{C}^n$ represent the positive orientation of the complex structure on the leaf. Taking in each point $p \in S_\epsilon^{2n-1}$ a unit vector oriented in that way yields a nowhere vanishing vector field on S_ϵ^{2n-1} whose flow, denoted by $\Phi_{\mathcal{F}}$, has integral curves coinciding with the leaves of $\mathcal{F} \cap S_\epsilon^{2n-1}$.

Definition 2.1. *Two real* 1*-dimensional foliations* $\mathcal{F}, \mathcal{G}$ *on the sphere* S^{2n-1} *are called topologically equivalent if there exists a homeomorphism* $\phi : S^{2n-1} \to S^{2n-1}$ *mapping the leaves of* $\mathcal{F}$ *onto the leaves of* $\mathcal{G}$*.*

In [Arn69], Arnol′d continued to show that for ϵ small enough the foliation $\mathcal{F} \cap B_\epsilon^{2n-1}$ is topologically equivalent to the foliation of B_ϵ^{2n-1} by the cones over the leaves of $\mathcal{F} \cap S_\epsilon^{2n-1}$ with vertex $0 \in \mathbb{C}^n$, and this equivalence is realised by a homeomorphism of B_ϵ^{2n-1} on itself. This obviously implies that topological equivalence of $\mathcal{F}$ and $\mathcal{G}$ follows from topological equivalence of $\mathcal{F} \cap S_\epsilon^{2n-1}$ and $\mathcal{G} \cap S_\epsilon^{2n-1}$.

It seems to be a well-accepted fact that the converse is also true, but because of lack of reference (and in particular of a recipe to construct the topological equivalence of the intersection foliations) we decided to present this proof and construction in all details.

Theorem 2.2 (Reconstruction Theorem)**.** *Two normalized holomorphic foliation germs* $\mathcal{F}, \mathcal{G}$ *with an isolated singularity in* $0 \in \mathbb{C}^n$ *of Poincaré type are topologically equivalent if, and only if the real intersection foliations* $\mathcal{F} \cap S_\epsilon^{2n-1}$ *and* $\mathcal{G} \cap S_\epsilon^{2n-1}$*,* $0 < \epsilon \ll 1$*, are topologically equivalent.*

Proof. As discussed above Arnol′d showed that $\mathcal{F} \cap B_\epsilon^{2n-1}$ respectively $\mathcal{G} \cap B_\epsilon^{2n-1}$ is topologically equivalent to the foliation $\widehat{\mathcal{F}}$ respectively $\widehat{\mathcal{G}}$ of B_ϵ^{2n-1} by the cones over the leaves of $\mathcal{F} \cap S_\epsilon^{2n-1}$ resp. $\mathcal{G} \cap S_\epsilon^{2n-1}$ with vertex $0 \in \mathbb{C}^n$. As already noticed that shows one direction of the Theorem.

To prove that a topological equivalence of $\mathcal{F}$ and $\mathcal{G}$ induces a topological equivalence of the intersection foliations $\mathcal{F} \cap S_\epsilon^{2n-1}$ and $\mathcal{G} \cap S_\epsilon^{2n-1}$, it is enough to construct the latter one from a topological equivalence of the cone foliations $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$, in the sense of Definition 1.2.

By possibly decreasing ϵ to ϵ' this topological equivalence can be realised by an embedding $H_C : B_{\epsilon'}^{2n} \hookrightarrow B_\epsilon^{2n}$ such that $H_C(0) = 0$, leaves of the foliation cone over $\widehat{\mathcal{F}} \cap S_{\epsilon'}^{2n-1}$ are mapped into leaves of the foliation cone over $\widehat{\mathcal{G}} \cap S_\epsilon^{2n-1}$ and the orientation of both the ambient space and the leaves is preserved. Composing H_C with the radial projection $r : B_\epsilon^{2n} - \{0\} \to S_\epsilon^{2n-1}$ produces a continuous map

$$h : S_{\epsilon'}^{2n-1} \overset{H_C}{\hookrightarrow} B_\epsilon^{2n} - \{0\} \overset{r}{\to} S_\epsilon^{2n-1}$$

Since H_C may map $S_{\epsilon'}^{2n-1}$ to a topological manifold in B_ϵ^{2n} intersecting the same radial line more than once, h may not be injective, and hence not the wanted homeomorphism. But from h we will be able to construct a homeomorphism $g : S_{\epsilon'}^{2n-1} \to S_\epsilon^{2n-1}$ defining a topological equivalence of $\widehat{\mathcal{F}} \cap S_{\epsilon'}^{2n-1}$ with $\widehat{\mathcal{G}} \cap S_\epsilon^{2n-1}$. This shows the theorem since the cone structure of $\widehat{\mathcal{F}}$ implies that $\widehat{\mathcal{F}} \cap S_{\epsilon'}^{2n-1}$ is topologically equivalent to $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon}$, and by construction, $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon} = \mathcal{F} \cap S^{2n-1}_{\epsilon}$ and $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon} = \mathcal{G} \cap S^{2n-1}_{\epsilon}$.

Let $L_{\widehat{\mathcal{F}},x}$ denote the leaf of $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ through $x \in S^{2n-1}_{\epsilon'}$, and $L_{\widehat{\mathcal{G}},y}$ the leaf of $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$ through $y \in S^{2n-1}_{\epsilon}$. Let $C_{\widehat{\mathcal{F}},x}$ denote the radial cone in $B^{2n}_{\epsilon'}$ with vertex in 0 over the leaf $L_{\widehat{\mathcal{F}},x} \subset S^{2n-1}_{\epsilon'}$, and similarly $C_{\widehat{\mathcal{G}},y}$ the radial cone in B^{2n}_{ϵ} with vertex in 0 over the leaf $L_{\widehat{\mathcal{G}},y} \subset S^{2n-1}_{\epsilon}$.

Claim 1. h is a surjective map, and $h(L_{\widehat{\mathcal{F}},x}) = L_{\widehat{\mathcal{G}},h(x)}$ for each $x \in S^{2n-1}_{\epsilon'}$.

Proof. There exists $\epsilon'' \ll \epsilon$ such that $B^{2n}_{\epsilon''} \subset H_C(B^{2n}_{\epsilon'})$, as H_C is an embedding fixing 0. Consequently, for every $y \in S^{2n-1}_{\epsilon}$, the segment $[y, 0] \subset B^{2n}_{\epsilon}$ intersects $H_C(B^{2n}_{\epsilon'})$ in a point $H_C(x)$, with $x \in S^{2n-1}_{\epsilon'}$. Hence $h(x) = y$, and the surjectivity of h is shown.

The equality $L_{\widehat{\mathcal{G}},h(x)} = h(L_{\widehat{\mathcal{F}},x})$ follows from the fact that by definition, the topological equivalence H_C maps $C_{\widehat{\mathcal{F}},x}$ bijectively onto $C_{\widehat{\mathcal{G}},h(x)} \cap H_C(B^{2n}_{\epsilon'})$. □

We want to relate h to the flows $\Phi_{\widehat{\mathcal{F}}} : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to S^{2n-1}_{\epsilon'}$ and $\Phi_{\widehat{\mathcal{G}}} : S^{2n-1}_{\epsilon} \times \mathbb{R} \to S^{2n-1}_{\epsilon}$ whose integral curves are the leaves of the real intersection foliations $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ and $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$. Note that in general, $\Phi_{\widehat{\mathcal{F}}}$ and $\Phi_{\widehat{\mathcal{G}}}$ will not commute with h, that is

$$(h \circ \Phi_{\widehat{\mathcal{F}}})(x, t) \neq \Phi_{\widehat{\mathcal{G}}}(h(x), t).$$

To obtain the correct relation, we lift $\Phi_{\widehat{\mathcal{F}}}$ to a flow $\Phi_{\widetilde{\mathcal{F}}}$ on $S^{2n-1}_{\epsilon'} \times \mathbb{R}$, by setting

$$\Phi_{\widetilde{\mathcal{F}}}((x, t'), t) := (\Phi_{\widehat{\mathcal{F}}}(x, t), t + t'), x \in S^{2n-1}_{\epsilon'}, t, t' \in \mathbb{R}.$$

The integral curves of $\Phi_{\widetilde{\mathcal{F}}}$ define a foliation $\widetilde{\mathcal{F}}$ on $S^{2n-1}_{\epsilon'} \times \mathbb{R}$ whose leaves project onto the leaves of $\widehat{\mathcal{F}}$ on $S^{2n-1}_{\epsilon'}$. Similarly,

$$\Phi_{\widetilde{\mathcal{G}}}((y, s'), s) := (\Phi_{\widehat{\mathcal{G}}}(y, s), s + s'), y \in S^{2n-1}_{\epsilon}, s, s' \in \mathbb{R}$$

defines a flow $\Phi_{\widetilde{\mathcal{G}}}$ and a foliation $\widetilde{\mathcal{G}}$ on $S^{2n-1}_{\epsilon} \times \mathbb{R}$ whose leaves project onto the leaves of $\widehat{\mathcal{G}}$ on S^{2n-1}_{ϵ}.

Let p_1, p_2 respectively q_1, q_2 denote the projections from $S^{2n-1}_{\epsilon'} \times \mathbb{R}$ respectively $S^{2n-1}_{\epsilon} \times \mathbb{R}$ to the first and second component. If $U \subset S^{2n-1}_{\epsilon'}$ respectively $V \subset S^{2n-1}_{\epsilon}$ are foliation charts of $\widehat{\mathcal{F}}$ respectively $\widehat{\mathcal{G}}$, then $p_1^{-1}(U)$ respectively $q_1^{-1}(V)$ are foliation charts of $\widetilde{\mathcal{F}}$ respectively $\widetilde{\mathcal{G}}$. Consequently, $h : S^{2n-1}_{\epsilon'} \to S^{2n-1}_{\epsilon}$ can be lifted exactly in one way to a continuous map

$$\widetilde{H} : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to S^{2n-1}_{\epsilon} \times \mathbb{R}$$

such that $p_2 = q_2 \circ \widetilde{H}$, $\widetilde{H}(x, 0) = (h(x), 0)$ for all $x \in S^{2n-1}_{\epsilon'}$ and the leaves of $\widetilde{\mathcal{F}}$ are mapped into the leaves of $\widetilde{\mathcal{G}}$. In particular, $q_1 \circ \widetilde{H} = h \circ p_1$, and since $(\Phi_{\widehat{\mathcal{F}}}(x, t), t) = \Phi_{\widetilde{\mathcal{F}}}((x, 0), t)$ the point $\widetilde{H}(\Phi_{\widehat{\mathcal{F}}}(x, t), t) \in S^{2n-1}_{\epsilon} \times \mathbb{R}$ must be in the same $\widetilde{\mathcal{G}}$-leaf as $\widetilde{H}(x, 0) = (h(x), 0)$. Hence there is an $s \in \mathbb{R}$ such that

$$\Phi_{\widetilde{\mathcal{G}}}((h(x), 0), s) = \widetilde{H}(\Phi_{\widehat{\mathcal{F}}}(x, t), t),$$

and the defining equations of $\Phi_{\widetilde{\mathcal{G}}}$ and $\widetilde{H}$ imply that

$$\Phi_{\widehat{\mathcal{G}}}((h(x), s), s) = (h(\Phi_{\widehat{\mathcal{F}}}(x, t)), (q_2 \circ \widetilde{H})(\Phi_{\widehat{\mathcal{F}}}(x, t), t)).$$

