Approximation of Degenerate Partial Differential Equations Arising in Finance

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To my family:
for the love, the example they are and for all the times that, even when they didn't understand my dreams, they were there to support me.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Liliana Angélica Costa Matos Pereira*)

Abstract

This thesis focuses on the discretization of degenerate partial differential equations arising in Finance.

In particular, the Cauchy problem for a second order linear parabolic PDE is discretized in the spatial variables for both the bounded and unbounded coefficient cases. The semi-discretization is considered for the general multi-dimensional version of the PDE and also for the particular one-dimensional case.

The approximation to the PDE problem solution is obtained by using basic finite difference methods in discrete Sobolev and weighted Sobolev spaces.

Existence and uniqueness results for the generalized solution to the semi-discretized problem are deduced. Finally, we give an estimate for the rate of convergence of the solution of the semi-discretized problem to the solution of corresponding the exact problem. Stronger results are deduced for the special case of one dimension on space.

Sumário

Esta dissertação estuda a discretização de equações diferenciais parciais degeneradas com aplicações às Finanças.

Em particular, o problema de Cauchy para uma equação diferencial parcial linear de segunda ordem é discretizado nas variáveis espaciais para os casos de coeficientes limitados e ilimitados. A semi-discretização é considerada para a versão multidimensional da EDP e também para o caso particular de uma dimensão espacial.

A aproximação à solução da EDP é obtida com recurso a métodos básicos de diferenças finitas em versões discretas de espaços de Sobolev e de Sobolev ponderados.

São deduzidos resultados de existência e unicidade para a solução generalizada do problema semi-discretizado. Finalmente, é dada uma estimativa para a taxa de convergência da solução do problema semi-discretizado para a solução do problema exacto correspondente. São obtidos resultados mais fortes para o caso especial de uma dimensão no espaço.

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Contents

Chapter 1

Introduction

Banks and other financial and non-financial institutions deal daily with credit and investment decisions. The mathematical problems involved in this financial decisions have been, in the last 50 years, object of increasing interest.

From the seminal contribution by Bachelier, the field of Financial Mathematics was built with the works of Fama, Cox, Black, Scholes, Merton and others. One of the major studies concerning quantitative modelling with the use of stochastic processes is the Black-Scholes model (1973) which allows pricing an option by solving a simple partial differential equation, the Black-Scholes equation.

Since then, the field of Financial Mathematics enjoyed an explosive expansion. With the markets' globalization, the financial system suffered a profound transformation and evolved to the present state of extreme product sophistication and complexity.

Financial derivatives are central to Financial Mathematics. Since there are not, in general, closed form solution to derivative prices, numerical analysis plays a major role in the field. Our interest goes to the approximation to the price of multi-asset options.

In this work, we consider the Cauchy problem for second order linear partial differential equation of parabolic type the multi-asset European option pricing can be cast into. The degenerate PDE is considered in its multidimensional version and its coefficients are allowed to grow in the spacial variables. The PDE problem is set in Sobolev and weighted Sobolev spaces and its solvability considered in the variational framework.

The main of this thesis is to approximate degenerate PDE linear parabolic of second order, both to unbounded and unbounded coefficient cases.

To achieve these goals we began by recalling some important classical results, in particular, we define the Cauchy problem, enunciate results about Sobolev and weighted Sobolev spaces and the we state conditions to the exact problems (both in degenerate and degenerate situations).

Previous works have been published on these subject, namely the works of [26, 27, 28, 31, 45]. In particular, in [26] developed a discretized problem in space variable and proved the existence and uniquenessof generalized solution to the nodegenerate problem (to bounded and unbounded case).

This thesis, adapted the same procedures but to degenerate case and proved the existence and uniqueness of this solution of the space discretized Cauchy problem. Is also proved the consistency of the scheme and obtained a rate of convergence of the solution to the problem in analysis to the corresponding exact problem.

With the new results and with the previous works on the one dimensional case to unbounded coefficients in the nondegenerate case, already mentioned, we applied this methodology to the degenerate problem both with bounded and unbounded coefficients (case of dimension d and one) and also applied to the one dimensional case to unbounded coeffients in the nondegenerate case.

Therefore, we begin by defining a spatial-discretized version of the PDE problem by using a basic finite-difference scheme. This new problem is considered in discrete versions of the Sobolev and weighted Sobolev spaces. Then, we prove existence and uniqueness results for the generalized solution to the semi-discretized problem and show that the scheme is stable. Next, we prove that the scheme is consistent. Finally, we deduce a convergence result and an estimate for the rate of convergence of the solution of the semi-discretized problem to the solution of corresponding the exact problem.

We treat separately the special case of one dimension in space for which stronger results are obtained.

We note that the usual procedure for obtaining numerical schemes for the PDE problem under study is to localize the exact problem to a bounded domain, and then to approximate the localized version of the problem (see, e.g., [8, 41, 57] and also [48], where the same technique is used for a more complex problem). If the procedure is adopted, then the PDE coefficient unboundedness is no longer a difficulty to tackle and the functional spaces to consider need not to be weighted.

If the alternative procedure of semi-discretizing the PDE problem in the whole spacial domain and then localizing the semi-discretized problem to a discrete bounded domain is chosen (see, e.g., [17, 18, 19]), then the coefficient unboundedness remains a problem to deal with. The present investigation is meaningful in this latter case.

Moreover, the study now developed extends the works [26, 27, 28, 30, 45] on the nondegenerate PDE case to the general degenerate case.

Next, we present this thesis contents.

In Chapter 2 - *Financial problem: stochastic modelling*, we summarize the stochastic and financial background for the Black-Scholes modelling of a multi-asset option of European type.

In Chapter 3 - *Approximation of PDEs with bounded coefficients*, we begin by presenting a Cauchy problem for a parabolic evolution equation in abstract spaces the PDE problem can be cast into. Classical existence and uniqueness results are given. Then, we present the initial value problem for a linear parabolic PDE, in both the nondegenerate and the degenerate cases, introduce the Sobolev spaces, and give classical existence and uniqueness results.

The PDE problem is then discretized in space by using a finite-difference scheme. The functional discrete Sobolev spaces are introduced. We establish the existence and uniqueness of the semi-discretized problem generalized solution, its stability, the scheme's consistency, and the convergence to the solution to the corresponding exact problem. A rate of convergence is estimated.

The special case of one dimension in space is dealt in separately, with stronger results.

In Chapter 4 - *Approximation of PDEs with unbounded coefficients*, we begin with the presentation of classical results for the existence and uniqueness of the generalized solution to a parabolic PDE initial value problem in a class of weighted Sobolev spaces. The PDE coefficients are allowed to grow and the PDE is considered in both the nondegenerate and the degenerate cases.

Then we discretize the PDE problem in space with the use of a finite difference scheme and introduce a discrete version to the weighted Sobolev spaces. Stability, consistency, and convergence results are proved.

As in the previous chapter, stronger results are obtained for the one dimensional case.

In Chapter 5 - *Conclusion and further research*, we briefly discuss our results and we outline future extensions of the present research.

Chapter 2

Financial problem: stochastic modelling

Financial analysis, in the first half of the 20th century based the price of the future options in the past information: the price variation in a certain period was based in variations in previous periods.

Bachelier, french mathematician already mentioned on the previous chapter, wrote in 1900 his PhD thesis under the theme "Théorie de la Spéculation" where, for the first time, the financial process is associated to stochastic process, but these results were only revealed sixty years later. Bachelier studied the french treasury bonds and concluded that the price behaviour is similar to random walk, which he studied in continuous time, known as Brownian motion.

2.1 Financial theory framework

To ensure the capacity of future prices it is fundamental to have a standardization of option prices. It is well known that random factors have a huge role in economics activity. Due to this fact, the process of pricing is random and any model used to describe this process has to be a stochastic process.

Random walk, martingale model and efficiency market theory.

Louis Bachelier, with his seminal work, associated the pricing process to randomness in his "Théorie de la Spéculation" thesis, due to the inexistence of memory in stochastic processes. One of the best examples is the random walk, the reason this dynamic was the first model used to describe the prices flutuation: in each moment the variation of prices (increasing or decreasing) is a random quantity between statistical independent moments.

Bachelier showed also that prices changes occur without any connection to external events, very often. So, using probability theory it is possible to establish laws that are verified by the prices and its variations. Bachelier modeled the sucessive changes in prices, using the Central Limit Theorem, obtaining a normal distribution to the prices flutuation and assuming that they were independent and identically distributed: assumption that formed the basis of the Theory of Efficient Markets. The later denominated random walk, was defined in Bachelier's thesis through the distribution function of the Wiener stochastic process (Brownian motion outset) and connecting with the diffusion equation.

Later, Albert Einstein presented the partial differential diffusion equation, using Brownian motion and defined an estimate to the molecula's size.

Until the middle of the 60's, efficiency of markets was about random walk theory, but since 1965 with Eugene Fama, the efficiency of financial markets comes up associated to the martingale model, accepting the predictability in expected variance conditioned of the profitability and the volatility in certain periods of time. Also Samuelson studied, in parallel with Fama, the random character of prices as the consequence of rational markets. The only difference between the two authors was the probabilistic model they used to describe the random variantion: Fama choose the Random Walk model and Samuelson introduced, for the first time, the Martingale model.

Fama revealed that asset's returns can variate in time in a predictable way and prices can be not random. So, Fama's efficiency model is based in the difference between the observed expected return and the foreseen expected return by a pricing model - the mean controled return to the asset risk, in order to get economic profit.

Eugene Fama (1970) defended that financial markets can have three efficiency stadiums: weak form, semi-strong form and strong form. In the weak form past price movements and volume data do not affect stock prices, in the semi-strong form all public information is calculated into a stock's current share price and in the strong form all information in a market, whether public or private, is accounted for in a stock's price.

2.2 Stochastic process background

In this section we will state some theorical results which are fundamental to this work.

Stochastic processes

As in [36], a stochastic process is a mathematical model for the occurrence, at each moment after the initial time, of a random phenomenon. Attending also to [46], we have:

Definition 2.2.1. $\{X_t(\omega), \omega \in \Omega, t \in T\}$ is said to be a stochastic process if it is a family of random variables defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with T the asset where the parameter t is defined. If $T = \mathbb{N}$ then the process is said to be in discrete time; if $T = [a, b] \subset \mathbb{R}$ or $T = \mathbb{R}$ the process is said to be in continuous time.

Remark 1. $\{X_t\}$ is the state of the process in instant t and $X_t(\omega)$ is called a trajectory of the process.

Remark 2. Consider the succession of independent random variables $\{Z_t, t \in \mathbb{N}\}$. Then, the Random Walk

$$X_t = Z_1 + Z_2 + ... + Z_t = X_{t-1} + Z_t$$

is a stochastic process in discrete time.

It is now important to establish the definition of continuity in mean, of a stochastic process.

Definition 2.2.2. Let $p \geq 1$. A stochastic process $\{X_t(\omega), \omega \in \Omega, t \in T\}$ with values in $\mathbb{R}$, where T is an interval of $\mathbb{R}$ and such that $E[|X_t|^p] < \infty$, is said to be continuous in mean of order p if, for all $t \in T$, we have

$$\lim_{s \to t} E[|X_t - X_s|^p] = 0$$

Remark 3. Continuity in mean of order p implies the continuity in probability.

Martingales

The martingale theory is very important on the modern theory of financial derivatives and requires knowledge on measure theory.

Owing to [46] the following results are established:

Definition 2.2.3. Let X be a random variable. The σ-algebra generated by X is the minor σ-algebra containing X and it is represented by $\{X^{-1}(B) : B \in \mathbb{B}_R\}$.

Definition 2.2.4. Assume the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the succession of σ-algebras $\{F_n, n \geq 0\}$. Also consider that $F_0 \subset F_1 \subset ... \subset F_n \subset F$. The succession $\{F_n, n \geq 0\}$ is called a filtration.

Definition 2.2.5. A stochastic process $\{M_n; n \geq 0\}$ is a martingale, in discrete time, in order to the filtration $\{F_n, n \geq 0\}$ if:

1. For each n, M_n is a random variable F_n measurable (M is a stochastic process adapted to the filtration $\{F_n, n \geq 0\}$);
2. For each n, $E[|M_n|] < \infty$;
3. For each n: $E[M_{n+1}|F_n] = M_n$.

Remark 4. In the definition of martingale we have that for every n,

$$E[M_{n+1}|F_n] = M_n$$

Instead, if we have for every n,

$$E[M_{n+1}|F_n] \geq M_n$$

then M_n is called a **submartingale**.
However, if we have for every n,

$$E[M_{n+1}|F_n] \leq M_n$$

then M_n is called a **supermartingale**.

Once again, it is important to state the definition in terms of continuity.

Definition 2.2.6. A stochastic process $\{M_t; t \geq 0\}$ is a martingale, in continuous time, in order to the filtration $\{F_t, t \geq 0\}$ if:

1. For all t, M_t is a random variable F_t measurable (M is a stochastic process adapted to the filtration $\{F_t, t \geq 0\}$);

2. For all $t > 0$, $E[|M_t|] < \infty$;

3. For all $s \leq t$: $E[M_t|F_s] = M_s$.

The following theorem is an important result for continuous martingales (see [46]).

Theorem 2.2.7 (Kolmogorov's submartingale inequality)**.** If M_n is a non-negative submartingale, then $P[max(M_1, ..., M_n) \leq a] \leq \dfrac{E[M_n]}{a}$ for $a > 0$.

Theorem 2.2.8 (Martingale Convergence Theorem)**.** If $\{M_n, n \geq 1\}$ is a martingale and $E[|M_n|] \leq M$ then, with probability 1, $\lim_{n\to\infty} M_n$ exists and is finite.

Brownian motion

The Brownian motion concept is associated to the botanist Robert Brown (1828) who observed an irregular motion in his pollen grain experience. Mathematically it is explained by the Brownian motion (see [46]).

Definition 2.2.9. A stochastic process $B = \{B_t; t \geq 0\}$ is a Brownian motion if:

1. $B_0 = 0$;

2. B has independent increments;

3. If $s < t$, then $B_t - B_s$ is a random variable with distribution $N(0, t - s)$;

4. The process B have continuous paths.

Remark 5. The Brownian motion has the following properties:

1. The Brownian motion is a Gaussian process;

2. $E[B_t] = 0$;

3. $E[B_s B_t] = min(s, t)$;

4. If B_t is a process satisfying the conditions (1), (2) and (3) then the distribution of B_t for each t must be normal;

5. Consider B_t a process satisfying the conditions (1), (2) and (3) and let m and σ^2 be the mean and the variance of B_1. Then $E[B_t] = tm$ and $Var[B_t] = t\sigma^2$. If $m = 0$ and $\sigma^2 = 1$ then B_t is called a standard Brownian motion.

Theorem 2.2.10 (Wiener)**.** There exists a Brownian motion on some probability space.

Some examples of Brownian motions are:

- **Geometric Brownian motion**: $X_t = e^{\mu t + \sigma B_t}$, where X has lognormal distribution;
- **Brownian motion with drift**: $Y_t = \mu t + \sigma B_t$ is a gaussian process;
- **Brownian bridge**: $Z_t = B_t - tB_1, t \in [0, 1]$, is also a gaussian process.

Next we introduce the definition of Brownian motions with filtrations.

Definition 2.2.11. Let F_t be a filtration. A stochastic process B_t is called an F_t-Brownian motion if:

1. Is a Brownian motion;
2. Is F_t adapted;
3. $B_t - B_s$ is independent of F_s for any $t > s$.

In order to clarify some properties, the following results connect martingales and Brownian motion.

Lemma 2.2.12. If B_t is an F_t- Brownian motion then it is an F_t-martingale.

Proposition 2.2.13. If $B = \{B_t; t \geq 0\}$ is a Brownian motion and $\{F_t^B, t \geq 0\}$ is the filtration generated by B, then the following processes are $\{F_t^B, t \geq 0\}$-martingales:

1. B_t;
2. $B_t^2 - t$;
3. $exp(aB_t - \frac{a^2 t}{2})$.

Stochastic integral

To state the existence of the stochastic integral (known as Itô process), it is necessary to impose some conditions.

Definition 2.2.14. Consider a measurable space (Ω, F) equipped with a filtration F_t. A random time T is a stopping time of the filtration, if the event $\{T \leq t\}$ belongs to the σ-field F_t, for every $t \geq 0$. A random time is an optional time of the filtration if $\{T < t\} \in F_t$, for every $t \geq 0$.

Lemma 2.2.15. If $\mathbb{T}$ is optional and θ is a positive constant then $\mathbb{T} + \theta$ is a stopping time.

Lemma 2.2.16. If $\mathbb{T}$ and $\mathbb{S}$ are stopping times then so are $\mathbb{T} \wedge \mathbb{S}$, $\mathbb{T} \vee \mathbb{S}$ and $\mathbb{T} + \mathbb{S}$.

Definition 2.2.17. A process X is said to be simple if there exists a strickly increasing sequence of real numbers $0 = t_0 < t_1 < ... < t_n = T$ and a set of random variables $\{\varepsilon_n\}$ with $\sup_{n \geq 0} \varepsilon_n(\omega) \leq C < \infty$, for every $\omega \in \Omega$, such that ε is F_{t_n}-measurable for every $n \geq 0$ and

$$X_t(\omega) = \varepsilon_0(\omega) I_{\{0\}}(t) + \sum_{i=0}^{\infty} \varepsilon_i(\omega) I_{(t_i, t_{i+1}]}(t),$$

$0 \leq t < \infty, \omega \in \Omega$. The class of all simple processes will be denoted by L_0 and we have $L_0 \subset L^*(M) \subset L(M)$.

The stochastical integral with respect to a Brownian motion is defined next.

Definition 2.2.18. Suppose that $X \in L_0$. The stochastic integral of the simple process X, with respect to a Brownian motion, B_t, is defined as

$$I_t(X) = \sum_{i=0}^{n-1} \varepsilon_i (B_{t \wedge t_{i+1}} - B_{t \wedge t_i}), \quad 0 \leq t < \infty,$$

where $n \geq 0$ is the unique integer for which $t_n < t < t_{n+1}$

Proposition 2.2.19 (Itô's Isometry)**.** Consider X a simple process and the Brownian motion B_t. X verifies the isometry property:

$$E\left[\left(\int_0^T X_t dB_t\right)^2\right] = E\left[\int_0^T X_t^2 dt\right].$$

The definition of Itô Integral follows.

Definition 2.2.20. Consider the Brownian motion B_t and the stochastic process X_t, such that:

1. X_t is F_t-measurable;
2. X_t is adapted;
3. $E\left[\int_0^T X_t^2 dt\right] < \infty$;

Then, the Itô Integral is defined by

$$\int_0^T X_t dB_t = \lim_{n\to\infty} \int_0^T X_t^{(n)} dB_t$$

where $X_t^{(n)}$ satisfies $\lim_{n\to\infty} E\left[\int_0^T \left(X_t - X_t^{(n)}\right)^2 dt\right] = 0$ and the limit is considered in L^2.

Now we relate stochastic integrals and martingales.

Proposition 2.2.21. Let X be a process satisfying the conditions:

1. $\int_a^b E[X_s^2]ds < \infty$
2. X is adapted to the F_t-filtration

Then the following relations hold:

- $E\left[\int_a^b X_s dB_s\right] = 0$
- $E\left[\left(\int_a^b X_s dB_s\right)^2\right] = \int_a^b E[X_s^2]ds$
- $\int_a^b X_s dB_s$ is F_b^B-measurable.

Proposition 2.2.22. For any process $g \in L^2[s,t]$ that is $E\left[\int_s^t g_u dB_u | F_s\right] = 0$.

Corollary 2.2.23. For any process $g \in L^2$, the process X, defined by

$$X(t) = \int_0^t g_s dB_s$$

is an (F_t)-martingale. It means that every stochastic integral is a martingale.

Extensions of the Stochastic Integral

The stochastic integral can be defined for a larger class of integrands processes. It is necessary to make some changes in the definition of stochastic integral. Therefore, the first and third conditions in Definition (2.2.20) can be relaxed for:

- There exists an increasing family of σ-algebras $\{H_t : t \geq 0\}$ such that:
    1. B_t is a martingale with respect to H_t and
    2. X_t is H_t-adapted.
- $P\left[\int_0^T X_t^2 dt < \infty\right] = 1$.

Definition 2.2.24. A continuous and F_t-adapted stochastic process $\{X_t, 0 \leq t \leq T\}$ is called an Itô Process if it can be expressed in the form

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$$

where $u, v \in L^2$.
As a shorthand notation, it can be written by

$$dX_t = udt + vdB_t.$$

Theorem 2.2.25 (One-dimensional Itô formula)**.** Let X_t be an Itô process given by

$$dX_t = udt + vdB_t.$$

Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$ (i.e. g is twice continuous differentiable on $[0, \infty) \times \mathbb{R}$). Then

$$Y_t = g(t, X_t)$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t).(dX_t)^2,$$

where $(dX_t)^2 = (dX_t).(dX_t)$ is computed according to the rules

$$dt.dt = dt.dB_t = dB_t.d_t = 0, \quad dB_t.dB_t = dt.$$

Theorem 2.2.26 (Integration by parts)**.** Suppose $f(s, w)$ is continuous and of bounded variation with respect to $s \in [0, t]$, for a.a.w. Then

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s.$$

Consider now with the multi-dimensional Itô formula. Let

$$B(t,w) = (B_1(t,w), ..., B_m(t,w))$$

denote m-dimensional Brownian motion. If each of the processes $u_i(t,w)$ and $v_{ij}(t,w)$ satifies the conditions given in the extension and definition of Itô process $(1 \le i \le n, 1 \le j \le m)$, then it is possible to form the following n-Itô processes:

$$\begin{cases} dX_1 = u_1 dt + v_{11} dB1 + ... + v_{1m} dB_m \\ \vdots \\ dX_n = u_n dt + v_{n1} dB_1 + ... + v_{nm} dB_m \end{cases}$$

Or, in matrix notation simply

$$dX(t) = udt + vdB(t),$$

where
$$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_{11} & ... & v_{1m} \\ \vdots & & \cdots \\ V_{n1} & ... & v_{nm} \end{bmatrix}, dB(t) = \begin{bmatrix} dB_1(t) \\ \vdots \\ dB_m(t) \end{bmatrix}.$$

Definition 2.2.27. A process $X(t)$ in the conditions above is called an n- dimensional Itô process (or simply an Itô process).

