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On dividends and other quantities of interest in the dual risk model

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To my family

Acknowledgements

Most people could believe that the difficult part of obtaining a PhD degree is to get original and good results. I would dare to say that this is not completely true. I have found myself struggling much more writing the PhD thesis than I ever did for any mathematical proof. Perhaps the process of organization and synthesis of the information is of a more complex nature, or I least, it has been for me. I have to thank my supervisor Professor Alfredo Egidio dos Reis, for his patience, and the great support that he has provided to me during all these years, as a teacher, as a colleague, as a running partner in many races and, especially, as a friend. I am grateful to CEMAPRE - the Centre for Applied Mathematics and Economics for all the financial support that gave to me, providing the means to work (like the laptop which I am using to write this right now) and allowing me to participate in a multitude of congresses in several countries.

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Resumo

Nesta dissertação trabalhamos em teoria do risco. Damos principal ênfase principal nos modelos de risco e teoria de ruína, dedicando a nossa atenção a algumas das mais interessantes e relevantes quantidades da área: a probabilidade da ruína, a transformada de Laplace e os dividendos descontados esperados. Os modelos de risco têm o objetivo de resolver, ou pelo menos, fornecer uma solução aproximada, a problemas que aparecem na prática do negócio dos seguros. Os desenvolvimentos que produzimos nesta dissertação têm a mesma finalidade. A nossa intenção é apresentar novas ferramentas para o cálculo das quantidades mencionadas acima, e uma melhor compreensão delas na prática.

Consideramos o modelo dual de risco quando os tempos entre ganhos seguem uma distribuição exponencial matricial e, quando for possível, dar exemplos dos nossos resultados para casos particulares, como as distribuições Phase–Type e Erlang. Mostramos, na maioria dos casos, fórmulas e fazemos uso de técnicas matemáticas de várias áreas, como a teoria da probabilidade, a teoria das equações integro–diferenciais, álgebra linear, análise complexa, entre outras.

PALAVRAS-CHAVE: modelo de risco dual; distribuição exponencial matricial; probabilidade da ruína; transformada de Laplace; dividendos descontados esperados antes da ruína.

Abstract

In this manuscript we work on risk theory. The main emphasis is on risk models and ruin theory, devoting our attention to some of the most interesting and relevant quantities in this area: ruin probabilities, Laplace transforms and expected discounted dividends.

Risk models are meant to solve or, at least, provide an approximate solution, to problems that appear in the practice of the insurance business. The developments we produce in this dissertation have the same goal. Our aim is to present new tools for computation of the quantities mentioned above, and a better understanding of them in the practice.

We consider the dual risk model when the interclaim times follow a matrix exponential distribution and, whenever possible, we give examples of our findings for particular cases, like the Phase–Type, the Generalized Erlang and the Erlang distributions. We show, in most cases, explicit formulas and we make use of mathematical techniques from several areas, like probability theory, the theory of integro–differential equations, linear algebra, complex analysis, among others.

KEYWORDS: Dual risk model; Matrix exponential distribution; ruin probability; Laplace transforms; expected discounted dividends prior to ruin.

List of Symbols

$N(t)$	counting process
$S(t)$	aggregate claims process
T_u	time of ruin
W_i	interclaim time i
X_i	claim amount i
$\psi(u)$	ruin probability
c	premium income
$i.i.d.$	independent and identically distributed
$k(t)$	density of interclaim times
$p(x)$	density of claim amounts
u	initial capital

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Chapter 1

Introduction

In the 20th century many of the necessary tools for dealing with matters of insurance were developed. These consist of probability theory, statistics and stochastic processes. The Swedish mathematicians Filip Lundberg and Harald Cramér were pioneers in those areas. They realized in the first half of the 20th century that the theory of stochastic processes provides the most appropriate framework for modelling the claims arriving in an insurance business. Nowadays, the Cramér–Lundberg model is one of the backbones of non–life insurance mathematics. It has been modified and extended in very different directions and, moreover, has motivated research in various other fields of applied probability theory, such as queuing theory, branching processes, renewal theory, reliability, dam and storage models, extreme value theory, and stochastic networks.

In 1903 Lundberg laid the foundations of modern risk theory. Risk theory is a synonym for non–life insurance mathematics, which deals with the modeling of claims that arrive in an insurance business and which give insight on how much a premium has to be charged in order to avoid bankruptcy (ruin) of an insurance company.

One of the Lundberg's main contributions is the introduction of a simple model which is capable of describing the basic dynamics of a homogeneous insurance portfolio. By this we mean a portfolio of *contracts* or *policies* for similar risks such as car insurance for a particular kind of car, insurance against theft in households or insurance against water damage of family homes. There are three assumptions in the model:

- Claims happen at the times W_i satisfying $0 \leq W_1 \leq W_2 \leq \cdots$. We call them claim arrivals, claim times or claim arrival times, and constitute a sequence of i.i.d (independent and identically distributed) non–negative random variables.

- The i–th claim arriving at time W_i causes the claim size or claim severity X_i. The sequence $\{X_i\}$ constitutes as well an *i.i.d.* sequence of non–negative random variables.

- The claim size process $\{X_i\}$ and the claim arrival process $\{W_i\}$ are mutually independent.

The *i.i.d.* property of the claim sizes, X_i, reflects the fact that there is a homogeneous probabilistic structure in the portfolio. The assumption that the claim sizes and the claim times be independent is very natural from an intuitive point of view. But the independence of claim sizes and claim arrivals also makes the life of the mathematician much easier, i.e., this assumption is made for mathematical convenience and tractability of the model.

Risk theory has been an active research area in Actuarial Science since the 20th century. In the heart of risk theory is ruin theory, which discusses how an insurance portfolio may be expected to vary with time. Ruin is said to occur if the insurer's surplus drops under a specific lower bound. The probability that ruins occurs, commonly referred as the ruin probability, is a very important measure of risk.

Much of the literature on ruin theory is concentrated on the classical risk theory, where an insurer starts with an initial surplus u, collects premiums continuously at a positive constant rate of c, while the aggregate claim process follows a compound Poisson process. The main research interest is the calculation of finite and infinite time ruin probabilities. Later on, actuarial researchers considered more components related to the time of ruin, like the surplus prior to ruin, the severity or deficit at ruin and its maximum, the probability of attaining a given upper barrier before ruin and the expected discounted dividends. Many results involving those quantities have been found during the recent years.

Gerber and Shiu (1998) considers the evaluation of the expected discounted penalty function, giving a unified treatment to the surplus before ruin, the deficit at ruin and the time to ruin.

Great part of the results in the classical risk model, like the results of Gerber *et al.* (1987), Dufresne and Gerber (1988a), Dufresne and Gerber (1988b), Dickson (1992) and Dickson and Egídio dos Reis (1996), are obtained as particular cases when the discount factor is zero, and almost all the previous results in classical ruin theory can be extended to the case with a positive discounting factor.

Lin and Willmot (1999) proposed an approach to solve the defective renewal equation, in which the discounted penalty function is expressed in terms of a compound geometric tail. Lin and Willmot (2000) further used it to derive the moments of the surplus before ruin, the deficit at ruin and the time of ruin.

During the last decades there have been a great interest in more general surplus processes, like surplus models with stochastic premium income processes, classical surplus processes under an economical environment (investment and inflation), surplus processes with dependent claim amounts and claim inter–occurrence times, surplus processes in which aggregate claims come from some classes of dependent or independent businesses, surplus processes with general claim number processes, or classical risk models perturbed by an independent diffusion process.

Sparre-Andersen (1957) in a paper to the International Congress of Actuaries in New York proposed a generalization of the classical (Poisson) risk theory, instead of assuming just exponentially distributed independent inter–occurrence (interclaim) times, he introduced a more general distribution function but retained the assumption of independence. He let claims occur according to a more general renewal process and derived an integral equation for the corresponding ruin probability. Since then the Sparre–Andersen model has been studied by many authors. In addition, random walks and queueing theory have provided a more general framework, which has led to explicit results in the case where the waiting times or the claim severities have distributions related to the Erlang (see Borovkov (1976)).

Another generalization of the surplus model that have deserved some attention in the past years is what is called the *dual model*. One of the innovations introduced by this model, is that it considered the claims as "gains", i.e., positive jumps at random times, and pays "costs" at a constant rate. Several authors have addressed the dual model. We can go back to Gerber (1979), pages 136–138, who called it the negative claims model. We can even go further back in time to authors like Cramér (1955), Takács (1967), Seal (1969) and Bühlmann (1970).

The dual model has a simple but illustrative interpretation, the surplus can be considered as the capital of an economic activity like research and development where gains are random, at random instants, and costs are certain. More precisely, the company pays expenses which occur continuously along time for the research activity and gets occasional profits according to a renewal process.

Recent works on the dual model focus on the study of dividends. When we consider the payment of dividends by means of a barrier or a threshold strategy the time to ruin becomes finite, i.e., ruin is certain. Therefore there is great interest in the study of dividend strategies in order to maximize the expected amount of "income" that can be attained prior to ruin.

We can mention Avanzi *et al.* (2007), who works on an idea proposed by de Finetti (1957), to find the dividend-payment strategy that maximizes the expected discounted value of dividends which are paid to the shareholders until the company is ruined or bankrupt, assuming that the surplus or shareholders' equity is a Lévy process which is skip-free downwards and a barrier strategy. Later on Avanzi and Gerber (2008) they extend their results to an aggregate gains process composed by a shifted compound Poisson process and an independent Wiener process.

On Avanzi (2009), the authors make a a taxonomical synthesis of the 50 years of actuarial research on different dividend strategies that followed de Finetti's original paper de Finetti (1957).

Afonso *et al.* (2013) considered a compound Poisson dual risk model with an upper dividend barrier. By establishing a proper and crucial connection between the original Cramér–Lundberg model and the dual model they study different ruin and dividend probabilities, such as the calculation of the probability of a dividend, distribution of the number of dividends, expected and amount of dividends as well as the time of getting a dividend.

Hua (2013) investigated the dual of a Sparre–Andersen model perturbed by diffusion under a barrier strategy, in which innovation inter-arrival times have a generalized Erlang distribution. Integro–differential equations with certain boundary conditions for the expected total discounted dividends are derived.

Ji and Zhang (2014) considered the generalized Erlang risk model and its dual model. By using a conditional measure-preserving correspondence between the two models, they derive an identity for two interesting conditional probabilities. Applications to the discounted joint density of the surplus prior to ruin and the deficit at ruin are also discussed.

Bayraktar and Egami (2008) considered the dual risk model with capital investments and Bayraktar *et al.* (2013) generalized further their results using the fluctuation theory of spectrally positive Lévy processes to show optimality of barrier strategies.

Ng (2009) considered the dual of the compound Poisson model under a threshold dividend strategy. They derive a set of two integro-differential equations satisfied by the expected total discounted dividends until ruin and show how the equations can be solved by using only one of the two integro-differential equations. Then Ng (2010) considered again the dual of the compound Poisson model but assuming that the gains follow a Phase-Type distribution. By using the property of the Phase-Type distribution, two pairs of upcrossing and downcrossing barrier probabilities are derived.

In the same line of research of threshold strategies, Zeng and Xu (2013) make contributions on the moment-generation function of the present value of total dividends until ruin. Sendova and Yang (2014) considered times between positive gains independent and identically distributed following a generalized Erlang distribution to derive an explicit expression for the Laplace transform of the ruin time and the expected discounted dividends when the threshold-dividend strategy discussed by Ng (2009) is implemented under the Sparre–Andersen model with Erlang distribution of the inter-event times.

Wen (2011a) considered the dual of compound Poisson model with diffusion under a threshold dividend strategy and Wen (2011b) considered the dual of the generalized Erlang risk model under a threshold dividend strategy. In both articles they derive an integro–differential equation satisfied by the expectation of the discounted dividends until ruin.

Cheung (2012) studied the dual risk model to investigate the fair price of a perpetual insurance which pays the expenses whenever the available capital reaches zero; the probability of recovery by the first gain after default if money is borrowed at the time of default; and the Laplace transforms of the time of recovery and the first duration of negative capital.

Liu and Liu (2014) studied the dual risk model with a barrier strategy under the concept of bankruptcy, in which one has a positive probability to continue business despite temporary negative surplus. Integro–differential equations for the expectation of the discounted dividend payments and the probability of bankruptcy are derived.

Dong and Liu (2010) considered an extension to a dual model under a barrier strategy, in which the innovation sizes depend on the innovation time via the FGM (Farlie–Gumbel–Morgenstern) copula. They first derived a renewal equation for the expected total discounted dividends until ruin. Some differential equations and closed-form expressions are given for exponential innovation sizes. Then the optimal dividend barrier and the Laplace transform of the time to ruin are considered.

Chen and Xiao (2010) considered the ruin probability of a kind of dual risk model with a threshold. They assumed that the expenses with a constant rate,and the aggregate positive gains is a compound process. Besides, the gain size depends on the inter-arrival time. The integro-differential equations satisfied by the ruin probability are derived.

Zhu and Yang (2008) devoted to studying a dual Markov-modulated risk model for the calculation of both the finite and infinite horizon ruin probabilities. Upper and lower bounds of Lundberg type are derived for these ruin probabilities. They also obtained a time–dependent version of Lundberg type inequalities.

Ma *et al.* (2010) considered the dividend problem in a two-state Markov-modulated dual risk model, in which the gain arrivals, gain sizes and expenses are influenced by a Markov process. A system of integro-differential equations for the expected value of the discounted dividends until ruin is derived. In the case of exponential gain sizes, the equations are solved and the best barrier is obtained via numerical example.

Liu *et al.* (2013) studied the dual risk model in which periodic taxation are paid according to a loss-carry-forward system and dividends are paid under a threshold strategy. They gave an analytical approach to derive the expression of the Laplace transform of the first upper exit time. They discussed the expected discounted tax payments for this model and obtain its corresponding integro-differential equations.

Outlining this dissertation we present developments in a dual risk model when the times between gains follow a matrix exponential distribution. Our work involves developments in Lundberg's equations, the ruin probability, the Laplace transform of the time to ruin, the expected discounted dividends, the probability of a dividend, distribution of the number of dividends, expected and amount of dividends as well as the time of getting a dividend.

Chapter 2 reviews the relevant results and techniques in the literature on the classical and the dual risk model and gives the mathematical preliminaries to the thesis.

In Chapter 3 we consider developments in the ruin probability and the Laplace transform of the time to ruin.

Chapter 4 is completely devoted to the expected discounted dividends.

Chapters 3 and 4 are the main core of this thesis where new developments are presented. Numerical examples are discussed in the parts of the work related to the Lundberg's equations, the ruin probability and the expected discounted dividends for particular distributions, like the Phase–Type and the Erlang.

At last, some conclusions and comments on further research are set out in Chapter 5.

This thesis is based on the following papers:

1. In Rodríguez-Martínez *et al.* (2015) we study the dual risk model when the times between gains are Erlang distributed. Using the roots of the fundamental and the generalized Lundberg's equations, we get expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution. Furthermore, we compute expected discounted dividends, as well as higher moments, when the individual common gains follow a Phase–Type distribution.

2. In Bergel *et al.* (2016) we consider the dual risk renewal model when the waiting times are Phase–Type distributed. Using the roots of the fundamental and the generalized Lundberg's equations, we get expressions for the ruin probability and the Laplace transform of the time of ruin for an arbitrary single gain distribution. Then, we address the calculation of expected discounted future dividends particularly when the individual common gains follow a Phase–Type distribution. We further show that the optimal dividend barrier does not depend on the initial reserve.

As far as the roots of the Lundberg equations and the time of ruin are concerned, we address the existing formulae in the corresponding Sparre-Andersen insurance risk model for the first hitting time of an upper barrier, shown by Li (2008a) and Li (2008c), and we generalize them to cover also the situations where we have multiple roots. We do that working a new approach and technique, which we also use for working the dividends. Unlike others, it can be also applied for every situation. For the Erlang model there is no multiplicity, e.g. see Bergel and Egídio dos Reis (2015), for the generalised Erlang we can have double roots, see Bergel and Egídio dos Reis (2016). In other Phase–Type and matrix exponential models we can have higher multiplicity.

3. Last but not least, we are currently working on a paper that will be submitted in the future, that generalizes several results from Afonso *et al.* (2013) for the case of a matrix exponential distribution.

Chapter 2

Risk models

In this chapter we set out the main characterization of the models and the concepts of risk theory that we consider in this manuscript.

In Section 2.1 we describe the dual risk model and denote the aggregate gains as a random process $S(t)$.

Section 2.2 introduces the definitions of ruin probability, the Laplace transform of the time to ruin and the expected discounted dividends.

In Section 2.3 we present the existing connection between the dual risk model and the Cramér–Lundberg risk model. We define the Lundberg's equations and talk about their solutions.

Section 2.4 is devoted to introduce the integro–differential equations that are satisfied by the quantities in Section 2.2.

Finally, in Section 2.5 we focus on defining the matrix exponential distribution which we use along this manuscript.

2.1 The dual risk model

In a dual risk process, an insurer's surplus at a fixed time $t > 0$ is determined by three quantities: the amount of surplus at time 0, the amount of costs up to time t and the amount received as gains up to time t. The only one of these three which is random is the gains income, so we start by describing the aggregate gains process, which we denote by

$\{S(t)\}_{t\geq 0}$.

Let $\{N(t)\}_{t\geq 0}$ be a counting process for the number of gains, so that for a fixed value $t>0$, the random variable $N(t)$ denotes the number of gains that occur in the fixed time interval $(0,t]$.

The individual gain amounts are modeled as a sequence of independent and identically distributed random variables $\{X_i\}_{i=1}^{\infty}$, so that X_i denotes the amount of the i–th gain, with cumulative distribution function $P(x)$ and density $p(x)$. We assume the existence of $\mu_1 = E[X_1]$. We denote the Laplace transform of $p(x)$ by $\hat{p}(s)$.

Let the times between gains, or gain inter–occurrence times, be denoted by the sequence of random variables $\{W_i\}_{i=1}^{\infty}$, that we assume i.i.d. and independent from sequence $\{X_i\}$. Then we have $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$. The cumulative distribution function of the W_i is denoted by $K(t)$ with density $k(t)$.

We say that the aggregate gains amount up to time t, denoted $S(t)$, is

$$S(t) = \sum_{i=1}^{N(t)} X_i. \tag{2.1.1}$$

If $N(t)=0$ then $S(t)=0$.

In the compound Poisson dual risk model it is assumed that $\{N(t)\}_{t\geq 0}$ is a Poisson process and therefore the gain inter–occurrence times are exponentially distributed. In this case the aggregate gains process $\{S(t)\}_{t\geq 0}$ is a compound Poisson process.

In a more general renewal dual risk model the distribution of the gain inter–occurrence times is not necessarily exponential, and it is more difficult to determine the nature of the counting process $\{N(t)\}_{t\geq 0}$ for every possible distribution. For the total gains amount $S(t)$ the expectation can be easily calculated by exploiting the independence of $\{X_i\}$ and $N(t)$, provided $E[N(t)]$ and $E[X_1]$ are finite

$$E[S(t)] = E\left[E\left(\sum_{i=1}^{N(t)} X_i \,|N(t)\right)\right] = E[N(t)E[X_1]] = E[N(t)]E[X_1].$$

The expectation does not tell us too much about the distribution of $S(t)$. We learn more about the order of magnitude of $S(t)$ if we combine the information about $E[S(t)]$ with the variance $Var[S(t)]$.

Assume that $Var[N(t)]$ and $Var[X_1]$ are finite. Conditioning on $N(t)$ and exploiting the independence of $\{X_i\}$ and $N(t)$, we obtain

$$\begin{aligned} Var\left[\sum_{i=1}^{N(t)}(X_i - E[X_1])\,|N(t)\right] &= \sum_{i=1}^{N(t)} Var[X_i|N(t)] \\ &= N(t)Var[X_1|N(t)] = N(t)Var[X_1], \end{aligned}$$

and we can conclude that

$$\begin{aligned} Var[S(t)] &= E[N(t)Var[X_1]] + Var[N(t)E[X_1]] \\ &= E[N(t)]Var[X_1] + Var[N(t)](E[X_1])^2. \end{aligned}$$

Now we can describe the surplus process, denoted by $\{U(t)\}_{t\geq 0}$, as

$$U(t) = u - ct + S(t), \tag{2.1.2}$$

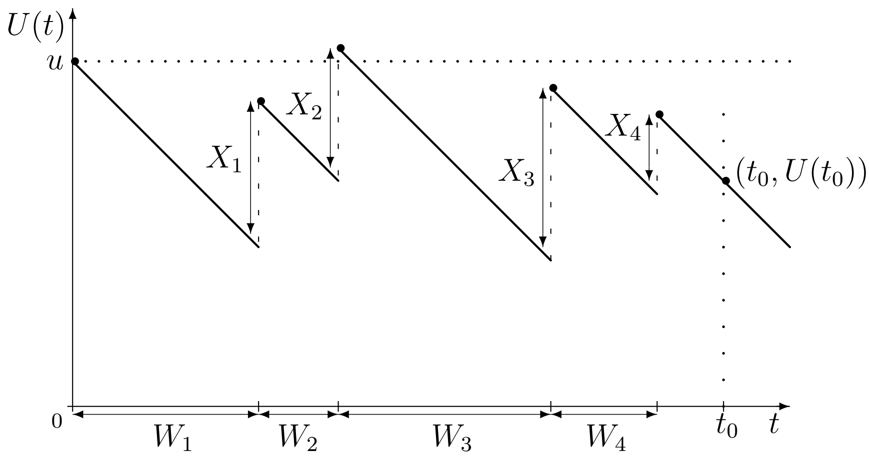
where u is the surplus at time 0 and c is the rate of costs per unit time, which we assume to be paid continuously.

Whenever the moment generating function of X_1 exists, we denote it by M_X and we assume that when it exists, there exists some quantity γ, $0 < \gamma \leq \infty$, such that $M_X(r)$ is finite for all $r < \gamma$ with

$$\lim_{r\to\gamma^-} M_X(r) = \infty$$

A graphical interpretation of the surplus process is given in Figure 2.1, where we see how the surplus process starts from the initial capital u at time $t = 0$, then decreases at the constant cost rate of c paid until the time W_1 when the first gain arrives, and continues over time. By the time t_0 the surplus process already had four received gains, so the counting process is equal to four.

A model of the form (2.1.2) seems to be natural for companies that have occasional gains



$$\mathbf{U(t_0) = u - ct_0 + X_1 + X_2 + X_3 + X_4, \quad N(t_0) = 4}$$

Figure 2.1: The surplus process

whose amount and frequency can be modelled by the process (2.1.1). For pharmaceutical or petroleum companies, the gain or "upward jump" should be interpreted as the net present value of future income from an invention or discovery. Other examples are commission–based business, such as real estate agent offices or brokerage firms that sell mutual funds or insurance products with a front–end load. Some authors have postulated that the model might be appropriate for an annuity or pension fund (see Cramér (1955), Takács (1967) and Seal (1969)). In summary, the surplus can be considered as the capital of an economic activity like research and development where gains are random, at random instants, and costs are certain.