Setting $\tau := q_2 \circ \widetilde{H} \circ (\Phi_{\widehat{\mathcal{F}}} \times p_2) : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to \mathbb{R}$ and comparing the second and the first components yield $s = \tau(x,t)$ and

$$h(\Phi_{\widehat{\mathcal{F}}}(x,t)) = \Phi_{\widehat{\mathcal{G}}}((h(x), \tau(x,t)). \tag{1}$$

This is the requested relation between h, $\Phi_{\widehat{\mathcal{F}}}$ and $\Phi_{\widehat{\mathcal{G}}}$.
By construction we have

$$\tau(x,0) = 0. \tag{2}$$

To obtain further properties of τ we need to investigate the leaves $L_{\widehat{\mathcal{F}},x}$ and $L_{\widehat{\mathcal{G}},y}$ of the real intersection foliations $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ and $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$ in more details. First of all, we must carefully distinguish between the *leaf topology* on $L_{\widehat{\mathcal{F}},x} = \Phi_{\widehat{\mathcal{F}}}(\{x\} \times \mathbb{R})$ and $L_{\widehat{\mathcal{G}},y} = \Phi_{\widehat{\mathcal{G}}}(\{y\} \times \mathbb{R})$ defined as the finest topology such that $\Phi_{\widehat{\mathcal{F}}|\{x\}\times\mathbb{R}}$ respectively $\Phi_{\widehat{\mathcal{G}}|\{y\}\times\mathbb{R}}$ are continuous, and the *inclusion topology* induced by the inclusion in $S^{2n-1}_{\epsilon'}$ respectively S^{2n-1}_{ϵ}. The leaf topology is always finer than the inclusion topology, and the two topologies only coincide if the leaf is locally closed in $S^{2n-1}_{\epsilon'}$ respectively S^{2n-1}_{ϵ}. If $L_{\widehat{\mathcal{F}},x}$ respectively $L_{\widehat{\mathcal{G}},y}$ are not bijective images of $\{x\} \times \mathbb{R}$ respectively $\{y\} \times \mathbb{R}$ under $\Phi_{\widehat{\mathcal{F}}}$ respectively $\Phi_{\widehat{\mathcal{G}}}$ then $\Phi_{\widehat{\mathcal{F}}|\{x\}\times\mathbb{R}}$ respectively $\Phi_{\widehat{\mathcal{G}}|\{y\}\times\mathbb{R}}$ are periodic maps, and the images $L_{\widehat{\mathcal{F}},x}$ respectively $L_{\widehat{\mathcal{G}},y}$ are compact both in leaf topology and inclusion topology. In particular, in that case $L_{\widehat{\mathcal{F}},x}$ respectively $L_{\widehat{\mathcal{G}},y}$ are embedded circles in $S^{2n-1}_{\epsilon'}$ respectively S^{2n-1}_{ϵ}. Note also that $\Phi_{\widehat{\mathcal{F}}|\{x\}\times\mathbb{R}}$ and $\Phi_{\widehat{\mathcal{G}}|\{y\}\times\mathbb{R}}$ are universal coverings of $L_{\widehat{\mathcal{F}},x}$ and $L_{\widehat{\mathcal{G}},y}$ endowed with the leaf topology.

Claim 2. The leaf $L_{\widehat{\mathcal{F}},x} \subset S^{2n-1}_{\epsilon'}$ is an embedded circle if, and only if the leaf $L_{\widehat{\mathcal{G}},h(x)} \subset S^{2n-1}_{\epsilon}$ is an embedded circle.

Proof. Assume that $L_{\widehat{\mathcal{F}},x}$ is an embedded circle, hence compact. Since $h(L_{\widehat{\mathcal{F}},x}) = L_{\widehat{\mathcal{G}},h(x)}$ by Claim 1 and h is continuous in the inclusion topology, $L_{\widehat{\mathcal{G}},h(x)}$ must be compact, hence closed. Then leaf and inclusion topology on $L_{\widehat{\mathcal{G}},h(x)}$ coincide, so $L_{\widehat{\mathcal{G}},h(x)}$ cannot be homeomorphic to $\mathbb{R}$ in leaf topology. Consequently, $L_{\widehat{\mathcal{G}},h(x)}$ is an embedded circle.
On the other hand, if $L_{\widehat{\mathcal{G}},h(x)}$ is an embedded circle then the cone leaf $C_{\widehat{\mathcal{G}},h(x)}$ and hence the intersection $C_{\widehat{\mathcal{G}},h(x)} \cap H_C(S^{2n-1}_{\epsilon'})$ is compact. But the topological equivalence H_C^{-1} maps $C_{\widehat{\mathcal{G}},h(x)} \cap (H_C(S^{2n-1}_{\epsilon'})$ onto $L_{\widehat{\mathcal{F}},x}$. So $L_{\widehat{\mathcal{F}},x}$ is compact in the inclusion topology, hence compact in the coinciding leaf topology, and hence an embedded circle, not a line. □

Using (1) and the functorial property of the flows $\Phi_{\widehat{\mathcal{F}}}$ and $\Phi_{\widehat{\mathcal{G}}}$ we calculate

$$\begin{aligned}
\Phi_{\widehat{\mathcal{G}}}(h(x), \tau(x,t) + \tau(\Phi_{\widehat{\mathcal{F}}}(x,t), t')) &= \Phi_{\widehat{\mathcal{G}}}(\Phi_{\widehat{\mathcal{G}}}(h(x), \tau(x,t)), \tau(\Phi_{\widehat{\mathcal{F}}}(x,t), t')) = \\
&= \Phi_{\widehat{\mathcal{G}}}(h(\Phi_{\widehat{\mathcal{F}}}(x,t)), \tau(\Phi_{\widehat{\mathcal{F}}}(x,t), t')) = \\
&= h(\Phi_{\widehat{\mathcal{F}}}(\Phi_{\widehat{\mathcal{F}}}(x,t), t')) = h(\Phi_{\widehat{\mathcal{F}}}(x, t+t')) = \\
&= \Phi_{\widehat{\mathcal{G}}}(h(x), \tau(x, t+t')).
\end{aligned}$$

If $\Phi_{\widehat{\mathcal{G}}}(h(x), \cdot)$ is injective this implies

$$\tau(x, t+t') = \tau(x,t) + \tau(\Phi_{\widehat{\mathcal{F}}}(x,t), t'). \tag{3}$$

If $L_{\widehat{\mathcal{F}},x}$ and $L_{\widehat{\mathcal{G}},h(x)}$ are embedded circles then $\Phi_{\widehat{\mathcal{F}}|\{x\}\times\mathbb{R}}$ and $\Phi_{\mathcal{G}|\{h(x)\}\times\mathbb{R}}$ are periodic maps with periods $T_{\widehat{\mathcal{F}},x}$ and $T_{\mathcal{G},h(x)}$. Consequently,

$$\tau(x, t+t') = \tau(x,t) + \tau(\Phi_{\widehat{\mathcal{F}}}(x,t), t') + k(t,t') \cdot T_{\mathcal{G},h(x)},$$

where $k(t,t')$ is an integer continuously depending on t, t', hence a constant k. Setting $t = t' = 0$ we obtain $k = k(0,0) = 0$ and thus (3).
In this situation, $\tau(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is the lifting of $h : L_{\widehat{\mathcal{F}},x} \to L_{\widehat{\mathcal{G}},h(x)}$ to the universal coverings of the leaves along the flows $\Phi_{\widehat{\mathcal{F}}} : \{x\} \times \mathbb{R} \to L_{\widehat{\mathcal{F}},x}$ and $\Phi_{\widehat{\mathcal{G}}} : \{h(x)\} \times \mathbb{R} \to L_{\widehat{\mathcal{G}},h(x)}$. Since liftings preserve fibers of the coverings this implies

$$\tau(x, t + T_{\widehat{\mathcal{F}},x}) = \tau(x,t) + l \cdot T_{\widehat{\mathcal{G}},h(x)}.$$

Since $H_C(L_{\widehat{\mathcal{F}},x})$ is an embedded circle in $C_{\widehat{\mathcal{G}},h(x)}$ with 0 in its interior, $h : L_{\widehat{\mathcal{F}},x} \to L_{\widehat{\mathcal{G}},h(x)}$ is homotopic to a homeomorphism. Since furthermore H_C preserves orientation, we conclude $l = 1$ and obtain:

$$\tau(x, t + T_{\widehat{\mathcal{F}},x}) = \tau(x,t) + T_{\widehat{\mathcal{G}},h(x)}. \tag{4}$$

As a last property of τ we show:

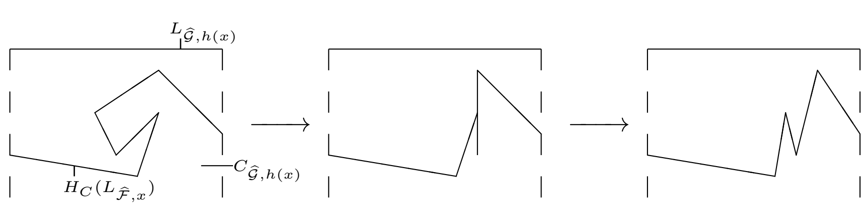
$$\lim_{t\to\infty} \tau(x,t) = \infty \text{ and } \lim_{t\to-\infty} \tau(x,t) = -\infty : \tag{5}$$

If $L_{\widehat{\mathcal{F}},x}$ and $L_{\widehat{\mathcal{G}},h(x)}$ are embedded circles, this follows from (4). Otherwise, both $\Phi_{\widehat{\mathcal{F}}|\{x\}\times\mathbb{R}}$ and $\Phi_{\widehat{\mathcal{G}}|\{h(x)\}\times\mathbb{R}}$ are bijective. In that case, for all $y \in L_{\widehat{\mathcal{G}},h(x)}$ the set of $t \in \mathbb{R}$ such that

$$y = h(\Phi_{\widehat{\mathcal{F}}}(x,t)) = \Phi_{\widehat{\mathcal{G}}}(h(x), \tau(x,t))$$

is bounded because the intersection of the line segment $[y, 0]$ with $H_C(L_{\widehat{\mathcal{F}},x})$ equals $[y,0] \cap H_C(S^{2n-1}_{\epsilon'})$, hence is compact. On the other hand, $|\tau(x,t)|$ may be arbitarily large, as $h(L_{\widehat{\mathcal{F}},x}) = L_{\widehat{\mathcal{G}},h(x)}$. Both facts together contradict $\lim_{t\to\pm\infty} |\tau(x,t)| \neq \infty$. The signs are again as claimed because H_C preserves orientation.

