# Absence Perception and the Philosophy of Zero

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**Abstract**

Zero provides a challenge for philosophers of mathematics with realist inclinations. On the one hand it is a bona fide number, yet on the other it is linked to ideas of nothingness and non-being. This paper provides an analysis of the epistemology and metaphysics of zero. We develop several constraints and then argue that a satisfactory account of zero can be obtained by integrating recent work in numerical cognition with a philosophical account of absence perception.

## Introduction

Zero is an intriguing number both mathematically and philosophically. Mathematically, it plays an important role in our theories of natural, integer, and real numbers. For instance, when considering an algebraic structure (e.g. a group) under addition, zero often serves as the identity element (since for any number $n$, $n + 0 = n$). Philosophically, our understanding of zero is tied up with classical questions concerning the status of non-being, finding consideration already in the work of Parmenidies, Plato, and Aristotle, through the 'Continental' tradition (e.g. Sartre), right up to contemporary philosophical debates.

The core issue concerning zero (that we explain in more detail below) is that zero exhibits something of a dual nature. On the one hand, we regard it as a perfectly legitimate number, on a par with other mathematical entities, and which which we can meaningfully compute. On the other, it represents 'nothingness': While I can have an experience of two or three objects, an experience of zero objects seems difficult to conceptualise—there would simply be nothing to experience in such a situation. We might then wonder how we should think of zero, both epistemologically and ontologically. One response would be to be an extreme kind of formalist or fictionalist about mathematics, and then let our philosophy of notation and proof-theory or fictions deal with the question of zero. While this is a possibility, in this paper I want to consider how we might account for zero on *Realist* perspectives, since the problem of accounting for zero's null-like nature then becomes more acute. Therefore we make the following two-part assumption from the get-go:

**Mathematical Realism.**

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(i) Mathematics is about a mind-independent realm of mathematical 'entities' (we do not commit to their being bona fide objects[1]) which require an epistemology of how we get to know them.

(ii) A mathematical sentence is true or false depending on whether or not the relevant entities match the syntax used (i.e. *some* sort of correspondence theory of truth is appropriate for mathematics).[2]

The puzzle facing us then is the following:

**Main Question.** How should we should understand zero as a mathematical entity? In particular:

1. How should we conceive of it ontologically?
2. How is it able to represent 'nothingness'?
3. How does it fulfill the technical roles it does?
4. How can we provide an adequate epistemology for zero?

In this paper we argue for the following claims:

(1.) We can think of zero as the *numerosity* property corresponding to an *absence* of positive cardinality.

(2.) This yields an account of zero on which it can be viewed as fundamentally of the same ontological kind as other numbers, with a similar epistemology, yet on which its distinctive null-like features and technical roles are accounted for.

Here's the plan: First (§1), we'll argue for some desiderata on a realist account of zero. Specifically we argue that (i) zero should be ontologically similar to the other numbers, (ii) we should have an account of how zero is phenomenologically able to represent nothingness, (iii) our theory of zero should explain how zero-like number concepts can fulfill similar technical roles in different contexts, and (iv) we should provide an epistemology for zero that incorporates these features. Next (§2), we'll argue against some otherwise tempting accounts of zero; (i) as a position in a structure, (ii) as arising out of counting procedures, and (iii) as a set. Each we will argue fails to meet at least one of the desiderata of §1. We will then argue (§3) that an epistemological and ontological story for zero can be obtained by regarding numbers as properties of collections, and that zero can be viewed as the *numerical* property corresponding to an *absence* of positive cardinality. We do this by combining elements of the literature on numerical cognition with recent work in the philosophy of absence perception. We then (§4) consider some salient objections to the view we proposed, in particular relating to the relatively late historical and ontogenetic development of zero concepts. Finally (§5) we conclude with some open questions.

# 1 Desiderata on an account

We'll begin with desiderata on a Realist account of the ontology and epistemology of zero. It is often easiest to see desiderata by considering *clearly unsatisfactory* responses to our main question, and this is the strategy we shall initially adopt here.

[1] They could, for example, be properties or concepts.

[2] We are sensitive to the fact that the term 'Realism' is used in many and varied ways, and just make this assumption for the purposes of the current paper. See [Jenkins, 2008] (esp. Ch 1) for clarification of this issue.