2.2 The ruin probability, the Laplace transform of the time to ruin and the expected discounted dividends

In this moment we introduce some of the most common quantities of interest for the dual risk model. Specifically we talk about the ruin probability, the Laplace transforms of the time to ruin and the expected discounted dividends.

The probability of ruin in infinite time, also known as the ultimate ruin probability, is defined as

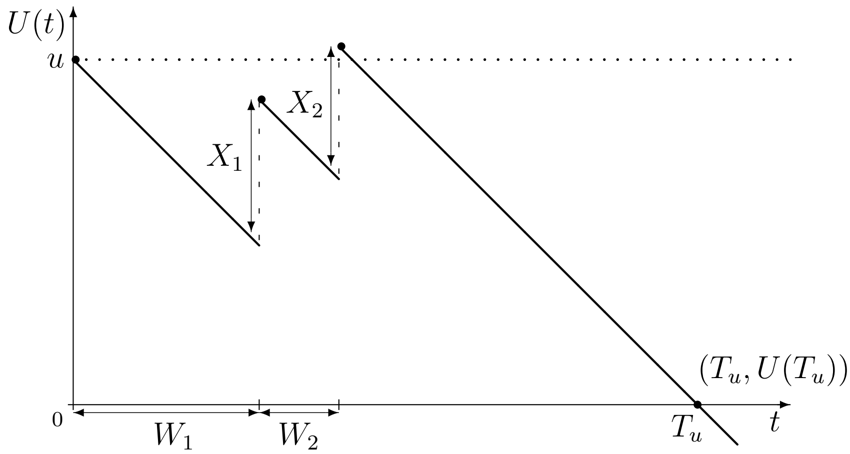
$$\psi(u) = Pr(U(t) \leq 0 \text{ for some } t > 0).$$

In words, $\psi(u)$ is the probability that the surplus falls below zero at some time in the future, that is when the constant costs exceeds the initial surplus plus the aggregate gains.

We denote the time to ruin, from initial surplus u, as the random variable T_u, so we have $T_u = \inf\{t > 0 : \ U(t) \leq 0\}, \ u \geq 0$, and $T_u = \infty$ if and only if $U(t) > 0 \ \ \forall t > 0$. Therefore, we can express the ruin probability as

$$\psi(u) = Pr(T_u < \infty).$$

Define $\phi(u) = 1 - \psi(u)$ to be the probability that ruin never occurs starting from initial surplus u. This probability is also known as the survival or non–ruin probability.



$\mathbf{U(T_u) \leq 0}$

Figure 2.2: The time to ruin

Figure 2.2 represents the time to ruin. In this example we have a surplus process with 2 received gains. Afterwards, no more gains were received and therefore the surplus fell at the constant rate c until it dropped to the level zero at the time $t = T_u$.

We also assume the so called net profit condition

$$cE[W_i] < E[X_i], \tag{2.2.1}$$

which means $cE[W_1] < \mu_1$, so that, per unit of time, the costs are less than the expected aggregate gain amount. This condition is very important and it brings an economical sense to the model. If this condition does not hold, then $\Psi(u) = 1$ for all $u \geq 0$. It is often convenient to write $cE[W_1] = (1-\theta)\mu_1$, so that $\theta > 0$ is the cost loading factor. During the interval of time W_i the costs paid are given by cW_i and the gain is X_i. The net profit $X_i - cW_i$ might be positive or negative, but on average it has to be positive. We show this in the Figure 2.3.

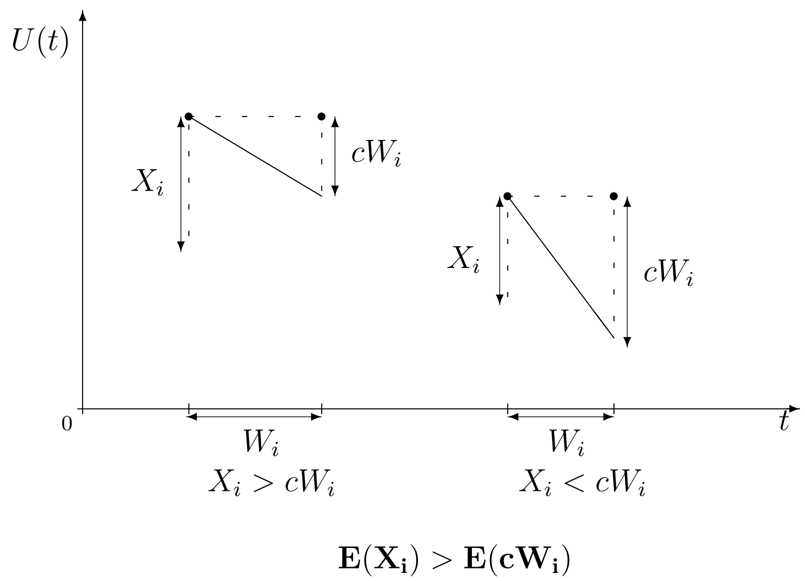


Figure 2.3: The net profit condition

The Laplace transform of the time to ruin is defined by

$$\psi(u,\delta) = E[e^{-\delta T_u}\mathbb{I}(T_u < \infty) \mid U(0) = u],$$

where $\delta > 0$ and $\mathbb{I}(.)$ is the indicator function. This Laplace transform can be interpreted as the expected value of one monetary unit received at the time of ruin discounted at the constant force of interest δ.

In particular, we can obtain the ultimate ruin probability $\psi(u)$ as a limiting case of the Laplace transform of the time to ruin, since

$$\lim_{\delta \to 0} \psi(u,\delta) \quad = \quad \psi(u). \tag{2.2.2}$$

We introduce an upper barrier into the dual model, let b denote its level. This barrier means a dividend payment level whose i-th single amount is going to be denoted by the random variable D_i explained as follows. Each time the surplus process upcrosses level b the excess gain is paid out immediately to the capital holders as a dividend, prior to ruin. Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u,b)$ be the aggregate discounted dividends, at force of interest δ and from initial surplus u. Let τ_i be the arrival time of D_i, then

$$D(u,b) = \sum_i e^{-\delta\tau_i} D_i.$$

We denote by $V_k(u,b) = E[D(u,b)^k]$, $k \geq 1$, the k-th order moment of $D(u,b)$. For simplicity denote $V(u,b) = V_1(u,b) = E[D(u,b)]$, the expected value of $D(u,b)$.

Note that

$$V(u,b) = E[u - b + D(b,b)] = u - b + V(b,b), \quad u \geq b. \tag{2.2.3}$$

If no gain arrives before the time $t_0 = u/c$ the surplus process arrives to ruin, and no dividends are paid. This is illustrated in the Figure 2.4.

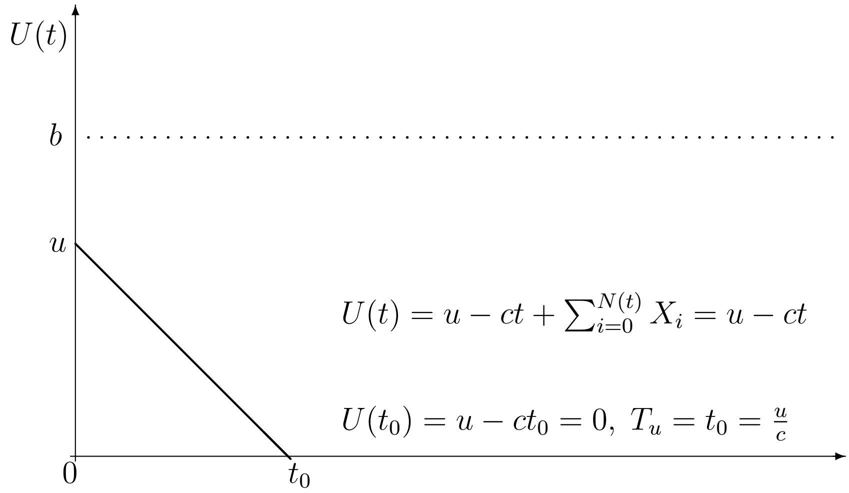


Figure 2.4: The dual risk model with an upper barrier level b.

On the contrary, if gains are received before the time $\frac{u}{c}$, the situation looks like in Figure 2.5.

2.3 The primal and the dual model. The Lundberg's equations.

In this section we make some connections of interest between the Cramér-Lundberg insurance risk model and the dual model. We could call the first as the *classical* or *standard risk model* however, often the literature when referring to the classical model it means the compound Poisson risk model, which is a particular case of the Sparre–Andersen risk model. So, we chose to call it simply the primal model.

The **primal model** is driven by an equation similar to (2.1.2)

$$U_P(t) = u + ct - \sum_{i=0}^{N(t)} X_i,\ t \geq 0,\ u \geq 0$$

where $U_P(t)$ represents the surplus of a portfolio of insurance risks at time t. For convenience we keep the same notation but note that the quantities involved have different meanings,

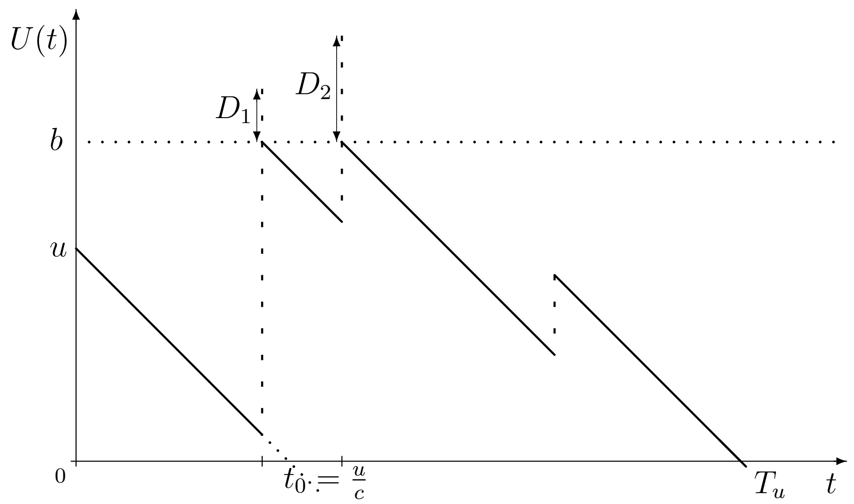


Figure 2.5: The dual risk model with an upper barrier level b.

particularly c and X_i, respectively premium rate and individual claim size i. Here, it is assumed a net profit condition $cE(W_i) > E(X_i)$ (reversed in comparison with (2.2.1)), which brings an economical sense to the model: it is expected that the income until the next claim is greater than the size of the next claim. The net income between the $(i-1)$-th and the i-th claims is $cW_i - X_i$. In this model it is well known the notion of the adjustment coefficient, provided that the moment generating function of X_1 exists, $M_X(\cdot)$. The adjustment coefficient, denoted as R, is the unique positive real root of the equation, developed as follows,

$$\begin{aligned} E\left[e^{-r(cW_1 - X_1)}\right] = 1 \quad &\Leftrightarrow \quad E\left[e^{-rcW_1}\right] E\left[e^{rX_1}\right] = 1 \\ &\Leftrightarrow \quad M_X(r) = \frac{1}{E\left[e^{-rcW_1}\right]}, r \in \mathbb{R}. \end{aligned} \tag{2.3.1}$$

We note that expectation $E\left[e^{rX_1}\right]$ exists at least for $r < 0$. Expectation $E\left[e^{-RcW_1}\right]$ is a Laplace transform and W_1 follows, on this manuscript, a light tail distribution (matrix exponential, and especial cases Phase–type or Erlang). The lefthand side of the starting equation above can be regarded as the expected discounted profit for each *waiting arrival period*. So that the adjustment coefficient R, provided that it exists, makes the expected discounted profit even (considering that premium income and claim costs come together). Constant R is then seen as an interest force. The equation (2.3.1) is known as the fundamental Lundberg's equation.

Now, let's take a similar perspective for the **dual model** case, driven by equation (2.1.2). The fundamental Lundberg's equation is now given as

$$E\left[e^{-s(X_1-cW_1)}\right]=1 \Leftrightarrow E\left[e^{scW_1}\right]E\left[e^{-sX_1}\right]=1 \Leftrightarrow \hat{p}(s)=\frac{1}{E\left[e^{scW_1}\right]}, \tag{2.3.2}$$

where the corresponding net income per *waiting arrival period* i is given by the reversed difference $X_i - cW_i$. In either case the fundamental Lundberg's equation has the same form, but here we do not have to assume the existence of the moment generating function of X_1, if we consider $s > 0$, and the definition of a similar constant to the one of the adjustment coefficient is not needed, we would indeed need the existence of expectation $E\left[e^{scW_1}\right]$ if a general distribution of W_1 were considered. As a final remark, obviously, if we set $r = -s$, equations (2.3.1) and (2.3.2) look the same.

In practice, the Lundberg's equation in the primal and the dual model are almost the same, the only difference between them lies in the nature of their roots, which depend on different choices of parameters. In the primal model, the roots will depend on any choice of parameters such the net profit condition $cE(W_i) > E(X_i)$ is satisfied. In the dual model, the choice of parameters must satisfied the reversed net profit condition $cE(W_i) < E(X_i)$.

Next, we extend the domain for $s \in \mathbb{C}$.

Since $E\left[e^{-sX_1}\right] = M_X(-s) = \hat{p}(s)$, where $\hat{p}$ denotes the Laplace transform of the gain amounts density p, we could even go further with this notation and write $E\left[e^{scW_1}\right] = \hat{k}(-cs)$, where $\hat{k}$ denotes the respective Laplace transform of the gain inter–occurrence times density k. Thus, the fundamental Lundberg's equation becomes

$$\hat{p}(s)=\frac{1}{\hat{k}(-cs)}, \quad \text{or} \quad \hat{k}(-cs)\hat{p}(s)=1. \tag{2.3.3}$$

From now on every time when we refer to the fundamental Lundberg's equation we refer to the equation (2.3.3).

A generalization of each of (2.3.2) was introduced to the actuarial literature and became known as the generalized Lundberg's equation. It takes the following form, for a constant

$\delta > 0$ (see e.g. Landriault and Willmot (2008)):

$$E\left[e^{-\delta W_1}e^{-s(X_1-cW_1)}\right] = 1,$$

In our notation, it becomes

$$\hat{p}(s) = \frac{1}{\hat{k}(\delta - cs)}, \quad \text{or} \quad \hat{k}(\delta - cs)\hat{p}(s) = 1. \tag{2.3.4}$$

From now on every time when we refer to the generalized Lundberg's equation we refer to the equation (2.3.4).

This last equation can be found in Gerber and Shiu (2005) and Ren (2007). We can think of (2.3.3) as the limiting case of (2.3.4) when $\delta \to 0^+$.

In the barrier and dividend problems that we treat later in this thesis, it is introduced the notion of $\delta > 0$ as an interest rate. For the calculation of dividends and the Laplace transform of the time to ruin we use the generalized Lundberg's equation (2.3.4). For the calculation of the ruin probability we use the fundamental Lundberg's equation (2.3.3).

2.3.1 Roots of the Lundberg's equations

The roots of the Lundberg's equations play an important role in the calculation of many quantities that are fundamental in risk and ruin theory. Namely, the ultimate and finite time ruin probabilities, the Laplace transform of the ruin time, the expected discounted future dividends, among others. All those calculations depend on the nature of the roots of the Lundberg's equation, particularly those roots with positive real parts. A study on the multiplicity of the roots can be found in Bergel and Egídio dos Reis (2014) and Bergel and Egídio dos Reis (2016). For the Erlang(n) model there is no multiplicity, e.g. see Bergel and Egídio dos Reis (2015), for the generalised Erlang(n) we can have double roots, see Bergel and Egídio dos Reis (2016). In other $Ph(n)$ and more general models we can have higher multiplicity.

In Rodríguez-Martínez *et al.* (2015) we showed that both the fundamental and the generalized Lundberg's equations have exactly n roots with positive real parts in a dual risk model with Erlang(n) distributed gain inter–occurrence times. We compared with the Sparre–Andersen risk model with Erlang(n) distributed interclaim times, where the situation is different, i. e., the generalized Lundberg's equation has still n roots with positive real parts, but the fundamental only has $n-1$ of these roots (see Li and Garrido (2004)). We illustrate this on Figure 2.6 for the case of Erlang(n) distributed times, and we give a brief description of it in the following paragraphs below.

First, we used Rouché's theorem to prove that the generalized Lundberg's equation has exactly n roots with positive real parts. This fact is true for more general distributions, like the Phase–Type (see Albrecher and Boxma (2005)). Then with considered the limiting case $\delta \to 0^+$ and showed that all these roots remain with positive real parts. This proves that the fundamental Lundberg's equation also has exactly n roots with positive real parts.

Let $\rho_1(\delta), \ldots, \rho_n(\delta)$ denote the roots of the generalized Lundberg's equation with positive real parts. At least one of them is real, say $\rho_n(\delta)$.

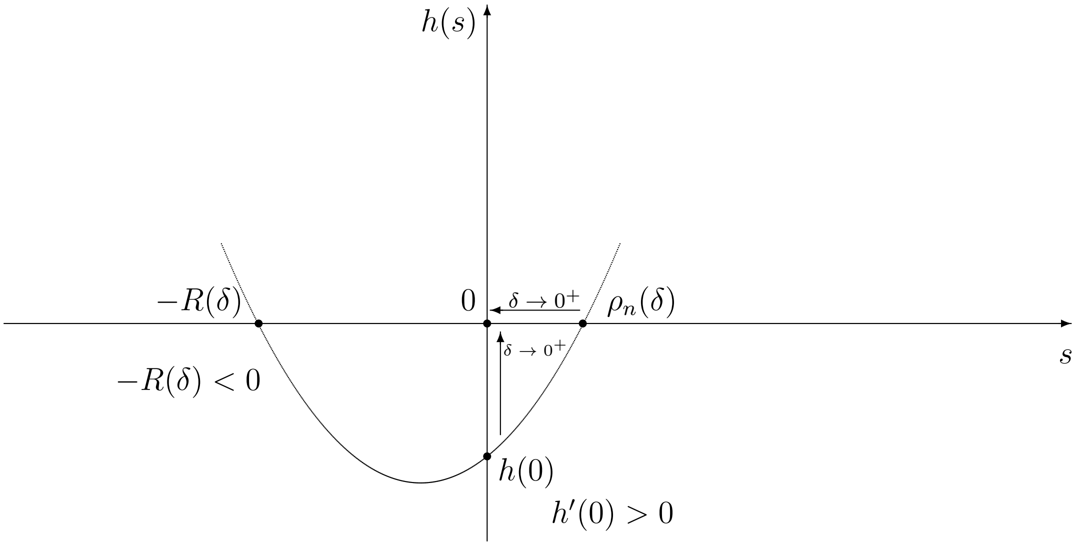
We have at least one negative real root, we denote the largest by $-R(\delta)$ (this is the adjustment coefficient for a Sparre–Andersen or primal risk model).

On the one hand, using the net profit condition $cE(W_i) > E(X_i)$ (primal model) we arrive at the limiting case:

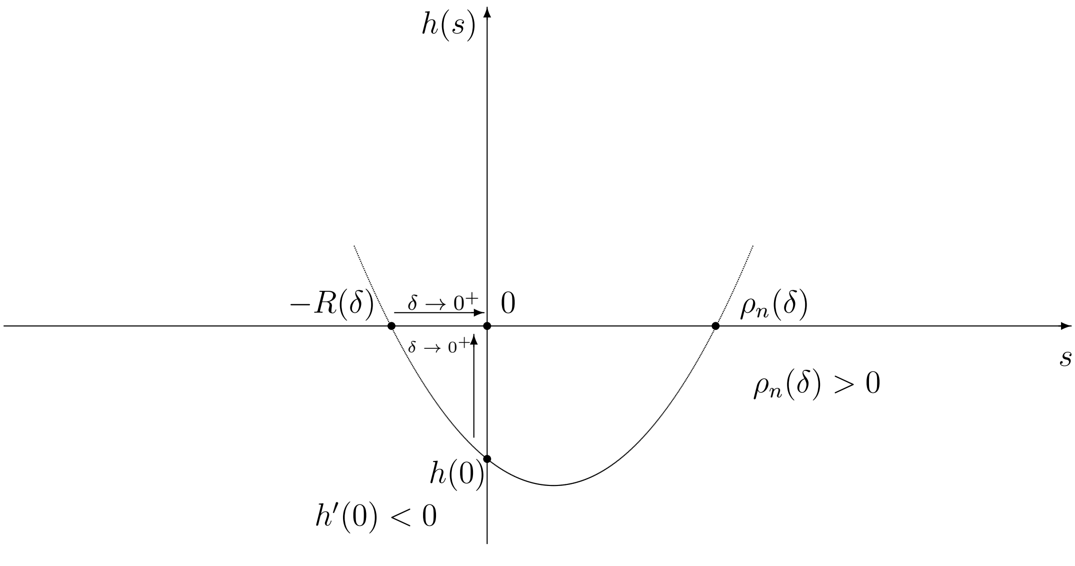
$$\begin{aligned} \lim_{\delta \to 0^+} (-R(\delta)) &< 0, \\ \lim_{\delta \to 0^+} \rho_n(\delta) &= 0, \\ \lim_{\delta \to 0^+} Re(\rho_i(\delta)) &> 0, i = 1, \ldots, n-1. \end{aligned}$$

On the other hand, using the net profit condition $cE(W_i) < E(X_i)$ (dual model) we arrive at the limiting case:

$$\begin{aligned} \lim_{\delta \to 0^+} (-R(\delta)) &= 0 \\ \lim_{\delta \to 0^+} \rho_n(\delta) &> 0 \\ \lim_{\delta \to 0^+} Re(\rho_i(\delta)) &> 0, i = 1, \ldots, n-1. \end{aligned}$$



Primal Model: $h(s) = \hat{p}(s) - \frac{1}{\hat{k}(\delta - cs)}$ when $\delta \to 0^+$.



Dual Model: $h(s) = \hat{p}(s) - \frac{1}{\hat{k}(\delta - cs)}$ when $\delta \to 0^+$.

Figure 2.6: The behavior at $\delta \to 0^+$ of the roots of the Lundberg's equations

Thus, in the dual model, both the fundamental and the generalized Lundberg's equation have n roots with positive real parts. For simplicity we will denote $\rho_i(\delta)$ by ρ_i, $i = 1, ..., n$, unless stated otherwise.

2.4 Integro–differential equations

In this section we present the renewal and integro–differential equations satisfied by three important quantities: the ruin probability, the Laplace transform of the time to ruin and the expected discounted dividends.