The aim is now to modify τ to a continuous map $\sigma : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to \mathbb{R}$ which is strictly increasing and surjective for fixed $x \in S^{2n-1}_{\epsilon'}$ but still satisfies a functorial property analogous to (3). We use σ to modify h to a topological equivalence g of $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ and $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$.
The modification of τ to σ and hence from h to g is done in two steps: First, we cut off any "moving backwards" of the image of the leaf $L_{\widehat{\mathcal{F}},x}$ on the leaf $L_{\widehat{\mathcal{G}},h(x)}$ by keeping the map stationary whenever such a backwards move starts. Then we smoothen the stationary intervals to obtain a bijective map. For the image $H_C(L_{\widehat{\mathcal{F}},x})$ in $C_{\widehat{\mathcal{G}},h(x)}$ these steps may locally be visualized as follows:



Continuity of τ and (5) imply that $\mu(x,t) := \max_{t'\leq t}\{\tau(x,t')\}$ defines a continuous function $\mu : S^{2n-1}_{\epsilon'} \times \mathbb{R} \to \mathbb{R}$ which is surjective and increasing for fixed x. It holds that

$$\begin{aligned} \mu(x,t+t') &= \max_{t''\leq t+t'}\{\tau(x,t'')\} = \max_{t'''\leq t'}\{\tau(x,t'''+t)\} = \\ &= \max_{t'''\leq t'}\{\tau(\Phi_{\widehat{\mathcal{F}}}(x,t),t''') + \tau(x,t)\} = \\ &= \mu(\Phi_{\widehat{\mathcal{F}}}(x,t),t') + \tau(x,t). \end{aligned} \tag{6}$$

$\mu(x,\cdot)$ is not necessarily strictly increasing. To modify μ to a strictly increasing function without destroying (6) we introduce the growth function

$$\gamma_\delta(x,t) := \min_{t<t'}\{t' : \tau(x,t') = \tau(x,t) + \delta\} - t > 0,$$

for a fixed $\delta > 0$. It is continuous on $S^{2n-1}_{\epsilon'} \times \mathbb{R}$, hence averaging μ by γ_δ leads to the continuous function

$$\sigma(x,t) := \frac{1}{\gamma_\delta(x,t)} \int_t^{t+\gamma_\delta(x,t)} \mu(x,t')dt'$$

which is strictly increasing and surjective onto $\mathbb{R}$ for fixed x, hence continuously invertible. Using (3) we see that $\gamma_\delta(x,t+t') = \gamma_\delta(\Phi_{\widehat{\mathcal{F}}}(x,t),t')$ and together with (6) this implies

$$\sigma(x,t+t') = \sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),t') + \tau(x,t). \tag{7}$$

Claim 3. The map $g : S^{2n-1}_{\epsilon'} \to S^{2n-1}_{\epsilon}, x \mapsto \Phi_{\widehat{\mathcal{G}}}(h(x),\sigma(x,0))$ defines a homeomorphism inducing a topological equivalence of $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ and $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$.

Proof. If $\Phi_{\widehat{\mathcal{G}}}(h(x),\sigma(x,0)) = \Phi_{\widehat{\mathcal{G}}}(h(y),\sigma(y,0))$ then $h(y)$ is in the same $\widehat{\mathcal{G}}$-leaf as $h(x)$, hence y is in the same $\widehat{\mathcal{F}}$-leaf as x, hence there is $t \in \mathbb{R}$ such that $y = \Phi_{\widehat{\mathcal{F}}}(x,t)$. Using (1), (7) and the functorial properties of the flow $\Phi_{\widehat{\mathcal{G}}}$ we calculate

$$\begin{aligned} \Phi_{\widehat{\mathcal{G}}}(h(x),\sigma(x,0)) &= \Phi_{\widehat{\mathcal{G}}}(h(y),\sigma(y,0)) = \Phi_{\widehat{\mathcal{G}}}(h(\Phi_{\widehat{\mathcal{F}}}(x,t)),\sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),0)) = \\ &= \Phi_{\widehat{\mathcal{G}}}(\Phi_{\widehat{\mathcal{G}}}(h(x),\tau(x,t)),\sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),0)) = \\ &= \Phi_{\widehat{\mathcal{G}}}(h(x),\tau(x,t) + \sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),0)) = \\ &= \Phi_{\widehat{\mathcal{G}}}(h(x),\sigma(x,t)). \end{aligned}$$

If $\Phi_{\widehat{\mathcal{G}}|\{h(x)\}\times\mathbb{R}}$ is bijective, this implies $\sigma(x,0) = \sigma(x,t)$, hence $t = 0$ by injectivity of σ for fixed x, hence $y = \Phi_{\widehat{\mathcal{F}}}(x,0) = x$. If $\Phi_{\widehat{\mathcal{G}}|\{h(x)\}\times\mathbb{R}}$ is periodic with period $T_{\widehat{\mathcal{G}},h(x)}$ and hence $\Phi_{\widehat{\mathcal{F}}|\{x\}\times\mathbb{R}}$ is periodic with period $T_{\widehat{\mathcal{F}},x}$ then for some $k \in \mathbb{Z}$ we have

$$\sigma(x,t) = \sigma(x,0) + k \cdot T_{\widehat{\mathcal{G}},h(x)} = \sigma(x,0) + k \cdot \tau(x,T_{\widehat{\mathcal{F}},x}) = \sigma(x,k \cdot T_{\widehat{\mathcal{F}},x}),$$

by (4) and (7). Injectivity of σ implies $t = k \cdot T_{\widehat{\mathcal{F}},x}$, hence $y = \Phi_{\widehat{\mathcal{F}}}(x,k \cdot T_{\widehat{\mathcal{F}},x}) = x$. So g is injective.

If $y \in S^{2n-1}_{\epsilon}$ then there exists $x \in S^{2n-1}_{\epsilon'}$ such that $y = h(x)$, since h is surjective. Then $y = \Phi_{\widehat{\mathcal{F}}}(h(x),0)$. Since $\sigma(x,\cdot)$ is surjective onto $\mathbb{R}$ there exists $t \in \mathbb{R}$ such that

$$\begin{aligned} y = \Phi_{\widehat{\mathcal{G}}}(h(x), \sigma(x,t)) &= \Phi_{\widehat{\mathcal{G}}}(h(x), \tau(x,t) + \sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),0)) = \\ &= \Phi_{\widehat{\mathcal{G}}}(\Phi_{\widehat{\mathcal{G}}}(h(x), \tau(x,t)), \sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),0)) = \\ &= \Phi_{\widehat{\mathcal{G}}}(h(\Phi_{\widehat{\mathcal{F}}}(x,t)), \sigma(\Phi_{\widehat{\mathcal{F}}}(x,t),0)) = \\ &= g(\Phi_{\widehat{\mathcal{F}}}(x,t)), \end{aligned}$$

by (7), (1) and the functorial property of the flow $\Phi_{\widehat{\mathcal{G}}}$. Hence g is surjective.
As a bijective continuous map from a compact topological space to a Hausdorff space, g is a homeomorphism. g is also mapping leaves of $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ to leaves of $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$, so g is a topological equivalence of $\widehat{\mathcal{F}} \cap S^{2n-1}_{\epsilon'}$ and $\widehat{\mathcal{G}} \cap S^{2n-1}_{\epsilon}$. □

This finishes the proof of the theorem. □

3. The case of $\mathbb{R}$-linearly independent eigenvalues in dimension 2

In this section, we only consider holomorphic foliation germs $\mathcal{F}$ around $0 \in \mathbb{C}^2$ represented by vector fields of the form

$$\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \ \lambda \in \mathbb{C} - \mathbb{R}.$$

These foliation germs are invariant under the maps $\mathbb{C}^2 \to \mathbb{C}^2, (x,y) \mapsto r(x,y)$ for all $r \in \mathbb{R}_{>0}$. Hence there is a real intersection foliation $\mathcal{F} \cap S^3_\epsilon$ for all $\epsilon \in \mathbb{R}_{>0}$ as in Section 2, and we assume from now on $\epsilon = 1$.

Lemma 3.1. *Let $S^1 \times S^1$ act on S^3 by $(x,y) \mapsto (xe^{it_1}, ye^{it_2})$. Then the intersection foliation $\mathcal{F} \cap S^3$ is invariant under this action.*

Proof. The 1-form $ydx - \lambda xdy$ corresponding to $\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ is pulled back to

$$ye^{it_2} d(xe^{it_1}) - \lambda x e^{it_1} d(ye^{it_2}) = e^{i(t_1+t_2)}(ydx - \lambda xdy)$$

by the action of $S^1 \times S^1$. Hence the tangent directions of the intersection foliation $\mathcal{F} \cap S^3$ are not changed, and the foliation is invariant under the action. □

For $0 < \epsilon_x, \epsilon_y < 1$, denote the torus $\{(x,y) \in S^3 : |x| = \epsilon_x\}$ by $T^x_{\epsilon_x}$ and the torus $\{(x,y) \in S^3 : |y| = \epsilon_y\}$ by $T^y_{\epsilon_y}$. Then $T^x_{\epsilon_x} = T^y_{\sqrt{1-\epsilon_x^2}}$.