Theorem 2.2.28 (The general Itô formula)**.** Let

$$dX(t) = udt + vdB(t)$$

be an n-dimensional Itô process as above. Let $g(t,x) = (g_1(t,x), ..., g_p(t,x))$ be a C^2 map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$. Then the process:

$$Y(t,\omega) = g(t, X(t))$$

is again an Itô process, whose component number k, Y_k, is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t,X) + \sum_i \frac{\partial g_k}{\partial x_i}(t,X) dX_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X) dX_i dX_j$$

where $dB_i dB_j = \delta_{ij} dt, \quad dB_i dt = dt dB_i = 0.$

Theorem 2.2.29 (The Itô representation theorem)**.** Let $F \in L^2(\Omega, \mathcal{F}_T, P)$. There exists a unique stochastic process u, F_t-adapted and with $E\left[\int_0^T u_t^2 dt\right] < \infty$, such that:

$$F = E(F) + \int_0^T u_s dB_s :$$

Theorem 2.2.30 (The martingale representation theorem)**.** Consider $B(t)$ such that $B(t) = (B_1(t), ..., B_n(t))$ is n-dimensional. Suppose M_t is an $F_t^{(n)}$-martingale (w.r.t. P) and that $M_t \in L^2(P)$ for all $t \geq 0$.Then there exists a unique stochastic process $g(s,\omega)$ such that $g \in \mathcal{V}^{(n)}(0,t)$ for all $t \geq 0$ and

$$M_t(\omega) = E[M_0] + \int_0^t g(s,\omega)dB_s \quad \text{a.s. for all} \quad T \geq 0.$$

Stochastic differential equations

Consider a Brownian motion $\{B_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that $\{F_t, t \geq 0\}$ is a filtration such that B_t is F_t-adapted and for any $0 \leq s < t$, the increment $B_t - B_s$ is independent of F_s.

We aim to solve the stochastic differential equation:

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t, \quad b(t,x) \in \mathbb{R}, \quad \sigma(t,x) \in \mathbb{R}$$

With initial condition X_0 independent of B_t. The coefficients $b(t,x)$ and $\sigma(t,x)$ are called, respectively, drift and diffusion coefficient. W_t is one dimensional "white noise".

The SDE can be written in the integral form:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

Or in the differential form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

Let us define now the solution of a stochastic differential equation: diffusion process.

Definition 2.2.31. The solution of a stochastic differential equation is an Itô process X_t such as:

1. X_t is adapted to Browian motion with continuous path;
2. $E\left[\int_0^T (\sigma(s, X_s))^2 ds\right] < \infty$.

Then X_t, solution of the SDE, is called diffusion process.

We now state the existence and uniqueness solution for SDE.

Theorem 2.2.32. Let $T > 0$ and $b(\cdot, \cdot) : [0, t] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma(., .) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times m}$ be measurable functions satisfying

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|); \quad x \in \mathbb{R}^n, \quad t \in [0, T]$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}|^2$) and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|; \quad x, y \in \mathbb{R}^n, \quad t \in [0, T]$$

for some constant D.
Let Z be a random variable which is independent of the σ-algebra $F_\infty^{(m)}$ generated by $B_s(.), s \geq 0$ and such that

$$E[|Z|^2] < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, \quad X_0 = Z \tag{2.1}$$

has a unique t-continuous solution $X_t(\omega)$.
Also have the property that $X_t(\omega)$ is adapted to the filtration F_t^Z generated by Z and $B_s(.), s \leq t$ and

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty.$$

Remark 6.

1. The solution X_t defined above is called a strong solution, since the version B_t of Brownian motion is given in advance and the solution constructed from it is F_t^Z-adapted.
2. If only are given the functions $b(t, x)$ and $\sigma(t, x)$ and ask for a pair of processes $((\tilde{X}_t, \tilde{B}_t), H_t)$ on a probability space $(\Omega, \mathcal{H}, P)$ such that (2.1) holds, then the solution $(\tilde{X}_t, \tilde{B}_t)$ is called a weak solution.
   Note that H_t is a increasing family of σ-algebras such that $\tilde{X}_t$ is H_t-adapted and $\tilde{B}_t$ is an H_t-Brownian motion.

3. A strong solution is also a weak solution but the inverse is not in general true.

Diffusion Theory

Due to the diffusion process, a role of important results to the stochastic calculus are important to be recalled.

Definition 2.2.33. A n-dimensional stochastic process $\{X_t, t \geq 0\}$ is a Markov process if, for every, $s < t$ that is

$$E[f(X_t)|X_r, r \leq s] = E[f(X_t)|X_s]$$

for any bounded Borel function f on $\mathbb{R}^n$.

Theorem 2.2.34 (The Markov property to diffusion processes)**.** Assume a bounded Borel function from $\mathbb{R}^n$ to $\mathbb{R}$. Then, for $t, h \geq 0$

$$E^x[f(X_{t+h})|\mathcal{F}_t^{(m)}]_{(\omega)} = E^{X_t(\omega)}[f(X_h)].$$

Where $\mathcal{F}_\tau^{(m)}$ is the σ-algebra generated by $\{B_{s\wedge\tau}; s \geq 0\}$.

Definition 2.2.35. Let $\{\mathcal{N}_t\}$ be an increasing family of σ-algebras in Ω. A function $\tau : \Omega \to [0, \infty]$ is called a (strict) stooping time w.r.t. $\{\mathcal{N}_t\}$ if

$$\{\omega; \tau(\omega) \leq t\} \in \mathcal{N}_t, \text{for al} t \geq 0.$$

τ is trivially a stopping time w.r.t. any filtration.

Theorem 2.2.36 (Strong Markov property for diffusions processes)**.** Let f be a bounded Borel function on $\mathbb{R}^n$, τ a stopping time w.r.t. $\mathcal{F}_t^{(m)}$, $\tau < \infty$ a.s.. Then,

$$E^x[f(X_{\tau+h})|\mathcal{F}_\tau^{(m)}] = E^{X_\tau}[f(X_h)]$$

for all $h \geq 0$.

The following definition is very important to link a diffusion process X_t to a second order partial differential operator A, in order to A be the generator of the process X_t.

Definition 2.2.37. Let $\{X_t\}$ be a (time- homogeneous) diffusion process in $\mathbb{R}^n$. The (infinitesimal) generator A of X_t is defined by

$$Af(x) = \lim_{t\to 0} \frac{E^x[f(X_t)] - f(x)}{t}; \quad x \in \mathbb{R}^n.$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that exists the limit at x is denoted by $\mathcal{D}_A(x)$, while $\mathcal{D}_A$ denotes the set of functions for which the limits exists for all $x \in \mathbb{R}^n$.

The relation between the operator A and the diffusion process is due to the Itô's formula: let $f(t, x)$ be a function of class $C^{1,2}$. Then $f(t, X_t)$ is an Itô process with differential

$$df(t, X_t) = \left(\frac{\partial f}{\partial t}(t, X_t) + A_t f(t, X_t)\right) dt + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial f}{\partial x_i}(t, X_t)\sigma_{i,j}(t, X_t)dB_t^j.$$

As a consequence, if

$$E\left(\int_0^t \left|\frac{\partial f}{\partial x_i}(s, X_s)\sigma_{i,j}(s, X_s)\right|^2 ds\right) < \infty \tag{2.2}$$

for every $t > 0$ and every i, j, then the process

$$M_t = f(t, X_t) - \int_0^t \left(\frac{\partial f}{\partial s} + A_s f\right)(s, X_s)ds$$

is a martingale.

Remark 7.

1. A sufficient condition for (2.2) is that the partial derivatives $\frac{\partial f}{\partial x_i}$ have linear growth, that is
$$\left|\frac{\partial f}{\partial x_i}(s, x)\right| \le C(1 + |x|^N). \tag{2.3}$$

2. If f satisfies the equation $\frac{\partial f}{\partial t} + A_t f = 0$ and (2.3) holds, then $f(t, X_t)$ is a martingale.

3. The martingale property of this process leads to a probabilistic interpretation of a parabolic equation with fixed terminal value, i.e.,
$$\frac{\partial f}{\partial t} + A_t f = 0$$
$$f(T, x) = g(x).$$

Theorem 2.2.38. Let X_t be the diffusion process

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

If $f \in C_0^2(\mathbb{R}^n)$ then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Theorem 2.2.39 (Dynkin's formula)**.** Let $f \in C_0^2(\mathbb{R}^n)$. Suppose τ is a stopping time, $E^x[\tau] < \infty$. Then

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau Af(X_s)ds\right].$$

Remark 8. If τ is the first exit time of a bounded set, $E^x[\tau] < \infty$, then the previous theorem holds for any function $f \in C^2$.

In the following, we have some classical results on solutions of stochastic differential equations, beginning with the Kolmogorov's backward equation.

Theorem 2.2.40. Let $f \in C_0^2(\mathbb{R}^n)$. Define

$$u(t,x) = E^x[f(X_t)] \tag{2.4}$$

then $u(t,.) \in \mathcal{D}_A$ for each t and

$$\frac{\partial u}{\partial t} = Au,\ t > 0,\ x \in \mathbb{R}^n \tag{2.5}$$

$$u(0,x) = f(x);\ x \in \mathbb{R}^n \tag{2.6}$$

where the right hand side is to be interpreted as A applied to the function $x \to u(t,x)$. Moreover, if $w(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ is a bounded function satisfying (2.5), (2.6), then $w(t,x) = u(t,x)$, given by (2.4).

Theorem 2.2.41 (The Feynman-Kač formula)**.** Let $f \in C_0^2(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$. Assume that q is lower bounded. Put

$$v(t,x) = E^x\left[exp\left(-\int_0^t q(X_s)ds\right)f(X_t)\right]. \tag{2.7}$$

Then

$$\frac{\partial v}{\partial t} = Av - qv,\ t > 0,\ x \in \mathbb{R}^n \tag{2.8}$$

$$v(0,x) = f(x);\quad x \in \mathbb{R}^n \tag{2.9}$$

Moreover, if $w(t,x) \in C^{1,2}(\mathbb{R}\times\mathbb{R}^n)$ is bounded on $\mathbb{K}\times\mathbb{R}^n$ for each compact $\mathbb{K} \subset \mathbb{R}$ and w solves (2.8) and (2.9), then $w(t,x) = v(t,x)$ given by (2.7).

The following results are fundamental in the stochastic calculus, and its applications. The main result is the drift coefficient can be changed without transforming radically the process law. Moreover, the new Itô process will be continuous in order to the original law. The following results are based on [46].

Theorem 2.2.42 (The Lèvy characterization of Brownian motion)**.** Consider $X(t) = (X_1(t), ..., X_n(t))$ a continuous stochastic process on a probability space $(\Omega, \mathcal{H}, \mathbb{Q})$ with values in $\mathbb{R}^n$. Then, the following (1) and (2), are equivalent:

1. $X(t)$ is a Brownian motion w.r.t. $\mathbb{Q}$, i.e., the law of $X(t)$ w.r.t. $\mathbb{Q}$ is the same law of an n-dimensional Brownian motion.

2. (a) $X(t) = (X_1(t), ..., X_n(t))$ is a martingale w.r.t. $\mathbb{Q}$ (and w.r.t. its own filtration) and

   (b) $X_i(t)X_j(t) - \delta_{ij}$ is a martingale w.r.t. $\mathbb{Q}$ (and w.r.t. its own filtration), for all $i, j \in \{1, 2, ..., n\}$.

Theorem 2.2.43 (The Girsanov theorem I)**.** Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form

$$dY(t) = a(t, \omega)dt + dB(t); \quad t \leq T, \quad Y_0 = 0.$$

where $T \leq \infty$ is a given constant and $B(t)$ is n-dimensional Brownian motion. Put

$$M_t = exp\left(-\int_0^t a(s, \omega)dB_s - \frac{1}{2}\int_0^t a^2(s, \omega)ds\right); \quad 0 \leq t \leq T.$$

Assume that M_t is a martingale with respect to $\mathcal{F}_t^{(n)}$ and $\mathbb{P}$. Define the measure $\mathbb{Q}$ on $\mathcal{F}_T^{(n)}$ by

$$d\mathbb{Q}(\omega) = M_T(\omega)dP(\omega).$$

Then $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T^{(n)}$ and $Y(t)$ is an -dimensional Brownian motion w.r.t. $\mathbb{Q}$, for $0 \leq t \leq T$.

Theorem 2.2.44 (The Girsanov theorem II)**.** Let $Y(t)$ be an Itô process of the form

$$dY(t) = \beta(t, \omega)dt + \theta(t, \omega)dB(t), \quad t \leq T$$

where $B(t) \in \mathbb{R}^m, \beta(t, \omega) \in \mathbb{R}^n$ and $\theta(t, \omega) \in \mathbb{R}^{n\times m}$. Suppose that exist processes $u(t, \omega) \in \mathcal{W}_H^m$ and $\alpha(t, \omega) \in \mathcal{W}_H^n$ such that

$$\theta(t, \omega)u(t, \omega) = \beta(t, \omega) - \alpha(t, \omega).$$

Put

$$M_t = exp\left(-\int_0^t u(s,\omega)dB_s - \frac{1}{2}\int_0^t u^2(s,\omega)ds\right), \quad t \le T$$

and

$$dQ(\omega) = M_T(\omega)dP(\omega) \quad \text{on} \quad \mathcal{F}_T^{(m)}.$$

Assume that M_t is a martingale (w.r.t. $F_t^{(n)}$ and P). Then Q is a probability measure on $F_T^{(m)}$, the process

$$\hat{B}(t) = \int_0^t u(s,\omega)ds + B(t); \qquad t \le T$$

is a Brownian motion w.r.t. Q and in terms of $\hat{B}(t)$, the process $Y(t)$ has the stochastic integral representation

$$dY(t) = \alpha(t,\omega)dt + \theta(t,\omega)d\hat{B}(t).$$

2.3 Aplication of stochastic calculus to finance

The field of Mathematics applied to finances emerged with the results of Black and Scholes (1973) and Merton (1973), when the stochastic modelling of assets prices has been generalized. They proposed the first equation to model a European option, which allows to price an option by solving a simple PDE.

The Black-Scholes Option Pricing Formula

First, we will define some basic terminology in finances. Then we will show the relation between Black-Scholes formula for pricing and the partial differential equations.

Definition 2.3.1.

1. A (mathematical) market is an $\mathcal{F}_t^{(m)}$-adapted $(n+1)$-dimensional Itô process $X(t) = (X_0(t), X_1(t), ..., X_n(t)); \quad 0 \le t \le T$ which we will assume has the form

$$dX_0(t) = \rho(t,\omega)X_0(t)dt; \qquad X_0(0) = 1$$

and

$$dX_i(t) = \mu_i(t,\omega)dt + \sum_{j=1}^{m}\sigma_{ij}(t,\omega)dB_j(t) = \mu_i(t,\omega)dt + \sigma_i(t,\omega)dB(t);$$

with $X_i(0) = x_i$. σ_i is row number i of the $n \times m$ matrix $[a_{ij}]$; $1 \le i \le n \in \mathbb{N}$.

2. The market $\{X(t)\}_{t\in[0,T]}$ is called normalized if $X_0(t) = 1$.

3. A portfolio in the market $\{X(t)\}_{t\in[0,T]}$ is an $(n+1)$-dimensional (t,ω)-mesurable and $F_t^{(m)}$-adapted stochastic process

$$\theta(t,\omega) = (\theta_0(t,\omega), \theta_1(t,\omega), ..., \theta_n(t,\omega)); \qquad o \leq t \leq T.$$

4. The value at time t of a portfolio $\theta(t)$ is defined by

$$V(t,\omega) = V^\theta(t,\omega) = \theta(t).X(t) = \sum_{i=0}^{n} \theta_i(t)X_i(t)$$

where $\cdot$ denotes inner product in $\mathbb{R}^{n+1}$.

5. The portfolio $\theta(t)$ is called self-financing if

$$\int_0^T \{\left|\theta_0(s)\rho(s)X_0(s) + \sum_{i=1}^{n} \theta_i(s)\mu_i(s)\right| + \sum_{j=1}^{m} \left[\sum_{i=1}^{n} \theta_i(s)\sigma_{ij}(s)\right]^2 \}ds < \infty \quad \text{a.s.} \tag{2.10}$$

and

$$dV(t) = \theta(t).dX(t)$$

i.e.

$$V(t) = V(0) + \int_0^t \theta(s).dX(s) \qquad \text{for} \quad t \in [0,T].$$

Definition 2.3.2. A portfolio $\theta(t)$ which satisfies (2.10) and which is self-financing is called admissible if the corresponding value process $V^\theta(t)$ is (t,ω) $a.s.$ lower bounded, i.e., there exists $K = K(\theta) < \infty$ such that

$$V^\theta(t,\omega) \geq -K \quad \text{for a.a.}(t,\omega) \in [0,T] \times \Omega.$$

Definition 2.3.3. An admissible portfolio $\theta(t)$ is called an arbitrage (in the market $\{X_t\}_{t\in[0,T]}$) if the corresponding value process $V^\theta(t)$ satisfies $V^\theta(0) = 0$ and

$$V^\theta(T) \geq 0 \qquad \text{a.s. and} \quad P[V^\theta(T) > 0] > 0.$$

Remark 9. The existence of arbitrage means a lack of equilibrium in the market.

Assume that the price X_t of a risky asset (*stock*) at time t is given by the geometric Brownian motion:

$$X_t = f(t, B_t) = X_0 e^{(c-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

where $B = (B_t, t \geq 0)$ is a Brownian motion and X_0 is assumed to be independent of B. X is the unique strong solution of the linear stochastic differential equation

$$X_t = X_0 + c\int_0^t X_s ds + \sigma \int_0^t X_s dB_s$$

which can be written as

$$dX_t = cX_t dt + \sigma X_t dB_t.$$

The cdt is the linear trend, σdB_t is the stochastic noise term, $c > 0$ is the mean rate of return and $\sigma > 0$ is the volatility.

Assume, now, a non-risky asset bound. An initial investiment capital β_0 returns an amount at time t of

$$\beta_t = \beta_0 e^{rt}$$

where $r > 0$ is the interest rate and β satisfies

$$\beta_t = \beta_0 + r\int_0^t \beta_s ds.$$

The portfolio includes the amounts of share a_t in stock and b_t in the bound, both stochastic processes adapted to Brownian motion. So, (a_t, b_t), $t \in [0, T]$ is called trading strategy. The choice of this pair will define the existence of profit.
The value of portfolio V_t at time t is given by $V_t = a_t X_t + b_t \beta_t$.
If $a_t < 0$ means short sale of stock and if $b_t < 0$ then the money is borrowed at the bond's riskless interest rate r. We will suppose that the trading stategy (a_t, b_t) is self-financing (i.e. the variation of its value is only responsability of variation on asset prices x_t and β_t).

The self-financing condition in differential form is:

$$dV_t = d(a_t X_t + b_t \beta_t) = a_t dX_t + b_t d\beta_t,$$

and in Itô form is:

$$V_t - V_0 = \int_0^t d(a_s X_s + b_s \beta_s) = \int_0^t a_s dX_s + \int_0^t b_s d\beta_s.$$

An option is a type of derivatives and it is negociated in financial institutions or in Stock Exchange.

Consider T the time of maturity/expiration of the option and K the exercise/strike price. We have two types of options: call option and put option. The first gives the owner the right to buy and the second the right to sell the option, both during the contract life and at a fixed price. The options can be either European or American (the most important). European option can only be exercised at the expiration date and the American options can be at any moment until expiration.

The payoff function of a European call option is given by:

$$(X_t - K)^+ = \max(0, X_T - K)$$

The payoff function of a European put option is given by:

$$(K - X_t)^+ = \max(0, K - X_T)$$

In the following results we will consider the European call option.

At this time a question is relevant: *what is the fair price for a European call option at $t = 0$?*

The Black-Scholes Model

This model impose some assumptions. One is the option is European and market movements cannot be predicted. Also is assumed that no dividens are paid out during the option life and that there are no transitions costs in buying the option. Besides that, the risk-free rate and volatility of the underlying are known and constant and the return is normally distributed.

Therefore, the price of the stock (risky asset) is described by the stochastic differential equation

$$dX_t = cX_t dt + \sigma X_t dB_t, \qquad t \in [0, T],$$

where c is the mean rate of return, σ the volatility, B is the standard Brownian motion and T is the time of maturity of the option.

The price of the bond (riskless asset) is described by the deterministic differential equation

$$d\beta_t = r\beta_t dt, \qquad t \in [0, T],$$

where $r > 0$ is the interest rate of the bound.

The value of portfolio at time t is given by

$$V_t = a_t X_t + b_t \beta_t, \qquad t \in [0,T].$$

The portfolio is self-financing if

$$dV_t = a_t dX_t + b_t d\beta_t, \qquad t \in [0,T].$$

At maturity time, V_T is equal to the contingent claim $h(X_T)$ for a given function h. European options, in particular,for call options we have $h(x) = (x-K)^+$. For put options, that is $h(x) = (K-x)^+$.

Recalling the Girsanov Theorem and changing the underlying probability measure $\mathbb{P}$, the discounted price of one share of stock $\tilde{X}_t = e^{-rt}X_t, t \in [0,T]$, becomes a martingale under the new probability measure $\mathbb{Q}$.
Representing $f(t,x) = e^{-rt}x$ and applying Itô lemma we obtain

$$d\tilde{X}_t = \sigma \tilde{X}_t d\tilde{B}_t \tag{2.11}$$

where

$$\tilde{B}_t = B_t + \left[\frac{c-r}{\sigma}\right] t, \qquad t \in [0,T].$$

By Girsanov Theorem, $\tilde{B}$ is a standard Brownian motion and the solution of (2.11), given by

$$\tilde{X}_t = \tilde{X}_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \tilde{B}_t}, \quad t \in [0,T],$$

transform, under $\mathbb{Q}$, into a martingale with respect to the natural Brownian motion.

Finally we state the Black-Scholes formula.

Theorem 2.3.4. Assume in the Black-Scholes model that there exists a self-financing strategy (a_t, b_t) such that the value V_t of a portfolio at time t is given by

$$V_t = a_t X_t + b_t \beta_t, \quad t \in [0,T],$$

and that V_T is equal to the contingent claim $h(X_T)$. Then, the value of the portfolio at time t is given by

$$V_t = E_{\mathbb{Q}}\left[e^{-r(T-t)}h(X_t)|\mathcal{F}_t\right], \quad t \in [0,T], \tag{2.12}$$

where $E_{\mathbb{Q}}(\mathcal{A}|\mathcal{F}_t)$ denotes the conditional expectation of the random variable $\mathcal{A}$, given by $\mathcal{F}_t = \sigma(B_s, s \le t)$, under the new probability measure $\mathbb{Q}$.

Next we study the value V_t of the portfolio and the Black-Scholes price of a European option. Let

$$\theta = T - t \quad \text{for} \quad t \in [0, T].$$

By (2.12), the value V_t of the portfolio at time t, according to the contingent claim $V_T = h(X_T)$ is:

$$V_t = E_{\mathbb{Q}}\left[e^{-r\theta}h(X_T)|\mathcal{F}_t\right] = E_{\mathbb{Q}}\left[e^{-r\theta}h(X_t e^{(r-\frac{1}{2}\sigma^2)\theta+\sigma(\tilde{B}_T-\tilde{B}_t)})|\mathcal{F}_t\right].$$

Since $\sigma(X_t) \subset \mathcal{F}_t$, X_t is a function of B_t and, under $\mathbb{Q}$, $\tilde{B}_T - \tilde{B}_t$ is independent of $\mathcal{F}_t$ and has an Normal distribution with $\mu = 0$ and $\sigma = \theta$. So, considering

$$V_t = f(t, X_t)$$

with

$$f(t,x) = e^{-r\theta}\int_{-\infty}^{\infty} h(xe^{(r-\frac{1}{2}\sigma^2)\theta+\sigma y^{\frac{1}{2}}})\varphi(y)dy,$$

and $\varphi(y)$ is the standard Normal density function.