One possible 'solution' to the problem posed in the introduction would be to acknowledge that zero is problematic, and press that we should just be fictionalists or formalists about zero, whilst retaining realism concerning the other numbers.[3]

Why is this account bad? Well, aside from the fact that it seems rather ad hoc, regarding zero as fundamentally *ontologically* different from the other numbers will result in a less theoretically elegant correspondence theory of truth on the realist's picture. For example, consider the sentence:

> "There are exactly six natural numbers less than six."

This is true on the correspondence theory just in case there *really are* six natural numbers less than six. But, given the current account of zero, this is false: There are really five natural numbers less than six (the numbers 1–5). We would thus require a version of the correspondence theory on which some way of dealing with fictions is incorporated, reducing the simplicity of the theory. If an account could avoid these kinds of awkward changes it would (ceteris paribus) be a better theory. We therefore propose the following:

> **Ontological Constraint.** Our account of zero should have it ontologically 'on a par' with the other natural numbers, in that it is of the same metaphysical kind and meshes easily with a correspondence theory of truth.

As noted in the Introduction, a feature of zero important for its role in our thought is how it interacts with concepts of nothingness and non-being. For instance, if you have zero of a certain object you do not have any such objects at all. This contrasts with other numbers and quantities where if you have any positive number or amount of something you genuinely have some amount of the relevant quantity or object. This yields the following constraint:

> **The Phenomenological Constraint.** Any philosophy of zero should account for why it represents 'nothingness' within our phenomenological experience.

This 'phenomenological' role for zero in representing nothingness is backed up by its technical use. As remarked earlier, in algebraic structures under addition zero serves as the identity element (since for any number or quantity $x$, $x + 0 = x$). It also has interesting interactions with multiplication and division. Since zero represents nothingness, any multiplication of a number by zero yields zero (i.e. zero is an absorbing element under multiplication), and zero has no multiplicative inverse (since, viewing multiplication as repeated addition, any number of times you add nothingness to itself will fail to yield any positive quantity). Moreover, zero fulfils these roles in a variety of technical contexts: Acting as both the identity element (under addition) and an absorbing element (under multiplication) in each of the integer ring, and rational and real fields. This yields the following constraint:

> **The Technical Constraint.** Explain how zero fulfils a similar role in a variety of technical contexts.

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[3]This attitude to the (closely related to zero) empty set is surprisingly widespread in the foundations of mathematics, especially amongst the early set theorists. Zermelo referred to it as *uneigentlich* ('non-actual'), Dedekind ([Dedekind, 1888], p. 797) excluded its consideration for reasons linked to problems of non-existence (and duly received criticism, along with Schröder in [Frege, 1917]), and [Fraenkel et al., 1973] regard its use as mere notational convenience. See [Kanamori, 2003] for further discussion.

One immediate question that should be addressed in light of the Technical Constraint is the sense in which we will be using the term 'zero'. Does this term for us refer to the *natural number* zero, the *integer* zero, or the *real number* zero (and many more besides)? We shall primarily consider how zero features in natural number contexts, however we shall also argue later that the account we provide in terms of absence perception can be generalised to other mathematical contexts, satisfying the Technical Constraint.

Immediately there is something of a tension between the three constraints. For, on the one hand, we want our account of zero to put zero on an ontological par with the other numbers. However, on the other hand, we require an account that explains why zero fulfils the peculiar phenomenological and technical role that it does, in particular with respect to the fact that it somehow represents 'nothing'. This suggests that there will be deep issues in providing an epistemology for zero; the epistemological story must be both similar to the story for other numbers (since zero is to be of the same ontological kind), yet also in some sense unique (since zero has very distinctive phenomenological and mathematical properties). We therefore add the following constraint:

> **The Epistemological Constraint.** In providing an epistemological story for zero, do so in a way that accounts for its similarity to other numbers, but does so in a way that makes clear how it is unique.

As we shall see later, we think that there is an account of zero that satisfies these constraints by combining existing work on the philosophy of number cognition and absence perception. For now, however, we move on to consideration of the competition.

## 2 Other accounts

In this section, we'll consider various different existing accounts of zero (or at least zero-like entities), and explain why they fail to satisfy one or more of the constraints we outlined above.