Lemma 3.2. *$T^x_{\epsilon_x}$ intersects all the leaves of the intersection foliation $\mathcal{F} \cap S^3$ not lying on the coordinate axes exactly once and transversally.*

Proof. The real tangent vectors to the torus $T^x_{\epsilon_x}$ in a point (x,y) are those real tangent vectors that are annihilated by the real differential forms

$$d(x\overline{x}) = \overline{x}dx + xd\overline{x} \text{ and } d(y\overline{y}) = \overline{y}dy + yd\overline{y}.$$

The real tangent vectors to the leaf $L_{(x,y)}$ through $(x,y) \in T^x_{\epsilon_x}$ in (x,y) are the $\mathbb{R}$-linear combinations of the real and imaginary part of $\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$. Since

$$(\overline{y}dy + yd\overline{y})(\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) = \overline{y}y \in \mathbb{R} \text{ and } y \neq 0$$

only the imaginary part of $\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ can be tangent to $T^x_{\epsilon_x}$. Since

$$(\overline{x}dx + xd\overline{x})(\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) = \lambda \overline{x}x \text{ and } x \neq 0$$

this can only be the case if $\mathrm{Im}(\lambda) = 0$. But we assumed $\lambda \in \mathbb{C} - \mathbb{R}$, hence the real tangent spaces of $T^x_{\epsilon_x}$ and $L_{(x,y)}$ intersect transversally, hence the leaf $L_{(x,y)} \cap S^3$ of $\mathcal{F} \cap S^3$ and $T^x_{\epsilon_x}$ intersect transversally.
In particular, on a leaf of $\mathcal{F} \cap S^3$ different from $\{x = 0\}$ and $\{y = 0\}$ the absolute value of the x-coordinate must always strictly increase or decrease. Consequently, such a leaf intersects $T^x_{\epsilon_x}$ exactly once. □

For $0 < \epsilon_x, \epsilon_y < 1$ and $t \in \mathbb{R}$ denote the disk $\{(x, \sqrt{1 - |x|^2}e^{it}) \in S^3 : |x| < \epsilon_x\}$ by D^x_{t,ϵ_x} and the disk $\{(\sqrt{1 - |y|^2}e^{it}, y) \in S^3 : |y| < \epsilon_y\}$ by D^y_{t,ϵ_y}.

Lemma 3.3. *D^x_{t,ϵ_x} and D^y_{t,ϵ_y} intersect all the leaves of the intersection foliation $\mathcal{F} \cap S^3$ everywhere transversally.*

Proof. By the $S^1 \times S^1$-invariance of the leaves of $\mathcal{F} \cap S^3$ shown in Lemma 3.1 we can assume that $t = 0$. Since D^x_{0,ϵ_x} is an open subset of $\{y^2 = 1 - |x|^2\} \subset \mathbb{C}^2$, a smooth manifold for $|x| < 1$, the real tangent vectors to D^x_{0,ϵ_x} are exactly those annihilated by the real and the imaginary part of the differential form

$$\omega = d(y^2 + |x|^2 - 1) = 2ydy + \overline{x}dx + xd\overline{x}.$$

We have

$$\omega_{\mathrm{Re}} = ydy + \overline{y}d\overline{y} + \overline{x}dx + xd\overline{x} \text{ and } \omega_{\mathrm{Im}} = -i(ydy - \overline{y}d\overline{y}).$$

Let $\theta(x, y)$ denote the complex tangent vector $\lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ to the leaf $L_{(x,y)}$ through $(x, y) \in D^x_{0,\epsilon_x}$. Then $\omega_{\mathrm{Im}}(\theta(x, y)) = -iy^2 \in i\mathbb{R} - \{0\}$ since $y \in \mathbb{R} - \{0\}$. But the real part of $\theta(x, y)$ is not tangent to both $\{y^2 = 1 - |x|^2\}$ and $S^3 = \{\overline{x}x + \overline{y}y = 1\}$ either:

$$\omega_{\mathrm{Re}}(\theta(x, y)) = 2y^2 + \lambda x\overline{x} \text{ and } d(\overline{x}x + \overline{y}y)(\theta(x, y)) = \lambda x\overline{x} + y\overline{y},$$

hence the real part of the first number vanishes for $\mathrm{Re}\lambda = -\frac{2y^2}{|x|^2}$, the second for $\mathrm{Re}\lambda = -\frac{y\overline{y}}{|x|^2}$. Since $y \neq 0$ this cannot happen for the same λ. □

Figure 3.1 visualizes the behaviour of leaves of $\mathcal{F} \cap S^3$ in the cut-up solid torus $\bigcup_{0 \le \epsilon_x \le \epsilon} T^x_{\epsilon_x}$ (respectively $\bigcup_{0 \le \epsilon_y \le \epsilon} T^y_{\epsilon_y}$) as decribed by Lemmas 3.2 and 3.3.

Theorem 3.4. *Let $\lambda_1 x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and $\lambda_2 x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ represent two holomorphic foliation germs $\mathcal{F}_1, \mathcal{F}_2$ in $0 \in \mathbb{C}^2$, with $\lambda_1, \lambda_2 \in \mathbb{C} - \mathbb{R}$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent.*

Proof. We will construct a topological equivalence of the intersection foliations $\mathcal{F}_1 \cap S^3$ and $\mathcal{F}_2 \cap S^3$. Then the statement follows by the Reconstruction Theorem 2.2.
Lemmas 3.2 and 3.3 show that every leaf of $\mathcal{F}_i$ in the tubular torus $\{(x, y) \in S^3 : 0 < |x| \le \frac{1}{2}\}$ is parametrized on the one hand by the absolute value ϵ_x of the x-coordinate, on the other hand by the argument t of the y-coordinate.
The parametrisation by ϵ_x yields the homeomorphisms

$$\Phi^{(i)}_x : T_{1/2} \times (0, 1/2] \to \{(x, y) \in S^3 : 0 < |x| \le 1/2\}$$

mapping a pair $(x, y) \times \epsilon_x$ to the unique intersection point of the leaf of $\mathcal{F}_i \cap S^3$ through (x, y) with $T^x_{\epsilon_x}$.
Then $\Phi^{(2)}_x \circ (\Phi^{(1)}_x)^{-1}$ is a homeomorphism of the tubular torus $\{(x, y) \in S^3 : 0 < |x| \le \frac{1}{2}\}$ into itself but might not be extendable to a

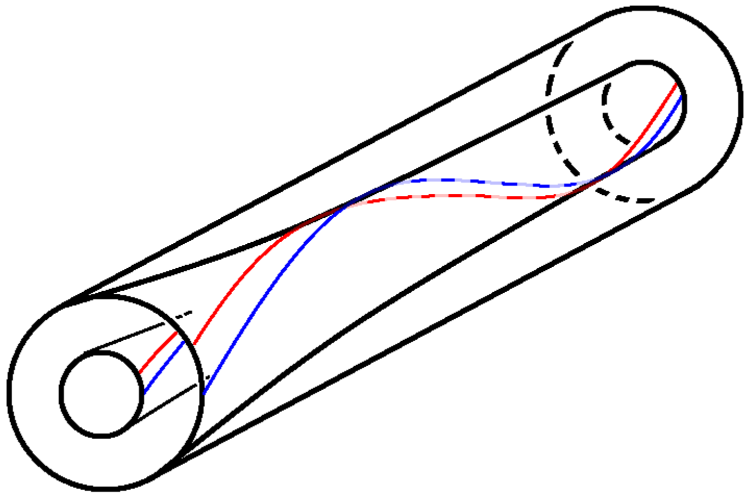


FIGURE 3.1.

homeomorphism of the solid torus $\{(x,y) \in S^3 : 0 \leq |x| \leq \frac{1}{2}\}$. To achieve that we reparametrize the ϵ_x-interval $(0, \frac{1}{2}]$ using the second parametrization by the argument of the y-coordinate: Every leaf $L_{(x,y)}$ through a point $(x,y) \in T^x_{\frac{1}{2}}$ defines an invertible function $\phi^{(i)}_x : (0, \frac{1}{2}] \to [0, \infty)$, mapping ϵ_x to $t - t_0$ where t is the argument of the y-coordinate of the intersection point of $L_{(x,y)}$ with $T^x_{\epsilon_x}$ and t_0 is the argument of y. These functions are the same for all such leaves because of the $S^1 \times S^1$-invariance, and we always have $\phi^{(i)}_x(\frac{1}{2}) = 0$. Then

$$\Phi^{(2)}_x \circ \left[\mathrm{id}_{T^x \frac{1}{2}} \times ((\phi^{(2)}_x)^{-1} \circ \phi^{(1)}_x)\right] \circ (\Phi^{(1)}_x)^{-1}$$

maps $D^x_{t,\frac{1}{2}}$ and $T^x_{\epsilon_x}$, $0 < \epsilon_x \leq \frac{1}{2}$ onto themselves. This implies that the identity map on $\{x = 0\} \cap S^3$ extends this composition of maps to a homeomorphism Φ_x of the solid torus $\{(x,y) \in S^3 : 0 < |x| \leq \frac{1}{2}\}$ mapping leaves of $\mathcal{F}_1 \cap S^3$ to leaves of $\mathcal{F}_2 \cap S^3$. Furthermore, the restriction of Φ_x to $T^x_{\frac{1}{2}}$ is the identity map.
In the same way we can construct a homeomorphism Φ_y of the solid torus $\{(x,y) \in S^3 : 0 < |y| \leq \frac{1}{2}\}$ mapping leaves of $\mathcal{F}_1 \cap S^3$ to leaves of $\mathcal{F}_2 \cap S^3$. Since again the restriction of Φ_y to $T^x_{\frac{1}{2}}$ is the identity map Φ_x and Φ_y glue to a topological equivalence of the intersection foliations $\mathcal{F}_1 \cap S^3$ and $\mathcal{F}_2 \cap S^3$. □

Remark 3.5. The theorem is Guckenheimer's result in dimension 2 [Guc72]. The proof above yields the construction of an explicit topological equivalence which is missing in Guckenheimer's original argument. Another explicit topological equivalence is constructed in [CKP78] using polycylinders instead of balls.
Independently of intersection foliations, there exist $a, b \in \mathbb{C}$ with $\mathrm{Re}(a), \mathrm{Re}(b) > -1$ such that $(x,y) \mapsto (x|x|^a, y|y|^b)$ is a topological equivalence of $\mathcal{F}_1$ and $\mathcal{F}_2$ if $\mathrm{Im}(\lambda_1)\mathrm{Im}(\lambda_2) > 0$ (otherwise, divide $\lambda_2 x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ by λ_2 and exchange x and y). This equivalence was pointed out by the anonymous referee.

4. The resonant case in dimension 2

In this section, we only consider holomorphic foliation germs $\mathcal{F}_m$ around $0 \in \mathbb{C}^2$ represented by vector fields θ_m of the form

$$(mx + y^m)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ m \geq 1.$$

Note that all these foliation germs are equal to the germs in Remark 1.6.(3) and (4), up to possibly rescaling the x-coordinate.