As in a European call option we have $h(x) = \max(0, x - K)$ it goes

$$f(t,x) = \int_{-z_2}^{\infty}\left[xe^{-\frac{1}{2}\sigma^2\theta+\sigma y\theta^{\frac{1}{2}}} - Ke^{-r\theta}\right]\varphi(y)dy$$

$$= x\Phi(z_1) - Ke^{-r\theta}\Phi(z_2),$$

with $\Phi(x)$ the standard Normal distribution,

$$z_1 = \frac{ln\left(\frac{x}{K}\right) + (r + \frac{1}{2}\sigma^2)\theta}{\sigma\theta^{\frac{1}{2}}} \quad \text{and} \quad z_2 = z_1 - \sigma\theta^{\frac{1}{2}}.$$

The Cauchy problem and a Feynman-Kač representation

Consider a solution to the stochastic integral equation

$$X_s^{(t,x)} = x + \int_t^s b(\theta, X_\theta^{(t,x)})d\theta + \int_t^s \sigma(\theta, X_\theta^{(t,x)})dW_\theta; \quad t \le s < \infty \tag{2.13}$$

The coefficients

$$b_i(t,x), \sigma_{ij}(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R} \tag{2.14}$$

are continuous ans satisfy the linear growth condition.

Since we have a stochastic problem, under some conditions is the solution of the partial differential equation. The equation (2.13) has a weak solution.

$$(X^{(t,x)}, W), (\Omega, \mathcal{F}, \mathbb{P}) \tag{2.15}$$

for every pair (t, x) and this solution is unique in the sense of probability law.

Consider, now, a fixed $T > 0$, the constants $L > 0$, $\lambda \geq 1$, and the functions $f(x) = \mathbb{R}^d \to \mathbb{R}$, $\quad g(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $k(t,x) : [0,T] \times \mathbb{R}^d \to [0,\infty)$ are continuous and satisfying one of the conditions

$$|f(x)| \leq L(1 + ||x||^{2\lambda}) \tag{2.16}$$

$$f(x) \geq 0; \quad \forall x \in \mathbb{R}^d \tag{2.17}$$

$$|g(t,x)| \leq L(1 + ||x||^{2\lambda}) \tag{2.18}$$

$$g(t,x) \geq 0; \quad \forall 0 \leq t \leq T, x \in \mathbb{R}^d.$$

Theorem 2.3.5. Under the preceding assumption (2.13)-(2.18), suppose that $v(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, of class $C^{1,2}([0,T) \times \mathbb{R}^d$ and satisfies the Cauchy problem

$$-\frac{\partial v}{\partial t} + kv = \mathcal{A}_t v + g; \quad \text{in } [0,T) \times \mathbb{R}^d,$$

$$v(T,x) = f(x); \quad x \in \mathbb{R}^d,$$

as well as the polynomial growth condition

$$\max_{0 \leq t \leq T} |v(t,x)| \leq M(1 + ||x||^{2\mu}); \quad x \in \mathbb{R}^d, \tag{2.19}$$

for some $M > 0, \mu \geq 1$. The $v(t,x)$ admits the stochastic representation

$$v(t,x) = E^{t,x}[f(X_T)exp\{-\int_t^T k(\theta, X_\theta)d\theta\} + \int_t^T g(s, X_s)exp\{-\int_t^s k(\theta, X_\theta)d\theta\}ds]$$

on $[0,T] \times \mathbb{R}^d$, in particular, such a solution is unique.

Remark 10. In the case of bounded coefficients, i.e.,

$$|b_i(t,x)| + \sum_{j=1}^{r} \sigma_{ij}^2(t,x) \leq \rho; \quad 0 \leq t < \infty, \quad x \in \mathbb{R}^d, \quad 1 \leq i \leq d,$$

the polynomial growth (2.19) in Theorem (2.3.5) may be replaced by

$$\max_{0 \leq t \leq T} |v(t,x)| \leq Me^{\mu ||x||^2}; \quad x \in \mathbb{R}^d$$

for some $M > 0$ and $0 < \mu < (\frac{1}{18}\rho T d)$.

Remark 11. A set of conditions *sufficient for the existence* of a solution v satisfying the polynomial growth condition (2.19) is:

1. *Uniform ellipticity*: Exists a positive constant δ such that

$$\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(t,x)\xi_i\xi_k \geq \delta||\xi||^2$$

   holds for every $\xi \in \mathbb{R}^d$ and $(t,x) \in [0,\infty) \times \mathbb{R}^d$;

2. *Boundedness*: The functions $a_{ik}(t,x), b_i(t,x), k(t,x)$ are bounded in $[0,T] \times \mathbb{R}^d$

3. *Smoothness*: The functions $a_{ik}(t,x), b_i(t,x), k(t,x)$ and $g(t,x)$ are uniformly Hölder-continuous in $[0,T] \times \mathbb{R}^d$

4. *Polynomial growth*: The functions $f(x)$ and $g(t,x)$ satisfy (2.16) and (2.18), respectively.

We aim to approximate by finite-difference methods, under some assumptions, the Cauchy problem:

$$\begin{aligned}
&L(t)u - \frac{du}{dt} + f(t) = 0 \quad in \quad [0,T] \times \mathbb{R}^d \\
&u(0,x) = g(x) \quad on \quad \mathbb{R}^d.
\end{aligned}$$

We will assume L, the second-order partial differential operator, such as

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x)$$

where f and g are real functions, $T \in (0,\infty)$, and the coefficients of second order partial derivatives have quadratic growth and the ones of first order have linear growth. The independent terms are bounded.

Chapter 3

Approximation of PDEs with bounded coefficients

As revealed earlier in this thesis, we aim to approximate degenerate PDEs when dealing with the Cauchy problem.

We begin by stating some of the most important results, for bounded coefficient case, on the solvability of parabolic PDE, essencial to set our problem.

3.1 Classical results

3.1.1 The Cauchy problem for a general parabolic evolution equation

Let V be a reflexive Banach space embedded into a Hilbert space with a fixed inner product. Let V^* be the dual of V.
Consider the initial version of the Cauchy problem:

$$L(t)u - \frac{du}{dt} + f(t) = 0 \quad \text{in} \quad [0,T], \quad u(0) = g \tag{3.1}$$

with $T \in (0, \infty), L(t)$ and $\frac{d}{dt}$ are linear operators from V to V^*, $\forall t \geq 0$,
$f \in L^2([0,T]; V^*)$ and $g \in H$.

It is important to define, at this moment,a generalized solution of the Cauchy problem and set, as well, some assumptions on asbtract spaces, so we can garantee the existence and uniqueness of a generalized solution to the problem above.

Assumption 3.1.1.1. *There exist constants $\lambda \geq 0, K, M$ and N such that*

1. $\langle L(t)v, v\rangle + \lambda |v|_V^2 \leq K|v|_H^2, \quad \forall v \in V$;

2. $|L(t)v|_{V^*} \le M|v|_V, \quad \forall v \in V;$

3. $\int_0^T |f(t)|^2_{V^*} dt \le N, \quad |g|_H \le N.$

Definition 3.1.1.2. $u \in C([0,T];H)$ is said to be a generalized solution of (3.1) on $[0,T]$ if

1. $u \in L^2([0,T];V)$;

2. For all $t \in [0,T]$

$$(u(t),v) = (g,v) + \int_0^t \langle L(s)u(s), v\rangle ds + \int_0^t \langle f(s), v\rangle ds$$

holds for all $v \in V$.

Theorem 3.1.1.3. *Under the conditions of Assumption (3.1.1.1), (3.1) has a unique generalized solution on* $[0,T]$*. Moreover,*

$$\sup_{t\in[0,T]} |u(t)|^2_H + \int_0^T |u(t)|^2_V dt \le N\left(|g|^2_H + \int_0^T |f(t)|^2_{V^*} dt\right),$$

where N *is a constant.*

3.1.2 The Sobolev spaces

In order to study the solvability of PDE with bounded coefficients, we have to introduce the Sobolev spaces and some elementary properties. With these concepts we are able to demonstrate the embbeding theorems, essential to our intended approximation.

To introduce the Sobolev spaces, we begin by defining the weak derivatives.

Definition 3.1.2.1. Let $v, w \in L^1_{loc}(U)$ (U is a domain in $\mathbb{R}^d$) and α is a multi-index. w is said to be the α^{th} weak partial derivative of v, denoted by $D^\alpha v = w$ if for all functions $\phi \in C_0^\infty(U)$:

$$\int_U vD^\alpha \phi dx = (-1)^{|\alpha|}\int_U w\phi dx.$$

Notation 3.1.2.2. $L^p_{loc}(U), \quad 1 \le p < \infty$ is the locally convex space of all the numeric functions u measurable in U, ϕ is a called a test function and C_0^∞ is the set of all infinitely differentiable functions on U with compact support.

In order to establish the framework to our problem, we state the following results (see [20]).

Lemma 3.1.2.3 (Uniqueness of weak derivatives)**.** *A weak α^{th}-partial derivative of v, if it exists, is uniquely defined up to a set of measure zero.*

Introducing, at this point, the Sobolev spaces.

Definition 3.1.2.4. Fix $1 \leq p \leq \infty$ and k as a nonnegative integer. The Sobolev space $W^{m,p}(U)$ is the group of all functions $u : U \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leq m, D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remark 12. If $p = 2, W^{m,2}(U)$ can be written as $H^m(U), (m = 0, 1, ...)$. The notation H is used to represent a Hilbert space, as we are going to see. Also consider $H^0(U) = L^2(U)$.

Definition 3.1.2.5. If $u \in W^{m,p}(U)$ the norm is given by

$$||u||_{W^{m,p}(U)} = \sum_{|\alpha| \leq m} \int_U (|D^\alpha u|^p dx)^{\frac{1}{p}} .$$

Definition 3.1.2.6. The closure of $C_c^\infty(U)$ in $W^{m,p}(U)$ is denoted by $W_0^{m,p}(U)$.

Next we state the elementary properties of weak derivatives, set in [20], so we can prove that partial derivatives are approximated by difference quotients.

Theorem 3.1.2.7 (Properties of weak derivatives)**.** *Assume $u, v \in W^{m,p}(U)$, $|\alpha| \leq m$. Then*

1. *$D^\alpha u \in W^{m-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$ for all multi-indices α, β with $|\alpha| + |\beta| \leq m$.*
2. *For each $\lambda, \mu \in \mathbb{R}, \lambda u + \mu v \in W^{m,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, |\alpha| \leq m$.*
3. *If V is an open subset of U, then $u \in W^{m,p}(V)$.*
4. *If $\zeta \in C_0^\infty(U)$, then $\zeta u \in W^{m,p}(U)$ and*

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta \zeta D^{\alpha-\beta} u \qquad \text{(Leibniz' formula)}$$

Theorem 3.1.2.8 (Sobolev spaces as function spaces)**.** *For $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(U)$ is a Banach space.*

Theorem 3.1.2.9. *$W^{m,p}(U)$ is separable if $1 \leq p < \infty$ and is uniformly convex and reflexive if $1 < p < \infty$. In particular, $W^{m,2}(U)$ is a separable Hilbert space with inner product*

$$(u, v)_m = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v),$$

where $(u, v) = \int_U u(x)\overline{v(x)}dx$ is the inner product on $L^2(U)$.

The Sobolev embedding theorem, see [2] for the results, states the existence of embeddings of $W^{m,p}(U)$ (or $W_0^{m,p}(U)$) into Banach spaces of the following types.

1. $W^{j,q}(U)$, where $j \leq m$ and in particular $L^q(U)$;

2. $W^{j,q}(U_k)$, where, for $1 \leq k < d$, U_k is the intersection of U with a k-dimensional plane in $\mathbb{R}^d$;

3. $C_B^j(U)$, the space of functions having bounded, continuous derivatives up to order j on U, normed by:

   $$||u; C_B^j(U)|| = \max_{0 \leq |\alpha| \leq j} \sup_{x \in U} |D^\alpha u(x)|.$$

4. $C^j(\bar{U})$, the closed subspace of $C_B^j(U)$ consisting of functions having bounded, uniformly continuous derivatives up to order j on U, with the same norm as $C_B^j(U)$:

   $$||\phi; C^j(\bar{U})|| = \max_{0 \leq |\alpha| \leq j} \sup_{x \in U} |D^\alpha \phi(x)|.$$

   This space is smaller than $C_B^j(U)$ due to the fact that its elements must be uniformly continues on U.

5. $C^{j,\lambda}(\bar{U})$, the closed subspace of $C^j(\bar{U})$ consisting of functions whose derivatives up to order j satisfy Hölder conditions of exponent λ in U. The norm on $C^{j,\lambda}(\bar{U})$ is:

   $$||\phi; C^{j,\lambda}(\bar{U})|| = ||\phi; C^j(\bar{U})|| + \max_{0 \leq |\alpha| \leq j} \sup_{x,y \in U; x \neq y} \frac{|D^\alpha \phi(x) - D^\alpha \phi(y)|}{|x - y|^\lambda}.$$

Remark 13 (The cone condition). U satisfies the *cone condition* if there exists a finite cone C such that each $x \in U$ is the vertex of a finite cone C_x contained in U and congruent to C (C_x can be obtained from C by rigid motion).

Remark 14 (Strong local Lipschitz property). U has the strong local Lipschitz property if there exists positive δ and M, a local finite open cover U_i of boundary U and for each U_j a real-valued function f_j of $n-1$ variables, such that the following conditions hold:

1. For some finite R, every collection of $R+1$ of the sets U_j has empty intersection;

2. For every pair of points $x, y \in U_j = x \in U : dist(x, bdryU) < \Omega$ such that $|x-y| < \delta$, there exists j such that

$$x, y \in \mathcal{V}_j = x \in U_j : dist(x, bdryU_j) > \delta;$$

3. Each function f_j satisfies a Lipschitz condition with constant M:

$$|f(\xi_1, ...\xi_{n-1}) - f(\eta_1, ..., \eta_{n-1})| \leq M|(\xi_1 - \eta_1, ..., \xi_{n-1} - \eta_{n-1}|;$$

4. For some cartesian coordinate system $(\xi_{j,1}, ..., \xi_{j,n})$ in U_j the set $\Omega \cap U_j$ is represented by the inequality

$$\xi_{j,n} < f_j(\xi_{j,1}, ..., \xi_{j,n-1}).$$

Theorem 3.1.2.10 (The Sobolev embedding theorem)**.** *Let U be a domain in $\mathbb{R}^d$ and let U^k be the k-dimensional domain obtaining by intersecting U with a k-dimensional plane in $\mathbb{R}^d, 1 \leq k \leq d$. Let j and m be non-negative integers and let p satisfy $1 \leq p < \infty$.*

Part I *If U has te cone property, then there exist the following embeddings:*

Case A *Suppose $mp < d$ and $d - mp < k \leq d$. Then*

$$W^{j+m,p}U \to W^{j,q}(U^k), \quad p \leq q \leq kp/(d-mp), \tag{3.2}$$

and in particular,

$$W^{j+m,p}U \to W^{j,q}(U), \quad p \leq q \leq dp/(d-mp),$$

or

$$W^{m,p}(U) \to L^q(U), \quad p \leq q \leq dp/(d-mp).$$

Moreover, if $p = 1$, so that $m < d$, embedding (3.2) also exists for $k = d - m$.

Case B *Suppose $mp = d$. Then for each k, $1 \le k \le d$;*

$$W^{j+m,p}(U) \to W^{j,q}(U^k) \quad p \le q < \infty, \tag{3.3}$$

so that, in particular,

$$W^{m,p}(U) \to L^q(U), \quad p \le q < \infty. \tag{3.4}$$

Moreover, if $p = 1$ so that $m = d$, embeddings (3.3) and (3.4) exist with $q = \infty$ as well. More,

$$W^{j+n,1}(U) \to C_B^j(U).$$

Case C *Suppose $mp > d$. Then*

$$W^{j+m,p}(U) \to C_B^j(U).$$

Part II *If U has the strong local Lipschitz property, then Case C of Part I can be refined as:*

Case C' *Suppose $mp > d > (m-1)p$. Then*

$$W^{j+m,p}(U) \to C^{j,\lambda}(\bar{U}), \quad 0 < \lambda \le m - (d/p).$$

Case C" *Suppose $d = (m-1)p$. Then*

$$W^{j+m,p}(U) \to C^{j,\lambda}(\bar{U}), \quad 0 < \lambda < 1. \tag{3.5}$$

Also, if $d = m-1$ and $p = 1$, then (3.5) holds for $\lambda = 1$ as well.

Part III *All the conclusions of Parts I and II are valid for arbitrary domains provided the W-spaces undergoing embedding are replaced with the corresponding W_0-spaces.*

The next Sobolev Embedding Theorem is based on [2] and [26].

Theorem 3.1.2.11. *Let U be a bounded domain in $\mathbb{R}^d$ with a C^1 boundary. Let $v \in W^{m,2}(U)$.*
If $m > \dfrac{d}{2}$ then $v \in C^{(m-[\frac{d}{2}]-1)+\delta}(U)$, where

$$\delta = \begin{cases} \left[\dfrac{d}{2}\right] + 1 - \dfrac{d}{2}, & \text{if } \dfrac{d}{2} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \dfrac{d}{2} \text{ is an integer.} \end{cases}$$

Moreover,

$$|v|_{(m-[\frac{d}{2}]-1)+\delta;U} \le N|v|_{W^{m,2}(U)},$$

with N a constant only depending on m, d, δ and U.

3.1.3 A parabolic PDE problem - the nondegenerate case

Based on previous works [26, 27, 30], we state the conditions to the existence and uniqueness of generalized solution of PDEs in the nondegenerate case for the exact problem.

Consider the second-order parabolic partial differential equation problem, with second order operator L, such that:

$$L(t,x) = a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x)$$

with a_{ij}, b_i, c are real valued functions on $[0,T] \times \mathbb{R}^d$

Consider now the Cauchy Problem:

$$\begin{aligned} &Lu - u_t + f = 0 \text{ in } \mathbb{Q} \\ &u(0,x) = g(x) \text{ in } \mathbb{R}^d \end{aligned} \tag{3.6}$$

with $T \in (0,\infty); \mathbb{Q} = [0,T] \times \mathbb{R}^d$ and f and g functions.

We consider now the Cauchy case where (3.6) is assumed to be nondegenerate.

We will use the notation, $C([0,T];W)$ for the space of continuous W-valued functions on $[0,T]$ and $L^2([0,T];W)$ the space of continuous W-valued functions ω on $[0,T]$, with the norm $||\omega||_{L^2([0,T];W)} = (\int_0^T ||\omega||^2 dt)^{1/2} < \infty$.

We assume the following assumption.

Assumption 3.1.3.1. *Let $m \geq 0$ be an integer. There exist constants $\lambda > 0, K$ such that*

1. $\sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \geq \lambda \sum_{i=1}^d |\xi_i|^2$, *for all* $t \geq 0, x, \xi \in \mathbb{R}^d$;
2. $|D_x^\alpha a_{ij}| \leq K$ *for all* $|\alpha| \leq m \vee 1, |D_x^\alpha b_i| \leq K, |D_x^\alpha c| \leq K$ *for all* $|\alpha| \leq m$, *where* D_x^α *denotes the* α^{th} *partial derivative operator with respect to* x;
3. $f \in L^2([0,T];W^{m-1,2}), g \in W^{m,2}$.

We define the generalized and classical solution of the problem (3.6).

Definition 3.1.3.2. $u \in C([0,T];L^2)$ is said to be a generalized solution of the problem (3.6) on $[0,T]$ if:

1. $u \in L^2([0,T];W^{1,2})$;

2. $\forall t \in [0, T]$

$$(u(t), \phi) = (g, \phi) + \int_0^t \{-(a_{ij}(s)D_i u(s), D_j\phi) + (b(s)D_i u(s) \\ - D_j aij(s)D_i u(s), \phi) + (c(s)u(s), \phi) + \langle f(s), \phi\rangle\}ds,$$

for all $\phi \in C_0^\infty(\mathbb{R}^d)$.

Remark 15.
Above, $(\cdot, \cdot)$ denots the inner product in L^2 and $||\cdot||$ is the norm space in W.

Definition 3.1.3.3. A given problem for a partial differential equation is well-posed in a classical sense if:

1. The problem has a solution;
2. This solution is unique;
3. The solution depends continuously on the data given in the problem.

By solving a PDE in the classical sense we need a definition of classical solution that holds the previous conditions (1) - (3).

Definition 3.1.3.4. $u(t, x) \in [0, T] \times \mathbb{R}^d$ is called a classical solution of the problem (3.6) if:

1. $u \in C^{0,2}([0, T] \times \mathbb{R}^d)$
2. For all $x \in \mathbb{R}^d$, for all $t \in [0, T]$

$$u(t, x) = g(x) + \int_0^t \{\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}u(s, x) + a_{i0}u(s, x) + f_i(s, x)) \\ + b_i\frac{\partial}{\partial x_i}u(s, x) + cu(s, x)\}ds.$$

Next we state the conditions of existence and uniqueness, see e.g. [31] and [26].

Theorem 3.1.3.5. *Under (1)–(3) in Assumption (3.1.1.1), problem (3.6) has a unique generalized solution on* $[0, T]$. *Moreover,*

$$u \in C([0,T]; W^{m,2}) \cap L^2([0,T]; W^{m+1,2})$$

and

$$\sup_{t\in[0,T]} \|u(t)\|^2_{W^{m,2}} + \int_0^T \|u(t)\|^2_{W^{m+1,2}} dt \leq N\Big(\|g\|^2_{W^{m,2}} + \int_0^T \|f(t)\|^2_{W^{m-1,2}} dt\Big),$$

where N *is a constant.*

3.1.4 A parabolic PDE problem - the degenerate case

Consider the problem (3.6) and assume the situation where the operator L is degenerate in the spatial variables. Beginning to establish some assumptions, see e.g. [26, 31], we will state the conditions to the existence and uniqueness of generalized solution to the exact degenerate problem.

Assumption 3.1.4.1. *Let* $m \geq 0$ *be an integer. There exist constants* $K \geq 0$ *such that*

1. $\sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \geq 0, \quad \forall t \geq 0, x \in \mathbb{R}^d$;
2. $|D^\alpha_{x^i} a_{ij}| \leq K$ *for all* $|\alpha| \leq m \vee 1, |D^\alpha_{x^i} b_i| \leq K, |D^\alpha_{x^i} c| \leq K$ *for all* $|\alpha| \leq m$;
3. $f \in L^2([0,T]; W^{m-1,2}), \quad g \in W^{m,2}$.

Definition 3.1.4.2. $u \in C([0,T]; L^2)$ is a generalized solution of (3.6) on $[0, T]$ if:

1. $u \in L^2([0,T]; W^{1,2})$;
2. $\forall t \in [0, T]$

$$\begin{aligned}(u(t), \varphi) =& (g, \varphi) + \int_0^t \{-(a_{ij}(s)D_i u(s), D_j\varphi) \\ &+ (b(s)D_i u(s) - D_j aij(s) D_i u(s), \varphi) \\ &+ (c(s)u(s), \varphi) + \langle f(s), \varphi\rangle\} ds,\end{aligned}$$

$\forall \varphi \in C_0^\infty(\mathbb{R}^d)$

Considering [26, 31], and adapting to the case of bounded coefficients, we state the next result, to existence and uniqueness of solution.