### 2.1 A position in a structure

One suggestion would be to identify zero with a position in a structure. One might hold the following view:

> **Structuralism.** Mathematical talk should be understood as fundamentally about *structures*, and mathematical reference should be understood as to positions in these structures.[4]

For simplicity, we can consider structures as given *graph-theoretically*.[5] On this view, structures are to be conceived of as directed unlabelled graphs. For our discussion, not much hangs on the details of the view, we simply make it to give visual content to mathematical structuralism.[6] For example, we can visualise parts of the

[4]Views of this kind are [Hellman, 1989] and [Shapiro, 1997]. It does not matter for current purposes whether the structuralism considered is *ante rem* or *in re*.

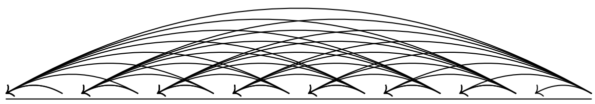
[5]We thank [name removed for blind review] here.

[6]In any case, such a view is roughly inter-translatable with other theories of mathematical structure, say by conceiving of the structure of the set-theoretic hierarchy as a particular directed unlabelled graph, with a node for every set and a directed edge between two such nodes representing sets $x$ and $y$ just in case $x \in y$.

Figure 1: A visual graph-theoretic representation of an initial part of the natural number structure under successor

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$$

Figure 2: A visual graph-theoretic representation of part of the real number structure under less-than



natural number structure under successor (Figure 1), and part of the real number structure under the less-than relation (Figure 2).

We can then identify zero with the relevant position in the structures. So, for example '0' denotes the initial position in a natural number structure, or a point in the real number structure with the other required relationships (e.g. being such that $x + 0 = x$ in the addition structure).[7]

Such a view performs well with respect to the *Ontological Constraint*. Zero is to be conceived of as a position in a (relevant kind of) structure, much as the other numbers and quantities are. Moreover, we have an account of how zero can play a similar role in different contexts—the available attendant isomorphisms on the structuralist picture facilitate the consideration of how zero can play the same role in different structural frameworks. For example, there will be an isomorphism between the structure of the integer ring and a substructure of the real field, on which $0$ in the integer ring will be mapped onto $0$ in the field structure.

However such a picture fails to account for how zero has any sort of null-like qualities. Conceiving of structures graph-theoretically (so as composed of points with arrows between them), $0$-positions in structures fail to have any non-being-like qualities. The initial position in the natural number structure, for instance, is just an initial position, but has no null-like features.

The case of the $0$-position on the real line is possibly even more acute, there it is *entirely arbitrary* which position we pick as $0$ (similarly with the integers). The same goes for *any* conception of structuralism; while we have picked an easily visually representable version of structuralism, other structuralist pictures also take away the 'distinctive' qualities of zero (aside from its particular place in a structure).

The structuralist sees this (possibly rightly) as a virtue of their view; it *is* arbitrary (in a sense) which of the available positions we designate as the null one. For the brand of realism we are considering, this will not do, however. We need to say more and explain *why* zero has the null-like features that it does, rather than simply *stipulating* that the relevant position has them. In this sense, the kind of realism we have adopted in this paper is unabashedly non-structuralist—for this kind of realist there is more to numbers (especially zero), than mere structural relationships. Later,

[7]There is a slight challenge of how to visualise binary relations graph-theoretically, however it can be done, say by factoring the relevant structure through the set-theoretic structure mentioned in the previous footnote.

when we provide our positive philosophy of zero, we will argue that we can account for the arbitrary selection of zero-position whilst at the same time providing null-like qualities for zero. For now though, we reject the structuralist position as violating the Phenomenological Constraint (given our initial set up).

## 2.2 Through counting

Instead, we might try and account for zero via the generalisation of a counting procedure. We might argue as follows: An abstract understanding of counting procedures is key to mathematical understanding, in particular our understanding of natural number.[8] An abstract understanding of counting procedure might then be argued to provide an account of zero. We begin with knowledge of one, learn to count, and slowly learn the numbers two, three, four and so on.[9] Given some arithmetical competence, we then realise that the successor function has an inverse; one can also subtract $1$ as well as add $1$ to yield the predecessor instead of the successor. We then consider what the predecessor of $1$ might be, and realise that it must be zero (since $1 - 1 = 0$). Thus we can have zero arise out of a understanding of counting practices; it is the number that would have to precede $1$.[10] One might then generalise this account of zero to other areas (such as its role in the real numbers).