Lemma 4.1. *The leaves of $\mathcal{F}_m$ intersect all spheres S^3_ϵ, $0 < \epsilon \leq 1$, transversally. In particular the real intersection foliation $\mathcal{F}_m \cap S^3$ on $S^3 = S^3_1$ exists.*

Proof. The leaf of $\mathcal{F}_m$ through $p = (x, y)$ does not intersect S^3_ϵ transversally in p if, and only if the holomorphic tangent vector $\theta_m(p)$ is tangent to S^3_ϵ in p if, and only if

$$(\overline{x}dx + \overline{y}dy + xd\overline{x} + yd\overline{y})\left((mx + y^m)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \overline{x}x + \overline{y}y + (m-1)\overline{x}x + \overline{x}y^m = 0.$$

But this is impossible since $\overline{x}x + \overline{y}y = \epsilon^2$, $(m-1)\overline{x}x \geq 0$ and

$$|\overline{x}y^m| = |x| \cdot |y|^m < \epsilon^{m+1} \leq \epsilon^2,$$

since $|x|, |y| \leq \epsilon$ but never $|x| = |y| = \epsilon$. □

Next, we analyse the leaves of the intersection foliations $\mathcal{F}_m \cap S^3$.

Proposition 4.2. *The only closed leaf of $\mathcal{F}_m \cap S^3$ is $\{y = 0\} \cap S^3$. The closure of any leaf $L_{(a,b)}$ through a point $(a, b) \in S^3 - \{y = 0\}$ is $L_{(a,b)} \cup (\{y = 0\} \cap S^3)$. For a certain $\epsilon_y = \epsilon_y(L_{(a,b)})$ with $0 < \epsilon_y \leq 1$ the leaf $L_{(a,b)}$ intersects a torus $T^y_{\epsilon'_y}$*

- *in two distinct points if $0 < \epsilon'_y < \epsilon_y$,*
- *in one point if $\epsilon'_y = \epsilon_y$ and*
- *not at all if $\epsilon'_y > \epsilon_y$.*

Proof. The holomorphic map

$$\lambda_{(a,b)} : \mathbb{C} \to \mathbb{C}^2,\ t \mapsto ((a + b^m t)e^{mt}, be^t)$$

defines the integral curve of the vector field $(mx + y^m)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ through $\lambda_{(a,b)}(0) = (a, b)$, that is the leaf of $\mathcal{F}_m$ through (a, b). If $(a, b) \in S^3$ the leaf of $\mathcal{F}_m \cap S^3$ through (a, b) is the $\lambda_{(a,b)}$-image of the branch through $t = 0$ of the curve in $\mathbb{C}$ implicitely given by

$$1 = e^{m(t+\bar{t})}(a\overline{a} + b^m\overline{a}t + a\overline{b}^m\bar{t} + (b\bar{b})^m t\bar{t}) + b\bar{b}e^{t+\bar{t}}.$$

Decomposing $t = t_R + it_I$ into real and imaginary part and rearranging the equation we obtain

$$(*)\ \ (b\bar{b})^m t_I^2 + 2\mathrm{Im}(a\overline{b}^m)t_I + a\overline{a} + 2\mathrm{Re}(b^m\overline{a})t_R + (b\bar{b})^m t_R^2 + b\bar{b}e^{2(1-m)t_R} - e^{-2mt_R} = 0.$$

This is a quadratic equation in t_I, with coefficients of t_I^2 and t_I independent of t_R.

Claim. For $t_R \leq 0$ the constant term of $(*)$ is increasing with t_R.

Proof. When we derive the constant term with respect to t_R we obtain the gradient

$$2\mathrm{Re}(b^m\overline{a}) + 2(b\overline{b})^m t_R + 2(1-m)b\overline{b}e^{2(1-m)t_R} + 2me^{-2mt_R}$$

which is $> 2e^{-2mt_R} + 2t_R + 2(1-m)e^{2t_R} - 2$ for $t_R \leq 0$ since $(a,b) \in S^3$ implies $|a|, |b| \leq 1$ and $|\mathrm{Re}(b^m\overline{a})| < 1$. But the function $x \mapsto e^{-2mx} + x + (1-m)e^{2x} - 1$ has derivative $-2me^{-2mx} + 2(1-m)e^{2x} + 1 < 0$ for $x \leq 0$, hence the gradient is always > 0 for $t_R \leq 0$, as it is > 0 for $t_R = 0$. $\square$

If $t_R \to \infty$ the constant term also tends to ∞. So we conclude: There exists a $t_0 \geq 0$ such that for all $t_R < t_0$ there are two solutions t_I to the equation $(*)$ symmetric to $-\frac{\mathrm{Im}(a\overline{b}^m)}{(b\overline{b})^m}$, and one solution $t_I = -\frac{\mathrm{Im}(a\overline{b}^m)}{(b\overline{b})^m}$ if $t_R = t_0$.
In particular, if $t_R \to -\infty$ we have $y = be^t \to 0$ which implies the claim on the closure of $L_{(a,b)}$. Since this leaf intersects a torus $T^y_{\epsilon'_y}$ in all points (a', b') of the leaf where $|b'| = \epsilon'_y$ the last claim follows. In particular, the maximal ϵ_y such that $L_{(a,b)} \cap T^y_{\epsilon_y} \neq \emptyset$ is given by $\epsilon_y = |e^{t_0}b|$. $\square$

Corollary 4.3. *All the leaves of $\mathcal{F}_m \cap S^3$ away from $\{y = 0\}$ are uniquely parametrised by the points of the set*

$$\{(a,b) \in S^3 : \mathrm{Im}(a\overline{b}^m) = 0, b \neq 0\}.$$

Proof. Since there is only one point on a leaf $L_{(a,b)}$ with maximal distance $\epsilon_y(L_{(a,b)})$ to $\{y = 0\}$, these points uniquely parametrise all leaves of $\mathcal{F}_m$ away from $\{y = 0\}$. Furthermore, (a,b) is such a point on $L_{(a,b)}$ if for $t = 0$ the linear and constant term of $(*)$ vanish. This is exactly the case when $\mathrm{Im}(a\overline{b}^m) = 0$ since $a\overline{a} + b\overline{b} = 1$. $\square$

Theorem 4.4. *The intersection foliations $\mathcal{F}_m \cap S^3$ are not topologically equivalent for different $m = 1, 2, \ldots$.*

Proof. Assume that $\Phi : S^3 \to S^3$ is a topological equivalence of $\mathcal{F}_{m_1} \cap S^3$ with $\mathcal{F}_{m_2} \cap S^3$. Then Φ maps the only closed leaf of $\mathcal{F}_{m_1}$ to the only closed leaf of $\mathcal{F}_{m_2}$, that is, $\{y = 0\} \cap S^3$ to itself. Hence Φ maps the open complement $U_1 := S^3 - \bigcup_{0 \leq \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$ of the solid torus $\bigcup_{0 \leq \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$ to an open set $\Phi(U_1)$ in S^3 not intersecting $\{y = 0\} \cap S^3$ but containing $\{x = 0\} \cap S^3$ if ϵ_1 is small enough, by a compactness argument.
Let $\overline{U_1}$ be the union of all leaves of $\mathcal{F}_{m_1} \cap S^3$ intersecting U_1. Then the complement $V_1 := S^3 - (\overline{U_1} \cup \{y = 0\})$ consists of leaves of the foliation $\mathcal{F}_{m_1} \cap S^3$. Corollary 4.3 shows that these leaves are uniquely parametrised by points of $\{\mathrm{Im}(a\overline{b}^{m_1}) = 0\} \cap \bigcup_{0 < \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$.
Note that for $0 < \epsilon_y < 1$ the intersection $\{\mathrm{Im}(a\overline{b}^m) = 0\} \cap T^y_{\epsilon_y}$ consists of m connected curves given by $m\arg(b) - \arg(a) \in \pi \cdot \mathbb{Z}$ on the torus $T^y_{\epsilon_y}$, each of them of homology class $(m, 1)$ with respect to the generating cycles $\{\arg(x) = 0\} \cap T^y_{\epsilon_y}$ and $\{\arg(y) = 0\} \cap T^y_{\epsilon_y}$. These curves are visualized in Figure 4.1 when $m = 2$, as the red and the blue curve on the torus $T^y_{\epsilon_y}$ cut up along a disk D^y_t. Hence $\{\mathrm{Im}(a\overline{b}^m) = 0\} \cap \bigcup_{0 < \epsilon_y \leq \epsilon_1} T^y_{\epsilon_y}$ has m connected components, and all of them can be retracted to a curve of homology class $(m, 1)$ in $T^y_{\epsilon_1}$. Since $S^3 - \{y = 0\}$ can be retracted to $S^3 \cap \{x = 0\}$, the homology class of this curve in $S^3 - \{y = 0\}$ is m times the generator represented by $S^3 \cap \{x = 0\}$.

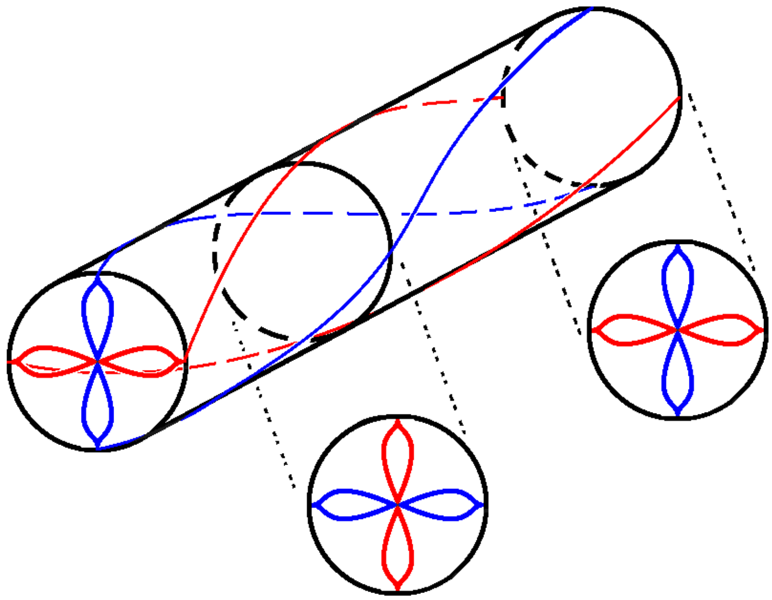


FIGURE 4.1.