2.4.1 The ruin probability and the Laplace transform of the time of ruin

In the dual risk model with exponentially distributed gain inter–occurrence times, i.e. $k(t) = \lambda e^{-\lambda t}$, the ruin probability satisfies the following renewal equation

$$\psi(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{\infty} p(x)\psi(u - ct + x)dx\, dt, \tag{2.4.1}$$

where $t_0 = u/c$ is the time of ruin without any gain arrival. This can be found in Afonso *et al.* (2013), and it is derived by conditioning on the time and amount of the first gain.

Differentiating with respect to u and rearranging, we get an integro–differential equation for $\psi(u)$ given by

$$\psi(u) + \left(\frac{c}{\lambda}\right) \frac{d}{du}\psi(u) = \int_0^{\infty} p(x)\psi(u + x)dx.$$

We can write this equation as

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D}\right) \psi(u) = \int_0^{\infty} p(x)\psi(u + x)dx, \tag{2.4.2}$$

where $\mathcal{I}$ is the identity operator and $\mathcal{D}$ is the differentiation operator, $\mathcal{D} = d/du$.

We can extend the previous method for more general distributions, like the distributions belonging to the matrix exponential family, i. e. Phase–Type, Erlang, among others. The renewal equation corresponding to (2.4.1) becomes

$$\psi(u) = 1 - K(t_0) + \int_0^{t_0} k(t) \int_0^{\infty} p(x)\psi(u - ct + x)dx\, dt. \tag{2.4.3}$$

The integro–differential equation analogous to (2.4.2) will be given in the following chapter.

In a similar way, conditioning on the time and the amount of the first gain, we find that the Laplace transform of the time to ruin for the dual risk model satisfies the renewal equation

$$\psi(u,\delta) = \left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \int_0^{\frac{u}{c}} k(t)e^{-\delta t} \int_0^{\infty} p(x)\psi(u - ct + x, \delta)dx\, dt.$$

Note that the above equation is valid for any renewal model with density k and distribution K. Changing variables $s = u - ct$, we get

$$\psi(u,\delta) = \left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^{u} k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} W_{\psi}(s,\delta)\, ds, \tag{2.4.4}$$

where $W_{\psi}(s,\delta) = \int_0^{\infty} p(x)\psi(s + x, \delta)dx$.

The integro–differential equations for the Laplace transform of the time to ruin will be considered in the next chapter.

2.4.2 The expected discounted dividends

The expected discounted dividends $V(u, b)$ satisfy the following renewal equation:

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} k(t)e^{-\delta t} \left[\int_0^{b-u+ct} V(u - ct + y, b)p(y)dy \right.\\ &\quad \left. + \int_{b-u+ct}^{\infty} \widetilde{V}(u - ct + y, b)p(y)dy\right] dt, \quad \text{for } u < b, \end{aligned}$$

with

$$\widetilde{V}(x, b) = E[D(x, b)] = E[x - b + D(b, b)] = x - b + V(b, b), \quad x \geq b.$$

The k-th ordinary moment of the discounted dividends $V_k(u,b)$ satisfies the renewal equation

$$\begin{aligned} V_k(u,b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-\delta kt}\left[\int_0^{b-u+ct} V_k(u-ct+y,b)p(y)dy+\right. \\ &\left.\int_{b-u+ct}^{\infty} \widetilde{V}_k(u-ct+y,b)p(y)dy\right]dt, \end{aligned}$$

with

$$\widetilde{V}_k(x,b) = \sum_{j=0}^{k}\binom{k}{j}(x-b)^j V_{k-j}(b,b), \quad x \geq b.$$

In the above expression we have $V_0(u,b) \equiv 1$.

The integro–differential equations for the expected discounted dividends will be studied in the next chapter.

2.5 The matrix exponential distribution

In probability theory, the matrix-exponential distribution is an absolutely continuous distribution with rational Laplace–Stieltjes transform (see Asmussen and O'Cinneide (2006)). They were first introduced by David Cox in 1955 as distributions with rational Laplace–Stieltjes transforms, see Bean *et al.* (2008).

The probability density function is

$$f(x) = \boldsymbol{\alpha} e^{\mathbf{B}x}\mathbf{b}^{\mathsf{T}} \quad \text{for} \quad x \geq 0$$

(and 0 when $x < 0$) where

$$\boldsymbol{\alpha} \in \mathbb{R}^{1\times n},\ \mathbf{B} \in \mathbb{R}^{n\times n},\ \mathbf{b} \in \mathbb{R}^{1\times n}.$$

There are no restrictions on the parameters $\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b}$ other than that they correspond to a probability distribution, see He and Zhang (2007). There is no straightforward way to ascertain if a particular set of parameters form such a distribution (Bean *et al.* (2008)). The dimension of the matrix $\mathbf{B}$ is the order of the matrix-exponential representation (Asmussen and O'Cinneide (2006)).

The distribution is a generalisation of the Phase–Type distribution.

In the following chapters we will work with distributions belonging to the matrix–exponential family, namely de Phase–Type and the Erlang. Next, we set our notation for these distributions.

2.5.1 The Phase–Type distribution

Phase–type distributions are the computational vehicle of much of modern applied probability. Typically, if a problem can be solved explicitly when the relevant distributions are exponentials, then the problem may admit an algorithmic solution involving a reasonable degree of computational effort, if one allows for the more general assumption of phase–type structure, and not in other cases. A proper knowledge of phase–type distributions seems therefore a must for anyone working in an applied probability area like risk theory.

We say that a distribution K on $(0, \infty)$ is Phase–Type(n) if K is the distribution of the lifetime of a terminating continuous time Markov process $\{J(t), t \geq 0\}$ with finitely many states and time homogeneous transition rates. More precisely, we define a terminating Markov process $\{J(t), t \geq 0\}$ with state space $E = \{1, 2, \ldots, n\}$ and intensity matrix $\mathbf{B}$ $(n \times n)$ as the restriction to E of a Markov process $\{\bar{J}(t), 0 \leq t < \infty\}$ on $E_0 = E \cup \{0\}$ where 0 is some extra state which is absorbing, that is, $Pr(\bar{J}(t) = 0 | \bar{J}(0) = i) = 1$ for all $i \in E$ and where all states $i \in E$ are transient. This implies in particular that the intensity matrix for $\{\bar{J}(t)\}$ can be written in block–partitioned form as

$$\left(\begin{array}{c|c} \mathbf{B} & \mathbf{b}^{\mathsf{T}} \\ \hline \mathbf{0} & 0 \end{array}\right). \tag{2.5.1}$$

The $(1 \times n)$ vector $\mathbf{b} = (b_1, \ldots, b_n)$ is the exit rate vector, i.e., the i–th component b_i gives the intensity in state i for leaving E and going to the absorbing state 0.

Note that since (2.5.1) is the intensity matrix of a non–terminating Markov process, the rows sums to zero which in matrix notation can be written as $\mathbf{b}^{\mathsf{T}} + \mathbf{B}\mathbf{1}^{\mathsf{T}} = \mathbf{0}$ where $\mathbf{1} = (1, 1, \ldots, 1)$ is the $(1 \times n)$ vector with all components equal to one. In particular we have

$$\mathbf{b}^{\mathsf{T}} = -\mathbf{B}\mathbf{1}^{\mathsf{T}}.$$

The intensity matrix $\mathbf{B}$ is denoted by $\mathbf{B} = (b_{ij})_{i,j=1}^{n}$. This matrix satisfies the conditions: $b_{ii} < 0$, $b_{ij} \geq 0$ for $i \neq j$, and $\sum_{j=1}^{n} b_{ij} \leq 0$ for $i = 1, \ldots, n$. The vector of entry probabilities is given by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$, so $Pr(\bar{J}(0) = i) = \alpha_i$. We illustrate the Phase–Type distribution in Figure 2.7.

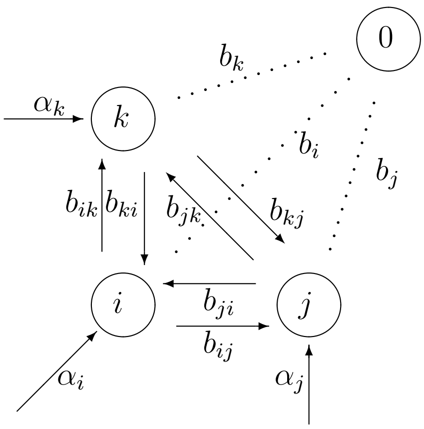


Figure 2.7: The Phase–Type distribution

Below we list expressions of most of the quantities of interest related to K, density, distribution, Laplace transform, mean and the j-th derivative of $k(t)$ at 0:

$$\begin{aligned}
k(t) &= \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{b}^{\mathsf{T}}, \quad t \geq 0, \\
K(t) &= 1 - \boldsymbol{\alpha} e^{\mathbf{B}t}\mathbf{1}^{\mathsf{T}}, \quad t \geq 0, \\
\hat{k}(s) &= \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \\
E[W_1] &= -\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}}, \\
k^{(j)}(0) &= \boldsymbol{\alpha}\mathbf{B}^{j}\mathbf{b}^{\mathsf{T}}, \quad j \geq 0,
\end{aligned} \tag{2.5.2}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.

It is important to notice that we can write the corresponding *net profit condition* (2.2.1) as

$$-c\boldsymbol{\alpha}\mathbf{B}^{-1}\mathbf{1}^{\mathsf{T}} < \mu_1. \tag{2.5.3}$$

We call the parameters $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b})$ the representation of the Phase–Type distribution.

2.5.2 The Erlang distribution

The Erlang(n) distribution is a particular example of the Phase – Type(n) distribution. The corresponding representation $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b})$ is given by

$$\boldsymbol{\alpha} = (1, 0, \ldots, 0)$$

$$\mathbf{b} = (0, 0, \ldots, \lambda)$$

$$\mathbf{B} = \begin{pmatrix} -\lambda & \lambda & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda & \lambda \\ 0 & 0 & \cdots & 0 & -\lambda \end{pmatrix}$$

Therefore we obtain the probability density function and the cumulative distribution function

$$\begin{aligned} k_n(t) &= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \ t \geq 0, \\ K_n(t) &= 1 - \sum_{i=0}^{n-1} \frac{1}{i!} e^{-\lambda t} (\lambda t)^i, \ t \geq 0. \end{aligned}$$

Some interesting properties of the Erlang(n) probability density function (for $n \geq 2$), are the following

$$\begin{aligned} k_n'(t) &= \lambda(k_{n-1}(t) - k_n(t)), \\ k_n^{(i)}(0) &= 0, \quad i = 0, \ldots, n-2, \\ k_n^{(n-1)}(0) &= \lambda^n. \end{aligned}$$

These properties do not hold for a general Phase–Type distribution. They will allow us to obtain easier formulas for ruin probabilities, as we shall see in the next chapter.

2.6 Final remarks

In this chapter we have learned about some of the most important quantities of interest in Risk Theory. We have taken a closer look at the renewal risk models, specifically the Sparre–Andersen and the dual risk models and the relation between the two of them. We presented the Lundberg's equations and described the probability distributions that will be used in the remaining of this thesis. In the next chapter we will evaluate ruin probabilities and the Laplace transform of the time of ruin for a dual risk model with gain inter–occurrence times distributed according to a matrix exponential distribution, specifically, a Phase–Type or Erlang.

Chapter 3

Lundberg's equation, ruin probabilities and Laplace transforms

3.1 Introduction

In this chapter we study the ruin probability and the Laplace transform of the time to ruin for a dual risk model. The distribution of the gain inter–occurrence times is matrix exponential. However, for simplicity and elegance of the formulas we concentrate on the Phase–Type and the Erlang distributions.

In Section 3.2 we devote attention to the Erlang case. For that purpose we go back to the Lundberg's equations, which will be useful to obtain closed formulas for the ruin probability and the Laplace transform of the time to ruin.

In Section 3.3 we do the same as in Section 3.2 but considering the Phase–Type distribution instead the Erlang distribution. Given the proximity between the Phase–Type and the more general matrix exponential distributions, the methods used can be easily extended to the latter.

We provide examples of our results whenever possible.

3.2 The Erlang case

The fundamental Lundberg's equation from the previous chapter (2.3.3) can be written in the form

$$\left(\frac{\lambda}{c} - s\right)^n = \left(\frac{\lambda}{c}\right)^n \hat{p}(s). \tag{3.2.1}$$

This equation is considered for the calculation of ruin probabilities.

On the other hand we have the generalized Lundberg's equation (2.3.4) expressed as

$$\left(\frac{\lambda+\delta}{c} - s\right)^n = \left(\frac{\lambda}{c}\right)^n \hat{p}(s), \tag{3.2.2}$$

where $\delta > 0$ is the force of interest. We use this equation for the calculation of the Laplace transform of the time to ruin.

We recall from Chapter 2 the roots of the fundamental (generalized) Lundberg's equation with positive real parts: $\rho_1, \rho_2, \ldots, \rho_n \in \mathbb{C}$. In the case of the generalized Lundberg's equation, those roots depend on δ. However, we will use the same notation for the roots of (3.2.1) and (3.2.2).

We assume the net profit condition (2.2.1), which in our case becomes

$$cE[W_i] < E[X_i] \Longleftrightarrow c\frac{n}{\lambda} < \mu_1. \tag{3.2.3}$$

3.2.1 The ruin probability

We already know that the ruin probability is a limiting case of the Laplace transform of the time to ruin when the interest force δ tends to zero. However, we pay attention to both the former and the latter separately to make some important remarks.

For Erlang(n) distributed gain inter–occurrence times, the renewal equation satisfied by the ruin probability is

$$\psi(u) = 1 - K_n(t_0) + \int_0^{t_0} k_n(t) \int_0^{\infty} p(x)\psi(u - ct + x)dx\, dt. \tag{3.2.4}$$

The integro–differential satisfied by the ruin probability is given in following theorem.

Theorem 1 In the Erlang(n) dual risk model the ruin probability satisfies the integro–differential equation

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi(u)=\int_0^\infty p(x)\psi(u+x)dx, \tag{3.2.5}$$

with boundary conditions

$$\psi(0)=1 \text{ and } \left.\frac{d^i}{du^i}\psi(u)\right|_{u=0}=0,\ i=1,\dots,n-1. \tag{3.2.6}$$

Proof. We proceed taking successive derivatives of the ruin probability using the renewal equation (3.2.4). Changing the variable, $u-ct=s$, the renewal equation can be rewritten in the form

$$\psi(u)=1-K_n\left(\frac{u}{c}\right)+\frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right)W(s)ds,$$

where $W(s)=\int_0^\infty \psi(s+x)p(x)dx$.

After applying the operator $(\mathcal{I}+(c/\lambda)\mathcal{D})$ to the ruin probability we get

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi(u)=1-K_{n-1}\left(\frac{u}{c}\right)+\frac{1}{c}\int_0^u k_{n-1}\left(\frac{u-s}{c}\right)W(s)ds.$$

Following an inductive argument, it is easy to show that

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^i\psi(u)=1-K_{n-i}\left(\frac{u}{c}\right)+\frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)W(s)ds,$$

for $i=1,\dots,n-1$. Particularly, we have

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi(u)=1-K_1\left(\frac{u}{c}\right)+\frac{1}{c}\int_0^u k_1\left(\frac{u-s}{c}\right)W(s)ds.$$

Applying the operator once more, we get

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n\psi(u)=W(u).$$

This proves equation (3.2.5). We have used here some very known properties of the Erlang(n)

probability density function (for $n \geq 2$), namely

$$\begin{aligned} k_n'(t) &= \lambda(k_{n-1}(t) - k_n(t)), \\ k_n^{(i)}(0) &= 0, \quad i = 0, \ldots, n-2, \\ k_n^{(n-1)}(0) &= \lambda^n. \end{aligned}$$

We now prove the boundary conditions. Clearly, $\psi(0) = 1$. We find the remaining conditions by computing directly the derivatives of $\psi(u)$ and evaluating at $u = 0$,

$$\frac{d^i}{du^i}\psi(u) = -\left(\frac{1}{c}\right)^i k_n^{(i-1)}\left(\frac{u}{c}\right) + \left(\frac{1}{c}\right)^{i+1} \int_0^u k_n^{(i)}\left(\frac{u-s}{c}\right) W(s)ds$$

for $i = 1, \ldots, n-1$. Hence, we obtain

$$\left.\frac{d^i}{du^i}\psi(u)\right|_{u=0} = 0, \; i = 1, \ldots, n-1.$$

■

The solution for the integro-differential equation (3.2.5) with boundary conditions given by (3.2.1) is shown in the following theorem.

Theorem 2 The ultimate ruin probability can be written as a combination of exponential functions

$$\psi(u) = \sum_{k=1}^{n} \left[\prod_{i=1, i\neq k}^{n} \frac{\rho_i}{(\rho_i - \rho_k)} \right] e^{-\rho_k u}, \tag{3.2.7}$$

where $\rho_1, \ldots, \rho_n$ are the only roots of the fundamental Lundberg's equation (3.2.1) which have positive real parts.

Proof. Let's consider a general solution $f(u)$ for equation (3.2.5)

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u) = \int_0^\infty p(x) f(u+x)dx. \tag{3.2.8}$$

We now look for particular solutions of this equation. Let $f(u) = e^{-ru}$, for some $r \in \mathbb{C}$.

Then, for the left hand side of (3.2.8) we obtain

$$\left(\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n f(u)=\left(1-\left(\frac{c}{\lambda}\right)r\right)^n e^{-ru},$$

hence,

$$\left(1-\left(\frac{c}{\lambda}\right)r\right)^n=\hat{p}(r),$$

which means that r must be a root of the fundamental Lundberg's equation (3.2.1).

Define the functions $f_1(u)=e^{-\rho_1 u},\ldots,f_n(u)=e^{-\rho_n u}$. Since they are linearly independent we can write any solution of (3.2.8) as

$$f(u)=\sum_{i=1}^{n} a_i e^{-\rho_i u},$$

where a_i, $i=1,...,n$, are constants. To get a formula for $\psi(u)$ we must find the constants a_i using the boundary conditions (3.2.6). These can be determined by solving a system of n equations on the unknowns $a_1,...,a_n$. In matrix form we have

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{a}=\mathbf{P}^{-1}\mathbf{e},$$

where $\mathbf{P}=\mathbf{P}(\rho_1,\ldots,\rho_n)$ is a Vandermonde matrix, $\mathbf{a}'=(a_1,a_2,\ldots,a_n)$ and $\mathbf{e}'=(1,0,\ldots,0)$. The determinant of $\mathbf{P}$ is given by

$$Det\ \mathbf{P}=\prod_{1\le i<j\le n}(\rho_j-\rho_i),$$

and using Cramér's rule we get expressions for the coefficients

$$\begin{aligned} a_k &= \frac{(-1)^{k-1}(\prod_{i=1,\ i\neq k}^{n}\rho_i)(\prod_{1\le i<j\le n,\ i\neq k,\ j\neq k}(\rho_j-\rho_i))}{\prod_{1\le i<j\le n}(\rho_j-\rho_i)} \\ &= \frac{(-1)^{k-1}(\prod_{i=1,\ i\neq k}^{n}\rho_i)}{(\prod_{i=1}^{k-1}(\rho_k-\rho_i))(\prod_{j=k+1}^{n}(\rho_j-\rho_k))} = \prod_{i=1,i\neq k}^{n}\frac{\rho_i}{(\rho_i-\rho_k)}. \end{aligned}$$

■

Remarks:

1. Note that although some of the roots are complex, expression (3.2.7) is always a real number.

2. If we considered the net profit condition from the primal model, $cE(W_i) > E(X_i)$, recall that $\rho_n = 0$ as explained at the end of Section 2.3, then we would have $a_n = 1$ and all the remaining coefficients $a_k = 0$, $k = 1, ..., n-1$, therefore giving $\psi(u) = 1$ as expected.

Example 1 For $n = 1$ (exponential or Erlang(1) case): Gerber (1979) found that $\psi(u) = e^{-\rho_1 u}$, where ρ_1 is the unique positive root of the fundamental Lundberg's equation (3.2.1).

For $n = 2$:

$$\psi(u) = \frac{\rho_2}{\rho_2 - \rho_1} e^{-\rho_1 u} - \frac{\rho_1}{\rho_2 - \rho_1} e^{-\rho_2 u},$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $\left(1 - \left(\frac{c}{\lambda}\right) s\right)^2 = \hat{p}(s)$.

For $n = 3$:

$$\psi(u) = \frac{\rho_2\rho_3}{(\rho_3 - \rho_1)(\rho_2 - \rho_1)} e^{-\rho_1 u} - \frac{\rho_1\rho_3}{(\rho_3 - \rho_2)(\rho_2 - \rho_1)} e^{-\rho_2 u} + \frac{\rho_1\rho_2}{(\rho_3 - \rho_1)(\rho_3 - \rho_2)} e^{-\rho_3 u},$$

where ρ_1, ρ_2, ρ_3 are solutions of $\left(1 - \left(\frac{c}{\lambda}\right) s\right)^3 = \hat{p}(s)$; one root is real and the other two are complex conjugates.

3.2.2 The Laplace transform of the time to ruin

For the Erlang(n) case, the Laplace transform of the time to ruin satisfies the renewal equation

$$\psi(u, \delta) = (1 - K_n(t_0)) e^{-\delta t_0} + \int_0^{t_0} k_n(t) e^{-\delta t} \int_0^{\infty} p(x)\psi(u - ct + x, \delta) dx dt. \qquad (3.2.9)$$

with $t_0 = u/c$. The following theorem shows an integro–differential equation for $\psi(u, \delta)$.

Theorem 3 In the Erlang(n) dual risk model the Laplace transform of the time of ruin satisfies the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n \psi(u,\delta)=\int_0^\infty p(x)\psi(u+x,\delta)dx, \tag{3.2.10}$$

with boundary conditions

$$\psi(0,\delta)=1, \quad \left.\frac{d^i}{du^i}\psi(u,\delta)\right|_{u=0}=(-1)^i\left(\frac{\delta}{c}\right)^i, \; i=1,\ldots,n-1. \tag{3.2.11}$$

Proof. Using a similar procedure to that of Theorem 1 we take successive derivatives of (3.2.9). Then, changing variable the renewal equation can be rewritten in the form

$$\psi(u,\delta)=\left(1-K_n\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}+\frac{1}{c}\int_0^u k_n\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds,$$

where $W_\delta(s)=\int_0^\infty \psi(s+x,\delta)p(x)dx$.

