



## A forgotten variational principle

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**Abstract** : An old variational principle (Maupertuis principle), forgotten as inconvenient quirk of History has been reformulated into the modified variational principle (MP) along with its reciprocal principle (RMP) and both these principles have been derived classically as well as quantum mechanically. In order to demonstrate the scope of very wide applicability of these new principles, the RMP has been applied to simple problems of Linear Oscillator, Anharmonic Oscillator, Anisotropic 2D-Quartic Oscillator (chaotic system) and Central force problem.

**Keywords** : Variational principle, anharmonic oscillator, chaotic system

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### Plan of the Article

#### 1. Introduction

#### 2. Derivation of MP and RMP

*(a) Classical derivation*

*(b) Quantum mechanical derivation*

#### 3. Simple applications of MP and RMP

*(a) Linear oscillator*

*(b) Anharmonic oscillator*

*(c) Anisotropic 2D quartic oscillator (A chaotic system)*

*(d) Central force problem*

#### 4. Discussions

## 1. Introduction

A mechanical system of particles can be described by a Lagrangian  $L(q_i, \dot{q}_i, t)$  or by Hamiltonian  $H(q_i, p_i, t)$  where  $q_i$  are generalized coordinates and  $p_i$  are the generalized momenta. The transformation from  $(q_i, \dot{q}_i, L)$  to  $(q_i, p_i, H)$  is the Legendre transformation

$$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t). \quad (1.1)$$

Maupertuis proposed in 1744 a global integral quantity [1] 'Action' which is least along the true path and greater for the unphysical virtual paths. Maupertuis definition of action and his corresponding principle were little vague, which were modified by Euler and Lagrange who took the action as

$$W = \int_{q_i}^{q_f} \sum_j p_j dq_j \quad (1.2)$$

where the arbitrary path  $q_j(t')$  runs from an initial point  $q_i = q_j(0)$  to a final point  $q_f = q_j(t)$ . These end points are kept fixed but the duration  $t$  is path dependent. Euler and Lagrange showed that for the true trajectory  $W$  is stationary provided that the virtual trajectories  $q_j(t')$  all are restricted to have the same fixed energy  $E = H = T + V$ . Thus the Maupertuis principle of least action may be written as

$$(\delta W)_E = 0 \quad (1.3)$$

where  $\delta W$  denotes a first order variation and the subscript  $E$  denotes that the energy is held fixed during the variation. This constraint of fixed energy brings several weaknesses in this Maupertuis principle given by eq. (1.3). It makes the energy conservation an assumption (and not a consequence), makes it cumbersome to convert this principle into a differential equation for the trajectory, paths become awkward to be handled analytically, and the virtual trajectories allowed by this principle in one dimension differ from the true trajectory by instantaneous velocity reversals. Due to these weaknesses the Maupertuis action (1.2) and his principle (1.3) were forgotten altogether. However, this principle can be reformulated (modified) into very useful variational principle (MP) by relaxing the constraint of fixed energy for virtual path, allowing a larger class of trial trajectories and keeping the mean energy  $\bar{E}$  fixed (not necessarily conserving the energy).

The mean energy  $\bar{E}$  may be defined as

$$\bar{E} = (1/t) \int_0^t H(q_i, p_i, t') dt' \quad (1.4)$$

and then this modified principle (MP) becomes

$$(\delta W)_{\bar{E}} = 0. \quad (1.5)$$

This variation principle, free from weakness of old principle (1.3), has the additional merit of allowing the reciprocal transformation interchanging  $W$  and  $\bar{E}$  transforming it into Reciprocal Maupertuis Principle (RMP) with the same solution *i.e.*

$$(\delta\bar{E})_W = 0. \tag{1.6}$$

The reciprocal pair of variational principle MP, given by eq. (1.5), and RMP given by eq. (1.6), can also be stated in the unconstrained form *i.e.* Unconstrained Maupertuis Principle (UMP) as

$$(\delta W) = t\delta\bar{E} \tag{1.7}$$

where the time  $t$ , the duration of pure trajectory, is constant Lagrangian multiplier.