The flow on S^3 associated to $\mathcal{F}_{m_1}$ induces a retraction of V_1 to $\{\mathrm{Im}(a\bar{b}^{m_1}) = 0\} \cap \bigcup_{0<\epsilon_y\leq\epsilon_1} T^y_{\epsilon_y}$, hence V_1 consists of m_1 connected components $V_1', \dots, V_1^{(m_1)}$. These components are visualized in Figure 4.1 when $m_1 = 2$, as the two regions enclosed by the red and the blue surfaces in the cut-up solid torus $\bigcup_{0\leq\epsilon_y\leq\epsilon_1} T^y_{\epsilon_y}$. By construction, $\Phi(V_1)$ does not intersect the complement of a solid torus, $U_2 := S^3 - \bigcup_{0\leq\epsilon_y\leq\epsilon_2} T^y_{\epsilon_y}$ if ϵ_2 is close enough to 1. Constructing V_2 from U_2 using $\mathcal{F}_{m_2}$ as V_1 was constructed from U_1 using $\mathcal{F}_{m_1}$, this implies $\Phi(V_1) \subset V_2$, and $\Phi(V_1')$ lies in one of the m_2 connected components of V_2, say V_2'.
Consequently, we have a commutative diagram of homeomorphisms and embeddings,

$$\begin{array}{ccccc} V_1' & \overset{\Phi}{\to} & \Phi(V_1') & \subset & V_2' \\ \cap & & \cap & & \cap \\ S^3 - \{y=0\} & \overset{\Phi}{\to} & S^3 - \{y=0\} & = & S^3 - \{y=0\}. \end{array}$$

This diagram induces the commutative diagram of group homomorphisms of homology group

$$\begin{array}{ccccc} \mathbb{Z} & \overset{\cdot\pm1}{\longrightarrow} & \mathbb{Z} & \to & \mathbb{Z} \\ \cdot m_1 \downarrow & & & & \downarrow \cdot m_2 \\ \mathbb{Z} & = & \mathbb{Z} & = & \mathbb{Z} \end{array}$$

The left and right vertical homomorphism are given by multiplications with m_1 and m_2 because of the retractions constructed above, whereas the upper right homomorphism is given by multiplication with an arbitrary integer n.
Consequently, we obtain $\pm m_1 = \pm n \cdot m_2$, hence $m_1 \geq m_2$. Exchanging the roles of m_1 and m_2 we also obtain $m_1 \leq m_2$ and therefore $m_1 = m_2$. $\square$

Remark 4.5. The holomorphic foliation germs $\mathcal{F}_m$ discussed in this section are not of general type, in the terminology of [MM12]: One feature of plane holomorphic foliation germs of general type is that the singularities of the reduction are represented by vector fields without a linear part with eigenvalue 0. But from $(mx+y^m)\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ respectively the holomorphic 1-form $ydx-(mx+y^m)dy$ representing the same holomorphic foliation germ we obtain 1-forms

$$t(1-m+t^mx^{m-1})dx-(mx+t^mx^m)dt$$

respectively vector fields

$$(mx+t^mx^m)\frac{\partial}{\partial x}+t(1-m+t^mx^{m-1})\frac{\partial}{\partial t}$$

in the (x,t)-chart with $(x,y)=(x,xt)$ and

$$yds+(s(1-m)+y^{m-1})dy \text{ respectively } y\frac{\partial}{\partial y}+((m-1)s+y^{m-1})\frac{\partial}{\partial s}$$

in the (s,y)-chart with $(x,y)=(sy,y)$, by blowing up $\mathbb{C}^2$ in 0.
If $m=1$ the blown-up foliation in the (x,t)-chart is represented by $x\frac{\partial}{\partial x}+t^2\frac{\partial}{\partial t}$, yielding a reduced singularity in $(x,t)=(0,0)$ but not one of general type.
If $m\geq 2$ the blown-up foliation has a singularity of type $\mathcal{F}_{m-1}$ in $(s,y)=(0,0)$. Thus further reducing this singularity will finally lead to another reduced singularity not of general type.

5. The non-resonant case of $\mathbb{R}$-linearly dependent eigenvalues in dimension 2

In this section, we only consider holomorphic foliation germs $\mathcal{F}_\lambda$ around $0\in\mathbb{C}^2$ represented by vector fields of the form

$$\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ \lambda\in\mathbb{R}_{>0}.$$

As in Section 3 these foliation germs are invariant under rescaling with positive real constants. Hence it is enough to consider the real intersection foliations $\mathcal{F}_\lambda\cap S^3_1=\mathcal{F}_\lambda\cap S^3$.

Lemma 5.1. *Every leaf of the intersection foliation $\mathcal{F}_\lambda\cap S^3$ lies on a torus $T^x_{\epsilon_x}$, $0\leq\epsilon_x\leq 1$.*

Proof. The flow of the vector field $\lambda x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ is given by $(a,b,t)\mapsto(ae^{\lambda t},be^t)$. Since $\lambda\in\mathbb{R}_{>0}$ the intersection of the associated integral manifold through a point $(a,b)\in S^3$ with S^3 is parametrised by $t\mapsto(ae^{\lambda it},be^{it})$. Thus the leaf of $\mathcal{F}_\lambda\cap S^3$ through (a,b) lies on the torus $T_x(|a|)$. □

5.1. $\lambda\in\mathbb{Q}_{>0}$.

Assume that $\lambda=\frac{p}{q}$, where $p,q\in\mathbb{N}$ are relatively prime.

Proposition 5.2. *Every leaf of the intersection foliation $\mathcal{F}_\lambda\cap S^3$ is closed. A leaf on the torus $T_x(\epsilon_x)$, $0<\epsilon_x<1$, is a curve of type (p,q), where p describes the winding number of the leaf around $\{x=0\}\cap S^3$ and q the winding number around $\{y=0\}\cap S^3$. The holonomy in a point $(0,e^{it})\in\{x=0\}\cap S^3$ following the leaf in counter-clockwise direction is given by the germ of the map $D^t_x(\epsilon_x)\to D^t_x(\epsilon_x)$, $0<\epsilon_x\ll 1$, multiplying the x-coordinate by $e^{2\pi i\cdot\frac{p}{q}}$. Similarly, the holonomy of the leaf in a point $(e^{it},0)$ following the leaf in counter-clockwise direction is described by the germ of the map $D^t_y(\epsilon_y)\to D^t_y(\epsilon_y)$ multiplying the y-coordinate with $e^{2\pi i\cdot\frac{q}{p}}$.*

The holonomy in all points of S^3 *away from* $\{x = 0\} \cup \{y = 0\}$ *is the identity.*

Proof. $\mathcal{F}_\lambda$ is also represented by the vector field $px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$. The flow of this vector field is given by $(a, b, t) \mapsto (ae^{pt}, be^{qt})$, and the intersection of the associated integral manifold through (a, b) with S^3 is parametrised as $t \mapsto (ae^{pit}, be^{qit})$, $t \in \mathbb{R}$. These parametrisations are periodic, with period $\frac{2\pi}{\gcd(p,q)} = 2\pi$. The claims of the proposition follow. $\square$

Corollary 5.3. *Two foliation germs* $\mathcal{F}_\lambda$, $\mathcal{F}_\mu$, $\lambda, \mu \in \mathbb{Q}_{>0}$, *are topologically equivalent if, and only if* $\lambda = \mu$ *or* $= \frac{1}{\mu}$.

Proof. By the Reconstruction Theorem 2.2 we only have to decide whether the intersection foliations $\mathcal{F}_\lambda \cap S^3$ and $\mathcal{F}_\mu \cap S^3$ are topologically equivalent or not. Now, the topological types of the holonomy along closed paths on leaves of these real foliation are topologically invariant, in particular the order of the holonomy germ. Consequently, Proposition 5.2 implies that only $\mathcal{F}_{\frac{p}{q}} \cap S^3$ and $\mathcal{F}_{\frac{q}{p}} \cap S^3$ can be topologically equivalent, and in that case the equivalence is given by $(x, y) \mapsto (y, x)$. $\square$

5.2. $\lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$. As in the proof of Lemma 5.1 the leaf of the intersection foliation $\mathcal{F}_\lambda \cap S^3$ through a point $(a, b) \in S^3$ is parametrised by $t \mapsto (ae^{i\lambda t}, be^{it})$, hence lies on $T_x(|a|)$. Since λ is irrational the leaf is not closed but dense on the torus $T_x(|a|)$, for all $a \in \mathbb{C}$ such that $0 < |a| < 1$. Thus we can describe the leaves of $\mathcal{F}_\lambda \cap S^3$ as follows:

Lemma 5.4. *The intersection foliation* $\mathcal{F}_\lambda \cap S^3$ *has two closed leaves,* $\{x = 0\} \cap S^3$ *and* $\{y = 0\} \cap S^3$, *whereas the closure of every other leaf is a torus* $T_x(\epsilon_x)$, $0 < \epsilon_x < 1$. $\square$

Next, we consider the continuous map $f : S^3 \to [0, 1], (x, y) \mapsto |x|$. Its fibers are $f^{-1}(\epsilon_x) = T_x(\epsilon_x)$, $0 \leq \epsilon_x \leq 1$. Lemma 5.4 shows that a topological equivalence Φ of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$, $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$, induces a homeomorphism $\phi : [0, 1] \to [0, 1]$ such that $\phi \circ f = f \circ \Phi$, with $\phi(\{0, 1\}) = \{0, 1\}$. Note that $\phi(0) = 0$ and $\phi(1) = 1$ means that Φ maps the closed leaves $\{x = 0\} \cap S^3$ respectively $\{y = 0\} \cap S^3$ onto themselves, whereas $\phi(0) = 1$, $\phi(1) = 0$ indicates that Φ interchanges the closed leaves.

Furthermore, Φ maps the torus $T_x(\epsilon_x)$ homeomorphically onto the torus $T_x(\phi(\epsilon_x))$, $0 < \epsilon_x < 1$. Recall that the (extended) mapping class group of a 2-dimensional torus $T^2 \cong S^1 \times S^1$ is given by $GL(H_1(T^2), \mathbb{Z})$ [FM12, Thm.2.5]. Identifying the tori $T_x(\epsilon_x)$ for different $0 < \epsilon_x < 1$ by rescaling the x- and the y-coordinate the following statement makes sense:

Proposition 5.5. *If* $\Phi : S^3 \to S^3$ *is a topological equivalence of* $\mathcal{F}_\lambda \cap S^3$ *with* $\mathcal{F}_\mu \cap S^3$, $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ *then the restriction* $\Phi_{|T_x(\epsilon_x)} : T_x(\epsilon_x) \to T_x(\phi(\epsilon_x))$ *is of one of the types* $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ *in the mapping class group of a 2-dimensional torus, for all* $0 < \epsilon_x < 1$.

Proof. Interchanging the coordinates yields a homeomorphism $\Psi : S^3 \to S^3, (x, y) \mapsto (y, x)$ whose restriction to tori $T_x(\epsilon_x)$ is of type $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the mapping class group of a 2-dimensional torus. Composing Ψ with a topological equivalence Φ of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$ such that $\phi(0) = 1$, $\phi(1) = 0$ yields a topological equivalence Φ' of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_{\frac{1}{\mu}} \cap S^3$ such that $\phi'(0) = 0$, $\phi'(1) = 1$. Hence, from now on we will only consider that case.