The account fares reasonably well with respect to the Ontological Constraint. Zero is obtained by arithmetical procedures much as the other numbers, and can be a bona fide entity in this framework. Presumably the Phenomenological Constraint is also satisfied, since the predecessor of $1$ must have null-like qualities.

There are problems, however, when we dig into the details of the epistemological account. The core problem is that while the account *seems* appealing, it presupposes that we already have a concept of zero in order to have it be the predecessor of $1$. It is because we *already* know that the number that comes before $1$ is zero that we can obtain it from $1$ by subtraction.

This plays out with respect to how we generalise the concept of zero to other structures. A natural way of accounting for our move from the naturals to the integers is to consider a copy of the positive non-zero natural numbers inverted before $0$ (as depicted in Figure 3).

In order to make that move though, we *already* needed a concept of zero. Supposing then that we lacked a concept of zero, but had a concept of positive natural number. Our natural number structure would then start from $1$ (as depicted in Figure 4). What kind of structure would a similar inversion strategy yield with this conception of natural number? What should we say the predecessor sequence of $1$ is? When inverting, it is plausible to suppose that we would have the predecessor of $1$ as $-1$, since that is what one obtains if the structure is 'flipped' (as depicted in Figure 5).[11]

One could still define *versions* of arithmetical operations for this structure. For instance addition could be informally defined as "moving the number of places right in the number line corresponding to the numeral". With this in place we can define multiplication in the usual way as repeated addition. Under this conception, we would obtain a very different arithmetic (for instance, addition would not be commutative, since $1 + -1 = -1$ and $-1 + 1 = 1$). We would also lose various nice

[8] This is emphasised by numerous authors, but particularly in [Dehaene, 1997] and [Nieder and Dehaene, 2009], which we shall see some discussion of later.

[9] A story similar to this is given in [Buijsman, F].

[10] [Wellman and Miller, 1986] in fact use backwards counting songs ending in zero in their empirical studies of the ontogenetic development of zero-concepts in children.

[11] One might also obtain $-2$ as the predecessor of 1. Our criticisms would remain the same.

Figure 3: Moving from the naturals to the integers

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \cdots$$

$$\Downarrow$$

$$\cdots -4 \leftarrow -3 \leftarrow -2 \leftarrow -1 \longleftarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \cdots$$

Figure 4: The natural number structure without $0$

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \cdots$$

Figure 5: Trying to generate the integers without $0$

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \cdots$$

$$\Downarrow$$

$$\cdots -4 \leftarrow -3 \leftarrow -2 \leftarrow -1 \longleftarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 \cdots$$

properties of the integers (such as the existence of an identity element for addition). Nonetheless the fact remains that such a mathematical structure is perfectly legitimate, and admits of *some* meaningful arithmetic (even if very different from our own). However, without *already* possessing the concept of zero one would fail to define a zero concept for this structure. Thus, while it seems initially appealing, filling in the details of the account from counting seems a deeply problematic.

### 2.3 A set

A different option would be to identify zero with the empty set, and rely on the representative qualities of set theory and an attendant epistemology of sets to do the work. We might, for example, hold the following strong view.

> **Strong Set-Theoretic Reductionism.** Every mathematical object is a set.

Then we could simply identify zero with the empty set, after all, in nearly all theories of the ordinals (including our canonical one: the von Neumann ordinals) the empty set is the representative for $0$. We would then easily satisfy the Ontological Constraint (the empty set is, after all, a *set* just like the other numbers), and we would, prima facie, satisfy the Phenomenological Constraint (since the empty set is, after all, *empty*) *and* set theory provides a clear mathematical picture of how it plays the role it does.

Aside from the fact that we might dispute such a strong set-theoretic reductionism, it is when we examine the Epistemological Constraint that this account begins to appear problematic. The simple difficulty is that by reducing zero to the empty set, we have substituted one problem of non-being with another—it seems equally tricky to explain how we get to know the empty set as it does the number zero.