In his first paper on wave mechanics, Schrodinger [2] tried to derive the variational principle of wave mechanics from something like Reciprocal Maupertuis Principle (RMP). This principle for special cases of periodic and quassi periodic motions is equivalent to Percival's variational's principle of invariant tori [3]. Gray, Karl and Novikou have established [4] the links of MP and RMP with the usual Hamiltonian variational principle HP and its reciprocal principle RHP and applied these four principles to some problems of physical interest. In the present paper we shall derive the variational principles MP and RMP classically as well as quantum mechanically and demonstrate the applications of these principles to simple problems of Linear Oscillator, Anharmonic oscillator, chaotic systems (Anisotropic 2D-Quartic Oscillator) and Central force problem.

## 2. Derivation of MP and RMP

These variational principles may be derived classically as well as quantum mechanically in the following manner.

(a) *Classical derivation :*

Let us consider the function  $J[q(t')]$  such that

$$J[q(t')] = \int_0^t F(q_1, q_2, \dots; \dot{q}_1, \dot{q}_2, \dots, t') dt' \tag{2.1}$$

where  $q_1, q_2, \dots$  are generalized coordinates and  $\dot{q}_1, \dot{q}_2, \dots$  etc. are the generalized velocities of the system of any number of particles. Taking the variation of virtual trajectory

$$q_j(t') \rightarrow q_j(t') + \delta q_j(t')$$

where the end point variations of  $q_j^s$  are zero but there is an arbitrary final end point variation in  $t$ , *i.e.*

$$t \rightarrow t + \delta t.$$

Then applying the first variation theorem of calculus of variations, we have

$$\delta J = \int_0^t \sum_i \delta q_i [\partial F / \partial q_i - d/dt' (\partial F / \partial \dot{q}_i)] dt' + [F(t) - \sum_i \dot{q}_i(t) (\partial F / \partial \dot{q}_i)_i] \delta t \tag{2.2}$$

where  $\dot{q}_i = dq_i / dt$  and  $F(t)$  and  $(\partial F / \partial q_i)_i$  are the final time values.

Let us apply this result on the action given by eq. (1.2) which may also be written as

$$J = W = \int_0^t \sum_j p_j(t') \dot{q}_j(t') dt'$$

i.e.

$$F = \sum_j p_j(t) \dot{q}_j(t).$$

Then eq. (2.2) gives

$$\begin{aligned} \delta J = \delta W &= \sum_j \int_0^t \delta q_j (-\dot{p}_j(t')) dt' + \sum_j p_j(t') \dot{q}_j(t') - \sum_j \dot{q}_j(t') p_j(t') \delta t \\ &= -\sum_j \int_0^t \delta q_j \dot{p}_j \delta t' \end{aligned} \quad (2.3)$$

If  $H = H(q_j, p_j, t)$  is the Hamiltonian for the system, then the average energy is

$$\bar{E} = (1/t) \int_0^t H(q_j, p_j, t') dt'$$

and hence we have

$$\begin{aligned} \delta \bar{E} &= -\delta t / t^2 \int_0^t H(q_j, p_j, t') dt' + 1/t \delta \int_0^t H(q_j, p_j, t') dt' \\ &= -(\delta t / t) \bar{E} + 1/t \delta \int_0^t (\sum_j p_j \dot{q}_j - L) dt'. \end{aligned}$$

Using relation (2.2), this equation may be written as

$$\begin{aligned} t \delta \bar{E} &= -\bar{E} \delta t + \int_0^t \sum_j [-\partial L / \partial q_j - \dot{p}_j + d/dt(\partial L / \partial \dot{q}_j)] \delta q_j dt' \\ &\quad + \left[ \sum_j p_j \dot{q}_j - L - \sum_j p_j \dot{q}_j + (\dot{q}_j \partial L / \partial \dot{q}_j)_t \right] \delta t \\ &= -\bar{E} \delta t - \int_0^t \sum_j \dot{p}_j \delta q_j dt' - L \delta t + \sum_j p_j \dot{q}_j \delta t \end{aligned} \quad (2.4)$$

where we have used the Lagrangian equation

$$\partial L / \partial q_j - d/dt(\partial L / \partial \dot{q}_j) = 0$$

and the relation [6]

$$p_j = \partial L / \partial \dot{q}_j$$

for the holonic conservative system.