For all $0 < \epsilon_0 < 1$ the topological equivalence Φ maps the solid torus $\bigcup_{0 \leq \epsilon_x \leq \epsilon_0} T_x(\epsilon_x)$ homeomorphically onto the solid torus $\bigcup_{0 \leq \epsilon_x \leq \epsilon_0} T_x(\phi(\epsilon_x))$ and $\bigcup_{\epsilon_0 \leq \epsilon_x \leq 1} T_x(\epsilon_x)$ onto $\bigcup_{\epsilon_0 \leq \epsilon_x \leq 1} T_x(\phi(\epsilon_x))$, always mapping the tori $T_x(\epsilon_x)$ onto $T_x(\phi(\epsilon_x))$. The fundamental groups of these solid tori are generated by $L_x := \{x = 0\} \cap S^3$ respectively $L_y := \{y = 0\} \cap S^3$, and a curve of type (p, q) on the torus $T_x(\epsilon_x)$ (for the notation, see Proposition 5.2) is mapped to the class of $q \cdot L_x$ respectively $p \cdot L_y$ by the inclusion into the solid tori. Consequently, the homeomorphism Φ must map a curve of type (p, q) on $T_x(\epsilon_x)$ to a curve of type $(\pm p, \pm q)$ on $T_x(\phi(\epsilon_x))$. This implies the claim on the isotopy classes of $\Phi_{|T_x(\epsilon_x)}$. □

To finally classify the holomorphic foliation germs $\mathcal{F}_\lambda$, $\lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$, we consider Kronecker foliations F_λ, $\lambda \in \mathbb{R}_{>0}$, on the 2-dimensional torus $T^2 = S^1 \times S^1$. These foliations are given by the orbits of the flow

$$t \cdot_\lambda (e^{ia}, e^{ib}) = (e^{i(a+\lambda t)}, e^{i(b+t)}),\ t, a, b \in \mathbb{R}.$$

Proposition 5.6. *Two Kronecker foliations F_λ and F_μ, $\lambda, \mu \in \mathbb{R}$, are topologically equivalent if $\mu = \frac{a\lambda+b}{c\lambda+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$.*

Proof. Let $Q := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$. Then

$$\phi_Q : T^2 \to T^2, (e^{ix}, e^{iy}) \mapsto (e^{i(ax+by)}, e^{i(cx+dy)})$$

is a homeomorphism with inverse map $\phi_{Q^{-1}}$. Since for $s = (c\lambda + d)t$,

$$\begin{aligned} \phi_Q(t \cdot_\lambda (e^{ix}, e^{iy})) &= (e^{i(ax+by+(a\lambda+b)t)}, e^{i(cx+dy+(c\lambda+d)t)}) = \\ &= (e^{i(ax+by+\mu s)}, e^{i(cx+dy+s)}) = \\ &= s \cdot_\mu \phi_Q(e^{ix}, e^{iy}), \end{aligned}$$

ϕ_Q is a topological equivalence of F_λ and F_μ. □

If $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ the converse is also true, as the following theorem shows:

Theorem 5.7. *Let $\phi : T^2 \to T^2$ be a topological equivalence of Kronecker foliations F_λ and F_μ, $\lambda, \mu \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$. If ϕ has the homotopy type $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the mapping class group $GL(2, \mathbb{Z})$ of T^2 then $\mu = \frac{a\lambda+b}{c\lambda+d}$.*

Proof. First of all, we may assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, that is, ϕ is isotopic to the identity: If not, Proposition 5.6 shows that F_μ is topologically equivalent to $F_{Q^{-1} \cdot \mu}$ where Q^{-1} is the inverse matrix of $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Furthermore, the topological equivalence $\phi_{Q^{-1}}$ is of homotopy type $Q^{-1} \in GL(2, \mathbb{Z})$, so the topological equivalence $\phi_{Q^{-1}} \circ \phi$ between F_λ and $F_{Q^{-1} \cdot \mu}$ is of homotopy type $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently, if we show that $\lambda = Q^{-1} \cdot \mu$, then as claimed

$$\mu = Q \cdot \lambda = \frac{a\lambda + b}{c\lambda + d}.$$

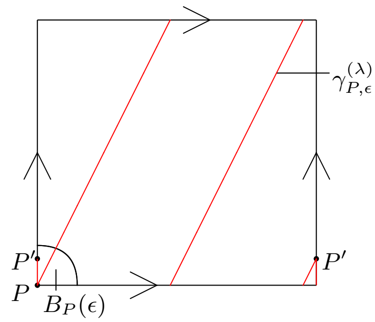


FIGURE 5.1.

For a given $\lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$ and a point $P = (e^{ia}, e^{ib}) \in T^2$, let

$$L_P^{(\lambda)} := \{(e^{i(a+\lambda t)}, e^{i(b+t)})|t \in \mathbb{R}\} \subset T^2$$

be the leaf of F_λ through P. Following ideas from ergodic theory we express the "slope" of the leaf $L_P^{(\lambda)}$ as a quotient of its topological intersection numbers with two curves representing generators of $H_1(T^2, \mathbb{Z})$. To this purpose we need arbitrarily long pieces of the leaf $L_P^{(\lambda)}$ starting in P and ending in P' arbitrarily close to P. Here, we measure distances on T^2 using the metric induced by the Euclidean metric on the universal covering $\mathbb{R}^2$.

So consider the preimage $p^{-1}(B_P(\epsilon))$ of a ball $B_P(\epsilon)$, $0 < \epsilon \ll 1$ under the parametrisation $p : \mathbb{R} \to L_P^{(\lambda)} \subset T^2$ given by $t \mapsto (e^{i(a+\lambda t)}, e^{i(b+t)})$. Since λ is irrational, $L_P^{(\lambda)}$ is dense in T^2, hence $p^{-1}(B_P(\epsilon))$ consists of infinitely many intervals in arbitrarily large distances to $0 \in \mathbb{R}$. One of the intervals in $p^{-1}(B_P(\epsilon))$, say I_0, contains 0, whereas the images of all the other intervals have a non-zero distance to P. In particular, if $\epsilon \to 0$ then the boundaries of all the intervals not containing 0 tend to $\pm\infty$. This observation holds for the intervals in the preimage of an arbitrary neighborhood basis of P.

Let I_1 be the interval in $p^{-1}(B_P(\epsilon))$ closest to the right to I_0. As indicated in Figure 5.1 we can construct a closed path $\gamma_{P,\epsilon}^{(\lambda)} : [0, 1] \to T^2$ starting and ending in P, by following the leaf $L_P^{(\lambda)}$ to a point $P' \in p(I_1)$ and connecting P' to P by a path inside $B_P(\epsilon)$.

Note that the homotopy class of $\gamma_{P,\epsilon}^{(\lambda)}$ depends neither on the choice of P' nor on the path connecting P' and P. Hence we can even construct a smoothly embedded path in that way. By construction, this path covers arbitrarily long segments of the leaf $L_P^{(\lambda)}$ if ϵ is small enough.

Next, set $C_1 := \{(e^{it}, 1) : t \in \mathbb{R}\}$ and $C_2 := \{(1, e^{is}) : s \in \mathbb{R}\}$. The closed curves $C_1, C_2 \subset T^2$ represent generators $[C_1], [C_2] \in H_1(T^2, \mathbb{Z})$ intersecting exactly once in the point $(1, 1) \in T^2$. Let $[\gamma_{P,\epsilon}^{(\lambda)}] \in H_1(T^2, \mathbb{Z})$ denote the homological 1-class represented by $\gamma_{P,\epsilon}^{(\lambda)}$, and consider the topological intersection numbers $[\gamma_{P,\epsilon}^{(\lambda)}] \cdot [C_i]$ (see [SZ94, 14.6]).

Claim: $\lambda = \lim_{\epsilon \to 0} \frac{[\gamma_{P,\epsilon}^{(\lambda)}] \cdot [C_2]}{[\gamma_{P,\epsilon}^{(\lambda)}] \cdot [C_1]}$.

Proof. We calculate the intersection numbers using their differential-topological interpretation, for smoothly embedded paths $\gamma^{(\lambda)}_{P,\epsilon}$ (see [Hir94, 5.2]). Since $L^{(\lambda)}_P$ intersects C_1 and C_2 everywhere with the same orientation, we just need to count the intersection points in $\gamma^{(\lambda)}_{P,\epsilon} \cap C_i$. Assuming for the moment that $P \notin C_1 \cup C_2$, for small enough ϵ we only need to count the intersection points of the part of $\gamma^{(\lambda)}_{P,\epsilon}$ lying on $L^{(\lambda)}_P$ with C_i. This part is the image $p([0, b_\epsilon])$ of an interval $[0, b_\epsilon] \subset \mathbb{R}$ under the parametrisation $p : \mathbb{R} \to L^{(\lambda)}_P$ introduced above. Then

$$\left[\frac{\lambda b_\epsilon}{2\pi}\right] \leq |p([0, b_\epsilon]) \cap C_2| \leq \left[\frac{\lambda b_\epsilon}{2\pi}\right] + 1 \text{ and } \left[\frac{b_\epsilon}{2\pi}\right] \leq |p([0, b_\epsilon]) \cap C_1| \leq \left[\frac{b_\epsilon}{2\pi}\right] + 1,$$

where $[x]$ denotes the maximal integer $\leq x \in \mathbb{R}$ and $|p([0, b_\epsilon]) \cap C_i|$ the number of intersection points of $p([0, b_\epsilon])$ and C_i. As discussed above, $b_\epsilon \to \infty$ if $\epsilon \to 0$, and the claim follows.
If $P \in C_1 \cup C_2$ the path in $\gamma^{(\lambda)}_{P,\epsilon}$ connecting P' with P can be chosen to intersect C_i only in a number of points bounded from above independently of ϵ. Hence the claim also holds in that case. □

Now, we calculate:

$$\lambda = \lim_{\epsilon \to 0} \frac{[\gamma^{(\lambda)}_{P,\epsilon}] \cdot [C_2]}{[\gamma^{(\lambda)}_{P,\epsilon}] \cdot [C_1]} = \lim_{\epsilon \to 0} \frac{[\phi(\gamma^{(\lambda)}_{P,\epsilon})] \cdot [\phi(C_2)]}{[\phi(\gamma^{(\lambda)}_{P,\epsilon})] \cdot [\phi(C_1)]} = \lim_{\epsilon \to 0} \frac{[\phi(\gamma^{(\lambda)}_{P,\epsilon})] \cdot [C_2]}{[\phi(\gamma^{(\lambda)}_{P,\epsilon})] \cdot [C_1]},$$

by the Claim and since $\phi : T^2 \to T^2$ is a homeomorphism assumed to be homotopic to the identity. But $\phi(L^{(\lambda)}_P) = L^{(\mu)}_{\phi(P)}$, hence $\phi(\gamma^{(\lambda)}_{P,\epsilon})$ is a path constructed as above for the leaf $L^{(\mu)}_{\phi(P)}$ of F_μ and the neighborhood basis $U_\epsilon := \phi(B_P(\epsilon))$ of $\phi(P)$, so the above limit is equal to

$$\lim_{\epsilon \to 0} \frac{[\gamma^{(\mu)}_{\phi(P),U_\epsilon}] \cdot [C_2]}{[\gamma^{(\mu)}_{\phi(P),U_\epsilon}] \cdot [C_1]} = \mu,$$

once again by the Claim. □

Theorem 5.8. *Two holomorphic foliation germs $\mathcal{F}_\lambda, \mathcal{F}_\mu$, $\mu, \lambda \in \mathbb{R}_{>0} - \mathbb{Q}_{>0}$, are topologically equivalent if, and only if $\lambda = \mu$ or $= \frac{1}{\mu}$.*