Theorem 3.1.4.3. *Assume conditions on Assumption (3.1.4.1). Let K be a constant and σ a matrix-valued function $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d_2}$ such that*

1. $\sigma_{in}\sigma_{jn} = A$
2. $|\sigma^{in}_{x^j}(t,x)| \leq K$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^d, \quad i = 1,...,d \quad n = 1,...,d_2.$

Then, there exists a unique generalized solution $(u(t))_{t\in[0,T]}$ of the problem (3.6). Moreover,

$$u \in C([0,T]; W^{m,2}) \cap L^2([0,T]; W^{m+1,2})$$

and

$$\sup_{t\in[0,T]} |u(t)|^2_{W^{m,2}} + \int_0^T |u(t)|^2_{W^{m+1,2}} dt \leq N\Big(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt\Big),$$

for N constant.

3.2 Finite-difference approximation

We will now discretize our degenerate problem (3.6) in the spatial variables in order to approximate its solution.

Based on the discrete framework defined by Gonçalves in [26] for the nondegenerate case, with bounded coefficients, we will adapte this framework and spaces to the degenerate case.

3.2.1 The discrete Sobolev spaces

We introduce the discretized version of Sobolev spaces $W^{0,2}$ and $W^{1,2}$.

$l^{0,2}$: We have the function space $l^{0,2} = \{v : Z^d_h \to \mathbb{R} : |v|_{l^{0,2}} < \infty\}$
with the inner product: $(v,w)_{l^{0,2}} = \sum_{x\in Z^d_h} v(x)w(x)h^d$
and the norm induced by the inner product: $|v|_{l^{0,2}} = (v,v)^{1/2}_{l^{0,2}} = \Big(\sum_{x\in Z^d_h} |v(x)|^2 h^d\Big)^{1/2}.$

$l^{1,2}$: Now we have the function space $l^{1,2} = \{v : Z^d_h \to \mathbb{R} : |v|_{l^{1,2}} < \infty\}$
with the inner product $(v,w)_{l^{1,2}} = (v,w)_{l^{0,2}} + \sum_{i=1}^d (\partial^+_i v, \partial^+_i w)_{l^{0,2}}$
and the the norm induced by the inner product, with v,w functions in $l^{1,2}$
$|v|_{l^{1,2}} = |v|^2_{l^{0,2}} + \sum_{i=1}^d |\partial^+_i v|^2_{l^{0,2}}.$

In these conditions, $l^{1,2}$ is densely embedded into $l^{0,2}$ and its dual (owing to the properties of the inner product defined above, we will maintain as $l^{0,2}$) is also densely embedded in the dual $(l^{1,2})^*$. Thus, we have the normal triple $l^{1,2} \hookrightarrow l^{0,2} \hookrightarrow (l^{1,2})^*$.

According to [26] the following results, can be stated and ensure that some of the conditions to the existence of solution is garantee. For the completeness we give brief proofs on that.

Proposition 3.2.1.1. *The functions spaces $l^{0,2}$ and $l^{1,2}$ are Hilbert Spaces.*

Proof. The first step of this proof is to prove that the space $l^{0,2}$ with the inner product defined is complete, i.e., that $l^{0,2}$ is a Banach space with a inner product, therefore a Hilbert space.

Assume (u_n) as a Cauchy sequence in $l^{0,2}$. Then,$\forall \epsilon > 0$ $\exists N$ that for $m, n > N$

$$|u_m - u_n|_{l^{0,2}} = \left(\sum_{x \in Z_h^d} |u_m(x) - u_n(x)|^2 h^d \right)^{\frac{1}{2}} < \epsilon \tag{3.7}$$

So, for every $x \in Z_h^d$, for $m, n > N$ there is,

$$|u_m(x) - u_n(x)|^2 h^d < \epsilon^2. \tag{3.8}$$

Fix $x = x_0$. Owing to (3.8) $(u_1(x_0), ..., u_m(x_0))$ is a Cauchy sequence of numbers in $\mathbb{R}$. Consequently, $u_m(x_0)$ is convergent to $u(x_0)$. Let $u = u(x_0), \forall x \in Z_h^d$.

Considering B a ball in Z_h^d and owing to (3.7), for $m, n > N$

$$\sum_{x \in B} |u_m(x) - u_n(x)|^2 h^d < \epsilon^2.$$

For $n \to \infty$ and for $m > N$

$$\sum_{x \in B} |u_m(x) - u(x)|^2 h^d < \epsilon^2.$$

Considering the diameter of B to tend to ∞, for $m > N$ that is

$$\sum_{x \in Z_h^d} |u_m(x) - u(x)|^2 h^d < \epsilon^2.$$

Therefore we have that $u_m - u \in l^{0,2}$ and that u_m is convergent to u. By Minkowski inequality and as $l^{0,2}$ we have

$$u = u_m + (u - u_m) \in l^{0,2}.$$

We proved that any Cauchy sequence in $l^{0,2}$ is convergent in the space norm, which proves the result for $l^{0,2}$.

For $l^{1,2}$ the proof is similar.

□

Proposition 3.2.1.2. *The function space $l^{1,2}$ is separable.*

Proof. We have to prove that $l^{1,2}$ with the inner product has a compact subset that is dense. Let S be the set as $S = B \cup \{x + e_i : x \in B, i = 1, ..., d\}$, where B is a ball in Z_h^d. Assume l as the set of all functions $w(x) \in l^{1,2}$ with rational values when $x \in S$ and becoming zero outside S. l is countable.

Consider u an arbitrary function in $l^{1,2}$ and let $x \in B$. For some $\epsilon > 0$, it is possible to choose w that

$$\sum_x |u(x) - w(x)|^2 h^d + \sum_{i=1}^{d} \sum_x |\partial_i^+ (u(x) - w(x))|^2 h^d \tag{3.9}$$

$$= \sum_x |u(x) - w(x)|^2 h^d + \sum_{i=1}^{d} \sum_x |h^{-1}(u(x + he_i) - w(x + he_i) - (u(x) - w(x)))|^2 h^d$$
$$\leq \sum_x |u(x) - w(x)|^2 h^d + 2\sum_{i=1}^{d} \sum_x |u(x + he_i) - w(x + he_i)|^2 h^{d-2}$$
$$+ 2\sum_{i=1}^{d} \sum_x |u(x) - w(x)|^2 h^{d-2} < \frac{\epsilon^2}{2}.$$

As $|u|^2_{l^{1,2}}$ is a convergent series, for some $\epsilon > 0$ there exists a diameter of B that, for x outside B that

$$\sum_x |u(x)|^2 h^d + \sum_x |u(x) - w(x)|^2 h^d + \sum_{i=1}^{d} \sum_x |\partial_i^+ u(x)|^2 h^d < \frac{\epsilon^2}{2}. \tag{3.10}$$

By (3.9) and (3.10) that is $|u - w|_{l^{1,2}} < \epsilon$.

Therefore $l^{1,2}$ has a countable subset dense in $l^{1,2}$ and the result is proved.

□

Proposition 3.2.1.3. *The function space $l^{1,2}$ is densely embeddable in $l^{0,2}$.*

Proof. Let u be an arbitrary function such that $u \in l^{0,2}$. Consider B a ball in Z_h^d. Let $\in l^{1,2} w$ be a function defined as below

$$w(x) = \begin{cases} u(x), & x \in B \\ 0, & \text{otherwise.} \end{cases}$$

For some $\epsilon > 0$, for a diameter of B sufficiently large, that is

$$|u - w|_{l^{0,2}} < \epsilon.$$

Therefore, we proved that $\overline{l^{1,2}} = l^{0,2}$ and the result is showed.

□

3.2.2 The discretized problem

As mentioned befor, this discretization of the Cauchy problem is based on previous works but now, adapting to degenerate case, we set the discretization in spatial variables of the second order linear parabolic PDE for the bounded coefficient case.

Starting the discretization of Cauchy problem (3.6), we begin by defining the discretized framework.

Assume the h-grid on $\mathbb{R}^d$, with $h \in (0, 1]$. Assume also that e_i denotes the canonical basis of $\mathbb{R}^d$.

$$Z_h^d = \{x \in \mathbb{R}^d : x = \sum_{i=1}^{d} e_i n_i, n_i = 0, \pm 1, \pm 2, ...\},$$

establish the difference quotients in space:

- Forward: $\partial_i^+ u = \partial_i^+ u(t, x) = \dfrac{u(t, x + he_i) - u(t, x)}{h}$
- Backward: $\partial_i^- u = \partial_i^- u(t, x) = \dfrac{u(t, x) - u(t, x - he_i)}{h}$.

and consider the discrete operator L_h such that:

$$L_h(t, x) = a_{ij}(t, x)\partial_j^- \partial_i^+ + b_i(t, x)\partial_i^+ + c(t, x).$$

The discrete problem can be written as

$$L_h u - u_t + f_h = 0 \quad \text{in} \quad Q(h) = [0, T] \times Z_h^d \tag{3.11}$$

$$u(0, x) = g_h(x) \quad \text{in} \quad Z_h^d$$

with $T \in (0,\infty)$, f_h, g_h such that

$$f_h : Q(h) \to \mathbb{R} \quad \text{and} \quad g_h : Z_h^d \to \mathbb{R}$$

So, we have:

$$a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u - \frac{\partial u}{\partial t} + f_h(t,x) = 0.$$

Assumption 3.2.2.1. *For the discretization of problem (3.6), we assume*

1. $f_h \in L^2([0,T]; l^{0,2})$;
2. $g_h \in l^{0,2}$.

Remark 16. In previous Assumption (3.2.2.1), the first condition can be replaced by $f_h \in L^2([0,T]; (l^{1,2})^*)$, where $(l^{1,2})^*$ is the dual space of $l^{1,2}$, defining a weaker condition on this space.

Remark 17. By [26] we can state that

$$|\partial_i^+ a_{ij}(t,x)| = |h^{-1}(a_{ij}(t,x+he_i) - a_{ij}(t,x))| \leq \left| \frac{\partial}{\partial x_i} a_{ij}(t,x+\tau e_i) \right|,$$

with τ such that $0 < \tau < h$.

We define now the generalized solution of the problem (3.11), solution we want to prove that exists and is unique in this discrete problem.

Definition 3.2.2.2. u is said to be a generalized solution of the discrete problem (3.11) if, $\forall t \in [0,T]$

$$\begin{aligned}(u(t),\varphi) = (g_h,\varphi) + \int_0^t \{&-(a_{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi) \\ &+ (b_i(s)\partial_i^+ u(s) - \partial_j^+ a_{ij}(s)\partial_i^+ u(s), \varphi) \\ &+ (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle\} ds,\end{aligned}$$

$\forall \varphi \in (l^{1,2})^*$. $(\cdot,\cdot)$ denotes the inner product in $l^{0,2}$.

We now state the existence and uniqueness of a generalized solution to the discretized problem (3.11).

Theorem 3.2.2.3. *Suppose (2) in Assumption (3.1.4.1), (1)-(2) of Assumption (3.2.2.1) and conditions (1)-(2) in (3.1.4.3). Then the discretized problem (3.11) has a unique generalized solution $u(t)$ on $[0, T]$. Moreover,*

$$sup_{t\in[0,T]}|u(t)|^2_{l^{0,2}} + \int_0^T |u(t)|^2_{l^{1,2}} dt \leq N \left(|g_h|^2_{l^{0,2}} + \int_0^T |f_h(t)|^2_{l^{0,2}} dt \right)$$

where N is a constant independent of h.

Proof. Let us consider the new problem obtained of (3.11) by changing the coefficient a_{ij} by $a^{\lambda}_{ij}(t, x, \lambda) = a_{ij}(t, x) + \lambda, \lambda > 0$.

We begin by proving that this problem has a unique generalized solution. Since $l^{1,2}$ and $(L^{1,2})^*$ satisfies the normal triple $L_h(s)_{l^{1,2}} \to (L^{1,2})^*$ for the problem.

Let $L_h(s) : l^{1,2} \to (L^{1,2})^*$ be a discrete bilinear functional and consider $\varphi, \psi \in l^{1,2}$ such that

$$\langle L_h(s)\psi, \varphi\rangle = - \left((a_{ij}(s) + \lambda)\partial^+_i \psi, \partial^+_j \varphi\right) \\ \left(b_i(s)\partial^+_i \psi - \partial^+_j (a_{ij}(s) + \lambda)\partial^+_i \psi, \varphi\right) + (c(s)\psi, \varphi)$$

To state the uniqueness of solution to the given problem, we need to prove:

1. $|\langle L_h(s)\psi, \varphi\rangle| \leq k|\psi|_{l^{1,2}}|\varphi|_{l^{1,2}}$ for all $\varphi, \psi \in l^{1,2}$ and k constant
2. $\langle L_h(s)\psi, \psi\rangle \leq k|\psi|^2_{l^{0,2}} - \epsilon|\psi|^2_{l^{1,2}}$ for all $\psi \in l^{1,2}, \epsilon > 0, k$ constant

To prove the first inequality, owing to $|D^{\alpha}_x a_{ij}| \leq k, |D^{\alpha}_x b_i| \leq k$, $|D^{\alpha}_x c| \leq k$ by

(2) in Assumption (3.1.4.1) and $\lambda \in (0,1)$ we have:

$$
\begin{aligned}
|\langle L_h(s)\psi, \varphi\rangle| &= |\sum_{x\in Z_h^d}\sum_{i,j}(a_{ij}(s)+\lambda)\partial_i^+\psi\partial_j^+\varphi h^d + \sum_{x\in Z_h^d}\sum_i b_i(s)\partial_i^+\psi\varphi h^d \\
&- \sum_{x\in Z_h^d}\sum_{i,j}(\partial_j^+(a_{ij}(s)+\lambda)\partial_i^+\psi\varphi h^d + \sum_{x\in Z_h^d} c(s)\psi\varphi h^d| \\
&\le |\sum_{x\in Z_h^d}\sum_{i,j}(k+\lambda)|\partial_i^+\psi\partial_j^+\varphi|h^d \\
&+ k\sum_{x\in Z_h^d}\sum_i |\partial_i^+\psi\varphi|h^d + k\sum_x |\psi\varphi|h^d \\
&\le (k+\lambda)\sum_i |\partial_i^+\psi|_{l^{0,2}}\sum_j |\partial_j^+\varphi|_{l^{0,2}} \\
&+ k\sum_i |\partial_i^+\psi|_{l^{0,2}}|\varphi|_{l^{0,2}} + k|\psi|_{l^{0,2}}|\varphi|_{l^{0,2}} \\
&\le (k+\lambda)\sum_i |\partial_i^+\psi|_{l^{0,2}}\sum_j |\partial_j^+\varphi|_{l^{0,2}} + k\sum_i |\partial_i^+\psi|_{l^{0,2}}|\varphi|_{l^{0,2}} \\
&+ |\psi|_{l^{0,2}}\sum_j |\partial_j^+\varphi|_{l^{0,2}} + k|\psi|_{l^{0,2}}|\varphi|_{l^{0,2}} \\
&\le k\sum_j |\partial_j^+\varphi|_{l^{0,2}}\left[\sum_i |\partial_i^+|_{l^{0,2}} + |\psi|_{l^{0,2}}\right] + k|\varphi|_{l^{0,2}}|\psi|_{l^{1,2}} \\
&\le k\sum_j |\partial_j^+\varphi|_{l^{0,2}}\left[\sum_i |\partial_i^+\psi|_{l^{0,2}} + |\psi|_{l^{0,2}}\right] + k|\varphi|_{l^{0,2}}|\psi|_{l^{1,2}} \\
&\le k\sum_j |\partial_j^+\varphi|_{l^{0,2}}|\psi|_{l^{1,2}} + k|\varphi|_{l^{0,2}}|\psi|_{l^{1,2}} \\
&\le k\sum_j |\partial_j^+\varphi|_{l^{0,2}}|\psi|_{l^{1,2}} + k|\varphi|_{l^{0,2}}|\psi|_{l^{1,2}} \\
&\le k|\psi|_{l^{1,2}}|\sum_j |\partial_j^+\varphi|_{l^{0,2}} + |\varphi|_{l^{0,2}}| \\
&\le k|\psi|_{l^{1,2}}|\varphi|_{l^{1,2}}
\end{aligned}
$$

For the second inequality, recalling the Cauchy-Schwartz, $2ab \le \delta a + \frac{1}{\delta}b$, $\delta > 0$.

$$
\begin{aligned}
&\langle L_h(s)\psi,\psi\rangle = \\
&\quad -((a_{ij}(s)+\lambda)\partial_i^+\psi,\partial_j^+\psi) \\
&\quad +(b_i(s)\partial_i^+\psi-\partial_j^+(a_{ij}(s)+\lambda)\partial_i^+\psi,\psi)+(c(s)\psi,\psi) \\
&\quad =-\sum_x\sum_{i,j}(a_{ij}(s)+\lambda)|\partial_i^+\psi|^2h^d+\sum_x\sum_{i,j}\left[b_i(s)-\partial_j^+(a_{ij}(s)+\lambda)\right] \\
&\quad \partial_i^+\psi\psi h^d+\sum_x c(s)|\psi|^2h^d \\
&\quad \le -(\epsilon+\lambda)\sum_x\sum_i|\partial_i^+\psi|^2h^d+(k+\lambda)\sum_x\sum_i|\partial_i^+\psi\psi|h^d+k\sum_x|\psi|^2h^d \\
&\quad \le (-\epsilon-\lambda)\sum_i|\partial_i^+\psi|^2_{l^{0,2}}+2k\sum_i|\partial_i^+\psi\psi|_{l^{0,2}}+k|\psi|^2_{l^{0,2}} \\
&\quad \le -\epsilon\sum_i|\partial_i^+\psi|^2_{l^{0,2}}+\epsilon|\psi|^2_{l^{0,2}}-\epsilon|\psi|^2_{l^{0,2}}+k|\psi|^2_{l^{0,2}}+2k\sum_i|\partial_i^+\psi\psi|_{l^{0,2}} \\
&\quad \le -\epsilon|\psi|^2_{l^{1,2}}+(\epsilon+k)|\psi|^2_{l^{0,2}}+2k|\partial_i^+\psi\psi|_{l^{0,2}} \\
&\quad \le -\epsilon|\psi|^2_{l^{1,2}}+(\epsilon+k)|\psi|^2_{l^{0,2}}+k\delta\sum_i|\partial_i^+\psi|+\frac{1}{\delta}k\sum_i|\psi|_{l^{0,2}} \\
&\quad \le -\epsilon|\psi|^2_{l^{1,2}}+k|\psi|^2_{l^{0,2}}
\end{aligned}
$$

We proved the discretized problem (3.11) has a unique solution.

The following steps are in to prove that the estimate in this theorem is valid.

$l^{0,2}$ and $l^{1,2}$ are Hilbert spaces, in particular they are complete spaces such that $l^{0,2}, l^{1,2} \subset L^2(\mathbb{R}^d)$. Also their weak derivatives are in $L^2(\mathbb{R}^d)$.

Consider $\lambda \in (0,1)$ and $a_{ij}^{\lambda}(t,x,\lambda) = a_{ij}(t,x)+\lambda$, instead of $a_{ij}(t,x)$. Let u_λ be the generalized solution of our problem. Then, $u_\lambda \in C([0,T],l^{0,2}) \cap L^2([0,T],l^{1,2})$. It is known that weak continuity in a Sobolev space implies strong continuity in its dual space.

Let L_h be a linear functional such that $L_h : l^{1,2} \to (l^{1,2})^*$, with inner product and norm defined as above in this proof. Assume the conditions on Assumption (3.1.4.1) and that $f_h \in L^2([0,T],l^{0,2})$ and $g_h \in l^{0,2}$.

By Definition (3.1.2.1) and Theorem (3.1.2.7), u_λ converges weakly to u in $C([0,T],l^{0,2}) \cap L^2([0,T],l^{1,2})$.

We have to prove that the estimate in this theorem is true to u_λ and independent of λ. Replacing in definition (3.11):

$$
\begin{aligned}
(u_\lambda(t),\varphi) = (g_h,\varphi) + \int_0^t \{ & -(a_{ij}(s)+\lambda)\partial_i^+ u_\lambda(s), \partial_j^+\varphi) \\
& + (b_i(s)\partial_i^+ u_\lambda(s) - \partial_j^+(a_{ij}(s)+\lambda)\partial_i^+ u_\lambda(s),\varphi) \\
& + (c(s)u_\lambda(s),\varphi) + \langle f_h(s),\varphi\rangle\} ds
\end{aligned}
$$

As $g_h \in l^{0,2} \subset W^{m,2} \subset L^2$ then $(g_h,\varphi)_{l^{0,2}} \leq (g,\varphi)_{W^{m,2}} \leq (g,\varphi)_{L^2}$ and $f_h \in L^2([0,T],l^{0,2}) \subset L^2$ then $(f_h,\varphi)_{l^{0,2}} \leq (f,\varphi)_{L^2}$.

Then we have

$$
\begin{aligned}
(u_\lambda(t),\varphi) \leq (g,\varphi)_{L^2} + \int_o^t \{ & -(a_{ij}(s)+\lambda)D_i u(s), D_j\varphi)_{L^2} \\
& + (b_i(s)D_i u(s) - D_j(a_{ij}(s)+\lambda)D_i u(s),\varphi)_{L^2} \\
& + (c(s)u(s)+f,\varphi)_{L^2}\} ds
\end{aligned}
$$

Since $\lambda \to 0$ then

$$
\begin{aligned}
(u_\lambda(t),\varphi) \leq (g,\varphi)_{L^2} + \int_o^t \{ & -a_{ij}(s)D_i u(s), D_j\varphi)_{L^2} \\
& + (b_i(s)D_i u(s) - D_j a_{ij}(s)D_i u(s),\varphi)_{L^2} \\
& + (c(s)u(s)+f,\varphi)_{L^2}\} ds
\end{aligned}
$$

We conclude that the estimate given is valid and in limit it is the solution to the nondegenerate problem.

Recalling Lemma (3.1.2.3), we found a upper bound to the left side of the estimate and, as $\lambda \to 0$, we have that $u_\lambda \to u$, i.e., the upper bound does not depend on λ.
By Theorem (3.1.4.3) the estimate in this theorem is valid for u and that the problem admits a unique generalized solution. □

3.2.3 Approximation results

Obtained the scheme it is necessary to prove that it is consisten. The following is a result to the consistency of the scheme.

Theorem 3.2.3.1. *Let m be an integer such that $m > \frac{d}{2}$. Let $u(t) \in W^{m+2,2}$, $v(t) \in W^{m+3,2}$, for all $t \in [0,T]$. Then there exists a constant N not depending on h such that*

1. $\sum_x |u_{x^i}(t,x) - \partial_i^+ u(t,x)|^2 h^d \leq h^2 N |u(t)|^2_{W^{m+2,2}}$.

2. $\sum_x |v_{x^ix^j}(t,x) - \partial_j^- \partial_i^+ v(t,x)|^2 h^d \le h^2 N |v(t)|^2_{W^{m+3,2}}$.

for all $t \in [0,T]$, $x \in Z_h^d$ *and* $\sum_x$ *is the summation over* Z_h^d

Proof. The proof follows the main steps of the proof in [26] for the corresponding result to the nondegeneracy case.