After applying the operator $\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)$ to the Laplace transform we get

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)\psi(u,\delta) &= \left(1-K_{n-1}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}\\ &\quad+\frac{1}{c}\int_0^u k_{n-1}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds.\end{aligned}$$

Similarly, following an inductive argument, we show that

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^i\psi(u,\delta) &= \left(1-K_{n-i}\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}\\ &\quad+\frac{1}{c}\int_0^u k_{n-i}\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds,\end{aligned}$$

for $i=1,\ldots,n-1$. In particular, we obtain

$$\begin{aligned}\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n-1}\psi(u,\delta) &= \left(1-K_1\left(\frac{u}{c}\right)\right)e^{-\delta(\frac{u}{c})}\\ &\quad+\frac{1}{c}\int_0^u k_1\left(\frac{u-s}{c}\right)e^{-\delta(\frac{u-s}{c})}W_\delta(s)ds.\end{aligned}$$

Applying once more the operator gives

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n}\psi(u,\delta)=W_{\delta}(u).$$

This proves equation (3.2.10).

For the boundary conditions, clearly $\psi(0,\delta)=1$. We find the remaining conditions computing directly the derivatives of $\psi(u,\delta)$ and evaluate at $u=0$,

$$\begin{aligned}\frac{d^{i}}{du^{i}}\psi(u,\delta) &= \left[\left(-\frac{\delta}{c}\right)^{i}\left(1-K_{n}\left(\frac{u}{c}\right)\right)-\frac{1}{c^{i}}\sum_{j=1}^{i}\binom{i}{j}(-\delta)^{i-j}\,k_{n}^{(j-1)}\left(\frac{u}{c}\right)\right]e^{-\delta\left(\frac{u}{c}\right)}\\ &\quad+\left(\frac{1}{c}\right)\int_{0}^{u}\left[\frac{1}{c^{i}}\sum_{j=0}^{i}\binom{i}{j}(-\delta)^{i-j}\,k_{n}^{(j)}\left(\frac{u-s}{c}\right)\right]e^{-\delta\left(\frac{u-s}{c}\right)}W_{\delta}(s)ds,\end{aligned}$$

for $i=1,\ldots,n-1$, so that we get $d^{i}\psi(u,\delta)/du^{i}|_{u=0}=(-\delta/c)^{i}$, $i=1,\ldots,n-1$. ■

The solution for $\psi(u,\delta)$ is given in the following theorem.

Theorem 4 The Laplace transform of the time of ruin can be written as a combination of exponential functions

$$\psi(u,\delta)=\sum_{k=1}^{n}\left[\prod_{i=1,i\neq k}^{n}\frac{(\rho_{i}-\frac{\delta}{c})}{(\rho_{i}-\rho_{k})}\right]e^{-\rho_{k}u}, \qquad (3.2.12)$$

where $\rho_{1},\ldots,\rho_{n}$ are the only roots of the generalized Lundberg's equation (3.2.2) which have positive real parts.

Proof. We use a similar procedure as in Theorem 2 to obtain formula (3.2.12). All the functions $e^{-\rho_{k}u}$, $k=1,...,n$, are particular solutions of the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^{n}f(u)=\int_{0}^{\infty}p(x)f(u+x)dx. \qquad (3.2.13)$$

Since these functions are linearly independent, we can write every solution of (3.2.13) as a linear combination of them. Therefore,

$$\psi(u,\delta)=\sum_{i=1}^{n}a_{i}e^{-\rho_{i}u},$$

where a_i, $i = 1, ...n$, are constants and solutions of the system of equations

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \rho_1 & \rho_2 & \cdots & \rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \rho_2^{n-1} & \cdots & \rho_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \frac{\delta}{c} \\ \vdots \\ \left(\frac{\delta}{c}\right)^{n-1} \end{pmatrix} \Leftrightarrow \mathbf{a} = \mathbf{P}^{-1}\boldsymbol{\Lambda},$$

in matrix form, where $\mathbf{P} = \mathbf{P}(\rho_1, \ldots, \rho_n)$ is a Vandermonde matrix, $\mathbf{a}' = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{\Lambda}' = (1, \delta/c, \ldots, (\delta/c)^{n-1})$.

Finally, we get expressions for the coefficients

$$\begin{aligned} a_k &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n}(\rho_i - \frac{\delta}{c}))(\prod_{1\leq i<j\leq n,\, i\neq k,\, j\neq k}(\rho_j - \rho_i))}{\prod_{1\leq i<j\leq n}(\rho_j - \rho_i)} \\ &= \frac{(-1)^{k-1}(\prod_{i=1,\, i\neq k}^{n}(\rho_i - \frac{\delta}{c}))}{(\prod_{i=1}^{k-1}(\rho_k - \rho_i))(\prod_{j=k+1}^{n}(\rho_j - \rho_k))} = \prod_{i=1, i\neq k}^{n} \frac{\left(\rho_i - \frac{\delta}{c}\right)}{(\rho_i - \rho_k)}. \end{aligned}$$

We note that δ/c is not a root of equation (3.2.2). Hence, we get the result. ■

Remarks:

1. The Laplace transform (3.2.12) shows an interesting form, it corresponds to Formula (2.12) found by Li (2008b), concerning the primal model and applied for the first hitting time that the surplus risk process, starting from zero, upcrosses a level $u > 0$. This result enhances the duality of the two models as explained by Afonso *et al.* (2013) who worked the compound Poisson, or Erlang(1), model. We mean, the first hitting time in the primal model corresponds to the ruin time in the dual model. It is interesting that the duality features shown for the classical Erlang(1) can be extended [see beginning of Section 3 of Afonso *et al.* (2013)]. Note that the net profit conditions in the two models are reversed. We refer to the explanations for the Lundberg's equations in Section 2.3. Formulae (3.2.12) above and (2.12) from Li (2008b) show the same *appearance* but parameter c have different admissible values.

2. Formula (3.2.7) is a limiting case, as $\delta \to 0^+$, of (3.2.12). We couldn't transpose directly to our model Formula (2.12) of Li (2008b) derived for the primal model because its

limit as $\delta \to 0^+$ would lead to a ruin probability of one. This is due to the reversed loading condition. The first hitting time in the primal model is a proper random variable whereas the time to ruin in the dual model is a defective one.

Example 2 For $n = 1$, the exponential case, Ng (2009) found that $\psi(u, \delta) = e^{-\rho_1 u}$, where ρ_1 is the unique positive real solution of $1 + \delta/\lambda - cs/\lambda = \hat{p}(s)$.

For $n = 2$:

$$\psi(u, \delta) = \frac{\rho_2 - \frac{\delta}{c}}{\rho_2 - \rho_1} e^{-\rho_1 u} - \frac{\rho_1 - \frac{\delta}{c}}{\rho_2 - \rho_1} e^{-\rho_2 u},$$

where $\rho_1, \rho_2 > 0$ are real, solutions of $(1 + \delta/\lambda - cs/\lambda)^2 = \hat{p}(s)$. The above formula corresponds to expression (2.1) of Dickson and Li (2013), for the Laplace transform of the first hitting time in the primal model.

For $n = 3$:

$$\psi(u, \delta) = \frac{(\rho_2 - \frac{\delta}{c})(\rho_3 - \frac{\delta}{c})}{(\rho_3 - \rho_1)(\rho_2 - \rho_1)} e^{-\rho_1 u} - \frac{(\rho_1 - \frac{\delta}{c})(\rho_3 - \frac{\delta}{c})}{(\rho_3 - \rho_2)(\rho_2 - \rho_1)} e^{-\rho_2 u} + \frac{(\rho_1 - \frac{\delta}{c})(\rho_2 - \frac{\delta}{c})}{(\rho_3 - \rho_1)(\rho_3 - \rho_2)} e^{-\rho_3 u},$$

where ρ_1, ρ_2, ρ_3 are solutions of $(1 + \delta/\lambda - cs/\lambda)^3 = \hat{p}(s)$. One root is real and positive, the other two are complex conjugates.

3.3 The Phase–Type case

In the dual risk model with Phase–Type distributed gain inter–occurrence times, the generalized Lundberg's equation is given by (2.3.4). We recall this expression

$$\hat{p}(s) = \frac{1}{\hat{k}(\delta - cs)}, \quad \text{or} \quad \hat{k}(\delta - cs)\hat{p}(s) = 1,$$

noting that the Laplace transform of the probability density k has the form

$$\hat{k}(\delta - cs) = \boldsymbol{\alpha}((\delta - cs)\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^{\mathsf{T}}, \tag{3.3.1}$$

Thus, in order to solve equations (2.3.4) numerically we need to determine a rational expression for the Laplace transform $\hat{k}(\delta - cs)$. The main difficulty is to compute the inverse matrix $((\delta - cs)\mathbf{I} - \mathbf{B})^{-1}$. Before we go further we give some definitions from linear algebra.

Definition 3.3.1 Let $\mathbf{A} = (a_{i,j})_{i,j=1}^n$ be a $n \times n$ matrix. Define, for the given subindexes $1 \leq i_1 < i_2 < \ldots < i_k \leq n$,

$$\mathbf{M}_{i_1,i_2\ldots i_k}(\mathbf{A}) = det \begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \ldots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ldots & a_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \ldots & a_{i_k,i_k} \end{pmatrix}, \ 1 \leq k \leq n.$$

These are the minors $k \times k$ of the matrix $\mathbf{A}$ obtained by deleting the row and the column that meet in a_{ii} for $i \notin \{i_1, i_2, \ldots, i_k\}$. Then

$$tr_k(\mathbf{A}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mathbf{M}_{i_1,i_2\ldots i_k}(\mathbf{A}).$$

We call $tr_k(\mathbf{A})$ the k-generalized trace of the matrix A. In particular, $tr_1(\mathbf{A}) = tr(\mathbf{A}) = trace(\mathbf{A})$, and $tr_n(\mathbf{A}) = det(\mathbf{A})$.

Using this definition enables us to express the characteristic polynomial of the matrix $\mathbf{B}$ as

$$det(s\mathbf{I} - \mathbf{B}) = \sum_{i=0}^{n} (-1)^{n-i} tr_{n-i}(\mathbf{B}) s^i.$$

Moreover, the inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$ can be obtained as follows:

Theorem 3.3.1 The inverse matrix $(s\mathbf{I} - \mathbf{B})^{-1}$ has the expression

$$(s\mathbf{I} - \mathbf{B})^{-1} = \frac{N(s, \mathbf{B})}{det(s\mathbf{I} - \mathbf{B})},$$

where the matrix $N(s, \mathbf{B})$ takes the form

$$N(s, \mathbf{B}) = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1-i} (-1)^j tr_j(\mathbf{B}) \mathbf{B}^{n-1-i-j} \right) s^i.$$

Proof. We prove that $(s\mathbf{I}-\mathbf{B})^{-1}(s\mathbf{I}-\mathbf{B})=\mathbf{I}$ or, equivalently, that

$$(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) = det(s\mathbf{I}-\mathbf{B})\mathbf{I}.$$

If we denote by $\mathbf{a}_i$ the $n\times n$ matrix given by

$$\mathbf{a_i} = \sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j},$$

then

$$\begin{aligned}(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) &= (s\mathbf{I}-\mathbf{B})\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)s^i \\ &= (s\mathbf{I}-\mathbf{B})\sum_{i=0}^{n-1}\mathbf{a}_i s^i \\ &= \mathbf{a}_{n-1}s^n + \sum_{i=1}^{n-1}(\mathbf{a}_{i-1}-\mathbf{a}_i\mathbf{B})s^i - \mathbf{a}_0\mathbf{B}.\end{aligned}$$

Now we can easily verify that $\mathbf{a}_{n-1}=\mathbf{I}$. Since

$$det(\mathbf{B}\mathbf{I}-\mathbf{B}) = \sum_{j=0}^{n}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-j} = 0,$$

we get $-\mathbf{a}_0\mathbf{B} = (-1)^n det(\mathbf{B})\mathbf{I}$, and

$$\begin{aligned}\mathbf{a}_{i-1}-\mathbf{a}_i\mathbf{B} &= \sum_{j=0}^{n-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-i-j} - \left(\sum_{j=0}^{n-1-i}(-1)^j tr_j(\mathbf{B})\mathbf{B}^{n-1-i-j}\right)\mathbf{B} \\ &= (-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I}.\end{aligned}$$

Therefore,

$$\begin{aligned}(s\mathbf{I}-\mathbf{B})N(s,\mathbf{B}) &= \mathbf{I}s^n + \sum_{i=1}^{n-1}((-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I})s^i + (-1)^n det(\mathbf{B})\mathbf{I} \\ &= \sum_{i=0}^{n}((-1)^{n-i}tr_{n-i}(\mathbf{B})\mathbf{I})s^i = det(s\mathbf{I}-\mathbf{B})\mathbf{I}.\end{aligned}$$

This completes the proof. ∎

From Theorem 3.1 and (3.3.1) we get the rational expression for the Lundberg's equations. The generalized Lundberg's equation for the Phase–Type(n) dual risk model becomes

$$\frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(\delta - cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}} = \hat{p}(s), \tag{3.3.2}$$

and we obtain the corresponding fundamental Lundberg's equation by setting $\delta = 0$ in equation (3.3.2)

$$\frac{det((-cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(-cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}} = \hat{p}(s). \tag{3.3.3}$$

Although the new expressions for the Lundberg's equations found in (3.3.2) and (3.3.3) are already in rational form, they are not adequate for our purposes. What we need are expressions that show a natural connection with other parts of this manuscript. The reason for this will be clear in Section 3.3.1 when we will calculate quantities like the Laplace transform of the time of ruin and the ruin probability using integro–differential equations. It turns out that these integro–differential equations can be expressed using polynomial forms, denoted as $B_\delta(\cdot)$ and $q_\delta(\cdot)$, and these polynomial forms can be used instead to rewrite the generalized and fundamental Lundberg's equations (3.3.2) and (3.3.3). This is shown next.

The generalized Lundberg's equation can be written as

$$B_\delta(-s) = q_\delta(-s)\hat{p}(s), \qquad s \in \mathbb{C}, \tag{3.3.4}$$

where B_δ and q_δ are polynomials in s given by

$$B_\delta(s) = \frac{det(\mathbf{B} - \delta\mathbf{I} - cs\mathbf{I})}{det(\mathbf{B})} = \sum_{i=0}^{n} B_i \left(s + \frac{\delta}{c}\right)^i$$

and

$$q_\delta(s) = \sum_{j=0}^{n-1} \tilde{B}_j \left(s + \frac{\delta}{c}\right)^j.$$

The equivalent fundamental Lundberg's equation (for $\delta = 0$) is

$$B(-s) = q(-s)\hat{p}(s), \qquad s \in \mathbb{C}. \tag{3.3.5}$$

The coefficients B_i and $\tilde{B}_j$ of the polynomials B and q, respectively, are given by the following expressions

$$B_i = (-c)^i \frac{tr_{n-i}(\mathbf{B})}{det(\mathbf{B})}, \qquad \tilde{B}_j = \sum_{i=j+1}^{n} B_i \left(\frac{1}{c}\right)^{i-j} k^{(i-1-j)}(0).$$

Theorem 3.3.2 Expressions (3.3.2) and (3.3.4) are equivalent forms of the generalized Lundberg's equation. Corresponding expressions (3.3.3) and (3.3.5) represent the fundamental Lundberg's equation.

Proof. The proof is simple and follows by rearranging and comparing the coefficients of the above mentioned versions of the Lundberg's equations. Namely, we have to prove that

$$\frac{det((\delta - cs)\mathbf{I} - \mathbf{B})}{\boldsymbol{\alpha} N(\delta - cs, \mathbf{B})\mathbf{b}^{\mathsf{T}}} = \frac{B_\delta(-s)}{q_\delta(-s)}. \tag{3.3.6}$$

From the left-hand side we have

$$B_\delta(-s) = \frac{det(\mathbf{B} - \delta\mathbf{I} + cs\mathbf{I})}{det(\mathbf{B})} = \frac{(-1)^n}{det(\mathbf{B})} det((\delta - cs)\mathbf{I} - \mathbf{B}),$$

and from the right-hand side:

$$\begin{aligned}
q_\delta(-s) &= \sum_{j=0}^{n-1} \tilde{B}_j \left(\frac{\delta}{c} - s\right)^j = \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} B_i \left(\frac{1}{c}\right)^{i-j} k^{(i-1-j)}(0) \left(\frac{\delta}{c} - s\right)^j \\
&= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^i c^i \frac{tr_{n-i}(\mathbf{B})}{det(\mathbf{B})} \left(\frac{1}{c}\right)^i \boldsymbol{\alpha} \mathbf{B}^{i-1-j} \mathbf{b}^{\mathsf{T}} c^j \left(\frac{\delta}{c} - s\right)^j \\
&= \boldsymbol{\alpha} \sum_{j=0}^{n-1} \sum_{i=0}^{n-j-1} (-1)^{n-i} \frac{tr_i(\mathbf{B})}{det(\mathbf{B})} \mathbf{B}^{n-1-i-j} (\delta - cs)^j \, \mathbf{b}^{\mathsf{T}} \\
&= \frac{(-1)^n}{det(\mathbf{B})} \boldsymbol{\alpha} N(\delta - cs, \mathbf{B}) \mathbf{b}^{\mathsf{T}}.
\end{aligned}$$

This proves (3.3.6). ■

Remark 3.3.1 Alternatively, we can write

$$B_\delta(s) = \frac{det(\mathbf{B} - \delta\mathbf{I} - cs\mathbf{I})}{det(\mathbf{B})} = \sum_{i=0}^{n} B_{i,\delta} \, s^i, \tag{3.3.7}$$

$$q_\delta(s) = \sum_{j=0}^{n-1} \tilde{B}_{j,\delta}\, s^j, \tag{3.3.8}$$

where the coefficients $B_{i,\delta}$ and $\tilde{B}_{j,\delta}$ are given by

$$\begin{aligned} B_{i,\delta} &= (-c)^i \frac{tr_{n-i}(\mathbf{B} - \delta \mathbf{I})}{det(\mathbf{B})}, \\ \tilde{B}_{j,\delta} &= \sum_{i=0}^{n-1-j} B_{1+i+j,\delta} \left(\sum_{l=0}^{i} \binom{i}{l} (-\delta)^l k^{(i-l)}(0) \right) \left(\frac{1}{c}\right)^{1+i}. \end{aligned}$$

These forms are going to be used in the following section.

3.3.1 The time to ruin and its Laplace transform

In this section we study the ruin probability and the Laplace transform of the time to ruin in the Phase–Type(n) dual risk model.

Conditioning on the time and the amount of the first gain, we find that the Laplace transform of the time to ruin for the Phase–Type(n) dual risk model satisfies the renewal equation

$$\psi(u,\delta) = \left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \int_0^{\frac{u}{c}} k(t) e^{-\delta t} \int_0^\infty p(x)\psi(u - ct + x, \delta) dx\, dt.$$

Note that the above equation is valid for any renewal model with density k and distribution K. Changing variables $s = u - ct$, we get

$$\psi(u,\delta) = \left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)} + \frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)} W_\psi(s,\delta)\, ds, \tag{3.3.9}$$

where $W_\psi(s,\delta) = \int_0^\infty p(x)\psi(s + x, \delta)dx$.

Before we continue further, we state the following lemma, which will be useful in a subsequent theorem.

Lemma 3.3.1 Let B_δ, q_δ be the polynomials described in (3.3.4) for the generalized Lundberg's equation, and consider the following differential operators

$$B_\delta(\mathcal{D}) = \sum_{i=0}^{n} B_i \left(\mathcal{D} + \frac{\delta}{c}\right)^i = \sum_{i=0}^{n} B_{i,\delta}\ \mathcal{D}^i, \quad q_\delta(\mathcal{D}) = \sum_{j=0}^{n-1} \tilde{B}_j \left(\mathcal{D} + \frac{\delta}{c}\right)^j = \sum_{j=0}^{n-1} \tilde{B}_{j,\delta}\ \mathcal{D}^j. \tag{3.3.10}$$

for $\mathcal{D} = \frac{d}{du}$. Then the following properties hold

$$\begin{aligned} B_\delta(\mathcal{D}) \left[k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)}\right] &= 0, \\ B_\delta(\mathcal{D}) \left[\left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)}\right] &= 0. \end{aligned}$$

Proof.

$$\begin{aligned} B_\delta(\mathcal{D}) \left[k\left(\frac{u-s}{c}\right) e^{-\delta\left(\frac{u-s}{c}\right)}\right] &= \sum_{i=0}^{n} B_{i,\delta}\mathcal{D}^i[\boldsymbol{\alpha} e^{\mathbf{B}\left(\frac{u-s}{c}\right)}\mathbf{b}^\mathsf{T} e^{-\delta\left(\frac{u-s}{c}\right)\mathbf{I}}] \\ &= \boldsymbol{\alpha}\left[\sum_{i=0}^{n} B_{i,\delta}\mathcal{D}^i(e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)})\right]\mathbf{b}^\mathsf{T} \\ &= \boldsymbol{\alpha}\left[\sum_{i=0}^{n} B_{i,\delta}\left(\frac{1}{c}\right)^i (\mathbf{B}-\delta\mathbf{I})^i e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)}\right]\mathbf{b}^\mathsf{T} \\ &= \boldsymbol{\alpha}\left[B_\delta\left(\frac{1}{c}(\mathbf{B}-\delta\mathbf{I})\right)\right] e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)}\mathbf{b}^\mathsf{T} \\ &= \boldsymbol{\alpha}\left[\frac{det(\mathbf{B}-\delta\mathbf{I}-c\mathbf{I}\left(\frac{1}{c}(\mathbf{B}-\delta\mathbf{I})\right))}{det(\mathbf{B})}\right] e^{(\mathbf{B}-\delta\mathbf{I})\left(\frac{u-s}{c}\right)}\mathbf{b}^\mathsf{T} = 0, \end{aligned}$$

Analogously, we can see $B_\delta(\mathcal{D})\left[\left(1 - K\left(\frac{u}{c}\right)\right) e^{-\delta\left(\frac{u}{c}\right)}\right] = 0.$ ■

We can obtain a formula for the Laplace transform of the time of ruin $\psi(u, \delta)$ solving the following integro–differential equation with boundary conditions:

Theorem 3.3.3 The Laplace transform of the time of ruin $\psi(u, \delta)$ satisfies the integro–differential equation

$$B_\delta(\mathcal{D})\psi(u, \delta) = q_\delta(\mathcal{D})W_\psi(u, \delta), \tag{3.3.11}$$

The boundary conditions of (3.3.11) are given by

$$
\begin{aligned}
\psi(0,\delta) &= 1, \\
\left.\frac{d^i}{du^i}\psi(u,\delta)\right|_{u=0} &= \left(-\frac{\delta}{c}\right)^i - \sum_{j=0}^{i-1}\frac{1}{c^i}\binom{i}{j}(-\delta)^j k^{(i-1-j)}(0) \\
&+ \sum_{j=0}^{i-1}\left(\sum_{l=0}^{i-1-j}\frac{1}{c^{i-j}}\binom{i-1-j}{l}(-\delta)^l k^{(i-1-j-l)}(0)\right)W_\psi^{(j)}(0,\delta), \\
& \quad i = 1,\dots,n-1.
\end{aligned}
\tag{3.3.12}
$$

Proof. We proceed taking successive derivatives of $\psi(u,\delta)$ using the renewal equation (3.3.9). We want to prove the equation $B_\delta(\mathcal{D})\psi(u,\delta) = q_\delta(\mathcal{D})W_\psi(u,\delta)$. The j-th derivative of $\psi(u,\delta)$ with respect to u is given by

$$
\begin{aligned}
\frac{d^j}{du^j}\psi(u,\delta) &= \left[\left(-\frac{\delta}{c}\right)^j\left(1-K\left(\frac{u}{c}\right)\right) - \sum_{i=0}^{j-1}\frac{1}{c^j}\binom{j}{i}(-\delta)^i k^{(j-1-i)}\left(\frac{u}{c}\right)\right]e^{-\delta\left(\frac{u}{c}\right)} \\
&+\sum_{i=0}^{j-1}\left(\sum_{l=0}^{j-1-i}\left(\frac{1}{c}\right)^{j-i}\binom{j-1-i}{l}(-\delta)^l k^{(j-1-i-l)}(0)\right)W_\psi^{(i)}(u,\delta) \\
&+\frac{1}{c}\int_0^u\left[\sum_{i=0}^{j}\frac{1}{c^j}\binom{j}{i}(-\delta)^i k^{(j-i)}\left(\frac{u-s}{c}\right)\right]e^{-\delta\left(\frac{u-s}{c}\right)}W_\psi(s,\delta)ds,
\end{aligned}
$$

for $j = 1,\dots,n-1$. Hence, we obtain

$$
\begin{aligned}
\left.\frac{d^j}{du^j}\psi(u,\delta)\right|_{u=0} &= \left(-\frac{\delta}{c}\right)^j - \sum_{i=0}^{j-1}\frac{1}{c^j}\binom{j}{i}(-\delta)^i k^{(j-1-i)}(0) \\
&+\sum_{i=0}^{j-1}\left(\sum_{l=0}^{j-1-i}\left(\frac{1}{c}\right)^{j-i}\binom{j-1-i}{l}(-\delta)^l k^{(j-1-i-l)}(0)\right)W_\psi^{(i)}(0,\delta),
\end{aligned}
$$

for $j = 1,\dots,n-1$.