Subtracting eq. (2.3) from eq. (2.4), we have

$$t \delta \bar{E} - \delta W = (H - \bar{E}) \delta t, \quad (2.5)$$

where we have used the relation

$$H = \sum_j p_j \dot{q}_j - L.$$

If  $H(t) = E$  is constant of motion, then we have

$$t\delta\bar{E} - \delta W = 0. \quad (2.6)$$

If we fix  $\bar{E}$ , i.e. we set  $\delta\bar{E} = 0$ , we get

$$(\delta W)_{\bar{E}} = 0 \quad (2.6a)$$

which is reformulated MP given by eq. (1.5), whereas for fixed  $W$  i.e.  $\delta W = 0$ , we get

$$(\delta\bar{E})_W = 0 \quad (2.6b)$$

which is RMP given by eqn (1.6).

From eq. (2.6), the reciprocal pairs of variational Principles MP and RMP can also be written in the unconstrained form UMP as follows;

$$\delta W = t\delta\bar{E} \quad (2.7)$$

where the time, the duration of the pure trajectory, may be treated as the constant Lagrangian multiplier.

(b) *Quantum mechanical derivation :*

Schrödinger tried [2] to derive the variational principle of wave mechanics from something like the RMP but in his second paper on wave mechanics [7] he described his heuristic argument used in first paper as incomprehensible and presented a second basis for Schrödinger equation based on the analogy between geometric and wave optics on one hand and particle and wave on the other hand. It is interesting to note that Klein and collaborators have derived [8] RMP principle directly from the matrix mechanics. Following the arguments presented by Gray and his coworkers [4], we shall demonstrate here that RMP is the classical limit of quantum variational principle

$$\delta[\langle n|H|n\rangle/\langle n|n\rangle] = 0 \quad (2.8)$$

which turns into RMP for large quantum numbers  $n$ . For simplicity we shall consider a one dimensional periodic motion, where the state  $|n\rangle$  corresponds to a classical periodic trajectory with precisely the same energy  $E_n$ .

Using the WKB approximation for all trial wave functions  $\psi_n(q) = |n\rangle$ , we have

$$\langle n|n\rangle = \int \psi_n^* \psi_n dq = C \int_{q_{\min}}^{q_{\max}} dq/v = C \int_0^T dt = CT \quad (2.9)$$

where  $v$  is the velocity,  $C$  is a constant and  $T$  is the period of motion. Under the same approximation we have

$$\langle n | \hat{H} | n \rangle = C \int_0^t H(q, p) dt. \quad (2.10)$$

Thus if we have the WKB wave functions for  $|n\rangle$ , then equation (2.8) reduces to

$$\delta [1/T \int_0^t H(q, p) dt] = 0. \quad (2.11)$$

For the one dimensional periodic motion, the constraint on the energy for an allowed state  $|n\rangle$  for large  $n$  is

$$W (\text{cycle}) = \oint pbq = nh \quad (2.12)$$

which is Bohr-Sommerfeld-Wilson quantization rule. It shows that for fixed  $n$  the action  $W$  is to be kept fixed. Hence for fixed large value of  $n$ , the quantum variational principle [2.8] reduces to [2.11] for fixed  $W$ , *i.e.*

$$(\delta \bar{E})_W = 0$$

showing that the quantum variational principle (2.8) transforms into RMP for one dimensional periodic motion. It may also be demonstrated [3] that such transformation is valid for quasi-periodic case also.

### 3. Simple applications of MP and RMP

Reformulated MP and RMP are the new and useful principles of classical mechanics. Gray, Karl and Novikov have established [4] the link of MP and RMP with Hamilton variational principle (HP) and its reciprocal RHP and applied RMP to some problems of physical interest. Here we shall apply RMP to some simpler problems to demonstrate the scope of its wider applicability.

(a) *Linear oscillator :*

For a linear oscillator the Hamiltonian is given by

$$H = (m\dot{q}^2 + kq^2)/2 \quad (3.1)$$

which corresponds to a simple pendulum, in square approximation of cosine term, for

$$q = l\theta$$

and

$$k = mg/l. \quad (3.2)$$

Let us choose the trial trajectory as

$$q(t) = A \sin \omega t \quad (3.3)$$

where  $q = 0$  at  $t = 0$  and  $t = 2\pi/\omega = T$  (at the end of cycle).