Proof. By the Reconstruction Theorem 2.2 it is enough to show the statement for the intersection foliations $\mathcal{F}_\lambda \cap S^3$ and $\mathcal{F}_\mu \cap S^3$.
Exchanging the coordinates yields a topological equivalence Φ of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_{\frac{1}{\lambda}} \cap S^3$. On the other hand, let Φ be a topological equivalence of $\mathcal{F}_\lambda \cap S^3$ with $\mathcal{F}_\mu \cap S^3$. As above, for $0 < \epsilon_x < 1$ the restriction $\Phi_{|T_x(\epsilon_x)}$ maps the torus $T_x(\epsilon_x)$ to another torus $T_x(\epsilon'_x)$ and induces a topological equivalence of the Kronecker foliations $F_\lambda = \mathcal{F}_{\lambda|T_x(\epsilon_x)}$ and $F_\mu = \mathcal{F}_{\mu|T_x(\epsilon'_x)}$. Proposition 5.5 shows that $\Phi_{|T_x(\epsilon_x)}$ must be of type $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$ in the mapping class group of a 2-dimensional torus. Then Theorem 5.7 implies that $\lambda = \mu$ or $\lambda = \frac{1}{\mu}$. □

6. Topological equivalence classes in dimension 2

In each of the Sections 3, 4, 5.1 and 5.2 we identified the topological equivalence classes of plane holomorphic foliation germs represented by vector fields of a certain type, and the list in Remark 1.6 shows that every plane holomorphic foliation germ is of one of these types. Consequently, the classification is completed by the following statement:

Theorem 6.1. *The topological equivalence classes determined in Sections 3, 4, 5.1 and 5.2 are pairwise distinct.*

Proof. If the eigenvalues of the linear part of the representing vector field are $\mathbb{R}$-linearly independent then there exists two closed leaves in the intersection foliation, and the closure of any other leaf of the intersection foliation consists of the leaf and these two closed leaves – see the results of Section 3. If the eigenvalues are $\mathbb{R}$-linearly dependent and have resonances then there is only one closed leaf in the intersection foliation – see the results of Section 4. If the eigenvalues are $\mathbb{Q}$-linearly dependent but the vector field is non-resonant then every leaf in the intersection foliation is closed – see the results of Section 5.1. Finally, if the eigenvalues are $\mathbb{R}$-linearly dependent but $\mathbb{Q}$-linearly independent then all leaves in the intersection foliation besides the two closed leaves have as closure a torus – see the results of Section 5.2.
Thus, in each of the four cases, there exist leaves of the intersection foliation with topological properties not occuring in the other cases. Hence the Reconstruction Theorem 2.2 shows the theorem. □

7. Topological equivalence classes in dimension ≥ 3

Guckenheimer's Stability Theorem generalizes Theorem 3.4 to arbitrary dimensions:

Theorem 7.1 ([Guc72])**.** *Let $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ and $\sum_{i=1}^n \mu_i z_i \frac{\partial}{\partial z_i}$ represent two holomorphic foliation germs with an isolated singularity in $0 \in \mathbb{C}^n$ such that $\lambda_1, \dots, \lambda_n$ respectively $\mu_1, \dots, \mu_n$ are in the Poincaré domain and pairwise $\mathbb{R}$-linearly independent. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent.*

Guckenheimer also showed that $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent if, under the same assumptions on the λ_i, the vector field θ_2 representing $\mathcal{F}_2$ is obtained from $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ representing $\mathcal{F}_1$ by a sufficiently small holomorphic perturbation. This implies the following classification result:

Proposition 7.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ represented by $[U_1, \theta_1]$ and $[U_2, \theta_2]$ such that the eigenvalues of the linear parts of the vector fields θ_1 respectively θ_2 are in the Poincaré domain and pairwise $\mathbb{R}$-linearly independent. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologically equivalent.*

Proof. Assume that $\theta_1 = \sum_{i=1}^n f_i(z) \frac{\partial}{\partial z_i}$ and $\theta_2 = \sum_{i=1}^n g_i(z) \frac{\partial}{\partial z_i}$. As discussed in Section 1 we can assume that the non-linear terms of the power series $f_i(z)$ and $g_i(z)$ consist of resonant monomials $z_1^{m_1} \cdots z_n^{m_n}$ with respect to the eigenvalues $\lambda_1, \dots, \lambda_n$ of the linear part of θ_1 respectively $z_1^{n_1} \cdots z_n^{n_n}$ with repect to the eigenvalues $\mu_1, \dots, \mu_n$ of the linear part of θ_n, that is, the $\lambda_1, \dots, \lambda_n$ respectively $\mu_1, \dots, \mu_n$ satisfy the resonance $\lambda_i = \sum_{j=1}^n m_j \lambda_j$ respectively $\mu_i = \sum_{j=1}^n n_j \mu_j$ for some integers $m_j, n_j \geq 0$. Since $\lambda_1, \dots, \lambda_n$ respectively $\mu_1, \dots, \mu_n$ are in the

Poincaré domain there are only finitely many of these resonant monomials, hence $f_i(z)$ and $g_i(z)$ are polynomials.
Possibly after a holomorphic coordinate change we can furthermore assume that the real parts of all the λ_i and μ_i are positive and that

$$0 < \text{Re}\lambda_1 < \cdots < \text{Re}\lambda_n \text{ respectively } 0 < \text{Re}\mu_1 < \cdots < \text{Re}\mu_n.$$

Thus, resonances $\lambda_i = \sum_{j=1}^n m_j\lambda_j$ respectively $\mu_i = \sum_{j=1}^n n_j\mu_j$ always satisfy $m_j = n_j = 0$ for $j \geq i$. Consequently, rescaling the ith coordinate z_i by a real factor ϵ_i such that $0 < \epsilon_1 \ll \epsilon_2 \ll \cdots \ll \epsilon_n$ changes the vector fields θ_1, θ_2 to vector fields with non-linear parts arbitarily close to 0.
So Guckenheimer's Stability Theorem implies that $\mathcal{F}_1$ respectively $\mathcal{F}_2$ are topologically equivalent to the foliations represented by the linear parts $\sum_{i=1}^n \lambda_i z_i \frac{\partial}{\partial z_i}$ respectively $\sum_{i=1}^n \mu_i z_i \frac{\partial}{\partial z_i}$ of θ_1 respectively θ_2, and these foliations are topologically equivalent by Theorem 7.1. □

Under the assumptions of the proposition the appearence of resonant monomials involving only $\mathbb{R}$-linearly independent eigenvalues does not influence the topological equivalence class. So for more general situations we introduce the following notion:

Definition 7.3. *Let $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be a set of complex numbers in the Poincaré domain. A resonance $\lambda_i = \sum_{j=1}^n m_j\lambda_j$ is called inessential if not all the $\lambda_j \in \mathbb{C}$ with $m_j \neq 0$ lie on the same real ray starting in the origin. Otherwise the resonance is called essential.*

The 2-dimensional classification in Sections 3 - 6 shows that $\mathbb{R}$-linear (in)dependence of the two eigenvalues of the linear part of a representing vector field distinguishes the topological equivalence class of holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^2$ of Poincaré type. In higher dimension we extend this dichotomy to the following invariant:

Definition 7.4. *The ray configuration of a tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is the ordered partition of this set into subsets consisting of those $\lambda_i \in \mathbb{C}$ lying on the same real ray starting in the origin, and the subsets are ordered by increasing angle of this ray with the positive real axis.*
Two ray configurations are called equivalent if the sizes of the partition subsets, in the order of the partition, are equal, or become equal after reversing the order of one of the partitions.

Finally, the 2-dimensional classification shows that topologically equivalent plane holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^2$ of Poincaré type having equivalent ray configurations are also holomorphically equivalent.
Combining all these observations we predict the following behaviour of such foliation germs in arbitrary dimensions:

Conjecture 7.5. *Two holomorphic foliation germs of rank 1 with an isolated singularity in $0 \in \mathbb{C}^n$ of Poincaré type are topologically equivalent if and only if the following two conditions are satisfied:*

1. *The ray configurations of the tuples of eigenvalues of the linear part of a vector field representing the foliation germs are equivalent.*

(2) *For every two corresponding partition subsets* $\{i_1, \ldots, i_k\}$, $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ *of the two ray configurations, the restrictions of the two foliation germs to the linear subspaces*

$$L_1 := \{z_l = 0 : l \neq i_1, \ldots, i_k\}, L_2 := \{z_m = 0 : m \neq j_1, \ldots, j_k\} \subset \mathbb{C}^n$$

are holomorphically equivalent.

In particular, the conjecture predicts in full generality that the appearence of inessential resonant monomials does not influence the topological equivalence class. In dimension 3 inessential resonances occur only in cases covered by Guckenheimer's Stability Theorem since eigenvalues on 3 different rays are needed to produce such a resonance.

In dimension 4 inessential resonances not described by Guckenheimer's Stability Theorem may occur with ray configuration $(1, 1, 2)$, of the form $\lambda_2 = n\lambda_1 + m\lambda_3 + l\lambda_4$ where n, m, l are integers with $n \geq 1$, $m, l \geq 0$ and $m + l \geq 1$.

Then the conjecture predicts that the holomorphic foliation germs given by the vector fields

$$\lambda_1 x \frac{\partial}{\partial x} + (\lambda_2 y + x^n z^m w^l) \frac{\partial}{\partial y} + \lambda_3 z \frac{\partial}{\partial z} + \lambda_4 w \frac{\partial}{\partial w}$$

and

$$\lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y} + \lambda_3 z \frac{\partial}{\partial z} + \lambda_4 w \frac{\partial}{\partial w}$$

are topologically equivalent. The conjecture also predicts that the topological equivalence class of the latter linear foliation germ only depends on the topological equivalence class of the plane holomorphic foliation germ given by $\lambda_3 z \frac{\partial}{\partial z} + \lambda_4 w \frac{\partial}{\partial w}$.

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