Now we apply the differential operator $B_\delta(\mathcal{D})$ to $\psi(u,\delta)$

$$
\begin{aligned}
B_\delta(\mathcal{D})\psi(u,\delta) &= \underbrace{B_\delta(\mathcal{D})\left[\left(1-K\left(\frac{u}{c}\right)\right)e^{-\delta\left(\frac{u}{c}\right)}\right]}_{=0} + B_\delta(\mathcal{D})\left(\frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}W_\psi(s,\delta)\,ds\right) \\
&= \sum_{j=0}^{n} B_{j,\delta}\mathcal{D}^j\left(\frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}W_\psi(s,\delta)ds\right) \\
&= \sum_{j=0}^{n} B_{j,\delta}\left[\sum_{i=0}^{j-1}\left(\sum_{l=0}^{j-1-i}\left(\frac{1}{c}\right)^{j-i}\binom{j-1-i}{l}(-\delta)^l k^{(j-1-i-l)}(0)\right)W_\psi^{(i)}(u,\delta)\right. \\
&\quad \left. + \frac{1}{c}\int_0^u\left(\sum_{i=0}^{j}\frac{1}{c^j}\binom{j}{i}(-\delta)^i k^{(j-i)}\left(\frac{u-s}{c}\right)\right)e^{-\delta\left(\frac{u-s}{c}\right)}W_\psi(s,\delta)ds\right] \\
&= \sum_{j=1}^{n} B_{j,\delta}\left[\sum_{i=0}^{j-1}\left(\sum_{l=0}^{i}\left(\frac{1}{c}\right)^{i+1}\binom{i}{l}(-\delta)^l k^{(i-l)}(0)\right)W_\psi^{(j-1-i)}(u,\delta)\right] \\
&\quad + \frac{1}{c}\int_0^u \underbrace{B_\delta(\mathcal{D})\left[k\left(\frac{u-s}{c}\right)e^{-\delta\left(\frac{u-s}{c}\right)}\right]}_{=0}W_\psi(s,\delta)ds \\
&= \sum_{j=0}^{n-1}\underbrace{\sum_{i=0}^{n-1-j}B_{1+i+j,\delta}\left(\sum_{l=0}^{i}\left(\frac{1}{c}\right)^{i+1}\binom{i}{l}(-\delta)^l k^{(i-l)}(0)\right)}_{=\tilde{B}_{j,\delta}}W_\psi^{(j)}(u,\delta) \\
&= \sum_{j=0}^{n-1}\tilde{B}_{j,\delta}W_\psi^{(j)}(u,\delta) = q_\delta(\mathcal{D})W_\psi(u,\delta).
\end{aligned}
$$

This completes the proof. ■

For the Phase–Type(n) dual risk model, we have found that the Laplace transform of the time of ruin can be written as follows

Theorem 3.3.4

$$\psi(u,\delta) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij,\delta}u^{j-1}e^{-\rho_i u}, \tag{3.3.13}$$

where $\rho_1,\ldots,\rho_L$ are the only roots of the generalized Lundberg's equation which have positive real parts, and ρ_i has multiplicity β_i, with $\sum_{i=1}^{L}\beta_i = n$.

Proof. It is very simple to verify that if ρ is a single root of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$ then the function $f(u) = e^{-\rho u}$ satisfies the integro–differential equation $B_\delta(\mathcal{D})f(u) = q_\delta(\mathcal{D})W_f(u)$, where $W_f(u) = \int_0^\infty p(x)f(u+x)dx$. Moreover, we can show that if ρ is a root of the generalized Lundberg's equation with multiplicity $\beta \geq 1$ then the functions $f(u) = u^{j-1}e^{-\rho u}$, $j = 1, \ldots, \beta$ are all solutions of the same integro–differential equation. We will prove that $B_\delta(\mathcal{D})f(u) = q_\delta(\mathcal{D})W_f(u)$. We have

$$f^{(k)}(u) = \sum_{l=0}^{j-1} \left(\prod_{m=0}^{l-1} (k-m) \right) (-\rho)^{k-l} \binom{j-1}{l} u^{j-1-l} e^{-\rho u}.$$

Then from the left-hand side

$$\begin{aligned}
B_\delta(\mathcal{D})f(u) &= \sum_{k=0}^{n} B_{k,\delta} f^{(k)}(u) \\
&= \sum_{k=0}^{n} B_{k,\delta} \sum_{l=0}^{j-1} \left(\prod_{m=0}^{l-1} (k-m) \right) (-\rho)^{k-l} \binom{j-1}{l} u^{j-1-l} e^{-\rho u} \\
&= \sum_{l=0}^{j-1} \left(\sum_{k=l}^{n} B_{k,\delta} \left(\prod_{m=0}^{l-1} (k-m) \right) (-\rho)^{k-l} \right) \binom{j-1}{l} u^{j-1-l} e^{-\rho u} \\
&= \sum_{l=0}^{j-1} B_\delta^{(l)}(-\rho) \binom{j-1}{l} u^{j-1-l} e^{-\rho u}. \qquad (3.3.14)
\end{aligned}$$

From the right-hand side, we have

$$\begin{aligned}
W_f(u) &= \int_0^\infty f(u+x)p(x)dx = \int_0^\infty (u+x)^{j-1} e^{-\rho(u+x)} p(x)dx \\
&= \int_0^\infty \sum_{i=0}^{j-1} \binom{j-1}{i} u^{j-1-i} x^i e^{-\rho x} e^{-\rho u} p(x)dx \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} u^{j-1-i} e^{-\rho u} \int_0^\infty x^i e^{-\rho x} p(x)dx \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} u^{j-1-i} e^{-\rho u} (-1)^i \hat{p}^{(i)}(\rho).
\end{aligned}$$

Therefore,

$$\begin{aligned}
q_\delta(\mathcal{D})W_f(u) &= \sum_{k=0}^{n-1} \tilde{B}_{k,\delta} W_f^{(k)}(u) \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} (-1)^i \hat{p}^{(i)}(\rho) \sum_{k=0}^{n-1} \tilde{B}_{k,\delta} \frac{d}{d^k}(u^{j-1-i}e^{-\rho u}) \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} (-1)^i \hat{p}^{(i)}(\rho) \sum_{l=0}^{j-1-i} q_\delta^{(l)}(-\rho) \binom{j-1-i}{l} u^{j-1-i-l} e^{-\rho u} \\
&= \sum_{i=0}^{j-1} \binom{j-1}{i} (-1)^i \hat{p}^{(i)}(\rho) \sum_{l=i}^{j-1} q_\delta^{(l-i)}(-\rho) \binom{j-1-i}{l-i} u^{j-1-l} e^{-\rho u} \\
&= \sum_{l=0}^{j-1} \left[\sum_{i=0}^{l} (-1)^i \binom{l}{i} q_\delta^{(l-i)}(-\rho) \hat{p}^{(i)}(\rho) \right] \binom{j-1}{l} u^{j-1-l} e^{-\rho u}. \qquad (3.3.15)
\end{aligned}$$

Since the root ρ has multiplicity $\beta \geq 1$, it satisfies the equations

$$B_\delta^{(l)}(-\rho) = \sum_{i=0}^{l} (-1)^i \binom{l}{i} q_\delta^{(l-i)}(-\rho) \hat{p}^{(i)}(\rho), \; l = 0, 1, \ldots \beta - 1, \qquad (3.3.16)$$

which implies that expressions (3.3.14) and (3.3.15) are identical, thus proving our statement. Since the functions $u^{j-1}e^{-\rho_i u}, \; i = 1, \ldots, L; \; j = 1, \ldots, \beta_i$ are linearly independent, any solution of $B_\delta(\mathcal{D})f(u) = q_\delta(\mathcal{D})W_f(u)$ can be expressed in the following way

$$f(u) = \sum_{i=1}^{L} \sum_{j=1}^{\beta_i} b_{ij} u^{j-1} e^{-\rho_i u}, \qquad (3.3.17)$$

for some constants b_{ij}.

Since the Laplace transform of the time of ruin satisfies an integro–differential equation of the same form, $B_\delta(\mathcal{D})\psi(u,\delta) = q_\delta(\mathcal{D})W_\psi(u,\delta)$, it can be written as

$$\psi(u,\delta) = \sum_{i=1}^{L} \sum_{j=1}^{\beta_i} a_{ij,\delta} u^{j-1} e^{-\rho_i u}.$$

Using the boundary conditions (3.3.12) we can determine the constants $a_{ij,\delta}$ that correspond to $\psi(u,\delta)$ in the following way

$$\begin{aligned}
\psi(0,\delta) &= \sum_{i=1}^{L} a_{i1,\delta} = 1, \\
\left.\frac{d^m}{du^m}\psi(u,\delta)\right|_{u=0} &= \left.\frac{d^m}{du^m}\sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij,\delta}u^{\,j-1}e^{-\rho_i u}\right|_{u=0} \\
& \left(-\frac{\delta}{c}\right)^m - \sum_{j=0}^{m-1}\frac{1}{c^m}\binom{m}{j}(-\delta)^j k^{(m-1-j)}(0) \\
&+ \sum_{j=0}^{m-1}\left(\sum_{l=0}^{m-1-j}\frac{1}{c^{m-j}}\binom{m-1-j}{l}(-\delta)^l k^{(m-1-j-l)}(0)\right) W_{\psi}^{(j)}(0,\delta), \\
& m = 1,\ldots,n-1.
\end{aligned}$$

where $W_{\psi}(u,\delta) = \int_0^{\infty} p(x)\left[\sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij,\delta}(u+x)^{\,j-1}e^{-\rho_i(u+x)}\right]dx$.
Regardless of multiplicities, this gives a system of n equations on the n unknowns constants $a_{ij,\delta}, i = 1,\ldots,L; j = 1,\ldots,\beta_i$, that can be solved using standard linear algebra methods. ■

Remark 3.3.2 If all the roots with positive real parts of the generalized Lundberg's equation are single (multiplicity 1), then we write the Laplace transform of the time of ruin in the following way

$$\psi(u,\delta) = \sum_{i=1}^{n} a_{i,\delta}e^{-\rho_i u},$$

and the constants $a_{i,\delta}$ can be found using the boundary conditions (3.3.12), which is equivalent to solving the following system of n equations on the n unknowns $a_{i,\delta}$:

$$\sum_{i=1}^{n} a_{i,\delta} = 1,$$

and

$$\begin{aligned}
&\sum_{i=1}^{n} a_{i,\delta}\left[(-\rho_i)^j - \hat{p}(\rho_i)\sum_{m=0}^{j-1}\left(\sum_{l=0}^{j-1-m}\frac{1}{c^{j-m}}\binom{j-1-m}{l}(-\delta)^l k^{(j-1-m-l)}(0)\right)(-\rho_i)^m\right] \\
&\qquad = \left(-\frac{\delta}{c}\right)^j - \sum_{m=0}^{j-1}\frac{1}{c^j}\binom{j}{m}(-\delta)^m k^{(j-1-m)}(0),\ j = 1,\ldots,n.
\end{aligned}$$

Example 3 For $n = 2$, the Laplace transform of the time of ruin in the Phase–Type(2) model has the expression

$$\begin{aligned}\psi(u,\delta) &= \frac{\rho_2 - \frac{\delta}{c} + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2)-1)}{\rho_2-\rho_1+\frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2)-\hat{p}(\rho_1))}e^{-\rho_1 u} \\ &\quad - \frac{\rho_1 - \frac{\delta}{c} + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_1)-1)}{\rho_2-\rho_1+\frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2)-\hat{p}(\rho_1))}e^{-\rho_2 u},\end{aligned}$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$.

3.3.2 The ruin probability

The ruin probability in the dual risk model with exponential inter–arrival times ($Ph(1)$, $k(t) = \lambda e^{-\lambda t}$) satisfies the following renewal equation

$$\psi(u) = e^{-\lambda(\frac{u}{c})} + \int_0^{\frac{u}{c}} \lambda e^{-\lambda t} \int_0^\infty p(x)\psi(u-ct+x)dx\, dt,$$

where u/c is the time of ruin if no gain arrives. See e.g. Afonso *et al.*(2013). Gerber (1979) found that $\psi(u) = e^{-\rho u}$, where ρ is the unique positive root of the fundamental Lundberg's equation ($n = 1$).

For the Ph(n) dual risk model the renewal equation becomes

$$\psi(u) = 1 - K\left(\frac{u}{c}\right) + \int_0^{\frac{u}{c}} k(t) \int_0^\infty p(x)\psi(u-ct+x)dx\, dt. \tag{3.3.18}$$

The corresponding integro–differential equation is given in the following theorem:

Corollary 3.3.1 The ruin probability $\psi(u)$ satisfies the following integro–differential equation

$$B(\mathcal{D})\psi(u) = q(\mathcal{D})W(u), \tag{3.3.19}$$

where $W(u) = \int_0^\infty p(x)\psi(u+x)dx$ and B, q are the same polynomials described before for the fundamental Lundberg's equation (3.3.5). The operator $\mathcal{D}$ is the differentiation with respect to u, as before.

The boundary conditions of (3.3.19) are given by

$$
\begin{aligned}
\psi(0) &= 1, \\
\left.\frac{d^j}{du^j}\psi(u)\right|_{u=0} &= -\frac{1}{c^j}k^{(j-1)}(0) + \sum_{i=0}^{j-1}\frac{1}{c^{i+1}}k^{(i)}(0)W^{(j-1-i)}(0), \qquad (3.3.20) \\
& j = 1, \ldots, n-1.
\end{aligned}
$$

Proof. This is a special case of Theorem 3.3.3 for $\delta \to 0$. ■

For the Phase–Type(n) dual risk model, we found that the ruin probability can be written as follows

Corollary 3.3.2 The ultimate ruin probability $\psi(u)$ can be written in the general form

$$
\psi(u) = \sum_{i=1}^{L}\sum_{j=1}^{\beta_i} a_{ij}u^{j-1}e^{-\rho_i u},
$$

where $\rho_1, \ldots, \rho_L$ are the only roots of the Fundamental Lundberg's equation which have positive real parts, and ρ_i has multiplicity β_i, with $\sum_{i=1}^{L}\beta_i = n$.

Proof. This is a special case of Theorem 3.3.4 for $\delta \to 0$. ■

Example 4 For $n = 2$, the ruin probability in the Phase–Type(2) model has the expression

$$
\begin{aligned}
\psi(u) &= \frac{\rho_2 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2) - \hat{p}(\rho_1))}e^{-\rho_1 u} \\
&\quad - \frac{\rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_1) - 1)}{\rho_2 - \rho_1 + \frac{1}{c}\boldsymbol{\alpha}\mathbf{b}^{\mathsf{T}}(\hat{p}(\rho_2) - \hat{p}(\rho_1))}e^{-\rho_2 u},
\end{aligned}
$$

where $\rho_1, \rho_2 > 0$ are real and solutions of $B(-s) = q(-s)\hat{p}(s)$.

3.4 Final remarks

As we have mentioned before, one of the fundamental purposes in insurance mathematics is to provide adequate methods to solve the problems that may appear in the actuarial practice. Throughout this chapter we have considered the calculation of ruin probabilities and the Laplace transform of the time to ruin in the case when the gain inter–occurrence times follow distributions belonging to the matrix exponential family, like the Phase–Type and the Erlang.

The presented methods can be extended for more general distributions. We will continue our work in the next chapter on the expected discounted dividends.

Chapter 4

Expected discounted dividends

4.1 Introduction

In this chapter we study the expected discounted dividends for a dual risk model with an upper barrier level. The distribution of the gain inter–occurrence times is matrix exponential. However, for simplicity and elegance of the formulas we concentrate on the Phase–Type and the Erlang distributions.

In Section 4.2 we devote attention to the Erlang case, where we study the expected discounted dividends and higher moments.

In Section 4.3 we work with the Phase–Type distribution. We show formulas for the expected discounted dividends and results on optimal dividend barriers.

In both Sections 4.2 and 4.3 we make the additional assumption that the claim amounts follow a Phase–Type distribution.

4.2 The Erlang case

From this section on we consider the existence of an upper dividend barrier b so that when the surplus upcrosses b the excess is paid as dividend. From that arrival/payment instant the process restarts from level b and that repeats whenever it occurs in the future, until ruin.

4.2.1 An integro-differential equation

In the Poisson case, exponentially distributed interjumps arrivals, see e.g. Afonso *et al.* (2013), the expected present value of the discounted dividends, $V(u,b)$, satisfies the renewal equation, for $u \leq b$,

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} \lambda e^{-(\lambda+\delta)t} \left\{ \int_0^{b-u+ct} V(u-ct+y,b)p(y)dy \right. \\ &\quad \left. + \int_{b-u+ct}^{\infty} \left[y+u-ct-b+V(b,b)\right] p(y)dy \right\} dt \end{aligned}$$

Note that $V(0,b)=0$, since for $u=0$ ruin immediately occurs, and that

$$V(u,b) = u - b + V(b,b), \quad \text{for} \quad u > b. \tag{4.2.1}$$

Changing variable, $s = u - ct$, and differentiating with respect to u we get

$$\left(\left(1 + \frac{\delta}{\lambda}\right) \mathcal{I} + \left(\frac{c}{\lambda}\right) \mathcal{D} \right) V(u,b) = W_\delta(u,b),$$

where

$$W_\delta(u,b) = \int_0^{b-u} V(u+y,b)p(y)dy + \int_{b-u}^{\infty} (y+u-b+V(b,b))p(y)dy. \tag{4.2.2}$$

In the Erlang(n) model ($n \geq 2$), the corresponding renewal equation is given by

$$\begin{aligned} V(u,b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-\delta t} \left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy + \right. \\ &\quad \left. \int_{b-u+ct}^{\infty} (y+u-ct-b+V(b,b))p(y)dy \right] dt. \end{aligned}$$

After a similar variable change, we can write it in the following form

$$V(u,b) = \frac{1}{c} \int_0^u k_n\left(\frac{u-s}{c}\right) e^{-\delta(\frac{u-s}{c})} W_\delta(s,b)ds. \tag{4.2.3}$$

The following theorem shows the final form of this equation.

Theorem 4.2.1 $V(u,b)$ satisfies the integro–differential equation

$$\left(\left(1+\frac{\delta}{\lambda}\right)\mathcal{I}+\left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b)=W_\delta(u,b), \tag{4.2.4}$$

where $W_\delta(u,b)$ is given by (4.2.2), with boundary conditions

$$\left.\frac{d^i}{du^i}V(u,b)\right|_{u=0}=0, \quad i=0,\ldots,n-1. \tag{4.2.5}$$

Proof. The proof follows exactly the method applied previously, taking successive derivatives of (4.2.3). ■

4.2.2 The annihilator of $p(x-u)$

Because of condition (4.2.1), we can not write the solutions of (4.2.4) as a linear combination of n exponential functions as we did before in the cases of the ruin probability and the Laplace transform of the time of ruin. Otherwise, conditions given by (4.2.5) would led to $V(u,b)\equiv 0$, which is a contradiction. We will need instead more than n exponential functions; the exact number needed will depend on the nature of the distribution of the single gains, $P(x)$. However, we can apply the annihilator approach known from the theory of ordinary differential equations to find the appropriate solutions, e.g. see Zill (2012), Section 4.5.

We can rewrite $W_\delta(u,b)$ in (4.2.2) as

$$\begin{aligned} W_\delta(u,b) &= \int_u^b V(x,b)p(x-u)dx+\int_b^\infty (x-b+V(b,b))p(x-u)dx \\ &= \int_u^b V(x,b)p(x-u)dx+\int_b^\infty \widetilde{V}(x,b)p(x-u)dx, \end{aligned} \tag{4.2.6}$$

with $\widetilde{V}(x,b)=x-b+V(b,b)$. The idea is to find a linear differential operator that will annihilate $p(x-u)$ (where the variable is u), so that when we apply this operator to the integro–differential equation (4.2.4) we obtain a linear homogeneous differential equation of a higher degree (and the integral term $W_\delta(u,b)$ vanish).