Using eq. (1.2) we have the action  $W$  as

$$\begin{aligned} W &= \int_0^t p dq = m \int_0^t \dot{q} dq = m \int_0^t \dot{q}^2 dt \\ &= m\omega^2 A^2 \int_0^t \cos^2 \omega t dt \end{aligned}$$

$$= (\frac{1}{2})m\omega^2 A^2 T = \pi\omega mA^2 \quad (3.4)$$

From eq. (1.4), the average energy  $\bar{E}$  for the present case may be written as

$$\bar{E} = (1/T) \int_0^T H dt = (1/2T) \int_0^T (m\dot{q}^2 + kq^2) dt$$

Using eq. (3.3) and carrying out the integration, we have

$$\bar{E} = (A^2/4) (m\omega^2 + k) = (W/4\pi m) [m\omega + k/\omega] \quad (3.5)$$

where we have used the relation (3.4).

Using RMP and treating  $\omega$  as a variational parameter in the relation

$$(\partial\bar{E}/\partial\omega)_W = 0,$$

we get

$$m - k/\omega^2 = 0$$

or

$$\omega = \sqrt{(k/m)} = \omega_0$$

which gives the well known relation for period of linear oscillator

$$T = 2\pi\sqrt{(m/k)} \quad (3.6)$$

which reduces to the standard relation for the time period of simple pendulum on using relation (3.2).

(b) *Anharmonic oscillator :*

The Hamiltonian of a one dimensional anharmonic oscillator (modified linear oscillator) is given by

$$H = (1/2) m\dot{q}^2 + (k/2)q^2 + \mu q^3 + \lambda q^4. \quad (3.7)$$

For  $q = l\theta$ ,  $k = mq/l = m\omega_0^2$ ,  $\mu = 0$  and  $\lambda = -k/(24l^2)$  it reduces to the Hamiltonian of a plane pendulum in quadratic approximation of cosine.

If we choose the same trial trajectory given by eq. (3.3), then we have

$$W = \int_0^T p dq = \pi m\omega A^2 \quad (3.8)$$

and

$$\begin{aligned} \bar{E} &= \frac{1}{T} \int_0^T H dt \\ &= W/(4\pi m) [m\omega + k/\omega] + (\mu A^3/T) \int_0^T \sin^3 \omega t dt + (\lambda A^4/T) \int_0^T \sin^4 \omega t dt \end{aligned}$$

where the first integral on the right hand side vanishes and we have

$$\begin{aligned}\bar{E} &= W/(4\pi m)[m\omega + k/\omega] + 3\lambda W^2/(8\pi^2 m^2 \omega^2) \\ &= (W/4\pi m)[m\omega + k/\omega + 3\lambda W/(2\pi m\omega^2)].\end{aligned}\quad (3.8a)$$

Then RMP,

$$(\partial\bar{E}/\partial\omega)_W = 0,$$

gives

$$\omega^2 = \omega_0^2 + 3\lambda A^2/m, \quad (3.9)$$

where  $\omega^2 = k/m$ . For  $\lambda = 0$ , we get the frequency of simple pendulum *i.e.*  $\omega = \omega_0$  where period is independent of the amplitude  $A$ , the action  $W$  and the energy  $\bar{E}$ .

For plane pendulum with  $q = l\theta$ ,  $\lambda = -m\omega_0^2/(24l^2)$  and  $\theta_{\max} = A/l = B$ ,

we get

$$\omega^2 = \omega_0[1 - B^2/8]^{1/2} \quad (3.10)$$

with the period given by

$$T = 2\pi/\omega = T_0[1 - B^2/8]^{-1/2} \quad (3.11)$$

where  $T_0 = 2\pi/\omega_0$ . This relation gives

$$T = T_0 [1 + B^2/16 + 3B^2/512 + \dots] \quad (3.12)$$

which is correct to the order  $B^2$  *i.e.* accuracy  $O(B^2)$ . For better accuracy one can take a more elaborate trial trajectory with more parameters.

(c) *Anisotropic 2D quartic oscillator (A chaotic system) :*

This case belongs to non-integral (chaotic) systems which have their own difficulties in finding their solutions. The Hamiltonian of this case is given by [9]

$$H = (m/2)[\dot{q}_1^2 + \dot{q}_2^2] + cq_1^2q_2^2 \quad (3.13)$$

where  $C$  is a constant.