Not many accounts of the empty set exist in the literature. One can be extracted from [Maddy, 1998], who suggests (for the physicallistically-inclined philosopher) that since we just require that the empty set have no members, we can just pick any old non-set physical urelement we like to do the duty of the empty set. The hierarchy then built off any such object will have all the regular mathematical properties we require. Moreover, since the Maddy of [Maddy, 1998] holds that sets are *perceivable* and *physically located*, there is *literally* a set-theoretic hierarchy located wherever an object is. The epistemology of the empty set then reduces to the (non-mysterious) epistemology of the relevant object we pick.

One key problem with this account is that it makes the choice of the empty set seems arbitrary and open to modal variation. Might we pick a contingently existing object? If so it seems like the necessity of mathematical truth becomes dubious, since the empty set (and hence all the relevant pure sets) may *fail* to exist at certain possible worlds. Indeed, if we pick a destructible object, it may fail to exist at *this* world at some point in the future.

This problem is partially assuaged by [Lewis, 1991]'s account of the empty set, somewhat similar to Maddy's in spirit (in that he picks a particular urelement), and on which he avoids these problems by being a staunch four-dimensionalist about time, a strong modal realist, and then identifying the empty set with the mereological fusion of absolutely everything. However, we might worry about making our account of the empty set dependent upon such strong metaphysical assumptions concerning the nature of temporal and modal space.

A different, less ontologically-loaded, response on behalf of the Maddy-style theorist is to argue that they were never trying to point to *the* identity of the empty set, but rather set theory could be viewed as explaining what *would* be true on any hierarchy of the correct kind, and claims about the empty set should be understood in this vain. Then worries of contingent existence disappear; we remain neutral on any *one* urelement being the empty set.

This response goes so far, but unfortunately does not pass muster. The key problem is that we would then have a slew of difficult philosophical questions to answer about the nature of out talk concerning the empty set, some of which are problematic from the perspective of the Ontological Constraint. For example, should we now say that there are many empty sets or just one? The former is *strictly speaking* correct, whereas the latter is what would be true given the 'relativised' interpretation of our mathematical talk. Unfortunately in such a case we have to modify the correspondence theory *slightly*, since we cannot take our talk at face-value (it has to be appropriately 'relativised'). A philosophy of zero that is able to avoid these sorts of deformations in the correspondance theory is thus (in this respect) preferable.

More seriously for the Ontological Constraint, since the empty set is now not (strictly speaking) a *set*, whereas the other sets *are* sets, we have to do additional violence (even within a particular relativisation) to the correspondence theory of truth. In particular, the empty set is not of the same fundamental kind as other mathematical objects (which are, on Maddy's view, bona fide *sets*), and so we require a work-around in the version of the correspondence theory we employ.

A closely related suggestion (famously championed by Frege) is that we should identify the empty set with the class abstract:

$$\{x|x \neq x\}$$

which has, as a matter of logic, no instances. We can then (following Frege) identify the empty set with the collection of all 0-membered concept extensions, i.e. $\{\{\}\}$.

There are technical and philosophical problems with implementing Frege's theory of number, since for $n \geq 1$ the relevant class of concept extensions is a proper class, necessitating a technical theory of classes and subsequent philosophical interpretation to implement.[12] However our main worry here concerns the Epistemological Constraint. On the Fregean picture, we lack a strong epistemological story of how we get to know such an entity. While our knowledge of logic is sufficient to explain how there might be a concept extension with exactly zero instances, it is unclear what kind of epistemological understanding we have of non-instantiable contradictory conditions. Such an answer will depend on an analysis of this question in the philosophy of logic, and we might not want our account of zero to be so dependent.[13]

It therefore seems that there are at least *difficulties* in implementing an account of zero that makes use of the empty set. Later, we will see that our account of zero can in fact *complement* the Fregean (or any) account of the empty set, elucidating a response to these worries, and yielding additional epistemological information about the empty set. In this sense, our complaint with Frege's account is not that it is *wrong*, but rather that it is *incomplete*.

[12] There are options here, we might interpret classes as property extensions (e.g. [Forster, 1995]) or via plural quantification (e.g. [Boolos, 1984], [Uzquiano, 2003]), or via mereological fusions (e.g. [Welch and Horsten, 2016]).

[13] Though, of course, Frege would be quite happy with this.