From this moment onwards we work the particular case when the single gains follow a distribution of the Phase-Type family, PH(m). Our notations and definitions are presented as usually done in this case. Denote by $\mathbf{B} = (b_{ij})_{1\leq i,j\leq m}$ the matrix of the transition rates between the transient states, let $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the vector of the initial probabilities, $\eta' = (\eta_1, \eta_2, \ldots, \eta_m)$ the vector of the exit rates to the absorbing state, and the $1 \times m$ vector $\mathbf{1}' = (1, 1, \ldots, 1)$. We have that $\eta = -\mathbf{B}\mathbf{1}$. Let $\mathbf{I}_m$ denote the identity matrix of order m. It is well known for this family that the probability and distribution functions are denoted as $p(x) = \alpha' e^{\mathbf{B}x}\eta$ and that $P(x) = 1 - \alpha' e^{\mathbf{B}x}\mathbf{1}$, respectively. Its Laplace transform is given by $\hat{p}(s) = \alpha' \left(s\mathbf{I}_m - \mathbf{B}\right)^{-1} \eta$. Let's consider the following theorem:

Theorem 4.2.2 One annihilator of degree m for $p(x-u)$ is $q_{\mathbf{B}}(-\mathcal{D})$, where $\mathcal{D} = \frac{d}{du}$ denote differentiation with respect to u and $q_{\mathbf{B}}(y) = Det(\mathbf{B} - y\mathbf{I}_m)$ is the characteristic polynomial of the matrix $\mathbf{B}$.

Proof. The proof is based on the Cayley–Hamilton theorem of linear algebra, which states that every square matrix satisfies its own characteristic equation [see e.g. Lang (2010)]. ■

Example 4.2.1 When we consider the exponential(β) distribution for the individual gain size, we have that $p(x) = \beta e^{-\beta x}$, then $\mathbf{B} = (-\beta)$, $\alpha' = (1)$, $\eta' = (\beta)$ and $\mathbf{1}' = (1)$. Hence,

$$q_{\mathbf{B}}(y) = Det(\mathbf{B} - y\mathbf{I}_1) = -\beta - y \quad \text{and} \quad q_{\mathbf{B}}(-\mathcal{D}) = \frac{d}{du} - \beta.$$

It is easy to check that $\left(\frac{d}{du} - \beta\right) p(x-u) = 0$.

For a more general case when the individual gain size follows an Erlang(m, β) distribution, we have that $p(x) = \beta^m x^{m-1} e^{-\beta x}/(m-1)!$, so that

$$\mathbf{B} = \begin{pmatrix} -\beta & \beta & \cdots & 0 & 0 \\ 0 & -\beta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\beta & \beta \\ 0 & 0 & \cdots & 0 & -\beta \end{pmatrix},$$

$\alpha' = (1, 0, \ldots, 0)$ and $\eta' = (0, 0, \ldots, 0, \beta)$. Then

$$q_{\mathbf{B}}(y) = Det(\mathbf{B} - y\mathbf{I}_m) = (-\beta - y)^m \quad \text{and} \quad q_{\mathbf{B}}(-\mathcal{D}) = \left(\frac{d}{du} - \beta\right)^m .$$

It is easily verified that $\left(\frac{d}{du} - \beta\right)^m p(x-u) = 0$.

Now, we want to apply $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation (4.2.4). We consider the polynomial expression of $q_{\mathbf{B}}(-\mathcal{D})$:

$$q_{\mathbf{B}}(-\mathcal{D}) = \sum_{i=0}^{m} q_i \frac{d^i}{du^i},$$

where q_i, $i = 0, 1, ..., m$, are constants (namely $q_0 = Det(\mathbf{B})$, $q_{m-1} = Trace\,(\mathbf{B})$, $q_m = 1$). Thus, we have the following theorem:

Theorem 4.2.3 After applying $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation (4.2.4) we get a linear homogeneous differential equation of degree $m+n$ of the following form

$$\begin{aligned} 0 \;=\; & \sum_{l=0}^{n+m} \left[\sum_{i+k=l} q_i \binom{n}{n-k} \left(1 + \frac{\delta}{\lambda}\right)^{n-k} \left(\frac{c}{\lambda}\right)^k \right] \frac{d^l}{du^l} V(u,b) \\ & + \sum_{j=0}^{m-1} \left[\sum_{k=j+1}^{m} q_k \alpha'(-\mathbf{B})^{k-j}\mathbf{1} \right] \frac{d^j}{du^j} V(u,b). \end{aligned} \tag{4.2.7}$$

Proof. Since, expanding the binomial, with $\frac{d^0}{du^0} = \mathcal{I}$,

$$\left(\left(1 + \frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b) = \sum_{k=0}^{n} \binom{n}{n-k} \left(1 + \frac{\delta}{\lambda}\right)^{n-k} \left(\frac{c}{\lambda}\right)^k \frac{d^k}{du^k} V(u,b),$$

then from one side we have

$$\begin{aligned} & q_{\mathbf{B}}(-\mathcal{D}) \left(\left(\left(1 + \frac{\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V(u,b) \right) = \\ & \qquad \sum_{l=0}^{n+m} \left[\sum_{i+k=l} q_i \binom{n}{n-k} \left(1 + \frac{\delta}{\lambda}\right)^{n-k} \left(\frac{c}{\lambda}\right)^k \right] \frac{d^l}{du^l} V(u,b) \,, \end{aligned}$$

and from the other side

$$\begin{aligned}
q_{\mathbf{B}}(-\mathcal{D})W_\delta(u,b) &= q_{\mathbf{B}}(-\mathcal{D})\left[\int_u^b V(x,b)p(x-u)dx + \int_b^\infty \widetilde{V}(x,b)p(x-u)dx\right] \\
&= \int_u^b V(x,b)q_{\mathbf{B}}(-\mathcal{D})p(x-u)dx - \sum_{k=1}^{m} q_k \sum_{j=0}^{k-1} \frac{d^j}{du^j}V(u,b)\left[\left.\frac{d^{k-1-j}}{du^{k-1-j}}p(x-u)\right|_{x=u}\right] \\
&\quad + \int_b^\infty \widetilde{V}(x,b)q_{\mathbf{B}}(-\mathcal{D})p(x-u)dx \\
&= -\sum_{j=0}^{m-1}\left[\sum_{k=j+1}^{m} q_k\alpha'(-\mathbf{B})^{k-j}\mathbf{1}\right]\frac{d^j}{du^j}V(u,b).
\end{aligned}$$

The result follows. ■

4.2.3 Expression for the expected discounted dividends

We look for solutions of (4.2.7) of the form

$$V(u,b) = \sum_{k=1}^{n+m} a_k e^{-r_k u} \tag{4.2.8}$$

that are also solutions of (4.2.4), for some coefficients a_k and some exponents r_k that are up to be determined. Solution is got replacing (4.2.8) in (4.2.4) and (4.2.6). The lefthand side of (4.2.4) comes

$$\sum_{k=1}^{n+m} a_k\left(1+\frac{\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_k\right)^n e^{-r_k u}\ .$$

For the righthand side $W_\delta(u,b)$, given by (4.2.6), we have for the first integral, that

$$\begin{aligned}
\int_u^b V(x,b)p(x-u)dx &= \int_u^b \sum_{k=1}^{n+m} a_k e^{-r_k x}\alpha' e^{\mathbf{B}(x-u)}\eta dx \\
&= \sum_{k=1}^{n+m} a_k\alpha' e^{-\mathbf{B}u}\left(\int_u^b e^{(\mathbf{B}-r_k\mathbf{I}_m)x}dx\right)\eta \\
&= \sum_{k=1}^{n+m} a_k\alpha'\left(\mathbf{B}-r_k\mathbf{I}_m\right)^{-1} e^{\mathbf{B}(b-u)}e^{-r_k b}\eta + \sum_{k=1}^{n+m} a_k\hat{p}(r_k)e^{-r_k u}\ .
\end{aligned} \tag{4.2.9}$$

The second integral in (4.2.6) comes

$$\begin{aligned}
\int_b^\infty \tilde{V}(x,b)p(x-u)dx &= \int_b^\infty \left(x-b+\sum_{k=1}^{n+m} a_k e^{-r_k b}\right) \alpha' e^{\mathbf{B}(x-u)}\eta dx \\
&= \alpha' e^{\mathbf{B}(b-u)} \left(\int_0^\infty x e^{\mathbf{B}x} dx\right)\eta + \sum_{k=1}^{n+m} a_k e^{-r_k b} \alpha' e^{-\mathbf{B}u} \int_b^\infty e^{\mathbf{B}x} dx \eta \\
&= \alpha' e^{\mathbf{B}(b-u)} \mathbf{B}^{-2}\eta - \sum_{k=1}^{n+m} a_k e^{-r_k b} \alpha' e^{\mathbf{B}(b-u)} \mathbf{B}^{-1}\eta \ .
\end{aligned}$$

Gathering the two integrals we have, knowing that $\eta = -\mathbf{B}\mathbf{1}$,

$$\alpha' \left[\sum_{k=1}^{n+m} a_k e^{-r_k b}\left((r_k\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B} + \mathbf{I}_m\right) - \mathbf{B}^{-1}\right] e^{\mathbf{B}(b-u)}\mathbf{1} + \sum_{k=1}^{n+m} a_k \hat{p}(r_k) e^{-r_k u} \ .$$

Equation (4.2.4) then comes

$$\begin{aligned}
0 &= \sum_{k=1}^{n+m} a_k \left[\left(1+\frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right) r_k\right)^n - \hat{p}(r_k)\right] e^{-r_k u} \\
&\quad - \alpha' \left[\sum_{k=1}^{n+m} a_k e^{-r_k b}\left((r_k\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B} + \mathbf{I}_m\right) - \mathbf{B}^{-1}\right] e^{\mathbf{B}(b-u)}\mathbf{1} \ .
\end{aligned} \tag{4.2.10}$$

Since equation (4.2.10) holds for any $u \geq 0$, the coefficients of $e^{-r_k u}$ and $e^{\mathbf{B}(b-u)}$ must be zero. This means that

$$\left(1+\frac{\delta}{\lambda} - \left(\frac{c}{\lambda}\right) r_k\right)^n - \hat{p}(r_k) = 0, \quad k = 1, \ldots, n+m,$$

so the exponents r_k, $k = 1, ..., n+m$, are all the $m+n$ roots of the generalized Lundberg's equation (**??**), where n roots have positive real parts, namely $\rho_1, \rho_2, \ldots, \rho_n$, and m have negative real parts, $\rho_{n+1}, \rho_{n+2}, \ldots, \rho_{n+m}$. Also, we must have

$$\alpha' \left[\sum_{k=1}^{n+m} a_k e^{-r_k b}\left((r_k\mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B} + \mathbf{I}_m\right) - \mathbf{B}^{-1}\right] = \mathbf{0}. \tag{4.2.11}$$

This gives a homogeneous system of m equations with $(m+n)$ unknown coefficients a_k. The remaining n equations that we need (to have a full system of $(m+n)$ equations with $(m+n)$

unknowns), are the n boundary conditions (4.2.5).

Example 4.2.2 Let's assume that the time between two consecutive jumps is Erlang(2) distributed and the individual jump amounts are Erlang$(2,\beta)$ distributed. Then, the negative loading condition is $c < \lambda/\beta$ and the generalized Lundberg's equation is given by

$$(\lambda+\delta-cs)^2(\beta+s)^2 = \lambda^2\beta^2. \tag{4.2.12}$$

Let

$$V(u,b) = \sum_{k=1}^{4} a_k e^{-\rho_k u}.$$

The exponents ρ_k, $k=1,...,4$, are the four roots of (4.3.12). Say, ρ_1, ρ_2 are the two roots with positive real parts and ρ_3, ρ_4 are those with negative real parts. From the two boundary conditions (4.2.5) we get

$$\sum_{k=1}^{4} a_k = 0 \text{ and } \sum_{k=1}^{4} a_k\rho_k = 0,$$

and from (4.3.10) we get

$$\sum_{k=1}^{4} a_k e^{-\rho_k b}\frac{\rho_k}{\rho_k+\beta} = -\frac{1}{\beta} \quad \text{and} \quad \sum_{k=1}^{4} a_k e^{-\rho_k b}\frac{\rho_k\beta}{(\rho_k+\beta)^2} = -\frac{1}{\beta},$$

so we have a system of four equations in the four unknowns $a_1,\dots,a_4$. In matrix form we have

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ e^{-\rho_1 b}\frac{\rho_1}{\rho_1+\beta} & e^{-\rho_2 b}\frac{\rho_2}{\rho_2+\beta} & e^{-\rho_3 b}\frac{\rho_3}{\rho_3+\beta} & e^{-\rho_4 b}\frac{\rho_4}{\rho_4+\beta} \\ e^{-\rho_1 b}\frac{\rho_1\beta}{(\rho_1+\beta)^2} & e^{-\rho_2 b}\frac{\rho_2\beta}{(\rho_2+\beta)^2} & e^{-\rho_3 b}\frac{\rho_3\beta}{(\rho_3+\beta)^2} & e^{-\rho_4 b}\frac{\rho_4\beta}{(\rho_4+\beta)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\beta} \\ -\frac{1}{\beta} \end{pmatrix}.$$

Now, set the values for the parameters $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then ρ_1 =1.831, $\rho_2 = 0.423$, $\rho_3 = -0.063$ and $\rho_4 = -1.471$. After computing the coefficients we obtain the values of the expected discounted dividends, for $u \in \{1,3,5,10,15,20\}$ and $b \in \{2,3,6,10,30,40\}$, that are shown in Table 4.1. This table was built similarly to Table

7.1 of Afonso *et al.* (2013). Also we notice that for a fixed u the value of $V(u,b)$ increases until a certain value of b and then decreases. This suggests an existing optimal value. This behavior is expected and corroborates the findings of Afonso *et al.* (2013) and Avanzi *et al.* (2007).

$u\backslash b$	2	3	6	10	30	40
1	1.049	1.301	1.856	1.781	0.526	0.279
3	3.492	4.533	6.451	6.189	1.826	0.972
5	5.492	6.533	9.374	8.993	2.653	1.412
10	10.492	11.533	14.501	13.829	4.081	2.172
15	15.492	16.533	19.501	18.829	5.647	3.006
20	20.492	21.533	24.501	23.829	7.746	4.123

Table 4.1: Expected discounted dividends

4.2.4 Higher moments of the discounted dividends

In the Erlang(n) model, the k-th ordinary moment of the discounted dividends $V_k(u,b)$ satisfies the renewal equation

$$\begin{aligned} V_k(u,b) &= \int_0^{\frac{u}{c}} k_n(t)e^{-\delta kt}\left[\int_0^{b-u+ct} V_k(u-ct+y,b)p(y)dy + \right. \\ &\left. \int_{b-u+ct}^{\infty} \widetilde{V}_k(u-ct+y,b)p(y)dy\right] dt, \end{aligned}$$

with

$$\widetilde{V}_k(x,b) = \sum_{j=0}^{k} \binom{k}{j} (x-b)^j V_{k-j}(b,b), \quad x \geq b.$$

In the above expression we have $V_0(u,b) \equiv 1$.

Theorem 4.2.4 $V_k(u,b)$ satisfies the integro–differential equation

$$\left(\left(1+\frac{k\delta}{\lambda}\right)\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n V_k(u,b) = W_{\delta k}(u,b), \tag{4.2.13}$$

with boundary conditions

$$\left.\frac{d^i}{du^i}V_k(u,b)\right|_{u=0} = 0, \quad i = 0,\ldots,n-1, \tag{4.2.14}$$

where

$$W_{k\delta}(u,b) = \int_u^b V_k(x,b)p(x-u)dx + \int_b^\infty \tilde{V}_k(x,b)p(x-u)dx.$$

Proof. The proof is similar to that of Theorem 4.2.1. ■

Assuming that gains follow a PH(m) distribution we can apply an analogous method to that used for $V(u,b)$ to find and an expression for $V_k(u,b)$ and numerical values in the same way. We apply the same annihilator $q_{\mathbf{B}}(-\mathcal{D})$ to the integro–differential equation (4.2.13) to obtain

$$\begin{aligned} 0 \;=\; & \sum_{l=0}^{n+m}\left[\sum_{i+j=l} q_i \binom{n}{n-j}\left(1+\frac{k\delta}{\lambda}\right)^{n-j}\left(\frac{c}{\lambda}\right)^j\right]\frac{d^l}{du^l}V_k(u,b) + \\ & \sum_{j=0}^{m-1}\left[\sum_{i=j+1}^{m} q_i\alpha'(-\mathbf{B})^{i-j}\mathbf{1}\right]\frac{d^j}{du^j}V_k(u,b). \end{aligned} \tag{4.2.15}$$

Therefore, we seek for solutions of (4.2.15) of the form

$$V_k(u,b) = \sum_{l=1}^{n+m} a_l e^{-r_l u}, \tag{4.2.16}$$

that are also solutions of (4.2.13), for some coefficients a_l and some exponents r_l that are up to be determined.

Similarly to the case $k=1$ in the previous subsection, replacing (4.3.9) in (4.2.13) we get

$$\begin{aligned} 0 \;=\; & \sum_{j=1}^{n+m} a_j\left[\left(1+\frac{k\delta}{\lambda}-\left(\frac{c}{\lambda}\right)r_j\right)^n - \hat{p}(r_j)\right]e^{-r_j u} \\ & -\alpha'\left[\sum_{j=1}^{n+m} a_j e^{-r_j b}\left((r_j\mathbf{I}_m-\mathbf{B})^{-1}\mathbf{B}+\mathbf{I}_m\right) + \sum_{j=1}^{k} j\binom{k}{j}V_{k-j}(b,b)(-\mathbf{B})^{-j}\right]e^{\mathbf{B}(b-u)}\mathbf{1}. \end{aligned} \tag{4.2.17}$$

Since (4.2.17) holds for any $u \geq 0$, we must have

$$\left(1 + \frac{k\delta}{\lambda} - \left(\frac{c}{\lambda}\right) r_j\right)^n - \hat{p}(r_j) = 0, \quad j = 1, \ldots, n+m,$$

so the exponents r_j, $j = 1, ..., n+m$, are all the $m+n$ roots of the generalized Lundberg's equation (**??**) (n roots have positive real parts and m roots with negative real parts). Furthermore,

$$\alpha' \left[\sum_{j=1}^{n+m} a_j e^{-r_l b} \left((r_j \mathbf{I}_m - \mathbf{B})^{-1}\mathbf{B} + \mathbf{I}_m\right) + \sum_{j=1}^{k} j \binom{k}{j} V_{k-j}(b,b)(-\mathbf{B})^{-j}\right] = \mathbf{0}. \qquad (4.2.18)$$

Example 4.2.3 (Example 4.2.2 cont'd) We want to compute $V_2(u,b)$. The generalized Lundberg's equation is now given by

$$(\lambda + k\delta - cs)^2(\beta + s)^2 = \lambda^2\beta^2. \qquad (4.2.19)$$

Let

$$V_2(u,b) = \sum_{j=1}^{4} a_j e^{-\rho_j u}.$$

The exponents ρ_j are the four roots of (4.2.19). From (4.2.14) and (4.2.18) we get $\sum_{j=1}^4 a_j = 0$, $\sum_{j=1}^4 a_j\rho_j = 0$ and

$$\begin{aligned}
\sum_{j=1}^{4} a_j e^{-\rho_j b} \frac{\rho_j}{\rho_j + \beta} &= -2V(b,b)\frac{1}{\beta} - \frac{2}{\beta^2}, \\
\sum_{l=1}^{4} a_j e^{-\rho_j b} \frac{\rho_j \beta}{(\rho_j + \beta)^2} &= -2V(b,b)\frac{1}{\beta} - \frac{4}{\beta^2}.
\end{aligned}$$

Therefore, in matrix form,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ e^{-\rho_1 b}\frac{\rho_1}{\rho_1+\beta} & e^{-\rho_2 b}\frac{\rho_2}{\rho_2+\beta} & e^{-\rho_3 b}\frac{\rho_3}{\rho_3+\beta} & e^{-\rho_4 b}\frac{\rho_4}{\rho_4+\beta} \\ e^{-\rho_1 b}\frac{\rho_1\beta}{(\rho_1+\beta)^2} & e^{-\rho_2 b}\frac{\rho_2\beta}{(\rho_2+\beta)^2} & e^{-\rho_3 b}\frac{\rho_3\beta}{(\rho_3+\beta)^2} & e^{-\rho_4 b}\frac{\rho_4\beta}{(\rho_4+\beta)^2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ -2V(b,b)\frac{1}{\beta} - \frac{2}{\beta^2} \\ -2V(b,b)\frac{1}{\beta} - \frac{4}{\beta^2} \end{pmatrix}.$$

Set $\lambda = \beta = 1$, $c = 0.75$, $\delta = 0.02$. Then $\rho_1 = 1.853$, $\rho_2 = 0.494$, $\rho_3 = -0.107$ and $\rho_4 = -1.467$.

Values for the standard deviation of $D(u, b)$, for $u = 1, 3, 5, 10, 15, 20$ and $b = 2, 3, 6, 10, 30, 40$ are shown in Table 4.2. Similar comments to those for Table 4.1 can be made. Note that the standard deviation does not depend on u for $u \geq b$.

$u \backslash b$	2	3	6	10	30	40
1	2.534	3.389	4.893	4.638	1.655	0.973
3	3.419	5.058	7.335	6.906	2.667	1.621
5	3.419	5.058	7.483	6.985	2.966	1.841
10	3.419	5.058	7.452	6.864	3.531	2.277
15	3.419	5.058	7.452	6.864	4.269	2.829
20	3.419	5.058	7.452	6.864	5.093	3.496

Table 4.2: Standard deviation of the discounted dividends

4.3 The Phase–Type case

In this section we consider a barrier strategy for dividend calculation in terms of a dividend barrier b. Although we just consider results for the expected discounted future dividends we could extend the presented methods to higher moments. Any time the regulated surplus upcrosses b the excess is paid as a dividend. From that payment instant the process restarts from level b and that repeats whenever it occurs in the future until ruin.

Let $\{D_i\}_{i=1}^{\infty}$ be the sequence of the dividend payments and let $D(u, b)$ be the aggregate discounted dividends, at force of interest δ. Let τ_i be the arrival time of D_i, then

$$D(u, b) = \sum_i e^{-\delta \tau_i} D_i.$$

We denote by $V(u, b) = E[D(u, b)]$, the expected value of $D(u, b)$.

Note that

$$V(u, b) = E[u - b + D(b, b)] = u - b + V(b, b), \quad u \geq b. \tag{4.3.1}$$

The expected discounted dividends $V(u, b)$ satisfy the following renewal equation:

$$
\begin{aligned}
V(u,b) &= \int_0^{\frac{u}{c}} k(t)e^{-\delta t}\left[\int_0^{b-u+ct} V(u-ct+y,b)p(y)dy \right.\\
&\quad \left. + \int_{b-u+ct}^{\infty} \widetilde{V}(u-ct+y,b)p(y)dy\right] dt, \quad \text{for } \ u<b,
\end{aligned}
$$

with

$$
\widetilde{V}(x,b) = E[D(x,b)] = E[x-b+D(b,b)] = x-b+V(b,b), \quad x \geq b.
$$

Differentiating the renewal equation with respect to u produces an integro–differential equation for $V(u,b)$.