Let us choose the following simple quasi periodic anisotropic harmonic trajectory so that the quantization semi-classically via the Einstein-Brillouin-Keller [EBK] rule [10] becomes simple;

$$\begin{aligned}q_1 &= A_1 \cos \omega_1 t, \\ q_2 &= A_2 \cos \omega_2 t\end{aligned}\quad (3.14)$$

where in general  $\omega_1$  and  $\omega_2$  are not equal.

Over a long period  $T$ , the action becomes

$$W = \int_0^T m[\dot{q}_1^2 + \dot{q}_2^2] dt$$



$$= m [A_1^2 \omega_1^2 + A_2^2 \omega_2^2] T/2 \quad (3.15)$$

$$= (T/2\pi)(\omega_1 W_1 + \omega_2 W_2) \quad (3.15a)$$

where  $W_1 = m\pi\omega_1 A_1^2$  and  $W_2 = m\pi\omega_2 A_2^2$  are the one cycle actions for  $q_1$  and  $q_2$  motions respectively. The average energy for this case is

$$\begin{aligned} \bar{E} &= 1/T \int_0^T H dt \\ &= (m/4)(\omega_1^2 A_1^2 + \omega_2^2 A_2^2) + (C/T) \int_0^T A_1^2 A_2^2 \cos^2 \omega_1 t \cos^2 \omega_2 t dt \\ &= (m/4)(\omega_1^2 A_1^2 + \omega_2^2 A_2^2) + (C/4)A_1^2 A_2^2 \end{aligned} \quad (3.16)$$

$$= (1/4\pi)[\omega_1 W_1 + \omega_2 W_2 + CW_1^2 W_2^2 / (\pi m^2 \omega_1 \omega_2)] \quad (3.16a)$$

For simple trajectory (3.14), the mean subactions  $\bar{W}_1$  and  $\bar{W}_2$  are

$$\bar{W}_1 = W_1, \quad \bar{W}_2 = W_2$$

and for quasi periodic trajectory like (3.14), extremizing  $\bar{E}$  at fixed  $W$  is equivalent to extremizing it at fixed  $\bar{W}_1$  and  $\bar{W}_2$  treating  $\omega_1$  and  $\omega_2$  as variational parameters and applying RMP or (3.16a), we have

$$\left( \frac{\partial \bar{E}}{\partial \omega_1} \right)_{W_1, W_2} = 0 \quad \text{and} \quad \left( \frac{\partial \bar{E}}{\partial \omega_2} \right)_{W_1, W_2} = 0$$

which give

$$W_1 - CW_1 W_2 / (\pi m^2 \omega_1^2 \omega_2) = 0 \quad (3.17)$$

and

$$W_2 - CW_1 W_2 / (\pi m^2 \omega_1 \omega_2^2) = 0$$

These equations give

$$\omega_1^2 \omega_2 = CW_2 / (\pi m^2)$$

and

$$\omega_2^2 \omega_1 = CW_1 / (\pi m^2) \quad (3.18)$$

or

$$\omega_1 / \omega_2 = W_2 / W_1 \quad (3.19)$$

Substituting this relation in to eq. (3.17), we have

$$\omega_1^3 = CW_2^2 / (\pi m^2 W_1)$$

and

$$\omega_2^3 = CW_1^2 / (\pi m^2 W_2).$$

Substituting these results into eq. (3.16a),

we get

$$\bar{E} = (3/4\pi)(C/\pi m^2)^{1/3}(W_1 W_2)^{2/3} \quad (3.20)$$

which is the same result (with slightly different numerical coefficient) as obtained by Martens *et al* [11] by using the adiabatic approximation.

Substituting

$$W_1 = (n_1 + 1/2)h$$

and

$$W_2 = (n_2 + 1/2)h$$

into eq. (3.20), we obtain the semiclassical quantization of energy in the following form

$$\bar{E} = E_{n_1, n_2} = (3/4\pi)(Ch^4/\pi m^2)^{1/3}(n_1 + 1/2)^{2/3}(n_2 + 1/2)^{2/3} \quad (3.21)$$

which is exactly the same result as obtained by Gray *et al* [4]. This relation when compared with exact results obtained by the method of numerical calculations [11] shows the exciting accuracy for the lowest fifty levels. Hence the method of RMP can be applied to get the approximate results even for non-integrable systems.