# 3 Absence perception in a cardinal context

None of the accounts we considered were able to fully meet the constraints of §1. However, there was something good about each. The structuralist perspective was able to easily account for the fact that the position which we select as zero in a real number line was more-or-less arbitrary. The perspective on which we argued that zero arose by counting revealed an important feature of its epistemology: It must mesh well with our counting procedures. And the idea that we could account for zero as a set showed that our account of zero should interact with our theory of sets. In this section we provide our positive proposal, specifically:

> **Zero as an absence.** Zero is the numerical property corresponding to an absence of positive cardinality.

Here is how we will argue for this claim. First, we'll argue that for the Realist, numbers conceived of as properties of sets (or pluralities) is an attractive position. Next we'll review some of the literature on absence perception. We'll then argue that there is reasonable evidence that zero should be understood as a numerical property corresponding to absences. Finally we'll argue that this satisfies the constraints we outlined earlier.

## 3.1 Numbers as properties

One response to the question of how we should understand cardinal numbers emerges out of the literature on number perception in the cognitive sciences. Specifically work by Brannon, Dehaene, Nieder, and others[14] argues for the conclusion that small numbers (four and downwards) can simply be perceived (through a process known as 'subitizing'). Larger numbers can be approximately discriminated from one another, and exhibit the so-called 'distance effect', two collections closer in cardinality are harder to discriminate into the larger and smaller than two collections close in cardinality. This ability is both not specific to humans (having been observed even in rats), and is multi-modal (it does not matter, for example, whether or not the array is visual or aural). Moreover, this perception of numerosities is correlated neurally with increased activation of the parietal cortex. It appears that numerosities are cognitively encoded on a numerical continuum, with the numbers 1–4 precisely discriminable, and higher numerosities approximately so.

The argument we can put forward then is that the numbers 1–4 are simply features of the world that we can perceive (we consider some counterarguments here in the section on objections). Knowledge of larger numbers can then be obtained by an analysis of our counting practices. This yields both an ontological answer for the realist (cardinal numbers are properties of collections), but also an epistemological one concerning the nature of our perception and mathematical enculturation.

## 3.2 Numerosity zero

Consider then the following (somewhat silly) example: Suppose that you are really hungry, and I tell you that there are either three or four sandwiches in the hamper next to me, and you are welcome to all of them (suppose I can't remember whether I packed the last sandwich I made). You are *really* hungry, and it *really* matters to

[14] See here [Dehaene, 1997] for a book-length treatment of the matter, and [Nieder and Dehaene, 2009] for a relatively recent survey of the literature, and [Dehaene and Brannon, 2011] for a relatively recent collection.

you whether or not there are two or three sandwiches there. You are thus primed to engage your ability to perceive small cardinalities. You open the hamper to find...no sandwiches! (I secretly ate them all on the bus and then lied.) Might we argue that in this case that you subitize the zero-cardinality of the set in question? This would then yield an epistemology of zero: Tokens of numerosity zero can then be experienced in perception (and subsequently combined with an understanding of our counting procedures), integrating it into the usual picture of a mental number line.

While it was long thought that this was not possible[15], recent studies on rhesus monkeys and children (of 4 years) have challenged this view. For example rhesus monkeys are able to discriminate empty sets in numerical matching tasks, showing similar distance effects and neuronal activation of the parietal cortex as with ordinary numerosity perception, with the distance effect explained by overlapping neuronal activation that decreases as the numerosity of the stimulus increases ([Ramirez-Cardenas et al., 2016], [Okuyama et al., 2015], [Merritt et al., 2009])[16]. These distance effects have been observed to transfer to 4 year old children on numerosity-matching tasks ([Merritt and Brannon, 2013]). Thus, it seems plausible that in fact numerosity zero is encoded by our faculty of subitizing in a similar way to the other numbers. We might simply argue that in fact the numbers 0–4 are part of our inherent numerical perception of the world, which can then be integrated with abstraction and counting procedures to yield a precise concept of natural number. Thus, we gain knowledge of the number zero by perceiving token instances of zero numerosity and integrating it within our wider epistemological and ontological story of numerical cognition and abstraction.

### 3.3 Absence perception

While we have identified literature from the cognitive sciences that indicates that zero is perceived as a numerical property of the world, it is a separate philosophical problem to explain how this data should be interpreted.