In order to prove the first inequality. Consider the mean-value theorem:

$$\partial_i^+ u(t,x) = h^{-1}(u(t,x+he_i) - u(t,x)) = u_{x^i}(t, x+\theta h e_i)$$

on the another hand,

$$u_{x^i}(t,x) - \partial_i^+ u(t,x) = u_{x^i}(t,x) - u_{x^i}(t,x+\theta he_i) = h_{x^ix^i}(t, x+\theta' he_i)$$

for some $0 < \theta' < \theta < 1$.

Let us consider the d-cells

$$R_h = (x^1, x^2, ..., x^d) \in \mathbb{R}^d : x_h^i < x^i < x_h^i + h, i = 1,2,...,d,$$

with $x_h = (x_h^1, x_h^2, ..., x_h^d) \in Z_h^d$.

$$\forall x_h \in Z_h^d, |u_{x^ix^i}(t, x_h + \theta' he_i)| \le \sup_{x\in R_h} |u_{x^ix^i}(t,x)|,$$

therefore

$$|u_{x^i}(t,x_h) - \partial^+ u(t,x_h)|^2 \le h^2 \sup_{x\in R_h} |u_{x^ix^i}(t,x)|^2. \tag{3.12}$$

For the particular case of the d-cell where $h = 1$ and $x_1 = (0,...,0)$ we will represent by R_1^0. Thus,

$$\sup_{x\in R_h} |u_{x^ix^i}(t, x_h + hx)|. \tag{3.13}$$

Now, fixing open balls B_h such that $B_h \supset R_h$, with vertices $x_h^i, x_h^i + h, i = 1,2,...,d$ on the boarder of the sphere. Let R_1^0 be contained in the B_1^0. Therefore,

$$\sup_{x\in R_1^0} |u_{x^ix^i}(t,x_h+hx)|^2 \le \sup_{x\in B_1^0} |u_{x^ix^i}(t,x_h+hx)|^2 \tag{3.14}$$

Owing to (1) of Theorem (3.1.2.7) and to Theorem (3.1.2.11), for $m > \frac{d}{2}$ that is:

$$
\begin{aligned}
\sup_{x\in B_1^0} |u_{x^ix^i}(t,x_h+hx)|^2 &\le N \sum_{|\alpha|\le m} \int_{B_1^0} |D_x^\alpha u_{x^ix^i}(t,x_h+hx)|^2 dx \\
&\le N \sum_{|\alpha|\le m+2} \int_{B_1^0} |D_x^\alpha u(t,x_h+hx)|^2 dx \\
&= N \sum_{|\alpha|\le m+2} \int_{B_h} |D_x^\alpha u(t,x)|^2 h^{-d} h^{2|\alpha|} dx
\end{aligned}
$$

$$
\le N \sum_{|\alpha|\le m+2} \int_{B_h} |D_x^\alpha u(t,x)|^2 h^{-d} dx. \tag{3.15}
$$

By (3.12), (3.13), (3.14) and (3.15) we have:

$$
\begin{aligned}
\sum_{x_h\in Z_h^d} |u_{x^i}(t,x_h) - \partial_i^+ u(t,x_h)|^2 h^d &\le Nh^2 \sum_{|\alpha|\le m+2} \sum_{x_h\in Z_h^d} \int_{B_h(x_h)} |D_x^\alpha u(t,x)|^2 dx \\
&\le Nh^2 \sum_{|\alpha|\le m+2} \sum_{x_h\in Z_h^d} \int_{R_h(x_h)} |D_x^\alpha u(t,x)|^2 dx \\
&\le h^2 N |u(t)|^2_{W^{m+2,2}},
\end{aligned}
$$

with $B_h(x_h) = B_h, R_h(x_h) = R_h$, and we just proved the first inequality. For the second inequality the process is similar. □

The following steps are in order to state the rate of convergence, attending to [26].

Theorem 3.2.3.2. *Let u be the solution of problem (3.6) in Theorem (3.1.4.3) and u_h the solution of (3.11) in Theorem (3.2.2.3). Consider m an integer such that $m > \frac{d}{2}$ and $u \in L^2([0,T]; W^{m+3,2})$. Then, for some constant N not depending on h,*

$$
\begin{aligned}
&sup_{t\in[0,T]} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}} dt \\
&\le h^2 N \int_0^T |u(t)|^2_{W^{m+3,2}} dt + N(|g-g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} dt).
\end{aligned}
$$

Proof. From (3.6) and (3.11), we have that $u - u_h$ satisfies the problem

$$
\begin{cases}
(u-u_h)_t = L_h(u-u_h) + (L-L_h)u + (f-f_h) & \text{in } Q(h) \\
(u-u_h)(0,x) = (g-g_h)(x) & \text{in } Z_h^d.
\end{cases}
$$

This result is already proved for the nondegenerate case and is valid independently of the degeneracy.
Under the conditions of theorem, there are modifications in x such that the data $f(t)$ and g are continuous in x, for every $t \in [0, T]$, we have that $f - f_h \in L^2([0, T]; l^{0,2})$ and $g - g_h \in l^{0,2}$.

With respect to the term $(L - L_h)u$, note that if $u(t) \in W^{m+3,2}$, for all $t \in [0, T]$,

$$\sum_{x \in Z_h^d} |(L - L_h)(t)u(t)|^2 h^d$$
$$= \sum_{x \in Z_h^d} \left| a_{ij}(t,x) \left(\frac{\partial^2}{\partial x_i \partial x_j} - \partial_j^- \partial_i^+ \right) u(t,x) + b_i(t,x) \left(\frac{\partial}{\partial x_i} - \partial_i^+ \right) u(t,x) \right|^2 h^d$$
$$\leq h^2 N \|u(t)\|^2_{W^{m+3,2}} < \infty,$$

Thus
$(L - L_h)(t)u(t) \in l^{0,2}$, for every $t \in [0, T]$. Moreover, $u \in L^2([0, T]; W^{m+3,2})$, we obtain immediately $(L - L_h)u \in L^2([0, T]; l^{0,2})$.

Holding the estimate, owing to Theorem (3.1.4.3)

$$\sup_{0 \leq t \leq T} \|u(t) - u_h(t)\|^2_{l^{0,2}} + \int_0^T \|u(t) - u_h(t)\|^2_{l^{1,2}} dt$$
$$\leq N \bigg(\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}} dt + \int_0^T \|(L - L_h)(t)u(t)\|^2_{l^{0,2}} \, dt \bigg).$$

Owing again to (2) in Assumption (3.1.4.1) and to Theorem (3.2.3.1), the result follows. □

Next corollary state the previous rate of convergence with a well structured statement.

Corollary 3.2.3.3. *Let u be the solution of problem (3.6) in Theorem (3.1.4.3) and u_h the solution of (3.11) in Theorem (3.2.2.3). Consider m an integer such that $m > \frac{d}{2}$ and $u \in L^2([0, T]; W^{m+3,2})$.*
If there is a constant N not depending on h,such that

$$|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} dt) \leq h^2 N (|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt),$$

then

$$sup_{t \in [0,T]} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}} dt$$
$$\leq h^2 N (\int_0^T |u(t)|^2_{W^{m+3,2}} dt + |g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt).$$

Proof. The result is a imediatly consequence of Theorem (3.2.3.2). □

3.2.4 The special one-dimensional case

Following the previous results and [26, 27, 28, 45] we now apply the same approach to the special one dimension in space, in degenerate case, to bounded coefficients.

Consider the Cauchy Problem in $\mathbb{R}$.

$$\begin{aligned} Lu - \frac{\partial u}{\partial t} + f = 0 \quad \text{in} \quad Q \\ u(0,x) = g(x) \quad \text{in} \quad \mathbb{R} \end{aligned} \tag{3.16}$$

where $Q = [0,T]\times\mathbb{R}$, T is a positive constant and L is the second-order partial differential operator with bounded coefficients in $\mathbb{R}$:

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x),$$

t with values in $[0,T]$ and f, g real valued functions.

The PDE theory to this special problem is a particularization of the theory presented above to the d-dimensional problem. Therefore we state the most important results for the one dimensional case.

Assumption 3.2.4.1. *Let the integer m be nonnegative. There exist constants k and $\lambda \geq 0$ such that:*

1. $a(t,x) \geq \lambda$, $\forall t \geq 0, \forall x \in \mathbb{R}$
2. $|D_x^\alpha a| \leq k, \quad \forall |\alpha| \leq m \vee 1$

   $|D_x^\alpha b| \leq k$, $|D_x^\alpha c| \leq k, \quad \forall |\alpha| \leq m$
3. $f \in L^2([0,T]; W^{m-1,2}), \quad g \in W^{m,2}$

where D_x^α is the α^{th}-partial derivative operator with respect to x.

With the previous assumptions it is possible to establish the definition of generalized and classical solutions of (3.16) in $\mathbb{R}$.

Definition 3.2.4.2. Under the conditions in Assumption (3.2.4.1), we say that $u \in L^2([0,T]; W^{1,2})$ is a generalized solution of (3.16) if:

1. $u \in L^2([0,T]; W^{1,2})$

2. $\forall t \in [0,T]$,

$$(u(t), \varphi) = (g, \varphi) + \int_0^t \{-a(s)D_x u(s), D_x\varphi) \\ + (b(s)D_x u(s) - D_x a(s)D_x u(s), \varphi) + ((c(s)u(s), \varphi) + \langle f(s), \varphi\rangle\}ds$$

for all $\varphi \in C_0^\infty(\mathbb{R})$.

Finally, the following results states the existence and uniqueness of solution of (3.16).

Theorem 3.2.4.3. *Under the conditions in Assumption (3.2.4.1) exists a generalized solution $(u(t))_{t\in[0,T]}$ of the problem (3.16).*
Moreover

$$u \in C([0,T]; W^{m,2}) \cap L^2([0,T]; W^{m+1,2})$$

and

$$\sup_{t\in[0,T]} |u(t)|^2_{W^{m,2}} + \int_0^T |u(t)|^2_{W^{m+1,2}} dt \\ \leq N(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt$$

for N constant.

Discrete framework

We now particularize the framework presented above for the d-dimensional case.

Consider the h-grid, on $\mathbb{R}$, with $h \in (0,1]$:

$$Z_h = \{x \in \mathbb{R} : x = nh, \quad n = 0, \pm 1, \pm 2, ...\}$$

and consider the difference quotients in space, for all $x \in Z_h$:

- Forward: $\partial^+ u = \partial^+ u(t,x) = \dfrac{u(t,x+h) - u(t,x)}{h}$;
- Backward: $\partial^- u = \partial^- u(t,x) = \dfrac{u(t,x) - u(t,x-h)}{h}$.

Let L_h be the discrete operator, such that:

$$L_h(t,x) = a(t,x)\partial^-\partial^+ + b(t,x)\partial^+ + c(t,x).$$

So, the discrete version of the second order parabolic Cauchy problem, can be written as:

$$\begin{aligned} &L_h u - u_t + f_h = 0 \quad \text{in} \quad \text{Q(h)} = [0,T] \times Z_h \\ &u(0,x) = g_h(x) \quad \text{in} \quad Z_h \end{aligned}$$

with $T \in (0,\infty)$ and f_h and g_h functions such that

$$f_h : \mathbb{Q}(h) \to \mathbb{R} \quad \text{and} \quad g_h : Z_h \to \mathbb{R}.$$

The particular discrete Sobolev space for the one dimensional case are

$l^{0,2} = \{v : Z_h \to \mathbb{R} : |v|_{l^{0,2}} < \infty\}$
with the inner product $(v,\omega)_{l^{0,2}} = \sum_{x\in Z_h} v(x)\omega(x)h$
and norm $|v|_{l^{0,2}} = (v,v)^{1/2}_{l^{0,2}} = (\sum_{x\in Z_h} |v(x)|^2 h)^{1/2}$.

$l^{1,2} = \{v : Z_h \to \mathbb{R} : |v|_{l^{1,2}} < \infty\}$
with the inner product $(v,\omega)_{l^{1,2}} = (v,\omega)_{l^{0,2}} + (\partial^+ v, \partial^+\omega)$
and norm $|v|_{l^{1,2}} = |v|^2_{l^{0,2}} + |\partial^+ v|^2_{l^{0,2}}$, with $v,\omega \in l^{1,2}$..

Assumption 3.2.4.4. *Assume that:*

1. $f_h \in L^2([0,T]; l^{0,2})$
2. $g_h \in l^{0,2}$.

As we are proving the existence of weak solution of (3.16), consider next the definition of generalized solution.

Definition 3.2.4.5. Consider $u \in C([0,T]; l^{0,2}) \cap L^2([0,T]; l^{1,2})$ and $\varphi \in l^{1,2}$. Under the conditions in Assumption (3.2.4.4), we say that u is a generalized solution of problem (3.16), if, for all $t \in [0,T]$:

$$\begin{aligned} (u(t),\varphi) = (g_h,\varphi) + \int_0^t &\{-(a(s)\partial^+ u(s), \partial^+\varphi) \\ &+ (b(s)\partial^+ u(s) - \partial^+ a(s)\partial^+ u(s), \varphi) + (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle\}ds \end{aligned}$$

where $(\cdot,\cdot)$ is the inner product in $l^{0,2}$.

Now, state the conditions of existence and uniqueness of solution to the problem in study. The proof is a consequence of the Theorem (3.2.2.3) for the d-dimensional case.

Theorem 3.2.4.6. *Assume the conditions on Assumptions (3.2.4.1) and (3.2.4.4). Then the problem (3.16) has a unique generalized solution u in $[0,T]$. Moreover,*

$$\sup_{t\in[0,T]} |u(t)|^2_{l^{0,2}} + \int_0^T |u(t)|^2_{l^{1,2}} dt \\ \leq N(|g_h|^2_{l^{0,2}} + \int_0^T |f_h(t)|^2_{l^{0,2}} dt)$$

with N a constant independent of h.

Approximation results

In what concerns consistency we can prove results sharper then the corresponding one for the d-dimensional cases.

Proposition 3.2.4.7. *Consider $u(t) \in W^{2,2}$, $v(t) \in W^{3,2}$ for all $t \in [0,T]$. There exists a constant N, independent of h, such that:*

1. $\sum_{x\in Z_h} \left|\frac{\partial}{\partial x}u(t,x) - \partial^+ u(t,x)\right|^2 h \leq h^2 |u(t)|^2_{w^{2,2}}$
2. $\sum_{x\in Z_h} \left|\frac{\partial^2}{\partial x^2}v(t,x) - \partial^-\partial^+ v(t,x)\right|^2 h \leq h^2 N |v(t)|^2_{W^{3,2}}$

for all $t \in [0,T], x \in Z_h$.

Proof. This proof follows the guidelines of [45] for the particular case. We will prove (1).

The forward difference quotient can be written

$$\partial^+ u(t,x) = h^{-1}(u(t,x+h) - u(t,x)) = \int_0^1 \frac{\partial}{\partial x} u(t,x+hq)dq.$$

Thus

$$\begin{aligned} \Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x) &= \int_0^1 \Big(\frac{\partial}{\partial x}u(t,x) - \frac{\partial}{\partial x}u(t,x+hq)\Big)dq \\ &= h\int_0^1 \int_0^1 q\frac{\partial^2}{\partial x^2}u(t,x+hqs)dsdq. \end{aligned} \tag{3.17}$$

From (3.17), using Jensen's inequality, we obtain

$$
\begin{aligned}
\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2 &\le h^2\int_0^1\int_0^1 q^2\Big|\frac{\partial^2}{\partial x^2}u(t,x+hqs)\Big|^2 dsdq\\
&= h\int_0^1\int_0^{hq} q\Big|\frac{\partial^2}{\partial x^2}u(t,x+v)\Big|^2 dvdq\\
&\le h\int_0^1 qdq\int_0^h \Big|\frac{\partial^2}{\partial x^2}u(t,x+v)\Big|^2 dv \qquad (3.18)\\
&= \frac{h}{2}\int_0^h \Big|\frac{\partial^2}{\partial x^2}u(t,x+v)\Big|^2 dv\\
&= \frac{h}{2}\int_x^{x+h} \Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2 dz.
\end{aligned}
$$

Observe also that from (3.18) and (**??**), by the mean value theorem for integration, using Hölder inequality and Assumption (3.2.4.1) we have, for any $\theta\in(0,1)$,

$$
\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2 \le hN\int_x^{x+h}\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2 dz. \qquad (3.19)
$$

Finally, summing up (3.19) over Z_h, we get

$$
\sum_{x\in Z_h}\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2 h \le h^2N\|u(t)\|^2_{W^{2,2}},
$$

with N a constant independent of h, and (1) is proved.

We now prove (2). By writing the forward and backward difference quotients

$$
\partial^+v(t,x)=h^{-1}(v(t,x+h)-v(t,x))=\int_0^1\frac{\partial}{\partial x}v(t,x+hq)dq
$$

and

$$
\partial^-v(t,x)=h^{-1}(v(t,x)-v(t,x-h))=\int_0^1\frac{\partial}{\partial x}v(t,x-hs)ds,
$$

respectively, we have for the second-order difference quotient

$$
\begin{aligned}
\partial^-\partial^+v(t,x)=\partial^-\int_0^1\frac{\partial}{\partial x}v(t,x+hq)dq &= \int_0^1\Big(\frac{\partial}{\partial x}\int_0^1\frac{\partial}{\partial x}v(t,x+hq-hs)dq\Big)ds\\
&=\int_0^1\int_0^1\frac{\partial^2}{\partial x^2}v(t,x+h(q-s))dsdq.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x) &= \int_0^1\int_0^1\Big(\frac{\partial^2}{\partial x^2}(t,x)-\frac{\partial^2}{\partial x^2}v(t,x+h(q-s))\Big)dsdq\\
&= h\int_0^1\int_0^1\int_0^1(q-s)\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))dvdsdq.
\end{aligned}
\qquad (3.20)
$$

From (3.20), by Jensen's inequality,

$$\begin{aligned}
\Big|\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\Big|^2 &\le h^2\int_0^1\int_0^1\int_0^1 |q-s|^2\Big|\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))\Big|^2 dvdsdq\\
&= h\int_0^1\int_0^1\int_0^{h(q-s)} (q-s)\Big|\frac{\partial^3}{\partial x^3}v(t,x+w)\Big|^2 dwdsdq\\
&\le h\int_0^1\int_0^1 |q-s|dsdq\int_0^h\Big|\frac{\partial^3}{\partial x^3}v(t,x+w)\Big|^2 dw\\
&\le h\int_0^h\Big|\frac{\partial^3}{\partial x^3}v(t,x+w)\Big|^2 dw = h\int_x^{x+h}\Big|\frac{\partial^3}{\partial z^3}v(t,z)\Big|^2 dz,
\end{aligned}$$

and, following the same steps as in the proof of (1), we finally obtain

$$\sum_{x\in Z_h}\Big|\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\Big|^2 h \le h^2N\|v(t)\|^2_{W^{3,2}},$$

with N a constant independent of h, and (2) is proved. □

Based on results of [26], where the result is proved for the one dimensional case with bounded coefficients but to the nondegenerate case, it is possible to define the rate of convergence of problem (3.16).

Theorem 3.2.4.8. *Let u be the solution of the problem (3.16) and u_h be the solution of the same problem discretized (3.11), with $d = 1$. Assume $u \in L^2([0,T]; W^{3,2})$. Then*

$$\begin{aligned}
\sup_{t\in[0,T]}|u(t)-u_h(t)|^2_{l^{0,2}}+&\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}}dt\\
&\le h^2N\int_0^T|u(t)|^2_{W^{3,2}}dt+N(|g-g_h|^2_{l^{0,2}}\\
&+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}}dt)
\end{aligned}$$

for a constant N not depending on h.

Proof. Consider u and u_h as in the conditions of the theorem. We have:

$L_h(u-u_h)-\frac{d}{dt}(u-u_h)+(L-L_h)u+(f-f_h)=0$ in $[0,T]\times\mathbb{Z}_h$

$(u-u_h)(0,x)=(g-g_h)(x)$ in $\mathbb{Z}_h$

We know that $(f-f_h)\in L^2([0,T],l^{0,2}),(g-g_h)\in l^{0,2}$ and $(L-L_h)u\in L^2([0,T],l^{0,2})$ since $u\in W^{3,2}$.

Owing to Theorem (3.2.4.6) and to definition of the operators:

$$\begin{aligned}(L-L_h)u(t) =&(a(t,x)+\lambda)(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+)u(t,x)\\&+b(t,x)(\frac{\partial}{\partial x}-\partial^+)u(t,x)\end{aligned}$$

As $\lambda \to 0$ we can have

$$\begin{aligned}(L-L_h)u(t) =&a(t,x)(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+)u(t,x)\\&+b(t,x)(\frac{\partial}{\partial x}-\partial^+)u(t,x)\end{aligned}$$

By definition,

$$\begin{aligned}\partial^-u(t,x) &= h^{-1}(u(t,x)-u(t,x-h))\\&=\int_0^1 \frac{\partial}{\partial x}u(t,x-hs)ds\\ \partial^+u(t,x) &= h^{-1}(u(t,x+h)-u(t,x))\\&=\int_0^1 \frac{\partial}{\partial x}u(t,x+hq)dq\end{aligned}$$

Then,

$$\begin{aligned}\partial^-\partial^+u(t,x) &= \partial^-\int_0^1 \frac{\partial}{\partial u}u(t,x+hq)dq\\&=\int_0^1 \frac{\partial}{\partial u}\int_0^1 (\frac{\partial}{\partial x}u(t,x+hq-hs)dq)ds\\&=\int_0^1\int_0^1 \frac{\partial^2}{\partial x^2}u(t,x+h(q-s))dsdq.\end{aligned}$$

And

$$\begin{aligned}(\frac{\partial}{\partial x}-\partial^+)u(t,x) &= \frac{\partial}{\partial x}u(t,x)-\partial^+u(t,x)\\&=\frac{\partial}{\partial x}u(t,x)-\int_0^1\frac{\partial}{\partial x}u(t,x+hq)dq\\&=\int_0^1\frac{\partial}{\partial x}u(t,x)-\frac{\partial}{\partial x}u(t,x+hq)dq\\&=h\int_0^1\int_0^1 q\frac{\partial^2}{\partial x^2}u(t,x+hqs)dsdq\end{aligned}$$

Applying the Jensen's inequality and making $v = hqs$,

$$
\begin{aligned}
|(\frac{\partial}{\partial x} - \partial^+)u(t,x)|^2 &= |h\int_0^1\int_0^1 q\frac{\partial^2}{\partial x^2}u(t,x+hqs)dsdq|^2 \\
&\leq h\int_0^1\int_0^1 q^2|\frac{\partial^2}{\partial x^2}u(t,x+hqs)|^2dsdq \\
&\leq h\int_0^1\int_0^{hq} q|\frac{\partial^2}{\partial x^2}u(t,x+v)|^2dvdq \\
&\leq \int_0^1 qdq\int_0^h q|\frac{\partial^2}{\partial x^2}u(t,x+v)|^2dvdq \\
&\leq \frac{h}{2}\int_0^h |\frac{\partial^2}{\partial x^2}u(t,x+v)|^2dv \\
&\leq \frac{h}{2}\int_x^{x+h} |\frac{\partial^2}{\partial z^2}u(t,z)|^2dz
\end{aligned}
$$

Then, by Proposition (3.2.3.1), we have

$$
\sum_{x\in\mathbb{Z}_h} |(\frac{\partial}{\partial x} - \partial^+)u(t,x)|^2h \leq h^2N|u(t)|^2_{W^{2,2}}.
$$

$$
\begin{aligned}
(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+)u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) - \partial^-\partial^+u(t,x) \\
&= \frac{\partial^2}{\partial x^2}u(t,x) - \int_0^1\int_0^1 \frac{\partial^2}{\partial x^2}u(t,x+h(q-s))dsdq \\
&= \int_0^1\int_0^1 \frac{\partial^2}{\partial x^2}u(t,x)\frac{\partial^2}{\partial x^2}u(t,x+h(q-s))dsdq \\
&= h\int_0^1\int_0^1\int_0^1 (q-s)\frac{\partial^3}{\partial x^3}u(t,x+hv(q-s))dvdsdq
\end{aligned}
$$

Once again, owing to Jensen's inequality and making $w = hv(q-s)$, we have:

$$
\begin{aligned}
|\left(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\right)u(t,x)|^2 &= |h\int_0^1\int_0^1\int_0^1 (q-s)\frac{\partial^3}{\partial x^3}u(t,x+hv(q-s)dvdsdq|^2 \\
&\leq h^2\int_0^1\int_0^1\int_0^1 |q-s|^2|\frac{\partial^3}{\partial x^3}u(t,x+hv(q-s))|^2dvdsdq \\
&\leq h^2\int_0^1\int_0^1\int_0^{h(q-s)} \frac{q-s}{h}||\frac{\partial^3}{\partial x^3}u(t,x+w)|^2dwdsdq \\
&\leq h^2\int_0^1\int_0^1 |q-s|dsdq\int_0^h |\frac{\partial^3}{\partial x^3}u(t,x+w)|^2dw \\
&\leq h\int_x^h |\frac{\partial^3}{\partial x^3}u(t,x+w)|^2dw \\
&\leq h\int_x^{x+h} |\frac{\partial^3}{\partial z^3}u(t,z)|^2dz.
\end{aligned}
$$

Then we have

$$\sum_{x\in\mathbb{Z}_h} |(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+)u(t,x)|^2 h \le h^2 N|u(t)|^2_{W^{3,2}},$$

with N independent of h.