Theorem 4.3.1 The expected discounted dividends $V(u,b)$ satisfy the integro–differential equation

$$
B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b), \quad u<b, \tag{4.3.2}
$$

where

$$
W(u,b) = \int_u^b V(x,b)p(x-u)dx + \int_b^{\infty} \widetilde{V}(x,b)p(x-u)dx.
$$

and $B_\delta(\mathcal{D}), q_\delta(\mathcal{D})$ are defined as in (3.3.10). The boundary conditions of (4.3.2) are given by

$$
\begin{aligned}
V(0,b) &= 0,\\
\left.\frac{d^i}{du^i}V(u,b)\right|_{u=0} &= \sum_{j=0}^{i-1}\left(\sum_{l=0}^{i-1-j}\frac{1}{c^{i-j}}\binom{i-1-j}{l}(-\delta)^l k^{(i-1-j-l)}(0)\right)W^{(j)}(0,b),\\
& \quad i=1,\ldots,n-1.
\end{aligned} \tag{4.3.3}
$$

Proof. The proof follows the same methodology as that of Theorem 3.3.3. ■

Because of the additional information of a barrier level b in $V(u,b)$, we cannot solve the equation

$$
B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b), \tag{4.3.4}
$$

to find an expression for $V(u,b)$ as we did for the Laplace transform of the time to ruin

$\psi(u, \delta)$. There, we did not need to specify a particular density function $p(x)$ for the gain amounts, here we do and we show this in the following remark:

Remark 4.3.1 Consider the conditions that must be met by ρ when we insert $f(u) = e^{-\rho u}$ in (4.3.4). On the left hand side we have

$$B_\delta(\mathcal{D})f(u) = B_\delta(-\rho)e^{-\rho u}. \tag{4.3.5}$$

On the right-hand side we get, denoting $W_f(u, b)$,

$$\begin{aligned} W_f(u,b) &= \int_0^{b-u} f(x+u)p(x)dx + \int_{b-u}^{\infty} (x+u-b+f(b))p(x)dx \\ &= \int_0^{b-u} e^{-\rho(x+u)}p(x)dx + \int_{b-u}^{\infty} (x+u-b+e^{-\rho b})p(x)dx \\ &= e^{-\rho u}\hat{p}(\rho) + \int_b^{\infty} (x-b+e^{-\rho b}-e^{-\rho x})p(x-u)dx, \end{aligned}$$

and

$$q_\delta(\mathcal{D})W_f(u,b) = q_\delta(-\rho)\hat{p}(\rho)e^{-\rho u} + \int_b^{\infty} (x-b+e^{-\rho b}-e^{-\rho x})q_\delta(\mathcal{D})p(x-u)dx. \tag{4.3.6}$$

Comparing equations (4.3.5) and (4.3.6) we obtain

$$(B_\delta(-\rho) - q_\delta(-\rho)\hat{p}(\rho))e^{-\rho u} = \int_b^{\infty} (x-b+e^{-\rho b}-e^{-\rho x})q_\delta(\mathcal{D})p(x-u)dx, \ \forall u \geq 0. \tag{4.3.7}$$

If ρ was a root of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$, the left hand side of (4.3.7) would be zero. On the other side, the right-hand side is not necessarily zero since $q_\delta(\mathcal{D})p(x-u)$ may not be zero.

Indeed, we have to assume a particular distribution for the gain amounts. For the rest of this manuscript, we assume that the gain amounts follow a Phase–Type(m) distribution and we use the annihilator method to find $V(u, b)$. See similar approach in Rodríguez-Martínez *et al.* (2015).

Following the notation in Section 2.5.1, consider the case when the gains X_i follow a

Phase-Type(m) distribution $P(x)$ with representation $(\boldsymbol{\alpha}', \mathbf{B}', \mathbf{b}')$. Let $\rho_1, \ldots, \rho_n$ be the roots of the generalized Lundberg's equation $B_\delta(-s) = q_\delta(-s)\hat{p}(s)$ with positive real parts, and $\rho_n, \ldots, \rho_{n+m}$ the roots with negatiFor simplicity, assume that all those roots are distinct (although this is not the case in general, see Bergel and Egídio dos Reis (2014) or Bergel and Egídio dos Reis (2016)).

Because of condition (4.3.1), we can not write the solutions of (4.3.2) as a linear combination of n exponential functions as we did before in the cases of the ruin probability and the Laplace transform of the time of ruin. We will need more than n exponential functions, the exact required number will depend on the nature of the distribution of the single gains $P(x)$. However, we can apply the annihilator approach known from the theory of ordinary differential equations to find the appropriate solutions.

We can rewrite $W(u,b)$ as

$$\begin{aligned} W(u,b) &= \int_u^b V(x,b)p(x-u)dx + \int_b^\infty (x-b+V(b,b))p(x-u)dx \qquad (4.3.8) \\ &= \int_u^b V(x,b)p(x-u)dx + \int_b^\infty \widetilde{V}(x,b)p(x-u)dx, \end{aligned}$$

with $\widetilde{V}(x,b) = x - b + V(b,b)$. The idea is to find a linear differential operator that will annihilate $p(x-u)$ (where the variable is u), so that when we apply this operator to the integro–differential equation (4.3.2) we obtain a linear homogeneous differential equation of a higher degree. We apply the annihilator operator, denoted as $A(\mathcal{D}) = Det(\mathbf{I_m}\mathcal{D} + \mathbf{B}')$, at both sides of the integro–differential equation

$$B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b),$$

where $\mathbf{I}_m$ is the identity $m \times m$ matrix, and we obtain an homogeneous integro–differential equation of degree $m+n$.

Theorem 4.3.2 When $P(x)$ is Phase-Type(m) the solution of $V(u,b)$ is of the form

$$V(u,b) = \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u}, \quad u < b, \qquad (4.3.9)$$

where $\rho_l, l = 1, \ldots, n, n+1, \ldots, n+m$ are the roots of the generalized Lundberg's equation, n with positive real parts and m with negative real parts, and the coefficients $a_l(b)$, depending on b, are found using the boundary n conditions (4.3.3), and the identity

$$\boldsymbol{\alpha'} \left[\sum_{l=1}^{n+m} a_l(b) e^{-\rho_l b} \left((\rho_l \mathbf{I_m} - \mathbf{B'})^{-1} \mathbf{B'} + \mathbf{I_m} \right) - \mathbf{B'}^{-1} \right] = \mathbf{0}, \tag{4.3.10}$$

which gives another m conditions. We obtain a system of $m+n$ equations on the $m+n$ unknowns $a_l(b)$.

Proof. Let $p(x-u) = \boldsymbol{\alpha'} e^{\mathbf{B'}(x-u)} \mathbf{b'}^{\mathsf{T}}$. The annihilator operator $A(\mathcal{D})$ can be expanded as

$$A(\mathcal{D}) = Det(\mathbf{I_m}\mathcal{D} + \mathbf{B'}) = \sum_{i=0}^{m} tr_{m-i}(\mathbf{B'})\mathcal{D}^i.$$

This operator annihilates $p(x-u)$

$$\begin{aligned} A(\mathcal{D})p(x-u) &= \sum_{i=0}^{m} tr_{m-i}(\mathbf{B'})\mathcal{D}^i \left(\boldsymbol{\alpha'} e^{\mathbf{B'}(x-u)} \mathbf{b'}^{\mathsf{T}} \right) \\ &= \boldsymbol{\alpha'} \left[\sum_{i=0}^{m} tr_{m-i}(\mathbf{B'})\mathcal{D}^i e^{\mathbf{B'}(x-u)} \right] \mathbf{b'}^{\mathsf{T}} \\ &= \boldsymbol{\alpha'} \left[\underbrace{\sum_{i=0}^{m} tr_{m-i}(\mathbf{B'})(-\mathbf{B'})^i}_{det(\mathbf{B'} - \mathbf{I_m}\mathbf{B'}) = A(-\mathbf{B'}) = 0} e^{\mathbf{B'}(x-u)} \right] \mathbf{b'}^{\mathsf{T}} = 0. \end{aligned}$$

Since

$$V(u,b) = \sum_{l=1}^{n+m} a_l(b) e^{-\rho_l u},$$

we will prove that $V(u,b)$ satisfies the homogeneous integro–differential equation

$$A(\mathcal{D})[B_\delta(\mathcal{D})V(u,b)] = A(\mathcal{D})[q_\delta(\mathcal{D})W(u,b)],$$

or equivalently,

$$B_\delta(\mathcal{D})[A(\mathcal{D})V(u,b)] = q_\delta(\mathcal{D})[A(\mathcal{D})W(u,b)]. \tag{4.3.11}$$

We have

$$
\begin{aligned}
W(u,b) &= \int_u^b V(x,b)p(x-u)dx + \int_b^\infty (x-b+V(b,b))p(x-u)dx \\
&= \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u}\hat{p}(\rho_l) + \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b}\boldsymbol{\alpha'}(\rho_l \mathbf{I_m} - \mathbf{B'})^{-1}\mathbf{B'}e^{\mathbf{B'}(x-u)}\mathbf{1}^\intercal \\
&\quad -\boldsymbol{\alpha'}(\mathbf{B'})^{-1}e^{\mathbf{B'}(x-u)}\mathbf{1}^\intercal + \boldsymbol{\alpha'}e^{\mathbf{B'}(x-u)}\mathbf{1}^\intercal \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b},
\end{aligned}
$$

so, in the right-hand side of (4.3.11), we have

$$
A(\mathcal{D})W(u,b) = \sum_{l=1}^{n+m} a_l(b)A(-\rho_l)\hat{p}(\rho_l)e^{-\rho_l u}.
$$

On the left hand side, we have

$$
A(\mathcal{D})V(u,b) = A(\mathcal{D})\sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l(b)A(-\rho_l)e^{-\rho_l u}.
$$

Then,

$$
\begin{aligned}
B_\delta(\mathcal{D})[A(\mathcal{D})V(u,b)] &= \sum_{l=1}^{n+m} a_l(b)A(-\rho_l)B_\delta(-\rho_l)e^{-\rho_l u}, \\
q_\delta(\mathcal{D})[A(\mathcal{D})W(u,b)] &= \sum_{l=1}^{n+m} a_l(b)A(-\rho_l)q_\delta(-\rho_l)\hat{p}(\rho_l)e^{-\rho_l u}.
\end{aligned}
$$

This proves that $V(u,b)$ satisfies (4.3.11), because $B_\delta(-\rho_l) = q_\delta(-\rho_l)\hat{p}(\rho_l)$ for the values $l = 1, \dots n+m$.

Now, we want $V(u,b)$ to be a solution of our integro–differential equation (4.3.2), as in Theorem 4.3.1. Since solutions of (4.3.11) include those of $B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b)$, we want to know which are the extra conditions that must be satisfied by the coefficients $a_l(b)$

of $V(u,b)$ for this purpose. Replacing $V(u,b)$ in $B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b)$ we obtain

$$\begin{aligned} 0 &= \sum_{l=1}^{n+m} a_l(b)[\underbrace{B_\delta(-\rho_l) = q_\delta(-\rho_l)\hat{p}(\rho_l)}_{=0}]e^{-\rho_l u} \\ &= \boldsymbol{\alpha'}\left[\sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b}\left((\rho_l\mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' + \mathbf{I_m}\right) - \mathbf{B}'^{-1}\right] q_\delta(-\mathbf{B}')e^{\mathbf{B}'(x-u)}\mathbf{1}^\intercal,\ \forall u \geq 0. \end{aligned}$$

This proves that the identity holds

$$\boldsymbol{\alpha'}\left[\sum_{l=1}^{n+m} a_l(b)e^{-\rho_l b}\left((\rho_l\mathbf{I_m} - \mathbf{B}')^{-1}\mathbf{B}' + \mathbf{I_m}\right) - \mathbf{B}'^{-1}\right] = \mathbf{0}.$$

Using this identity and the boundary conditions (4.3.3) we obtain a system of $m+n$ equations that allow us to find the $m+n$ coefficients $a_l(b)$ in $V(u,b)$. ■

Example 4.3.1 Assume that $K(t)$ is Ph(2) distributed ($n = 2$) and $P(x)$ is Ph(2) distributed ($m = 2$), with representations $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b})$ and $(\boldsymbol{\alpha'}, \mathbf{B}', \mathbf{b}')$, respectively.

The net profit condition is $-c\boldsymbol{\alpha}\mathbf{B}\mathbf{1}^\intercal < -\boldsymbol{\alpha'}\mathbf{B}'\mathbf{1}^\intercal$ and the generalized Lundberg's equation becomes

$$B_\delta(-s)\bar{B}(s) = q_\delta(-s)\bar{q}(s), \tag{4.3.12}$$

where

$$\begin{aligned} B_\delta(-s) &= 1 - c\frac{tr(\mathbf{B})}{det(\mathbf{B})}\left(\frac{\delta}{c} - s\right) + \frac{c^2}{det(\mathbf{B})}\left(\frac{\delta}{c} - s\right)^2, \\ q_\delta(-s) &= 1 + \frac{c}{det(\mathbf{B})}\boldsymbol{\alpha}\mathbf{b}^\intercal\left(\frac{\delta}{c} - s\right), \\ \bar{B}(s) &= 1 - \frac{tr(\mathbf{B}')}{det(\mathbf{B}')}s + \frac{1}{det(\mathbf{B}')}s^2, \\ \bar{q}(s) &= 1 + \frac{1}{det(\mathbf{B}')}\boldsymbol{\alpha'}\mathbf{b}'^\intercal s. \end{aligned}$$

Let

$$V(u,b) = \sum_{l=1}^{4} a_l(b)e^{-\rho_l u}.$$

The exponents ρ_l's are the four roots of (4.3.12). Assume that ρ_1, ρ_2 have positive real parts and ρ_3, ρ_4 have negative real parts. The coefficients $a_l(b)$'s are obtained using the corresponding boundary conditions (4.3.3)

$$\begin{aligned} V(0,b) &= \sum_{l=1}^{4} a_l(b) = 0, \\ \frac{d}{du}V(u,b)\Big|_{u=0} &= \frac{1}{c}k(0)W(0,b) = -\sum_{l=1}^{4} \rho_l a_l(b), \quad \text{or} \\ 0 &= \sum_{l=1}^{4} a_l(b)\left(\frac{k(0)}{c}\hat{p}(\rho_l) + \rho_l\right), \end{aligned}$$

and the additional constrains (4.3.10), giving

$$\sum_{l=1}^{4} a_l(b) e^{-\rho_l b} \rho_l \boldsymbol{\alpha'} (\rho_l \mathbf{I_2} - \mathbf{B'})^{-1} = \boldsymbol{\alpha'} \mathbf{B'}^{-1}, \quad \text{with} \quad \boldsymbol{\alpha'} = (\alpha'_1, \alpha'_2), \quad \mathbf{B'} = \begin{pmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{pmatrix},$$

or

$$\begin{aligned} \sum_{l=1}^{4} a_l(b)\left(\frac{e^{-\rho_l b}\rho_l(\alpha'_1(\rho_l - b'_{22}) + \alpha'_2 b'_{21})}{det(\rho_l \mathbf{I_2} - \mathbf{B'})}\right) &= \frac{\alpha'_1 b'_{22} - \alpha'_2 b'_{21}}{det(\mathbf{B'})}, \\ \sum_{l=1}^{4} a_l(b)\left(\frac{e^{-\rho_l b}\rho_l(\alpha'_1 b'_{12} + \alpha'_2(\rho_l - b'_{11}))}{det(\rho_l \mathbf{I_2} - \mathbf{B'})}\right) &= \frac{-\alpha'_1 b'_{12} + \alpha'_2 b'_{11}}{det(\mathbf{B'})}. \end{aligned}$$

If we set the values for the parameters $c = 1$, $\delta = 0.05$ and

$$\boldsymbol{\alpha} = (0.2, 0.8), \quad \mathbf{B} = \begin{pmatrix} -3 & 2 \\ 4 & -7 \end{pmatrix}, \quad \boldsymbol{\alpha'} = (0.7, 0.3), \quad \mathbf{B'} = \begin{pmatrix} -2 & 1 \\ 5 & -5 \end{pmatrix},$$

then $\rho_1 = 8.41055$, $\rho_2 = 0.949785$, $\rho_3 = -0.0374676$ and $\rho_4 = -6.22287$. In Table 5.1 we show numerical values for $V(u,b)$ for some choices of (u,b). We can observe that for a fixed u we have a maximal $V(u,b)$ for a value of b between 5 and 7. In the following, we devote our study to the optimal barrier level b and show that it is independent of u.

$u\backslash b$	3	5	6	7	8	10	15	20
2	14.0478	18.5161	18.6277	18.2447	17.6847	16.4611	13.656	11.3231
3	15.9118	20.9727	21.0991	20.6653	20.031	18.6451	15.4678	12.8254
5	17.9118	23.5692	23.7112	23.2237	22.5109	20.9533	17.3828	14.4132
10	22.9118	28.5692	28.7183	28.1731	27.3464	25.4359	21.1015	17.4967
15	27.9118	33.5692	33.7183	33.1731	32.3464	30.4359	25.4503	21.1026
20	32.9118	38.5692	38.7183	38.1731	37.3464	35.4359	30.4503	25.4504

Table 4.3: Values of $V(u,b)$

4.3.1 Optimal Dividends

For a given initial capital u, let b^* denote the optimal value of the barrier b that maximizes the expected discounted dividends $V(u,b)$. Avanzi *et al.* (2007) show that for a dual model with exponentially distributed inter-arrival times the value of b^* is independent of u. The same situation occurs for a dual model with Phase–type(n) distributed inter-gain times and Phase–Type(m) distributed gain amounts. Also, the optimal level is independent of the initial surplus.

Theorem 4.3.3 b^* is independent of the initial surplus u.

Proof. For a given initial surplus $u_0 \geq 0$ let b_0^* be the optimal barrier level that maximizes the expected discounted dividends, $V(u_0,b)$ is maximal at $b = b_0^*$ and

$$\frac{\partial}{\partial b} V(u_0,b)\bigg|_{b=b_0^*} = 0, \quad \text{for} \quad u = u_0.$$

The idea of this proof is to show that

$$\frac{\partial}{\partial b} V(u,b)\bigg|_{b=b_0^*} = 0, \quad \forall u \geq 0.$$

From (4.3.1), we have $\forall u \geq b_0^*$ that

$$\frac{\partial}{\partial b} V(u,b)\bigg|_{b=b_0^*} = 0 = -1 + \frac{d}{d\,b} V(b,b)\bigg|_{b=b_0^*} \quad \Rightarrow \frac{d}{d\,b} V(b,b)\bigg|_{b=b_0^*} = 1.$$

Since we have $V(0,b) \equiv 0$ then clearly

$$\frac{\partial}{\partial\, b} V(0,b)\bigg|_{b=b_0^*} = 0, \quad \text{for} \quad u = 0.$$

It only remains to show that

$$\frac{\partial}{\partial\, b} V(u,b)\bigg|_{b=b_0^*} = 0, \quad 0 < u < b_0^*.$$

Previously in Theorem 4.3.1 we have found that in the Phase–Type(n) dual risk model the expected discounted dividends $V(u,b)$ satisfy the integro–differential equation

$$B_\delta(\mathcal{D})V(u,b) = q_\delta(\mathcal{D})W(u,b),$$

where

$$W(u,b) = \int_u^b V(y,b)p(y-u)dy + \int_b^\infty (y - b + V(b,b))p(y-u)dy\,.$$

Moreover, assuming that the gain amounts follow another Phase–Type(m) distribution, with density function $p(x) = \boldsymbol{\alpha}' e^{\mathbf{B}'x}\mathbf{b}'^{\mathsf{T}}$, we were able to write an expression of $V(u,b)$ of the form (4.3.9)

$$V(u,b) = \sum_{l=1}^{n+m} a_l(b)e^{-\rho_l u}.$$

Since

$$\begin{aligned}
\frac{\partial}{\partial\, b} W(u,b)\bigg|_{b=b_0^*} &= \int_u^{b_0^*} \frac{\partial}{\partial\, b} V(y,b)\bigg|_{b=b_0^*} p(y-u)dy + \\
&\quad \underbrace{\left(-1 + \frac{d}{d\, b} V(b,b)\bigg|_{b=b_0^*}\right)}_{=0} \int_{b_0^*}^\infty p(y-u)dy \\
&= \int_u^{b_0^*} \frac{\partial}{\partial\, b} V(y,b)\bigg|_{b=b_0^*} p(y-u)dy,
\end{aligned}$$

then for $0 < u < b_0^*$ we have that

$$B_\delta(\mathcal{D})\,\frac{\partial}{\partial\, b}V(u,b)\bigg|_{b=b_0^*} = q_\delta(\mathcal{D})\,\frac{\partial}{\partial\, b}W(u,b)\bigg|_{b=b_0^*},$$

or equivalently

$$B_\delta(\mathcal{D})\,\frac{\partial}{\partial\, b}V(u,b)\bigg|_{b=b_0^*} = q_\delta(\mathcal{D})\left[\int_u^{b_0^*}\frac{\partial}{\partial\, b}V(y,b)\bigg|_{b=b_0^*}p(y-u)dy\right],\quad 0<u<b_0^*. \quad (4.3.13)$$

When we replace

$$\frac{\partial}{\partial\, b}V(u,b)\bigg|_{b=b_0^*} = \sum_{l=1}^{n+m} a_l'(b_0^*)e^{-\rho_l u}$$

in (4.3.13) we get an identity of exponential functions in terms of the coefficients $a_l'(b_0^*)$ which is valid for all u in $(0, b_0^*)$, as follows.

Let's define the function

$$F(u) = \frac{\partial}{\partial\, b}V(u,b)\bigg|_{b=b_0^*} = \sum_{l=1}^{n+m} a_l'(b_0^*)e^{-\rho_l u}.$$

Then (4.3.13) becomes

$$B_\delta(\mathcal{D})F(u) = q_\delta(\mathcal{D})\left[\int_u^{b_0^*}F(y)p(y-u)dy\right],\quad 0<u<b_0^*. \quad (4.3.14)$$

On the left-hand side of (4.3.14) we calculate $B_\delta(\mathcal{D})F(u)$,

$$B_\delta(\mathcal{D})F(u) = \sum_{l=1}^{n+m} a_l'(b_0^*)B_\delta(\mathcal{D})e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l'(b_0^*)B_\delta(-\rho_l)e^{-\rho_l u}. \quad (4.3.15)$$

On the right hand side of (4.3.14) we compute $q_\delta(\mathcal{D})\left[\int_u^{b_0^*}F(y)p(y-u)dy\right]$.