(d) *Central force problem :*

The central force problem is always a motion in a plane where the polar axis is taken along the angular momentum vector *i.e.* angular momentum is constant of motion;

$$l = mr^2\dot{\theta} = \text{const.}, \quad (3.22)$$

the Hamiltonian is given by [6]

$$H = (1/2)mr\dot{r}^2 + l^2/(2mr^2) + V(r) \quad (3.23)$$

and the radial velocity is

$$\dot{r} = \sqrt{\{(2/m)[E - V(r) - l^2/(2mr^2)]\}} \quad (3.24)$$

where  $V(r)$  is the central potential and  $E$  is the total energy (incident energy for scattering problem).

If we assume the initial value of  $r$  as  $r_0$  (at  $t = 0$ ), then eq. (3.24) gives

$$t = \int_{r_0}^r dr / \left\{ (2/m) [E - V(r) - l^2/2mr^2] \right\}^{1/2} \quad (3.25)$$

which gives  $t$  as a function of  $r$ .

From eq. (3.22) and (3.24) we have

$$\theta = \theta' - \int l du / \{ m \sqrt{[(2/m)(E - V - l^2 u^2/2m)]} \} \quad (3.26)$$

where  $u = 1/r$  and  $\theta'$  is the constant of integration. The energy of the two dimensional motion of a particle in a central force field  $V(r)$  given by eqn (3.23) may be treated as the conserved energy for the one dimensional motion of the particle in the effective potential field

$$V_{\text{eff}} = V(r) + l^2/(2mr^2) \quad (3.27)$$

where the second term on the right hand side is the centrifugal energy. Eq. (3.24) shows that for  $E = V_{\text{eff}}$  the radial velocity of the particle is zero. This does not mean that the particle comes to rest since its angular velocity given by eq. (3.22) is non-vanishing. The vanishing value of  $\dot{r}$  in such a case indicates the turning point of the path where  $r$  begins to decrease instead of increasing and *vice-versa*.

The circular orbits are possible in central force motion whenever the effective potential has the extremum value. The circular orbit will be stable if its radial value yields a minimum value for the effective potential and it will be unstable if it yields a maxima. Thus for any circular orbit we have

$$dV_{\text{eff}}/dr = dV/dr - l^2/(mr^3) = 0$$

showing that for a particle moving in a circular orbit, the force  $f = \partial V/\partial r = l^2/mr^3 = mr\dot{\theta}^2$ , which is the centripetal force. If the potential, for any central force, is given by

$$V(r) = Cr^n,$$

then the conditions of stable circular orbit becomes

$$Cn(n-1)r^{n-2} + 3l^2/mr^4 > 0 \quad (3.28)$$

and

$$Cnr^{n-1} = l^2/mr^3. \quad (3.29)$$

These equations give  $n > -2$  as the condition of stability of circular orbits.

The variational principle RMP can be used for estimating the scattering angle as the function of impact parameter and incident energy for central potentials which are proportional to an even power of  $r$  as will be demonstrated by the following two examples.

(i)  $V(r) = C/r^2$ ; where  $C$  is positive constant.

The orbit shape is described by the radial distance  $r(\theta)$  from the centre of the given repulsive central potential where  $\theta$  is the polar angle in the scattering plane measured from the incident asymptotic direction. The projectile starts at  $r = -\infty$  corresponding to  $\theta = 0$  and returns to  $r = \infty$  at  $\theta = \theta_{\text{max}}$ . The scattering angle is obviously given by

$$\phi = \pi - \theta_{\text{max}} \quad (3.30)$$

The angular momentum  $l$ , being constant of the motion, may also be written as

$$l = (2mE)^{1/2}b \quad (3.31)$$

where  $b$  is the impact parameter. We may also write the radial momentum as

$$p_r = m\dot{r} = -mr^2\dot{u} = -mr^2u'\dot{\theta} = -lu' \quad (3.32)$$

where  $u = 1/r$  and  $u' = du/d\theta$ . Thus we may write eq. (3.23) as

$$\begin{aligned} H &= (l^2/2m) (u'^2 + u^2) + Cu^2 \\ &= \frac{1}{2}mU'^2 + \frac{1}{2}kU^2 \end{aligned} \quad (3.33)$$

where  $U = lu/m$ ,  $k = m[1 + (2Cm)/l^2]$  and  $U' = du/d\theta$ .