Owing to Theorem (3.2.4.6) , we have the result:

$$\begin{aligned}
\sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}} dt \\
&\le N|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} dt + \int_0^T |(L - L_h)u(t)|^2 dt \\
&\le N|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} dt + \int_0^T h^2 N|u(t)|^2_{W^{3,2}} dt \\
&\le \int_0^T h^2 N|u(t)|^2_{W^{3,2}} dt + N|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} dt
\end{aligned}$$

□

Now the following is a consequence of the previous theorem.

Corollary 3.2.4.9. *Let u be the solution of the problem (3.16) and u_h be the solution of the same problem discretized (3.11), with $d = 1$. Assume $u \in L^2([0,T]; W^{3,2})$ and m a positive integer. If exists a constant N not depending on h, such that,*

$$|g - g_h|^2_{l^{0,2}} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}} dt \le h^2 N(|g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt)$$

then

$$\begin{aligned}
\sup_{t\in[0,T]} |u(t) - u_h(t)|^2_{l^{0,2}} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}} dt \\
&\le h^2 N \left(\int_0^T |u(t)|^2_{W^{3,2}} dt + |g|^2_{W^{m,2}} + \int_0^T |f(t)|^2_{W^{m-1,2}} dt \right).
\end{aligned}$$

Proof. This result is an immediate consequence of the previous theorem. □

Chapter 4

Approximation of PDEs with unbounded coefficients

The results obtained in the previous chapter are now adapted and presented to the corresponding unbounded coefficients case. We begin by state some classical results on PDEs with unbounded coefficients and then we present the results to the nondegenerate case.

4.1 Classical results for parabolic PDEs

Suppose now that the coefficients of operator L are unbounded.

Let r and ρ be real positive smooth functions. Then r and ρ are called *weights* on G. Consider $C_0^\infty(G)$ the space of infinitely differentiable functions with compact supports in G.

We state some results on the solvability in weighted Sobolev spaces.

4.1.1 The weighted Sobolev spaces

Now we introduce the concept of weighted Sobolev spaces as in [31, 49, 50, 51, 52], space where we will study our framework for the unbounded coefficients.

Definition 4.1.1.1. [Weighted Sobolev spaces] Consider r and ρ positive smooth functions on $\mathbb{R}^d$ and an integer $m \geq 0$. We call $W^{m,2}(r,\rho)$ the weighted Sobolev space on $\mathbb{R}^d$ to the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm:

$$|\varphi|_{W^{m,2}(r,\rho)} = \left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^d} r^2 |\rho^{|\alpha|} D^\alpha \varphi|^2 dx \right)^{1/2}$$

with $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Remark 18. The inner product in $W^{m,2}(r,\rho)$ is defined by

$$(v,\omega)_{W^{m,2}(r,\rho)} = \sum_{|\alpha|\leq m} \int_{\mathbb{R}^n} r^2 \rho^{2|\alpha|} D^\alpha v D^\alpha \omega dx$$

for $v, \omega \in W^{m,2}(r,\rho)$.

Owing to results in [31] we state:

Proposition 4.1.1.2. *$W^{m,p}(r,\rho)$ with the norm above are separable Banach spaces. Moreover, if $p > 1$ they are reflexive and if $p = 2$ they are Hilbert spaces.*

Assumption 4.1.1.3. *Let $m \geq 0$ be an integer, and $r > 0$, $\rho > 0$ smooth functions on $\mathbb{R}^d$. There exists a constant K such that*

1. *$|D^\alpha \rho| \leq K\rho^{1-|\alpha|}$ for all α such that $|\alpha| \leq m-1$ if $m \geq 2$;*
2. *$|D^\alpha r| \leq K\dfrac{r}{\rho^{|\alpha|}}$ for all α such that $|\alpha| \leq m$.*

Example 4.1.1.4. The following functions (taken from [31, 26]), satisfy Assumption (4.1.1.3):

1. $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\quad \rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
2. $r(x) = \exp(\pm(1+|x|^2)^\beta)$, $0 \leq \beta \leq \frac{1}{2}$; $\quad \rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2} - \beta$;
3. $r(x) = (1+|x|^2)^\beta$, $\beta \in \mathbb{R}$; $\quad \rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \in \mathbb{R}$;
4. $r(x) = (1+|x|^2)^\beta \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\quad \rho(x) = (1+|x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
5. $r(x) = (1+|x|^2)^\beta \ \ln^\mu(2+|x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = \ln^\gamma(2+|x|^2)$, $\gamma \geq 0$;
6. $\rho(x) = \exp(-(1+|x|^2)^\gamma)$, $\gamma \geq 0$; each weight function $r(x)$ in examples $(1)-(5)$.

4.1.2 A nondegenerate PDE problem

We now consider the problem of the previous chapter but applied to the case where the operator L in nondegenerate in spatial variables and its coefficients are unbounded.

Consider V a reflexive separable Banach space embedded continuously and densely into a Hilbert space H. Consider also the normal triple with continuous and dense embeddings $V \hookrightarrow H \equiv H^* \hookrightarrow V^*$, where H^* is the dual of H.

We have the Cauchy problem

$$L(t)u - \frac{du}{dt} + f(t) = 0 \quad \text{in} \quad [0,T], \quad u(0) = g$$

and $T \in (0, \infty)$, $f \in L^2([0,T]; V^*)$, $g \in H$ and $L(t), \dfrac{d}{dt}$ linear operators from V to V^* for all $t \geq 0$.

Consider the second-order parabolic partial differential equation problem, with second order operator L, such that:

$$L(t,x) = a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(t,x)\frac{\partial}{\partial x_i} + c(t,x)$$

with a_{ij}, b_i, c are real valued functions on $[0,T] \times \mathbb{R}^d$.

Consider now the Cauchy Problem:

$$\begin{array}{l} Lu - u_t + f = 0 \text{ in } \mathbb{Q} \\ u(0,x) = g(x) \text{ in } \mathbb{R}^d \end{array} \tag{4.1}$$

with $T \in (0, \infty); \mathbb{Q} = [0,T] \times \mathbb{R}^d$ and f and g functions.

Considering the operator L under a coercivity condition and some assumptions on the behaviour of the weights r and ρ, on the operators coeficients and on the free data f and g as in [27].

Assumption 4.1.2.1. *Let r and ρ be a positive smooth functions on $\mathbb{R}^d$ and an integer $m \geq 0$. There are constants $\lambda > 0, K$ such that*

1. $\sum_{i,j=1}^d a_{ij}(t,x)\xi_i\xi_j \geq \lambda\rho^2 \sum_{i=1}^d |\xi_i|^2$, *for all* $t \geq 0, x \in \mathbb{R}^d, \xi \in \mathbb{R}^d$;
2. $|D_x^\alpha a_{ij}| \leq K\rho^{2-|\alpha|}$ *for all* $|\alpha| \leq m \vee 1$, $\quad |D_x^\alpha b_i| \leq K\rho^{1-|\alpha|}$, $\quad |D_x^\alpha c| \leq K$ *for all* $|\alpha| \leq m$, *where* $|D_x^\alpha|$ *is the* α^{th}*-partial derivative operator with respect to* x;
3. $f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ *and* $g \in W^{m,2}(r,\rho)$.

Defining next the generalized solution of the problem, solution which we will state its existence and uniqueness in the conditions defined.

Definition 4.1.2.2. We say that $u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalized solution of problem (4.1) on $[0,T]$ if

1. $u \in L^2([0,T]; W^{1,2}(r,\rho))$;
2. For every $t \in [0,T]$,

$$\begin{aligned}(u(t),\varphi) =&(g,\varphi) + \int_0^t \big\{ -(a_{ij}(s)D_i u(s), Dj\varphi) \\ &+ (b(s)D_i u(s) - D_j a_{ij}(s) D_i u(s), \varphi) \\ &+ (c(s)u(s),\varphi) + \langle f(s),\varphi\rangle \big\} ds\end{aligned}$$

holds for all $\varphi \in C_0^\infty$.

Remark 19. The notation $(\cdot,\cdot)$ in the above definition stands for the inner product in $W^{0,2}(r,\rho)$. Alternatively to the infinite differentiability of φ in (2), it can be required that $\varphi \in W^{1,2}(r,\rho)$.

Definition 4.1.2.3. $u(t,x) \in [0,T] \times \mathbb{R}^d$ is called a classical solution of (4.1) if:

1. $u(t,x) \in C^{0,2}([0,T] \times \mathbb{R}^d)$
2. For all $x \in \mathbb{R}^d$, $\forall t \in [0,T]$

$$\begin{aligned}u(t,x) = g(x) + \int_0^t &\big\{ \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j} u(s,x) + a_{i0}u(s,x) \\ &+ f_i(s,x) + (b_i \frac{\partial}{\partial x_j} u(s,x) + cu(s,x)\big\} ds\end{aligned}$$

Owing to [27, 31] we have the result that states the existence and uniqueness of solution to (4.1).

Theorem 4.1.2.4 (Existence and uniqueness of generalized solution)**.** *Under (1)–(2) in Assumption (4.1.1.3), with $m+1$ in place of m, with $m \geq 0$ an integer, and (1)–(3) in Assumption (4.1.2.1), problem (4.1) admits a unique generalized solution u on $[0,T]$. Moreover*

$$u \in C([0,T]; W^{m,2}(r,\rho)) \cap L^2([0,T]; W^{m+1,2}(r,\rho))$$

and

$$\sup_{0\le t\le T}\|u(t)\|^2_{W^{m,2}(r,\rho)}+\int_0^T\|u(t)\|^2_{W^{m+1,2}(r,\rho)}dt$$
$$\le N\Big(\|g\|^2_{W^{m,2}(r,\rho)}+\int_0^T\|f(t)\|^2_{W^{m-1,2}(r,\rho)}dt\Big),$$

with N *a constant.*

Under the conditions of (4.1.1.3) and (3.1.3.1) and considering $m>\frac{d}{2}+n, n\ge 0$ there exists a unique generalized solution of (4.1) which has a modification in x that states the existence of classical solution of (4.1), as proved in [31].

4.1.3 A degenerate PDE problem

The approach to the degenerate problem is the main issue in this thesis. Consider the problem defined above with the operator L degenerate and the unbounded coefficients.

Consider V a reflexive separable Banach space embedded continuously and densely into a Hilbert space H. Consider also the normal triple with continuous and dense embeddings $V\hookrightarrow H\equiv H^*\hookrightarrow V^*$, where H^* is the dual of H.

We have the Cauchy problem

$$L(t)u-\frac{du}{dt}+f(t)=0\quad\text{in}\quad[0,T],\quad u(0)=g$$

and $T\in(0,\infty), f\in L^2([0,T];V^*), g\in H$ and $L(t),\dfrac{d}{dt}$ linear operators from V to V^* for all $t\ge 0$.

Consider the second-order parabolic partial differential equation problem, with second order operator L, such as:

$$L(t,x)=a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+b_i(t,x)\frac{\partial}{\partial x_i}+c(t,x)$$

with a_{ij},b_i,c are real valued functions on $[0,T]\times\mathbb{R}^d$.

Consider now the Cauchy Problem:

$$\begin{aligned}&Lu-u_t+f=0 \text{ in } \mathbb{Q}\\&u(0,x)=g(x) \text{ in } \mathbb{R}^d\end{aligned}\tag{4.2}$$

with $T\in(0,\infty);\mathbb{Q}=[0,T]\times\mathbb{R}^d$ and f, g functions.

We have to state the same results, based on [31] that we established above but to degenerate case.

In order to obtain a unique solution to problem (4.2) the coefficients must satisfy some regularity conditions, adapted from [31].

Assumption 4.1.3.1. *Let r, ρ be positive smooth functions on $\mathbb{R}^d, m \geq 0$ and k constant. Consider $l = 1, 2, ..., d$.*

1. *Exists a matrix valued function $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d_2}$ such that*

   $\sigma^{in}\sigma^{jn} = a_{ij}$

   $|\sigma_j^{in}(t, x)| \leq K\dfrac{\rho_i(x)}{\rho_j(x)}$ *for all* $(t, x) \in [0, T] \times \mathbb{R}^d$, $i, j = 1, 2, ..., d$ *and* $n = 1, 2, ..., d$.

2. *For $i, j = 1, 2, ..., d$ and $\xi \in \mathbb{R}^d$:*

   $|D^\alpha a_{ij}| \leq \dfrac{\xi \rho_i \rho_j}{\rho^\alpha}$ *for all* $|\alpha| \leq m + 1 \vee 2$

   $|D^\alpha b_i| \leq \dfrac{\xi \rho_i}{\rho^\alpha}$ *and* $|D^\alpha c| \leq \dfrac{\xi}{\rho^\alpha}$

3. $f \in L^2([0, T], W^{m-1,2}(r, \rho))$ *and* $g \in W^{m,2}(r, \rho)$

Definition 4.1.3.2. $(u(t))_{t \in [0,T]}$ is called a generalized solution of problem (4.2) if:

1. $u \in L^2([0, T]; W^{m,2}(r, \rho))$;
2. For every $t \in [0, T]$,

$$
\begin{aligned}
(u(t), \varphi) = (g, \varphi) + \int_0^t \big\{ &- (a_{ij}(s) D_i u(s), D_j \varphi) \\
&+ (b_i(s) D_i u(s) - D_j a_{ij}(s) D_i u(s), \varphi) \\
&+ (c(s) u(s), \varphi) + \langle f(s), \varphi \rangle \big\} ds
\end{aligned}
$$

   holds for all $\varphi \in C_0^\infty$.

Definition 4.1.3.3. We say that $u(t, x) \in [0, T] \times \mathbb{R}^d$ is a classical solution of problem (4.2) if:

1. $u \in C^{0,2}([0, T] \times \mathbb{R}^d)$;
2. For every $t \in [0, T]$,

$$
\begin{aligned}
u(t) = g + \int_0^t &(a_{ij}(s) D_{x_i x_j} u(s, x) \\
&+ b_i(s) D_{x_i} u(s, x) + c(s) u(s, x) + f(s, x)) ds
\end{aligned}
$$

The existence and uniqueness of solution to problem (4.2) is set in the next theorem.

Theorem 4.1.3.4. *Let $m \geq 1$ and assume the conditions in Assumption (4.1.3.1). Then, there is a generalized solution $(u(t))_{t\in[0,T]}$ of the problem (4.2). Moreover, $u \in L^2([0,T], W^{m,2}(r,\rho)) \cap C([0,T], W^{m-1,2}(r,\rho))$ and*

$$\sup_{0\leq t\leq T} \|u(t)\|^2_{W^{m',2}(r,\rho)} + \int_0^T \|u(t)\|^2_{W^{m',2}(r,\rho)} dt \leq N\Big(\|g\|^2_{W^{m',2}(r,\rho)} + \int_0^T \|f(t)\|^2_{W^{m',2}(r,\rho)} dt\Big),$$

with N a constant and $m' \in [0,m]$.

Proof. The proof can be seen in [31], with the adaptation to the present problem. □

4.2 Finite-difference approximation

At this point, we will define our discrete framework for the degenerate case with unbounded coefficients. First we need to set the spaces where this approach is developed: weighted discrete Sobolev spaces.

4.2.1 The weighted discrete Sobolev spaces

Consider the next function space in our framework.

$$l^{0,2}(r) = \{v : Z_h^d \to \mathbb{R} : |v|_{l^{0,2}(r)} < \infty\},$$

with the norm

$$|v|_{l^{0,2}(r)} = \left(\sum_{x\in Z_h^d} r^2|v(x)|^2h^d\right)^{1/2},$$

and with the inner product

$$(v,\omega)_{l^{0,2}(r)} = \sum_{x\in Z_h^d} r^2 v(x)\omega(x)h^d$$

for all $v, \omega \in l^{0,2}(r)$.

Define, also, another function space:

$$l^{1,2}(r,\rho) = \{v : Z_h^d \to \mathbb{R} : |v|_{l^{1,2}(r,\rho)} < \infty\},$$

with the norm

$$|v|_{l^{1,2}(r,\rho)} = |v|^2_{l^{0,2}(r)} + \sum_{i=1}^{d} |\rho\partial_i^+ v|^2_{l^{0,2}(r)}$$

and with the inner product

$$(v,\omega)_{l^{1,2}(r,\rho)} = (v,\omega)_{l^{0,2}}(r) + \sum_{i=1}^{d} (\partial_i^+ v, \partial_i^+ \omega)_{l^{0,2}(r)},$$

where v, ω are functions in $l^{1,2}(r,\rho)$.

Consider the functions v of $[0,T]$ in $\mathbb{R}^d$ such that, for all $t \in [0,T]$, $v : Q(h) \to \mathbb{R}$ and $\omega(t) = \{\omega(t,x) : x \in Z_h^d\}$. Define also the subspaces:

- $C([0,T]; l^{0,2}(r))$
- $L^2([0,T]; l^{1,2}(r,\rho)) = \{\omega : [0,T] \to l^{1,2}(r,\rho) : |\omega|_{L^2} < \infty\}$, with $|\omega|^2_{L^2} = \int_o^T |\omega(t)|^2_{l^{1,2}(r,\rho)} dt$.

The proof of following results are in [26, 31] and now we will set some important results on the new discretized weighted Sobolev spaces, which allows us to prove some of the most important results on this chapter.

Proposition 4.2.1.1. *$l^{0,2}(r)$ is an Hilbert space.*

Proposition 4.2.1.2. *$l^{1,2}(r,\rho)$ is a reflexive and separable Banach space.*

Proposition 4.2.1.3. *$l^{1,2}(r,\rho)$ is continuous and densely embedded into $l^{0,2}(r)$.*

Remark 20. As in the bounded coefficients case, f_h can have a weaker condition, such that, $f_h \in L^2([0,T]; (l^{1,2}(r,\rho))^*)$, with $(l^{1,2}(r,\rho))^*$ the dual of $l^{1,2}(r,\rho)$.

4.2.2 The discretized problem

In the case of the discretization with unbounded data it is necessary to define the framework on the new environment, considering the basis of the framework defined in previous chapter.

As in [26] and previous in this thesis, consider the new problem, discretized version of the second order parabolic Cauchy problem, in $\mathbb{R}^d$.

$$L_h u - u + f_h = 0 \quad \text{in} \quad Q(h) = [0,T] \times Z_h^d \tag{4.3}$$

$$u(0,x) = g_h(x) \quad \text{in} \quad Z_h^d$$

with $T \in (0,\infty)$, f_h and g_h such as

$$f_h : Q(h) \to \mathbb{R} \quad \text{and} \quad g_h : Z_h^d \to \mathbb{R}$$

So, we have:

$$a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u - \frac{\partial u}{\partial t} + f_h(t,x) = 0$$

.

Based on Assumption (4.1.3.1), where regularity conditions under coefficients in the exact problem are imposed, we have:

Assumption 4.2.2.1. *Consider the following conditions and let r and ρ be positive smooth functions on $\mathbb{R}^d$*

1. $f_h \in L^2([0,T]; l^{0,2}(r))$
2. $g_h \in l^{0,2}(r)$.

Remark 21. $|\partial_i^+ a_{ij}(t,x)| = |h^{-1}(a_{ij}(t,x+he_i) - a_{ij}(t,x))| \leq |\frac{\partial}{\partial x_i} a_{ij}(t,x+\tau e_i)|$, with $0 < \tau < h$.

Now we can define the generalized solution of (4.3), solution we want to prove that exists and is unique.

Definition 4.2.2.2. $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalized solution of the discrete problem (4.3) if, for all $t \in [0,T]$ and for all $\varphi \in l^{1,2}(r,\rho)$

$$\begin{aligned}(u(t),\varphi) =& (g_h,\varphi) + \int_0^t \{-(a_{ij}(s)\partial_i^+ u(s), \partial_j^+ \varphi) \\ &+ (b_i(s)\partial_i^+ u(s) - \partial_j^+ a_{ij}(s)\partial_i^+ u(s), \varphi) \\ &+ (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle\}ds\end{aligned}$$

with $(\cdot,\cdot)$ representing the inner product in $l^{0,2}(r)$.

Considering the previous assumptions and the framework described, we can state the existence and uniqueness of solution.

Theorem 4.2.2.3. *Considering the conditions* $(1)-(2)$ *of Assumption (4.1.3.1) and* $(1)-(2)$ *in Assumption (4.2.2.1), the discrete problem (4.3) has a unique generalized solution in* $[0,T]$.
Moreover,

$$\sup_{t\in[0,T]} |u(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)|^2_{l^{1,2}(r,\rho)}dt \leq N(|g_h|^2_{l^{0,2}(r)} + \int_0^T |f_h(t)|^2_{l^{0,2}(r)}dt)$$

where N *is a constant not depending on* h.

Proof. Consider a new a_{ij} such that $a^{\lambda}_{ij}(t,x,\lambda) = a_{ij}(t,x) + \lambda$ with $\lambda \in (0,1)$.