Recall that $p(y-u) = \boldsymbol{\alpha'} e^{\mathbf{B}'(y-u)}\mathbf{b'}^{\mathsf{T}}$, therefore

$$\begin{aligned}
\int_u^{b_0^*} e^{-\rho_l y} p(y-u)dy &= e^{-\rho_l u}\hat{p}(\rho_l) - e^{-\rho_l u}\int_{b_0^*-u}^{\infty} e^{-\rho_l y} p(y)dy \\
&= e^{-\rho_l u}\hat{p}(\rho_l) - e^{-\rho_l u}\int_{b_0^*-u}^{\infty} e^{-\rho_l y}\boldsymbol{\alpha'} e^{\mathbf{B}'(y)}\mathbf{b'}^{\mathsf{T}} dy \\
&= e^{-\rho_l u}\left[\hat{p}(\rho_l) - \int_{b_0^*-u}^{\infty} \boldsymbol{\alpha'} e^{(\mathbf{B}'-\rho_l\mathbf{I})y}\mathbf{b'}^{\mathsf{T}} dy\right] \\
&= e^{-\rho_l u}\left[\hat{p}(\rho_l) - \boldsymbol{\alpha'}\int_{b_0^*-u}^{\infty} e^{(\mathbf{B}'-\rho_l\mathbf{I})y} dy\ \mathbf{b'}^{\mathsf{T}}\right] \\
&= e^{-\rho_l u}\left[\hat{p}(\rho_l) + \boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1} e^{(\mathbf{B}'-\rho_l\mathbf{I})(b_0^*-u)}\ \mathbf{b'}^{\mathsf{T}}\right] \\
&= e^{-\rho_l u}\left[\hat{p}(\rho_l) + \boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1} e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*} e^{-\mathbf{B}'u}\ \mathbf{b'}^{\mathsf{T}}\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
q_\delta(\mathcal{D})\int_u^{b_0^*} e^{-\rho_l y} p(y-u)dy &= q_\delta(-\rho_l)e^{-\rho_l u}\hat{p}(\rho_l) + \\
&\quad \boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1} e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*} q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b'}^{\mathsf{T}},
\end{aligned}$$

and,

$$\begin{aligned}
q_\delta(\mathcal{D})\int_u^{b_0^*} F(y)p(y-u)dy &= \sum_{l=1}^{n+m} a_l'(b_0^*) q_\delta(\mathcal{D})\int_u^{b_0^*} e^{-\rho_l y} p(y-u)dy \\
&= \sum_{l=1}^{n+m} a_l'(b_0^*) q_\delta(-\rho_l)e^{-\rho_l u}\hat{p}(\rho_l) + \\
&\quad \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1} e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*} q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\mathbf{b'}^{\mathsf{T}}. \quad (4.3.16)
\end{aligned}$$

Expressions in (4.3.15) and (4.3.16) are equal,

$$\begin{aligned}
\sum_{l=1}^{n+m} a_l'(b_0^*) B_\delta(-\rho_l)e^{-\rho_l u} &= \sum_{l=1}^{n+m} a_l'(b_0^*) q_\delta(-\rho_l)e^{-\rho_l u}\hat{p}(\rho_l) + \\
&\quad \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1} e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*} q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b'}^{\mathsf{T}}.
\end{aligned}$$

So,

$$\sum_{l=1}^{n+m} a_l'(b_0^*)[B_\delta(-\rho_l)-q_\delta(-\rho_l)\hat{p}(\rho_l)]e^{-\rho_l u} = \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}}.$$

Since $\rho_1,\ldots,\rho_{m+n}$ are the roots of the generalized Lundberg's equation then $B_\delta(-\rho_l) = q_\delta(-\rho_l)\hat{p}(\rho_l)$. Thus,

$$\begin{aligned} 0 &= \sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}} \\ &= \left[\underbrace{\sum_{l=1}^{n+m} a_l'(b_0^*)\boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}}_{=0}\right] q_\delta(-\mathbf{B}')e^{-\mathbf{B}'u}\ \mathbf{b}'^{\mathsf{T}},\ \forall\, u\in(0,b_0^*), \end{aligned}$$

since the above identity is valid for all u in the interval $(0,b_0^*)$. For simplicity, we have assumed that the roots $\rho_1,\ldots,\rho_{m+n}$ are all distinct, then the vectors $\boldsymbol{\alpha'}(\mathbf{B}'-\rho_l\mathbf{I})^{-1}e^{(\mathbf{B}'-\rho_l\mathbf{I})b_0^*}$ are linearly independent and we obtain

$$a_l'(b_0^*) = 0,\ \forall\, l = 1,\ldots,m+n.$$

This proves that

$$\left.\frac{\partial}{\partial\, b}V(u,b)\right|_{b=b_0^*} = 0, \quad 0<u<b_0^*.$$

Therefore, we have proven that the optimal barrier level is independent of u. ■

Remark 4.3.2 The result holds if we assume multiplicities higher than 1 in the roots ρ_l's.

Example 4.3.2 In Example 4.3.1 the optimal value of the barrier level is $b^* = 5.61986$, with $V(b^*,b^*) = 24.3976$.

4.4 Final remarks

One of the important goals of the study of risk theory is to find exact numerical techniques which can become popular in insurance practice. In this chapter we have investigated the expected discounted dividends in the case when the gain inter–occurrence times follow distributions belonging to the matrix exponential family, like the Phase–Type and the Erlang. However, the results can be still generalized for other distributions.

Chapter 5

On the probability and amount of a dividend

5.1 Introduction

This chapter consists of some incipient ideas related to dividend problems for future research. We consider again the dual risk model with an upper barrier level, but this time we don't focus on the expected discounted dividends. Instead, we study other quantities such as the probability of reaching the upper barrier before ruin occurring and the dividend amount and its distribution function.

In Section 5.2 we present the definition of the quantities mentioned above. We follow the article by Afonso *et al.* (2013) who consider them for a dual risk model with exponentially distributed gain inter–ocurrence times.

In Section 5.3 we present two methods to calculate the expected discounted dividends and the distribution of a dividend amount for a case when the gain inter-arrival times follow an Erlang distribution. This method can be applied for the probability of reaching an upper barrier before ruin and extended for more general matrix exponential distributions.

Section 5.4 is devoted to mention some ideas of future research for more general distribution types.

5.2 Basic definitions

Let's consider an arbitrary upper level $b \geq u \geq 0$ and let

$$\tau_u = \inf\{t > 0 : U(t) > b \mid U(0) = b\}$$

be the time to reach b for the surplus process, allowing the process to continue even if it crosses the ruin level "0". Due to the net profit condition (2.2.1) τ_u is a proper random variable since the probability of crossing b is one.

We denote by $\chi(u,b)$ the probability of reaching an upper barrier level b before ruin occurring, for a process with initial surplus u, and $\xi(u,b) = 1 - \chi(u,b)$ is the probability of ruin before reaching b. We have $\chi(u,b) = Pr(\tau_u < T_u)$.

Because of the existence of the barrier b the ruin probability is one. The ruin level can be attained before or after the process is reflected on b. Then the probability of ultimate ruin is $\xi(u,b) + \chi(u,b) = 1$.

Let $D_u = \{U(\tau_u) - b\}$ and $\tau_u < T_u$ be the dividend amount and its distribution function be denoted as

$$\begin{aligned} G(u,b;x) &= Pr((\tau_u < T_u \text{ and } U(\tau_u) \leq b + x) \mid u,b) \\ &= Pr((\tau_u < T_u \text{ and } D_u \leq x) \mid u,b) \end{aligned}$$

with density $g(u,b;x) = \frac{d}{dx}G(u,b;x)$. $G(u,b;x)$ is a defective distribution function, clearly

$$\lim_{x\to\infty} G(u,b;x) = Pr(\tau_u < T_u) = \xi(u,b) < 1.$$

5.3 Some developments

In what follows we concentrate on finding a method to calculate $G(u,b;x)$ for a gain inter-arrival times cdf $K(t)$, with density $k(t)$, and a gain amounts cdf $P(x)$, with density $p(x)$. The method can be emulated to calculate other probabilities, like $\xi(u,b)$ (and therefore $\chi(u,b)$).

Let $t_0 = u/c$. If no gain arrives before t_0, we have

$$G(u,b;x) = Pr((\underbrace{\tau_u < T_u}_{\text{false}} \text{ and } D_u \leq x) \mid u,b) = 0.$$

If the first gain arrives before t_0, then we derive a defective renewal equation

$$\begin{aligned} G(u,b;x) &= \int_0^{t_0} k(t) \Bigg[\underbrace{\int_0^{b-(u-ct)} G(u-ct+y,b;x)p(y)dy}_{\text{first gain and no dividend}} \\ &= + \underbrace{\int_{b-(u-ct)}^{b+x-(u-ct)} p(y)dy}_{\text{first gain and dividend}} \Bigg] dt. \end{aligned}$$

If $b \leq u \leq b+x$ we have $D_u = u - b$ and $G(u,b;x) = 1$.

Now let's assume the gain inter-arrival times follow an Erlang distribution $k(t)$. With the change of variables $s = u - ct$ the defective renewal equation above becomes

$$G(u,b;x) = \frac{1}{c}\int_0^u k\left(\frac{u-s}{c}\right) W_G(s,b;x)ds, \quad 0 < u < b, \tag{5.3.1}$$

where

$$\begin{aligned} W_G(s,b;x) &= \int_0^{b-s} G(s+y,b;x)p(y)dy + \int_{b-s}^{b+x-s} p(y)dy \\ &= \int_s^b G(y,b;x)p(y-s)dy + \int_b^{b+x} p(y-s)dy \\ &= \int_s^b G(y,b;x)p(y-s)dy + P(b+x-s) - P(b-s). \end{aligned}$$

Taking successive derivatives of (5.3.6) we obtain an integro–differential equation for $G(u,b;x)$

$$\left(\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n G(u,b;x) = W_G(u,b;x), \quad 0 < u < b, \tag{5.3.2}$$

with boundary conditions

$$\mathcal{D}^i G(u,b;x)\mid_{u=0} = 0, \quad 0 \le i \le n-1. \tag{5.3.3}$$

In what follows, we present two alternative ways to calculate $G(u,b;x)$. The first makes use of the same methodology used in Chapters 3 and 4, using annihilators, and the second generalizes Afonso *et al.* (2013) for Erlang distributions, using Laplace transforms.

5.3.1 Expected Discounted Dividends

For a general compound renewal as set here, we can retrieve the formulae for the total expected dividends from Afonso *et al.* (2013). Although it was developed for the classical compound Poisson model it is easy to see that it is applicable more generally. Let $D(u,b,\delta)$ denote the aggregate amount of discounted dividends and $V_n(u;b) = \mathbb{E}[D(u,b,\delta)^n]$, $n \in \mathbb{N}$, be its k-th ordinary moment, with $V(u;b) = V_1(u;b)$ for simplicity sake. The total discounted dividends $D(u,b,\delta)$ is given by, see Section 4 and Formula (4.2) by Afonso *et al.* (2013),

$$\begin{aligned} V(u;b) &= \mathbb{E}[D(u,b,\delta)] = \mathbb{E}\left[\sum_{i=1}^{\infty} e^{-\delta\left(\sum_{j=1}^{i} T_{(j)}\right)} D_{(i)}\right] \ , 0 \le u \le b \ , \\ &= \mathbb{E}\left(e^{-\delta T_u} D_u\right) + \mathbb{E}\left(e^{-\delta T_u}\right) \frac{\mathbb{E}\left(e^{-\delta T_b} D_b\right)}{1-\mathbb{E}\left(e^{-\delta T_b}\right)}. \end{aligned}$$

For higher moments, $V_n(u;b)$ recursion given by (4.7-8) in Afonso *et al.* (2013) also applies. We reproduce, *

$$V_n(u;b,\delta) = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\left[e^{-n\delta T_u} {D_u}^k\right] V_{n-k}(b;b,\delta)\,, \tag{5.3.4}$$

with $V_0(b;b,\delta) = 1$ and

$$V_n(b;b,\delta) = \frac{\sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\left[e^{-n\delta T_b} {D_b}^k\right] V_{n-k}(b;b,\delta)}{1-\mathbb{E}\left[e^{-n\delta T_b}\right]}\,. \tag{5.3.5}$$

[We need to develop appropriate formulae for $\mathbb{E}\left[e^{-\delta T_u} {D_u}^k\right]$, $k \in \mathbb{N}_0$]

5.3.2 On the amount and the probability of a single dividend

In this section we present two methods to calculate the distribution of a dividend amount, $G(u,b;x)$, for a case when the gain inter-arrival times follow an Erlang(n) distribution. That is, we consider that $K(t) = 1 - \sum_{1=0}^{n-1} e^{-\lambda t} (\lambda t)^i / i!$, and $k(t) = \lambda^n t^{n-1} e^{\lambda t}/(n-1)!$, $t \geq 0$. This method can be applied for the probability of reaching an upper barrier before ruin and extended for more general matrix exponential distributions. A similar approach can then be applied for the more general family of phase-type(n) distributions.

Let $t_0 = u/c$. If no gain arrives before t_0 necessarily event $\{\tau_u < T_u\}$ is false, and

$$G(u,b;x) = Pr\{\tau_u < T_u \text{ and } D_u \leq x \mid u,b\} = 0.$$

If the first gain arrives before t_0, either it does or does not cross b, then we derive a defective renewal equation

$$\begin{aligned} G(u,b;x) &= \int_0^{t_0} k(t) \left(\int_0^{b-(u-ct)} G(u-ct+y,b;x) p(y) dy \right. \\ &\quad \left. + \int_{b-(u-ct)}^{b+x-(u-ct)} p(y) dy \right) dt\,, \end{aligned}$$

where the first inner integral represents the probability of having a first gain at fixed time t and which amount does not cross the dividend level, but hapening in the fture, and the second representing the probability of a dividend in the first gain at t.

If $b \leq u \leq b+x$ we have $D_u = u-b$ and $G(u,b;x) = 1$. With the change of variables $s = u - ct$ the defective renewal equation above becomes

$$G(u,b;x) = \frac{1}{c} \int_0^u k\left(\frac{u-s}{c}\right) W_G(s,b;x) ds, \quad 0 < u < b, \tag{5.3.6}$$

$$
\begin{aligned}
W_G(s,b;x) &= \int_0^{b-s} G(s+y,b;x)p(y)dy + \int_{b-s}^{b+x-s} p(y)dy \\
&= \int_s^{b} G(y,b;x)p(y-s)dy + \int_{b}^{b+x} p(y-s)dy \\
&= \int_s^{b} G(y,b;x)p(y-s)dy + P(b+x-s) - P(b-s)\,.
\end{aligned}
$$

Taking successive derivatives of (5.3.6) we obtain the integro–differential equation for $G(u,b;x)$ in the following theorem, $\mathcal{D}$ is the differential operator with $\mathcal{D}^0 = \mathcal{I}$.

Theorem 5.3.1

$$
\left(\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n G(u,b;x) = W_G(u,b;x), \quad 0 < u < b, \tag{5.3.7}
$$

with boundary conditions

$$
\mathcal{D}^i G(u,b;x)\mid_{u=0} = 0, \quad 0 \le i \le n-1\,. \tag{5.3.8}
$$

□

Calculation of a solution for $G(u,b;x)$ from equations above is not straightforward. We can do it in two ways: Either using the annihilator method used by Rodríguez-Martínez *et al.* (2015) or the Laplace transform method as used by Afonso *et al.* (2013). In the following subsections we work both.

5.3.3 Annihilator method

Let $A(\mathcal{D})$ be a polynomial operator in $\mathcal{D}$, such that

$$
A(\mathcal{D})p(y-u) = 0.
$$

Then $A(\mathcal{D})W_G(u,b;x) = 0$ and

$$
A(\mathcal{D})\left(\mathcal{I} + \left(\frac{c}{\lambda}\right)\mathcal{D}\right)^n G(u,b;x) = 0
$$

is an homogeneous integro-differential equation of some degree $m+n$, whose solutions can be written as combinations of exponential functions

$$G(u,b;x) = \sum_{i=0}^{n+m-1} C_i e^{-r_i u}, \tag{5.3.9}$$

for some constants r_i's and some coefficients C_i's, functions of b and x, all independent from u.

In order to determine the r_i's and C_i's we replace (5.3.9) in (5.3.7) leading to the result in the theorem that follows.

Theorem 5.3.2

$$G(u,b;x) = \sum_{i=0}^{n+m-1} a_i e^{-r_i u}, \tag{5.3.10}$$

where $r_i, i = 0, \ldots, n+m-1$ is a root of Lundberg's equation, and the a_i's are found using the boundary conditions (5.3.8) together with the additional constraint

$$\sum_{i=0}^{n+m-1} \left[a_i \int_{b-u}^{\infty} e^{-r_i y} p(y) dy \right] e^{-r_i u} - P(b+x-u) - P(b-u) = 0\,,\ \forall u > 0.$$

□

To find $G(u,b;x)$ we must specify a distribution $P(x)$ for the individual claim amounts, where n of the roots of Lundberg's equation have positive real parts and m with negative real parts.

5.3.4 Laplace transforms method

Now we follow the approach presented by Afonso *et al.* (2013). Denoting $z = b-u$, or $u = b-z$, define

$$\tilde{G}(z,b;x) = G(b-z,b;x) = G(u,b;x). \tag{5.3.11}$$

Therefore

$$\tilde{G}(z,b;x) = \frac{1}{c} \int_0^{b-z} k\left(\frac{b-z-s}{c}\right) W_G(s,b;x) ds,$$

and the integro-differential equation (5.3.7) becomes

$$\left(\mathcal{I} - \left(\frac{c}{\lambda}\right)\frac{d}{dz}\right)^n \tilde{G}(z,b;x) = W_{\tilde{G}}(b-z,b;x), \quad 0 < z < b, \tag{5.3.12}$$

with boundary conditions

$$\frac{d^i}{dz^i}\tilde{G}(z,b;x)\mid_{z=0} = 0, \quad 0 \le i \le n-1. \tag{5.3.13}$$

In order to apply Laplace transforms to the integro-differential equation (5.3.12), we extend the domain of $\tilde{G}(z,b;x)$, as a function of z, to $(0,\infty)$:

$$\rho(z,b;x) := \begin{cases} \tilde{G}(z,b;x), & 0 \le z \le b, \\ 0, & z > b, \end{cases}$$

Afterwards, we obtained

Theorem 5.3.3

$$\widehat{\rho}(s,b;x) = \frac{\sum_{i=0}^{n-1}\left[\sum_{j=i+1}^{n}\left(-\frac{c}{\lambda}\right)^j \binom{n}{j}\rho^{(j-1-i)}(0,b;x)\right]s^i + \mathcal{T}_s P(x) - \mathcal{T}_s P(0)}{\left(1-\frac{c}{\lambda}s\right)^n - \widehat{p}(s)}, \tag{5.3.14}$$

where $\mathcal{T}_s f(x) = \int_0^\infty e^{-st} f(t+x)dt$ is an integral operator over an integrable function f.

We note $\mathcal{T}_s f(x)$ in known in the actuarial literature as Dickson-Hipp operator, see Dickson and Hipp (2001).

If we want to obtain $\rho(z,b;x)$ and therefore, $G(u,b;x)$, we must specify a distribution $P(x)$ for the claim amounts. This allows to factor the denominator, using all the roots of the Lundberg's equation, separate the resulting expression into partial fractions and then invert the Laplace transform (5.3.14).

5.4 Extensions

The quantities described in the previous section were all considered by Afonso et al. (2013) for the case when the gain inter–occurrence times follow an exponential distribution.The same techniques shown in previous sections can be extended straightforwardly for the generalized Erlang or even for the Phase–Type distributions. Our idea is to obtain similar closed formulas, whenever possible, and to provide numerical examples that could allow us to verify how sensitive is the model to the change of distributions and compare figures and numerical examples with Afonso et al. (2013) for the exponential case.

Chapter 6

Final remarks

It is in general difficult, if not impossible, to achieve all the objectives of the work plan which we originally wanted to execute.

In many occasions, solving a problem, or answering a question, just opens up the door for many more questions, widening up the view of an even bigger and yet unexplored landscape.

Scientific research is never a complete perfect product of developments but a long path of successive improvements and exploration.

The work which we developed and wrote in the course of this PhD thesis is another proof of this fact. We have been able to find the answer to some problems - which are related to mathematical models of interest in the non-life insurance industry - just to realize right after that we can formulate those same problems on a more general setting, with a more practical view, or with more realistic assumptions, leading the path for possible future research in this matter.

Now we summarize all the work which was achieved during writing this thesis.

In Chapter 1 we gave a brief history of risk theory and an overview of several different articles that have been published in this area. We also outlined the contents of this dissertation.

In Chapter 2 we set out the main characterization of the models and the concepts of risk theory that we considered in this manuscript. We described the dual risk model and denoted the aggregate gains as a random process $S(t)$. We introduced the definitions of some important quantities, like the ruin probability, the Laplace transform of the time to ruin and the expected discounted dividends and presented the existing connection between the dual risk model and the Cramér–Lundberg risk model. We defined the Lundberg's equations and introduced the integro–differential equations that are satisfied by the quantities mentioned before. Finally we presented the probability distributions that were used in the remaining of this thesis.

In Chapter 3 we studied the ruin probability and the Laplace transform of the time to ruin for a dual risk model. The distribution of the gain inter–occurrence times is matrix exponential. However, for simplicity and elegance of the formulas we concentrated on the Phase–Type and the Erlang distributions.

For that purpose, we used the respective Lundberg's equations and their solutions, considering the situations of multiplicity one (simple roots) or multiplicity higher than one (multiple roots). The integro–differential equations satisfied by ruin probability and the Laplace transform of the time to ruin were solved, and we obtained closed formulas for these quantities as well as interesting comparisons between the primal and the dual risk models. We provided examples of our results whenever possible.

In Chapter 4 we studied the expected discounted dividends for a dual risk model with an upper barrier level. The distribution of the gain inter–occurrence times is matrix exponential. For simplicity and elegance of the formulas we concentrate on the Phase–Type and the Erlang distributions.

We studied the integro–differential equations satisfied by the expected discounted dividends and their higher moments, and we solved them using similar techniques as in Chapter 3. However, due to the fact that a single dividend payment is obtained at a random event, we needed to specify a distribution for the gain amounts, and we choose again the Phase–Type.

Finally we studied the problem of setting an optimal barrier level to maximize the expected discounted dividends prior to ruin, and discovered that such optimal value of the barrier is independent of the initial surplus.

Many open problems remain. For example, is it possible to extend the kind of techniques used on this thesis to other - more general - distributions?, which ones?, is it possible to relax some of the assumptions of the dual risk model in such a way that the results obtained in this dissertation still hold? what kind of quantities from the primal/dual risk models could be studied by looking at their counterparts in the dual/primal risk models, using symmetric or duality arguments? We don't know the answers at the moment, but we hope that in the near future, more lights could shine on this area of risk theory.

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