This Hamiltonian is the same as that given by eq. (3.1) for linear oscillator, where the role of  $t$  is played by  $\theta$ . Since  $U$  runs through half a cycle (0 to 0) as  $t$  runs from 0 to  $T/2$  the period  $T$  is  $2\theta_{\max}$  and hence we can write the scattering angle (3.30) as

$$\begin{aligned} \phi &= \pi - T/2 \\ &= \pi - \pi/\omega \end{aligned} \quad (3.34)$$

where  $\omega$ , obtained by applying RMP, is given by

$$\omega = \sqrt{(k/m)} = \sqrt{[1 + 2Cm/l^2]} = \sqrt{(1 + C/\bar{E}b^2)} \quad (3.35)$$

which shows that for no potential *i.e.* for  $C = 0$ ,  $\omega = \omega_0 = 1$  and  $\phi = 0$  *i.e.* no scattering.

If we choose the attractive potential *i.e.*

$$V(r) = -C/r^2,$$

then eq. (3.31) will become

$$\omega = \sqrt{(1 - 2Cm/l^2)}$$

and the scattering angle will be given by

$$\phi/\pi = 1 - 1/\sqrt{(1 - 2Cm/l^2)} = (1 - 1/\sqrt{(1 - C/\bar{E}b^2)}) \quad (3.36)$$

Carrying out the integration in eq. (3.26) and setting  $\theta' = 0$ , we get the following equation of motion for the given attractive potential :

(i) For  $C \leq l^2/2m$ ,

$$u = \sqrt{\{2mE/(l^2 - 2mC)\}} \cos[\theta\sqrt{(1 - 2Cm/l^2)}] \quad (3.37)$$

which is periodic circular motion. The condition (3.29) of circular orbit for the given potential also gives the limit  $C = l^2/2m = \bar{E}b^2$ . Eq. (3.32) also shows that for this value of the constant of potential there is no scattering under the influence of the given attractive potential.

(ii) For  $C > l^2/2m$ ,

$$u = \sqrt{[mE/(2mC - l^2)]} \sin h[\theta \sqrt{(2mC/l^2 - 1)}] \quad (3.38)$$

showing that circular orbit of eq. (3.37) is no more stable for this value of constant of potential. Rather the path becomes an equiangular spiral where the velocity at any point is equal to that in a circle at the same distance and under the same attraction.

In eqs. (3.37) and (3.38) we have taken the positive energy *i.e.*  $E > 0$ . If we have  $E < 0$ , then for  $C > l^2/2m$ , we have the following equation of path rather than eq. (3.38);

$$u = \sqrt{[2mE/(l^2 - 2mC)]} \cos h[\theta \sqrt{(2mC/l^2 - 1)}] \quad (3.39)$$

which also shows that the path is an equiangular spiral where the velocity at any point is inversely proportional to the distance of the point from the center of force

(ii)  $V(r) = C/r^4$ , where  $C$  is a positive constant

With this potential the Hamiltonian (3.23) of the central force problem becomes

$$H = \frac{1}{2}mU'^2 + \frac{1}{2}mU^2 + \lambda U^4 \quad (3.40)$$

where

$$U = lu/m, \quad U' = dU/d\theta \quad \text{and} \quad \lambda = Cm^4/l^4$$

This Hamiltonian is similar to that given by eq (3.7), for  $\mu = 0$  and  $k = m$ , for anharmonic oscillator In eq (3.40) the angle  $\theta$  plays the role of  $t$  and hence the scattering angle is given by eq. (3.34) with  $\omega$  given by (3.9), *i.e.*

$$\omega^2 = 1 + 3\lambda A^2/m \quad (3.41)$$

since here  $\omega_0 = k/m = 1$ . This equation shows that for no potential,  $\omega = \omega_0 = 1$ , and scattering angle  $\phi = 0$ .