Let $L^{\lambda}_h(s) : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$, for every $s \in [0,T]$. We define

$$\left\langle L^{\lambda}_h(s)\psi,\varphi\right\rangle := -(a_{ij}(s)\partial^+_i\psi,\partial^+_j\varphi) + (b_i(s)\partial^+_i\psi - \partial^+_j a_{ij}(s)\partial^+_i\psi,\varphi) + (c(s)\psi,\varphi),$$

for all $s \in [0,T]$, $\varphi,\psi \in l^{1,2}(r,\rho)$.

We have to prove that this new problem, with the change in coefficients, has a unique solution.

1. $\exists K,\lambda > 0$ constants : $\langle L_h(s)\psi,\psi\rangle + \lambda\|\psi\|_{l^{1,2}(r,\rho)} \leq K\|\psi\|_{l^{0,2}(r)}$
2. $\exists K$ constant : $|\langle L_h(s)\psi,\varphi\rangle| \leq K\|\psi\|_{l^{1,2}(r,\rho)} \cdot \|\varphi\|_{l^{1,2}(r,\rho)}$,

for all $s \in [0,T]$, $\varphi,\psi \in l^{1,2}(r,\rho)$.

For the first property, owing to (1) and (2) in Assumption (4.1.3.1), conditions under the regularity of coefficients, and with the previous inner product, we have

$$\begin{aligned}
\langle L_h(s)\psi,\psi\rangle =& -\sum_{i,j}\sum_x r^2 a_{ij}(s)\partial^+_i\psi\partial^+_j\psi h^d \\
&+ \sum_i\sum_x r^2(b_i(s) - \partial^+_j a_{ij}(s))\partial^+_i\psi\psi h^d + \sum_x r^2 c(s)\psi\psi h^d \\
\leq& -\lambda\sum_i\sum_x r^2\left|\rho\partial^+_i\psi\right|^2 h^d + 2K\sum_i\sum_x r^2\rho\left|\partial^+_i\psi\psi\right| h^d \\
&+ K\sum_x r^2|\psi|^2 h^d \\
=& -\lambda\sum_i\left\|\rho\partial^+_i\psi\right\|^2_{l^{0,2}(r)} + 2K\sum_i\sum_x r^2\rho\left|\partial^+_i\psi\psi\right| h^d \\
&+ K\|\psi\|^2_{l^{0,2}(r)},
\end{aligned} \tag{4.4}$$

where the variable $x \in Z^d_h$ is omitted, $\sum_x$ denotes the summation over Z^d_h and $\sum_i$, $\sum_j$ the summation over $\{1,2,\ldots,d\}$. We use the Cauchy's inequality on the second term in estimate (4.4), and obtain

$$\begin{aligned}\langle L_h(s)\psi,\psi\rangle \leq & -\lambda\sum_i \left\|\rho\partial_i^+\psi\right\|^2_{l^{0,2}(r)} + \varepsilon K\sum_i\sum_x r^2\left|\rho\partial_i^+\psi\right|^2 h^d \\ & + \frac{K}{\varepsilon}\sum_i\sum_x r^2|\psi|^2h^d + K\|\psi\|^2_{l^{0,2}(r)} \\ = & -\lambda\sum_i \left\|\rho\partial_i^+\psi\right\|^2_{l^{0,2}(r)} - \lambda\|\psi\|^2_{l^{0,2}(r)} + \varepsilon K\sum_i \left\|\rho\partial_i^+\psi\right\|^2_{l^{0,2}(r)} \\ & + \frac{K}{\varepsilon}\|\psi\|^2_{l^{0,2}(r)} + (K+\lambda)\|\psi\|^2_{l^{0,2}(r)} \\ \leq & -\lambda\|\psi\|^2_{l^{1,2}(r,\rho)} + K\|\psi\|^2_{l^{0,2}(r)},\end{aligned}$$

with $\lambda > 0, K$ constants, by taking ε sufficiently small. The first property is proved.

The second property follows from (2) in Assumption (4.1.3.1), conditions under the derivatives of the coefficients, and Cauchy-Schwarz inequality

$$\begin{aligned}&|\langle L_h(s)\psi,\varphi\rangle| \\ &= \Big| -\sum_{i,j}\sum_x r^2 a_{ij}(s)\partial_i^+\psi\partial_j^+\varphi h^d + \sum_i\sum_x r^2 b_i(s)\partial_i^+\psi\varphi h^d \\ &\quad -\sum_{i,j}\sum_x r^2\partial_j^+ a_{ij}(s)\partial_i^+\psi\varphi h^d + \sum_x r^2 c(s)\psi\varphi h^d\Big| \\ &\leq K\sum_{i,j}\sum_x r^2\left|\rho^2\partial_i^+\psi\partial_j^+\varphi\right| h^d + K\sum_i\sum_x r^2\left|\rho\partial_i^+\psi\varphi\right| h^d + K\sum_x r^2|\psi\varphi| h^d \\ &\leq K\sum_i \left\|\rho\partial_i^+\psi\right\|_{l^{0,2}(r)}\sum_j \left\|\rho\partial_j^+\varphi\right\|_{l^{0,2}(r)} + K\sum_i \left\|\rho\partial_i^+\psi\right\|_{l^{0,2}(r)}\|\varphi\|_{l^{0,2}(r)} \\ &\quad + K\|\psi\|_{l^{0,2}(r)}\|\varphi\|_{l^{0,2}(r)} \\ &\leq K\|\psi\|_{l^{1,2}(r,\rho)}\cdot\|\varphi\|_{l^{1,2}(r,\rho)},\end{aligned}$$

where the same writing conventions are kept.

Owing to Theorem (4.1.3.4) the result of the existence of generalized solution to the problem follows.

$l^{0,2}(r,\rho)$ and $l^{1,2}(r,\rho)$ are Hilbert spaces in $L^2(\mathbb{R})$ and the weak derivatives of these spaces are also in $L^2(\mathbb{R})$.

As we fixed, we have $a^{\lambda}_{ij}(t,x,\lambda) = a_{ij}(t,x)+\lambda, \lambda\in(0,1)$ through a change on the original $a_{ij}(t,x)$.

Consider u_λ the generalized solution of the discretized problem. Then

$$u_\lambda \in C([0,T], l^{0,2}(r,\rho)) \cap L^2([0,T], l^{1,2}(r,\rho)).$$

Let $f_h \in L^2([0,T], l^{0,2}(r,\rho))$ and $g_h \in l^{0,2}(r,\rho)$.

Consider the linear functional $L_h : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$ with the same inner product and norm defined above.

Assume the conditions in (4.1.2.1) on the behaviour of the weights r and ρ on the operator coefficients and on the free data f_h and g_h. We have that u_λ is weakly convergent to u in $C([0,T], l^{0,2}(r,\rho)) \cap L^2([0,T], l^{1,2}(r,\rho))$ and

$$\begin{aligned}(u_\lambda(t),\varphi) = (g_h,\varphi) + \int_0^t &\{-(a_{ij}(s)+\lambda)\partial^+ u_\lambda(s), \partial_j^+\varphi)\\ &(b_i(s)\partial_i^+ u_\lambda(s) - \partial_j^+(a_{ij}(s)+\lambda)\partial^+ u_\lambda(s),\varphi)\\ &+ (c(s)u_\lambda(s),\varphi) + \langle f_h(s),\varphi\rangle\}ds\end{aligned}$$

Due to properties:

$$\begin{aligned}&g_h \in l^{0,2}(r,\rho) \subset W^{m,2}(r,\rho) \subset L^2(r,\rho) \Rightarrow\\ &\Rightarrow (g_h,\varphi)_{l^{0,2}(r,\rho)} \le (g,\varphi)_{W^{m,2}(r,\rho)} \le (g,\varphi)_{L^2(r,\rho)} \;\; and\\ &f_h \in L^2(([0,T], l^{0,2}(r,\rho)) \subset L^2(r,\rho) \Rightarrow\\ &\Rightarrow (f_h,\varphi)_{l^{0,2}(r,\rho)} \le (f,h)_{L^2(r,\rho)}\end{aligned}$$

As a consequence we have:

$$\begin{aligned}(u_\lambda(t),\varphi) \le (g,\varphi)_{L^2(r,\rho)} + \int_0^t &\{-(a_{ij}(s)+\lambda)D_i u(s), D_j\varphi)_{L^2(r\rho)}\\ &+ (b_i(s)D_i u(s) - D_j(a_{ij}+\lambda)D_i u(s),\varphi)_{L^2(r,\rho)}\\ &+ (c(s)u(s)+f,\varphi)_{L^2(r,\rho)}\}ds\end{aligned}$$

Then, we just fixed a bound to the left hand of the estimative which is valid and in limit is the solution of the nondegenerate problem, not depending on λ. Moreover, since $\lambda \to 0^+, u_\lambda \to u$.

And our proof is now complete. □

4.2.3 Approximation results

To characterize our approximations results, in particular that the scheme is consistent, we have to define the rate of convergence of the solution to the problem (4.3)and approximate partial derivatives. A result corresponding to the one presented bellow is proved in [26] for the nondegenerate case.

Proposition 4.2.3.1. *Consider r and ρ positive functions on $\mathbb{R}^d$. Consider m an integer such that $m > \frac{d}{2}$. Consider, in Assumption (4.1.3.1), that the conditions $(1)-(2)$ are satisfied and also that $\rho(x) \geq C$ on $\mathbb{R}^d$, with $C > 0$ a constant. Let $u(t) \in W^{m+2,2}(r,\rho)$, $v(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$. Then there exists a constant N independent of h such that*

1. $\sum_{x \in Z_h^d} r^2(x) \left|u_{x^i}(t,x) - \partial_i^+ u(t,x)\right|^2 \rho^2(x) h^d \leq h^2 N |u(t)|^2_{W^{m+2,2}(r,\rho)}$

2. $\sum_{x \in Z_h^d} r^2(x) \left|u_{x^i x^j}(t,x) - \partial_j^- \partial_i^+ v(t,x)\right|^2 \rho^4(x) h^d \leq h^2 N |v(t)|^2_{W^{m+3,2}(r,\rho)}$,

for all $t \in [0,T]$.

Proof. The proof we now develop follows the main ideas of the corresponding proof on [26].

Let us prove (1). We define a suitable geometric setting, and then obtain an estimate for

$$r^2(x) \left|u_{x^i}(t,x) - \partial_i^+ u(t,x)\right|^2 \rho^2(x),$$

with $x \in Z_h^d$, using a Sobolev's inequality on a fixed ball.

Let us consider d-cells

$$R_h = \{(x^1, x^2, \ldots, x^d) \in \mathbb{R}^d : x_h^i < x^i < x_h^i + h, \quad i = 1, 2, \ldots, d\},$$

with $x_h = (x_h^1, x_h^2, \ldots, x_h^d) \in Z_h^d$ fixed. Consider the particular d-cell where $h = 1$ and $x_1 = (0, 0, \ldots, 0)$, and denote it R_1^0. Now, take open balls B_h such that $B_h \supset R_h$, with the vertices $\{x_h^i, x_h^i + h, \; i = 1, 2, \ldots, d\}$ laying on the limiting sphere. Denote B_1^0 the ball containing R_1^0.

For every $x_h \in Z_h^d$, recalling that the conditions of the theorem, function $u(t)$ (function $v(t)$) has a modification in x which is continuously differentiable in x up to the order 2 (up to the order 3), and the derivatives equal the weak derivatives, for every $t \in [0,T]$, we have, by the mean-value theorem,

$$\partial_i^+ u(t, x_h) = h^{-1}(u(t, x_h + h e_i) - u(t, x_h)) = u_{x^i}(t, x_h + \theta h e_i)$$

and

$$\begin{aligned} \left|u_{x^i}(t,x_h) - \partial_i^+ u(t,x_h)\right| &= \left|u_{x^i}(t,x_h) - u_{x^i}(t, x_h + \theta h e_i)\right| \\ &\leq h \left|u_{x^i x^i}(t, x_h + \theta' h e_i)\right|, \end{aligned} \tag{4.5}$$

for some $0 < \theta' < \theta < 1$. Clearly,

$$\left|u_{x^i x^i}(t, x_h + \theta' h e_i)\right| \leq \sup_{x \in R_h} \left|u_{x^i x^i}(t,x)\right|, \tag{4.6}$$

and then, from (4.5) and (4.6),

$$\left|u_{x^i}(t,x_h)-\partial_i^+ u(t,x_h)\right|^2 \le h^2 \sup_{x\in R_h} |u_{x^ix^i}(t,x)|^2 . \tag{4.7}$$

We change variable in order to have the supremum in (4.7) calculated over the fixed d-cell R_1^0:

$$\sup_{x\in R_h} |u_{x^ix^i}(t,x)| = \sup_{x\in R_1^0} |u_{x^ix^i}(t,x_h+hx)| . \tag{4.8}$$

As

$$\sup_{x\in R_1^0} |u_{x^ix^i}(t,x_h+hx)|^2 \le \sup_{x\in B_1^0} |u_{x^ix^i}(t,x_h+hx)|^2 , \tag{4.9}$$

from (4.7) – (4.9) we immediately obtain

$$\begin{aligned} r^2(x_h)&\left|u_{x^i}(t,x_h)-\partial_i^+ u(t,x_h)\right|^2\rho^2(x_h) \\ &\le h^2 \sup_{x\in R_1^0}\left(r^2(x_h+hx)\,|u_{x^ix^i}(t,x_h+hx)|^2\,\rho^2(x_h+hx)\right) \\ &\le h^2 \sup_{x\in B_1^0}\left(r^2(x_h+hx)\,|u_{x^ix^i}(t,x_h+hx)|^2\,\rho^2(x_h+hx)\right). \end{aligned} \tag{4.10}$$

If U, V are open subsets of $\mathbb{R}^d$ with $V\subset U$ and $w\in W^{m,2}(U)$ then $w\in W^{m,2}(V)$ and also, if $w\in W^{m,2}(U)$ and $\zeta\in C_0^\infty(U)$ then $\zeta\in W^{m,2}(U)$ and $\zeta w\in W^{m,2}(U)$ and we have, for $m>d/2$, by using a Sobolev's inequality

$$\begin{aligned} &\sup_{x\in B_1^0} |r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)|^2 \\ &\qquad\le N\sum_{|\alpha|\le m}\int_{B_1^0} |D_x^\alpha\left(r(x_h+hx)u_{x^ix^i}(t,x_h+hx)\rho(x_h+hx)\right)|^2\,dx, \end{aligned} \tag{4.11}$$

with N a constant independent of h. Observe that the Leibniz' formula

$$\begin{aligned} |D_x^\alpha(ru_{x^ix^i}\rho)| &= \left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta(r\rho)D_x^{\alpha-\beta}u_{x^ix^i}\right| \\ &= \left|\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left(\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma r D^{\beta-\gamma}\rho\right)D_x^{\alpha-\beta}u_{x^ix^i}\right| \end{aligned} \tag{4.12}$$

holds (the arguments of r, ρ and $u_{x^ix^i}$ are omitted). Also, keeping the same convention,

$$|D^\gamma r| \le Kr\rho^{-|\gamma|} \quad \text{and} \quad \left|D^{\beta-\gamma}\rho\right| \le K\rho^{1-(|\beta|-|\gamma|)},$$

with K a constant, and then

$$\left|\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma r D^{\beta-\gamma}\rho\right| \le N\sum_{\gamma\le\beta}\binom{\beta}{\gamma}r\rho^{-|\gamma|}\rho^{1-(|\beta|-|\gamma|)} \le Nr\rho^{1-|\beta|}, \tag{4.13}$$

with N a constant. From (4.11) − (4.13), we get

$$\begin{aligned}
&\sup_{x\in B_1^0} |r(x_h + hx)u_{x^ix^i}(t, x_h + hx)\rho(x_h + hx)|^2 \\
&\le N \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \int_{B_1^0} r^2(x_h + hx) \left|\rho^{1-|\beta|}(x_h + hx)\right|^2 . \\
&. \left|D_x^{\alpha-\beta} u_{x^ix^i}(t, x_h + hx)\right|^2 dx.
\end{aligned} \tag{4.14}$$

Owing to Hölder inequality and to the hypotheses on function ρ, the integral in (4.14) can be estimated by

$$\begin{aligned}
&\int_{B_1^0} r^2(x_h + hx) \left|\rho^{1-|\beta|}(x_h + hx) D_x^{\alpha-\beta} u_{x^ix^i}(t, x_h + hx)\right|^2 dx \\
&\quad \le N \int_{B_1^0} r^2(x_h + hx) \left|\rho^{2+(|\alpha|-|\beta|)}(x_h + hx) D_x^{\alpha-\beta} u_{x^ix^i}(t, x_h + hx)\right|^2 dx \\
&\quad \cdot \sup_{x\in B_1^0} \left|\rho^{-1-|\alpha|}(x_h + hx)\right|^2 \\
&\quad \le N \int_{B_1^0} r^2(x_h + hx) \left|\rho^{2+(|\alpha|-|\beta|)}(x_h + hx) D_x^{\alpha-\beta} u_{x^ix^i}(t, x_h + hx)\right|^2 dx.
\end{aligned} \tag{4.15}$$

Thus, from (4.14) and (4.15),

$$\begin{aligned}
&\sup_{x\in B_1^0} |r(x_h + hx)u_{x^ix^i}(t, x_h + hx)\rho(x_h + hx)|^2 \\
&\quad \le N \sum_{|\alpha|\le m} \sum_{\beta\le\alpha} \int_{B_1^0} r^2(x_h + hx) \Big|\rho^{2+(|\alpha|-|\beta|)}(x_h + hx) \\
&\qquad \cdot D_x^{\alpha-\beta} u_{x^ix^i}(t, x_h + hx)\Big|^2 dx \\
&\quad \le N \sum_{|\alpha|\le m} \int_{B_1^0} r^2(x_h + hx) \left|\rho^{2+|\alpha|}(x_h + hx) D_x^{\alpha} u_{x^ix^i}(t, x_h + hx)\right|^2 dx \\
&\quad \le N \sum_{|\alpha|\le m+2} \int_{B_1^0} r^2(x_h + hx) \left|\rho^{|\alpha|}(x_h + hx) D_x^{\alpha} u(t, x_h + hx)\right|^2 dx \\
&\quad = N \sum_{|\alpha|\le m+2} \int_{B_h} r^2(x) \left|\rho^{|\alpha|}(x) D_x^{\alpha} u(t, x)\right|^2 h^{-d} h^{2|\alpha|} dx \\
&\quad \le N \sum_{|\alpha|\le m+2} \int_{B_h} r^2(x) \left|\rho^{|\alpha|}(x) D_x^{\alpha} u(t, x)\right|^2 h^{-d} dx.
\end{aligned} \tag{4.16}$$

Finally, owing to the particular geometry of the framework we have set, from

(4.10) and (4.16) we obtain

$$\sum_{x\in Z_h^d} r^2(x)\big|u_{x^i}(t,x)-\partial_i^+u(t,x)\big|^2\rho^2(x)h^d$$
$$\le Nh^2\sum_{|\alpha|\le m+2}\sum_{x_h\in Z_h^d}\int_{B_h(x_h)} r^2(x)\left|\rho^{|\alpha|}(x)D_x^\alpha u(t,x)\right|^2dx$$
$$\le Nh^2\sum_{|\alpha|\le m+2}\sum_{x_h\in Z_h^d}\int_{R_h(x_h)} r^2(x)\left|\rho^{|\alpha|}(x)D_x^\alpha u(t,x)\right|^2dx$$
$$\le h^2N\|u(t)\|^2_{W^{m+2,2}(r,\rho)},$$

where $B_h(x_h):=B_h$, $R_h(x_h):=R_h$, and the proof for (1) is complete.

The proof for (2) follows the same steps. □

Now we finally state the rate of convergence.

Theorem 4.2.3.2. *Consider, in Assumption (4.1.3.1), that the conditions* (1) − (2) *are satisfied and also that* $\rho(x)\ge C$ *on* $\mathbb{R}^d$*, with* $C>0$ *a constant. Consider* m *an integer such that* $m>\frac{d}{2}$ *and let* u *be the solution of problem (4.2) in Theorem (4.1.3.4) and* u_h *the solution of (4.3) in Theorem (4.2.2.3). For* $u\in L^2([0,T];W^{m+3,2}(r,\rho))$*, we have*

$$sup_{t\in[0,T]}|u(t)-u_h(t)|^2_{l^{0,2}(r)}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)}dt$$
$$\le h^2N\int_0^T|u(t)|^2_{W^{m+3,2}(r,\rho)}dt+N(|g-g_h|^2_{l^{0,2}(r)}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}(r)}dt).$$

with N *a constant not depending on* h*.*

Proof. Fix $a_{ij}^\lambda(t,x,\lambda)=a_{ij}(t,x)+\lambda$, $\lambda\in(0,1)$.

From (3.6) and (4.3), we have that $u-u_h$ satisfies the problem

$$\begin{cases}(u-u_h)_t=L_h(u-u_h)+(L-L_h)u+(f-f_h) & \text{in } Q(h)\\ (u-u_h)(0,x)=(g-g_h)(x) & \text{on } Z_h^d.\end{cases}\tag{4.17}$$

Under the conditions of theorem, there are modifications in x such that the data f and g are continuous in x, for every $t\in[0,T]$. Then, we see that $f-f_h\in L^2([0,T];l^{0,2}(r))$ and $g-g_h\in l^{0,2}(r)$.

With respect to the term $(L - L_h^\lambda)u$, note that if $u(t) \in W^{m+3,2}(r,\rho)$, for all $t \in [0,T]$,

$$\sum_{x \in Z_h^d} r^2(x) \left|(L - L_h^\lambda)(t)u(t)\right|^2 h^d$$
$$= \sum_{x \in Z_h^d} r^2(x) \left|(a_{ij}(t,x) + \lambda)\left(\frac{\partial^2}{\partial x_i \partial x_j} - \partial_j^- \partial_i^+\right) u(t,x) + b_i(t,x)\left(\frac{\partial}{\partial x_i} - \partial_i^+\right) u(t,x)\right|^2 h^d$$
$$= \sum_{x \in Z_h^d} r^2(x) \left|a_{ij}(t,x)\left(\frac{\partial^2}{\partial x_i \partial x_j} - \partial_j^- \partial_i^+\right) u(t,x) + b_i(t,x)\left(\frac{\partial}{\partial x_i} - \partial_i^+\right) u(t,x)\right|^2 h^d$$
$$\leq h^2 N \|u(t)\|^2_{W^{m+3,2}(r,\rho)} < \infty,$$

owing to (2) in Assumption (4.1.3.1) and to Theorem (4.2.3.1). Thus $(L - L_h)(t)u(t) \in l^{0,2}(r)$, for every $t \in [0,T]$. Moreover, as by assumption $u \in L^2([0,T]; W^{m+3,2}(r,\rho))$, we obtain immediately $(L - L_h^\lambda)u \in L^2([0,T]; l^{0,2}(r))$.

As seen before, problem (4.17) satisfies the hypotheses of Proposition (4.2.2.3). Holding the estimate,

$$\sup_{0 \leq t \leq T} \|u(t) - u_h(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t) - u_h(t)\|^2_{l^{1,2}(r,\rho)} dt$$
$$\leq N\Bigg(\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}(r)} dt + \int_0^T \|(L - L_h)(t)u(t)\|^2_{l^{0,2}(r)}\, dt\Bigg).$$

Owing again to (2) in Assumption (4.1.3.1) and to Proposition (4.2.3.1), the result follows.

□

The following Corollary gives us the rate of convergence with a better structured statement.

Corollary 4.2.3.3. *Let the hypotheses of Theorem (4.2.3.3) be satisfied, and denote u the solution of (4.2) in Theorem (4.1.3.4) and u_h the solution of (4.3) in Theorem (4.2.2.3). If there is a constant N independent of h such that*

$$|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)} dt$$
$$\leq h^2 N \left(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt\right)$$

then

$$\sup_{0 \leq t \leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)} dt$$
$$\leq h^2 N \left(\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} dt + |g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt\right).$$

Proof. The result is an immediate consequence of Theorem (4.2.3.2). □

4.2.4 The special one-dimensional case

As in Chapter 3,we now applly our methodology to the one dimension on spatial variable, in degenerate case (our pretended goal) for the unbounded coefficients case. The results we will prove are stronger than the results in [28, 45].

Consider the Cauchy Problem in $\mathbb{R}$.

$$\begin{aligned} Lu - \frac{\partial u}{\partial t} + f = 0 \quad \text{in} \quad Q \\ u(0,x) = g(x) \quad \text{in} \quad \mathbb{R} \end{aligned} \tag{4.18}$$

where $Q = [0,T] \times \mathbb{R}$, T is a positive constant and L is the second-order partial differential operator with unbounded coefficients in $\mathbb{R}$:

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x),$$

t with values in $[0,T]$ and f, g real valued functions.

In order to study the existence and uniqueness of problem (4.18) we must recall the weighted Sobolev spaces, defined on (4.1) and the solvability conditions of PDEs with unbounded coefficients in those spaces, in the degenerated case.

We now present one particularization case of one presented before that we include for completeness.

Assumption 4.2.4.1. *Let r and ρ be positive smooth functions on $\mathbb{R}$, consider m a nonnegative integer and a constant k.*

1. *$a(t,x) \geq \lambda \rho^2(x)$, λ constant;*
2. *For all $t \in [0,T]$, for all $x \in \mathbb{R}$:*
   $|D^\alpha a| \leq k\rho^{2-|\alpha|}$, for all $|\alpha| \leq m+1 \vee 2$
   and $|D^\alpha b| \leq k\rho^{1-|\alpha|}$ and $|D^\alpha c| \leq k\rho^{-|\alpha|}$ for all $|\alpha| \leq m$, where D^α is the α^{th} partial derivative with respect to x;
3. *$f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.*

Assumption 4.2.4.2. *Consider m a positive integer and r, ρ positive smooth functions on $\mathbb{R}$. There is a constant k such that*

1. *$|D^\alpha \rho| \leq k\rho^{1-\alpha}$;*
2. *$|D^\alpha r| \leq \frac{kr}{\rho^\alpha}$;*

3. $\sup_{|x-y|<\epsilon}\left(\frac{r(x)}{r(y)}+\frac{\rho(x)}{\rho(y)}\right)=k$, *for* $\epsilon>0$ *and* $x,y\in\mathbb{R}$

Now, we can state the definition of generalized and classical solution of (4.18).

Definition 4.2.4.3. $(u(t))_{t\in[0,T]}$ is said to be a generalized solution of the problem (4.18) if:

1. $u\in L^2([0,T];W^{m,2}(r,\rho))$;
2. $\forall t\in[0,T]$,

$$\begin{aligned}(u(t),\varphi)=(g,\varphi)+\int_0^t\{&(-a(s)D_xu(s),D_x\varphi)\\&+(b(s)D_xu(s)-D_xa(s)D_xu(x),\varphi)\\&+(c(s)u(s),\varphi)+\langle f(s),\varphi\rangle\}ds.\end{aligned}$$

for all $\varphi\in C_0^\infty$ and $(\cdot,\cdot)$ representing the inner product in $W^{0,2}(r,\rho)$.

The following result defines the conditions to the existence and uniqueness of a generalized solution to the problem (4.18).

Theorem 4.2.4.4. *Let* $m\geq 1$ *and assume the conditions on (4.2.4.1) and on (4.2.4.2). Then, there exists a unique generalized solution* $u\in[0,T]$ *for the problem (4.18). Moreover,*

$$u\in L^2([0,T],W^{m,2}(r,\rho))\cap L^2([0,T];W^{m+1,2}(r,\rho))$$

and

$$\begin{aligned}sup_{t\in[0,T]}|u(t)|^2_{W^{m,2}(r,\rho)}+&\int_0^T|u(t)|^2_{W^{m+1,2}(r,\rho)}dt\\&\leq N(|g|^2_{W^{m,2}(r,\rho)}+\int_0^T|f(t)|^2_{W^{m-1,2}}(r,\rho)dt)\end{aligned}$$

with N *constant.*

Proof. This proof can be seen at [31]. □

Discrete framework

To adapt our results to the special one dimension case, we now define the h-grid, on $\mathbb{R}$, with $h \in (0,1]$:

$$Z_h = \{x \in \mathbb{R} : x = nh, \quad n = 0, \pm 1, \pm 2, ...\}$$

and consider the difference quotients in space, for all $x \in Z_h$:

- Forward: $\partial^+ u = \partial^+ u(t,x) = \dfrac{u(t,x+h) - u(t,x)}{h}$;
- Backward: $\partial^- u = \partial^- u(t,x) = \dfrac{u(t,x) - u(t,x-h)}{h}$.

Let L_h be the discrete operator, such that:

$$L_h(t,x) = a(t,x)\partial^-\partial^+ + b(t,x)\partial^+ + c(t,x).$$

So, the discrete version of the second order parabolic Cauchy problem, can be written as:

$$L_h u - u + f_h = 0 \quad \text{in} \quad \mathrm{Q(h)} = [0,T] \times Z_h \tag{4.19}$$

$$u(0,x) = g_h(x) \quad \text{in} \quad Z_h \tag{4.20}$$

with $T \in (0,\infty)$ and f_h and g_h functions such that

$$f_h : \mathbb{Q}(h) \to \mathbb{R} \quad \text{and} \quad g_h : Z_h \to \mathrm{R}.$$

To complete the framework we must define the discrete version of weighted Sobolev spaces. So, instead of $W^{0,2}(r,\rho)$ we will consider:

$$l^{0,2}(r) = \{v : |v|_{l^{0,2}(r)} < \infty\}$$

with norm

$$|v|_{l^{0,2}(r)} = \left(\sum_{x \in Z_h} r^2(x) |v(x)|^2 h \right)^{1/2}$$

and inner product

$$(v,\omega)_{l^{0,2}(r)} = \sum_{x \in Z_h} r^2(x) v(x) \omega(x) h, \quad \forall v, \omega \in l^{0,2}(r),$$

and the discrete version of the weighted Sobolev space $W^{1,2}(r,\rho)$:

$$l^{1,2}(r,\rho) = \{\omega : |\omega|_{l^{1,2}(r,\rho)} < \infty\}$$

with norm

$$|\omega|^2_{l^{1,2}(r,\rho)} = |\omega|^2_{l^{0,2}(r)} + |\rho\partial^+\omega|^2_{l^{0,2}}$$

with inner product

$$(\omega, z)_{l^{1,2}(r,\rho)} = (\omega, z)_{l^{0,2}(r)} + (\rho\partial^+\omega, \rho\partial^+ z)_{l^{0,2}(r)},$$

for all $\omega, z \in l^{1,2}(r,\rho)$.

Owing to (4.2) and to [27], we have the following properties:

- $l^{0,2}(r)$ and $l^{1,2}(r,\rho)$ are Hilbert spaces;
- $|v|_{l^{0,2}(r)} \leq |v|_{l^{1,2}(r,\rho)}$ for all $v \in l^{1,2}(r,\rho)$;
- $l^{1,2}(r,\rho)$ is a reflexive and separable Banach space, continuous and densely embedded into the Hilbert space $l^{0,2}(r)$ (proof follows from [26]).

Consider the spaces:

- $C([0,T]; l^{0,2}(r))$: space of continuous $l^{0,2}(r)$-valued functions on $[0,T]$;
- $L^2([0,T]; l^{m,2}(r,\rho)) = \{z : [0,T] \to l^{m,2}(r,\rho) : \int_0^T |z(t)|^2_{l^{m,2}(r,\rho)} dt < \infty\}$, with $m = 0, 1$ and z : Q(h) $\to \mathbb{R}$ functions such that $(z(t))(x) = z(t,x)$ for all $t \in [0,T]$, $x \in Z_h$.

Next Assumption gives us some conditions over the data f_h and g_h.

Assumption 4.2.4.5. *Let r be a smooth positive function on $\mathbb{R}$. We assume*

1. $f_h \in L^2([0,T]; l^{0,2}(r))$;
2. $g_h \in l^{0,2}(r)$.

Definition 4.2.4.6. $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalized solution of (4.18) if, for every $t \in [0,T]$,

$$\begin{aligned}(u(t),\varphi) = (g_h,\varphi) + \int_0^t \big\{ &- (a(s)\partial^+ u(s), \partial^+\varphi) \\ &+ (b(s)\partial^+ u(s) - \partial^+ a(s)\partial^+ u(s), \varphi) \\ &+ (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle \big\} ds\end{aligned}$$

holds for all $\varphi \in l^{1,2}(r,\rho)$.

Remark 22. Above, $(\cdot,\cdot)$ is the inner product in $l^{0,2}(r)$.

Based on Theorem (4.2.4.4), with the previous Definition and Assumption we can now ensure that the problem has a unique generalized solution.

Theorem 4.2.4.7. *Under (1)–(2) in Assumption (4.2.4.1) and (1)–(2) in Assumption (4.2.4.5), problem (4.18) has a unique generalized solution u in $[0,T]$. Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)|^2_{l^{1,2}(r,\rho)} dt \le N\Big(|g_h|^2_{l^{0,2}(r)} + \int_0^T |f_h(t)|^2_{l^{0,2}(r)} dt \Big),$$

with N a constant independent of h.

Proof. The corresponding proof for the d dimensional case is above. □

Approximations results

In this subsection we prove that the solution of the discrete problem approximates the solution of the exact problem. The results presented to the one dimension case are stronger than the ones to the multidimensional case.

Next Theorem states the consistency of the scheme.

Theorem 4.2.4.8. *Let r,ρ be positive functions on $\mathbb{R}$ and assume the conditions in (4.2.4.2), with $\rho(x) \ge C$ on $\mathbb{R}$ and with $C > 0$.*
Let $u(t) \in W^{2,2}(r,\rho)$, $v(t) \in W^{3,2}(r,\rho)$, for all $t \in [0,T]$. There exists a constant N such that, for all $t \in [0,T]$:

1. $\sum_{x\in Z_h} r^2(x)|D_x u(t,x) - \partial^+ u(t,x)|^2 \rho^2(x) h \le h^2 N |u(t)|^2_{W^{2,2}(r,\rho)}$

2. $\sum_{x\in Z_h} r^2(x)|D_{x^2}v(t,x) - \partial^-\partial^+ v(t,x)|^2\rho^4(x)h \le h^2 N|v(t)|^2_{W^{3,2}(r,\rho)}$

with $m = 0, 1$.

Proof. This proof follows the main steps for the degenerate case in [45].

Let us prove (1). Observe that the forward difference quotient can be written

$$\partial^+ u(t,x) = h^{-1}(u(t,x+h) - u(t,x)) = \int_0^1 \frac{\partial}{\partial x} u(t, x+hq)dq.$$

Thus

$$\begin{aligned}\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x) &= \int_0^1 \Big(\frac{\partial}{\partial x}u(t,x) - \frac{\partial}{\partial x}u(t,x+hq)\Big)dq \\ &= h\int_0^1\int_0^1 q\frac{\partial^2}{\partial x^2}u(t,x+hqs)dsdq.\end{aligned} \tag{4.21}$$

From (4.21), using Jensen's inequality, we obtain

$$\begin{aligned}\Big|\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x)\Big|^2 &\le h^2\int_0^1\int_0^1 q^2\Big|\frac{\partial^2}{\partial x^2}u(t,x+hqs)\Big|^2 dsdq \\ &= h\int_0^1\int_0^{hq} q\Big|\frac{\partial^2}{\partial x^2}u(t,x+v)\Big|^2 dvdq \\ &\le h\int_0^1 qdq\int_0^h \Big|\frac{\partial^2}{\partial x^2}u(t,x+v)\Big|^2 dv \\ &= \frac{h}{2}\int_0^h \Big|\frac{\partial^2}{\partial x^2}u(t,x+v)\Big|^2 dv \\ &= \frac{h}{2}\int_x^{x+h}\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2 dz.\end{aligned} \tag{4.22}$$

Observe also that from (4.22), using (3) in Assumption (4.2.4.2) we have, for any
$\theta \in (0,1)$,

$$\begin{aligned}&r^2(x)\Big|\Big(\frac{\partial}{\partial x} - \partial^+\Big)u(t,x)\Big|^2\rho^2(x) \\ &\qquad\le hNr^2(x+\theta h)\rho^2(x+\theta h)\int_x^{x+h}\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2 dz.\end{aligned} \tag{4.23}$$

As, by the mean value theorem for integration, for some $\theta \in (0,1)$,

$$\begin{aligned}&r^2(x+\theta h)\rho^2(x+\theta h)\int_x^{x+h}\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2 dz \\ &\qquad= \int_x^{x+h} r^2(z)\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2\rho^2(z)dz,\end{aligned} \tag{4.24}$$

from(4.23) and (4.24), using Hölder inequality, we obtain

$$\begin{aligned} r^2(x)\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2\rho^2(x) \\ &\le hN\int_x^{x+h} r^2(z)\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2\rho^4(z)dz\cdot\sup_{z\in[x,x+h]}|\rho^{-2}(z)| \\ &\le hN\int_x^{x+h} r^2(z)\Big|\frac{\partial^2}{\partial z^2}u(t,z)\Big|^2\rho^4(z)dz, \end{aligned} \tag{4.25}$$

owing to the hypotheses on the weights ρ.

Finally, summing up (4.25) over Z_h, we get

$$\sum_{x\in Z_h} r^2(x)\Big|\Big(\frac{\partial}{\partial x}-\partial^+\Big)u(t,x)\Big|^2\rho^2(x)h\le h^2N\|u(t)\|^2_{W^{2,2}(r,\rho)},$$

with N a constant independent of h, and (1) is proved.

We now prove (2). Writing the forward and backward difference quotients

$$\partial^+v(t,x)=h^{-1}(v(t,x+h)-v(t,x))=\int_0^1\frac{\partial}{\partial x}v(t,x+hq)dq$$

and

$$\partial^-v(t,x)=h^{-1}(v(t,x)-v(t,x-h))=\int_0^1\frac{\partial}{\partial x}v(t,x-hs)ds,$$

respectively, we have for the second-order difference quotient

$$\begin{aligned} \partial^-\partial^+v(t,x)=\partial^-\int_0^1\frac{\partial}{\partial x}v(t,x+hq)dq &= \int_0^1\Big(\frac{\partial}{\partial x}\int_0^1\frac{\partial}{\partial x}v(t,x+hq-hs)dq\Big)ds \\ &= \int_0^1\int_0^1\frac{\partial^2}{\partial x^2}v(t,x+h(q-s))dsdq. \end{aligned}$$

Thus

$$\begin{aligned} \Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x) &= \int_0^1\int_0^1\Big(\frac{\partial^2}{\partial x^2}(t,x)-\frac{\partial^2}{\partial x^2}v(t,x+h(q-s))\Big)dsdq \\ &= h\int_0^1\int_0^1\int_0^1(q-s)\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))dvdsdq. \end{aligned} \tag{4.26}$$

From (4.26), by Jensen's inequality,

$$\begin{aligned} \Big|\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\Big|^2 &\le h^2\int_0^1\int_0^1\int_0^1|q-s|^2\Big|\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))\Big|^2dvdsdq \\ &= h\int_0^1\int_0^1\int_0^{h(q-s)}(q-s)\Big|\frac{\partial^3}{\partial x^3}v(t,x+w)\Big|^2dwdsdq \\ &\le h\int_0^1\int_0^1|q-s|dsdq\int_0^h\Big|\frac{\partial^3}{\partial x^3}v(t,x+w)\Big|^2dw \\ &\le h\int_0^h\Big|\frac{\partial^3}{\partial x^3}v(t,x+w)\Big|^2dw=h\int_x^{x+h}\Big|\frac{\partial^3}{\partial z^3}v(t,z)\Big|^2dz, \end{aligned}$$

and, following the same steps as in the proof of (1), we finally obtain

$$\sum_{x\in Z_h} r^2(x)\Big|\Big(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\Big)v(t,x)\Big|^2\rho^4(x)h \le h^2N\|v(t)\|^2_{W^{3,2}(r,\rho)},$$

with N a constant independent of h, and (2) is proved. □

Next we will present the rate of convergence of the solution. For that it is necessary to impose to the solution of (4.18) problem to another regularity condition, remaining the Theorem (4.2.4.7) satisfied. As a basis, consider Theorem (4.2.3.2).

Theorem 4.2.4.9. *Ler r,ρ be positive functions on $\mathbb{R}$. Consider satisfied the conditions in Assumption (4.2.4.2) and let $\rho(x)\ge C$ on $\mathbb{R}$, with C constant. Let u be the solution of (4.18) in Theorem (4.2.4.4) and u_h the solution of the same problem but in conditions of Theorem (4.2.4.7). For $u\in L^2([0,T];W^{3,2}(r,\rho))$ we have*

$$\begin{aligned} sup_{t\in[0,T]}|u(t)-u_h(t)|^2_{l^{0,2}(r)} &+ \int_0^T |u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)}dt \\ &\le h^2N\int_0^T |u(t)|^2_{W^{3,2}(r,\rho)}dt + N(|g-g_h|^2_{l^{0,2}(r)} \\ &+ \int_0^T |f(t)-f_h(t)|^2_{l^{0,2}(r)}dt) \end{aligned}$$

with N a constant independent of h.

Proof. From (4.18) and (4.19), we have that $u-u_h$ satisfies the problem

$$\begin{cases}(u-u_h)_t = L_h(u-u_h)+(L-L_h)u+(f-f_h) & \text{in } Q(h)\\ (u-u_h)(0,x)=(g-g_h)(x) & \text{in } Z_h.\end{cases} \tag{4.27}$$

Owing to $f(t)$ and g are continuous in x for every $t\in[0,T]$, we have that $f-f_h\in L^2([0,T];l^{0,2}(r))$ and $g-g_h\in l^{0,2}(r)$. Consider $a^\lambda(t,x,\lambda)=a(t,x)+\lambda$, with $\lambda\in(0,1)$.

With respect to the term $(L - L_h^\lambda)u$, if $u(t) \in W^{3,2}(r,\rho)$ for all $t \in [0,T]$,

$$\sum_{x\in Z_h} r^2(x)|(L - L_h^\lambda)(t)u(t)|^2 h$$
$$= \sum_{x\in Z_h} r^2(x)\Big|(a(t,x)+\lambda)\big(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\big)u(t,x) + b(t,x)\big(\frac{\partial}{\partial x} - \partial^+\big)u(t,x)\Big|^2 h$$
$$= \sum_{x\in Z_h} r^2(x)\Big|a(t,x)\big(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\big)u(t,x) + b(t,x)\big(\frac{\partial}{\partial x} - \partial^+\big)u(t,x)\Big|^2 h$$
$$\leq h^2 N \|u(t)\|^2_{W^{3,2}(r,\rho)} < \infty,$$

owing to (2) in Assumption (4.2.4.1) and to Theorem (4.2.4.7). Thus $(L - L_h)(t)u(t) \in l^{0,2}(r)$, for every $t \in [0,T]$. Moreover, we have, by assumption, $u \in L^2([0,T]; W^{3,2}(r,\rho))$, we obtain immediately $(L - L_h^\lambda)u \in L^2([0,T]; l^{0,2}(r))$.

So, we have that problem (4.27) satisfies the hypotheses of Theorem (4.2.4.4), therefore holding the estimate

$$\sup_{0\leq t\leq T} \|u(t) - u_h(t)\|^2_{l^{0,2}(r)} + \int_0^T \|u(t) - u_h(t)\|^2_{l^{1,2}(r,\rho)} dt$$
$$\leq N\Big(\|g - g_h\|^2_{l^{0,2}(r)} + \int_0^T \|f(t) - f_h(t)\|^2_{l^{0,2}(r)} dt + \int_0^T \|(L - L_h)(t)u(t)\|^2_{l^{0,2}(r)} dt\Big).$$

So the result follows. □

As a consequence of the Theorem (4.2.4.9), we can state:

Corollary 4.2.4.10. *Consider satisfied the conditions is Theorem (4.2.4.9). Let u be the solution of (4.18) in (4.2.4.4) and u_h the solution of the same problem in (4.2.4.7). If exists*

$$|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)} dt \leq h^2 N\Big(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt\Big)$$

then

$$\sup_{0\leq t\leq T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)} dt$$
$$\leq h^2 N\Big(\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} dt + |g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)} dt\Big).$$

for $m = 0, 1$.

Chapter 5

Conclusion and further research

In this thesis we used the framework in the works [31, 32] and extended the studies [26, 27, 28, 30, 45] to the spacial approximation to the solution of the Cauchy problem for a degenerate multidimensional second order linear parabolic PDE both in the cases where the PDE coefficients are bounded and unbounded.

The same results have been presented before but to the case where the operator of the second order linear parabolic PDE is nondegenerate. This thesis presents the results applied to the degenerate case, thus completing the existing gap on this theory scheme. Therefore, attending to Chapter 3 and to [26, 27, 28, 31, 45] and to this dissertation, we now have constituted a complete theory on the numerical approximation, with finite-difference schemes, of partial differential equations arising in finance, for both degenerate and degenerate cases, whether considering bounded and unbounded coefficients.

For that, a semi-discretized version of the PDE problem was constructed with the use of a finite-difference scheme, in discrete Sobolev and weighted Sobolev spaces.

Existence and uniqueness results for the generalized solution to the semi-discretized problem were deduced, as well as for its stability and for the scheme's consistency. Finally a convergence result was proved and a convergence rate obtained.

The case where there is only one dimension in space was treated separately, since stronger results could be obtained.

Although we have only studied the semi-discretization in space, a full discretization can be easily obtained by combining the results in the present study with the one in [26, 29, 45] for the time discretization.

Also, in order to obtain implementable numerical schemes, there is the need to localize the semi-discretized problem to a discrete bounded domain, with the imposition of artificial discrete boundary conditions. In this connection, the works [18, 19]), where transparent discrete boundary conditions are imposed, are particularly meaningful.

Finally, the order of accuracy of the schemes we constructed are very low. Therefore, there is also the need to accelerate the schemes.

Thus, possible future research directions are:

- Obtaining a full discretization in the whole space;
- Localizing the discretized problem to a finite computational domain, by imposing transparent discrete boundary conditions;
- Accelerating the numerical scheme, namely with the construction of an ADI scheme.

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