Applying RMP on the Hamiltonian of eq. (3.7) we have obtained the average energy in the form given by eq. (3.8a) Substituting results (3.8) and (3.9) into this equation, we get

$$24\bar{E}\lambda/m^2 = (\omega^2 - 1)(3\omega^2 + 1)$$

which gives

$$\omega^2 = 1/3[1 \pm 2\sqrt{(1 + 18\bar{E}\lambda/m^2)}]. \quad (3.42)$$

Substituting this result into eq. (3.34), we get

$$\phi/\pi = 1 - [3/\{1 \pm 2\sqrt{(1 + 18\bar{E}\lambda/m^2)}\}]^{1/2}. \quad (3.43)$$

If we identify  $\bar{E}$  with the incident energy  $E$  then using eq. (3.31) in the value of  $\lambda$

*i.e.*  $\lambda = Cm^4/l^4$ , we get

$$\lambda \bar{E} = \lambda E = Cm^2/(4Eb^4).$$

Substituting this value in eq. (3.43) for larger root of eq. (3.42), we get

$$1 - \phi/\pi = 3/[1 + 2\sqrt{(1 + 9C/2Eb^4)}]^{1/2} \quad (3.44)$$

which agrees with exact numerical calculation [12] over the whole range of parameter  $\lambda \bar{E} > 0$ . It demonstrates the validity of variational principle RMP in the problem of scattering from the central force field. This formula is valid for attractive potential also up to the limit  $C/Eb^4 \leq 2/9$ . For stronger attractive potential such that  $C/Eb^4 > 2/9$ , the incident particle will be captured by the potential and it will move in a circular orbit of radius  $a = 1/[1\sqrt{(Cm/2)}]$  passing through origin. This circular orbit will be stable only when

$$E = -mC^2/2I^2.$$

#### 4. Discussions

The variational principles MP, RMP and UMP given by eqs. (1.5), (1.6) and (1.7) have been derived classically in the form of eqs. (2.6) and (2.7) using the  $\delta$ -variation and also quantum mechanically for one dimensional periodic motion as the classical limit of quantum mechanical principle (2.8). It may also be demonstrated [3] that such transformation is valid for quasi-periodic motion also. These new principles are the concise statements of laws of classical mechanism for instance the energy conservation is the consequence of MP given by eq. (1.5). The RMP given by eq. (1.6) is also a very useful principle of classical mechanics. Gray and coworkers [4] have established the links of MP and RMP with Hamilton variational principle HP and its reciprocal RHP and also demonstrated that for quasi-periodic motions the RMP is equivalent to Percival's principle for invariant Tori [3]. Eq. (3.6) shows that the variational principle RMP, when applied to the linear harmonic oscillator with the Hamiltonian given by eq. (3.1), gives the well known period which gives the standard relation for the time period of simple pendulum on using the relations (3.2). The RMP when applied to anharmonic oscillator with the Hamiltonian given by eq. (3.7), gives the expression for angular frequency in the form of eq. (3.9). This relation leads to eq. (3.12) for the time period of plane pendulum in quadratic approximation of cosine. This result when compared with that computed directly through an elliptic integral is correct to order  $B^2$ . This accuracy order  $O(B^2)$  is expected with the trial trajectory of eq. (3.3) which is correct to zero order in  $\lambda$ . For the better accuracy one can take a more elaborate trial trajectory with more parameters.

The relation (3.20), obtained for the average energy on applying the variational principle RMP on a chaotic system like anisotropic 2D quartic oscillator with the Hamiltonian given by eq. (3.13), is similar, with slightly different numerical coefficient, to that obtained by Martens *et al* [11] by using the adiabatic approximation. The

classically quantized energy given by eq. (3.21) for this chaotic system is exactly similar as obtained by Gray *et al* [4]. This result when compared with exact results obtained by the method of numerical calculations [11] shows the exciting accuracy for the lowest fifty levels. Thus the method of RMP can be applied to get the approximate results even for non-integrable systems (*i.e.* chaotic systems).

Applying the method of variational principle RMP on the central force problem with inverse square potential, we have obtained eq. (3.34) for the scattering angle. This result reduces to relation (3.36) for the attractive inverse square potential. It shows that for the constant of potential  $C \geq Eb^2$ , where  $b$  is the impact parameter and  $E$  is the incident energy, there is no scattering under this attractive potential and the path of the particle becomes an equiangular spiral with the eqs. of motion given by (3.37), (3.38) and (3.39).

Applying the method of variational principle RMP on the central force problem with the potential inversely proportional to the fourth power of distance from the center of potential, we get the expression (3.44) for the scattering angle; which agrees with the exact numerical calculation. It demonstrates the validity of variational principle RMP in the problem of scattering for the central force field. This relation is valid for attractive potential also up to the limit  $C \leq (2Eb^4)/9$ .

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