We suggest that we can understand the fact that zero numerosity appears to be perceived can be understood as providing us with a case of *absence* perception. We have seen that the cognitive science literature strongly supports the claim that zero-like properties are perceived as numerosities within the world and integrated into the usual numerical continuum. The question remains, however: How should we understand the *philosophical* character of the experience correlated with these neurological states? This is not a neurological question, but rather a philosophical one; we need to to explain how these sorts of experience are integrated into a wider theory of perception.

Since the neurological data substantially confirms the claim that zero numerosity can be perceived in the world, the claim that zero is a *numerical property* looks justified. We will shortly discuss satisfaction of the desiderata outlined in §1, but for now we should note that this seems to indicate a satisfaction of the Ontological Constraint (since zero is a *numerosity property*, just like the other natural numbers). However, the neurological data does not explain in and of itself why the Phenomenological,

[15] See here [Mou and vanMarle, 2014] and [Wynn, 1998].

[16] Merritt, Rugani, and Brannon are especially explicit here:

> "The studies reported here demonstrate the same type of conceptual understanding of zero. The rhesus monkey apparently treats the class of all empty sets as equivalent and appreciates that empty sets are smaller in numerical magnitude than nonempty sets. Most important, the current studies show that the monkey clearly appreciates empty sets as occupying a place on a numerical continuum."([Merritt et al., 2009], p. 12.)

Technical, and Epistemological Constraints should be satisfied. We have an incomplete *philosophical* story, why should zero have the phenomenological and technical character it does, given that it seems that neuronal activation is similar to the other numbers? Certainly, the neuronal activation is *different*, but what philosophical correlates are there of how the distinction between a perception of zero numerosity and one of two numerosity is of an importantly different kind compared to the distinction between a perception of two and four numerosity?

In all cases of perception of numerosity zero considered, we have a case where a subject has been primed to have a cardinal number perception, but where an *absence* is a possible stimulus. (In this respect, our 'silly' sandwich example from earlier turned out to be roughly analogous to many of the *actual* experimental set-ups, and is not so silly after all.) We thus make the following claim: Perception of zero is a kind of *numerosity* perception, but is also importantly correlated with *absence property* corresponding to a lack of positive cardinality, accounting for its distinctive phenomenological and technical role.

Certainly, the question of whether absences can be perceived at all is a tricky philosophical question, since on many theories of perception we only perceive present objects or scenes.[17] This, however, has recently been challenged in the literature on absence perception, for example in the work of Roy Sorensen[18] and Anna Farennikova[19]. It would take us too far afield to give a full defence of absence perception, however we can lend plausibility to the epistemological and metaphysical story we are providing by clarifying the exact nature of the absence perception we are proposing.

[Sorensen, 2008] provides a book-length treatment of absence perception. Many absences that Sorensen considers have determinate sensible qualities (such as holes, gaps, the cold, and shadows). However, other absences do not have this quality. *Silence* is a good example here. Thinking of silence as a (relative[20]) *absence* of sound, there is no concrete positive *sensation* accompanying a token experience of silence, as there is with seeing darkness or feeling cold. For this reason, Sorensen claims:

> "Hearing silence is the most negative of perceptions: there is nothing positive being sensed and no positive sensation representing that absence."
> [Sorensen, 2008], p. 272

It is interesting that zero shares many properties with silence. For example, instances of zero numerosity (like silence) can be located; I can both hear the silence in the cockpit of an unconscious pilot who has left their microphone on[21] and see zero sandwiches in the hamper. Moreover, zero is (like silence) both detectable and directly perceivable; if I install a device to the outside of a box that emits a high-pitched tone when the contents of a box contain no items of a certain kind (respectively, when the volume inside the box is below a certain level), I can detect zero numerosity (respectively silence) inside the box without perceiving it. Especially interesting,

[17]See [Sorensen, 2008] and [Farennikova, 2013] for further discussion of this point. Good examples of theories of perception (in the case of sight) of this kind are those found (as Farennikova notes) in [Marr, 1982], [Gibson, 1966], and [Dretske, 1969].

[18]See [Sorensen, 2008].

[19]See [Farennikova, 2013].

[20]We say relative, because it seems false that a token instance of silence must be accompanied by a *total* absence of sound waves. In music, for example, we can say that the performer(s) in John Cage's 4'33" are silent, despite the fact that they may be making slight noises. The important point is that they are silent *relative* to the usual conventions of a musical performance. Some (e.g. [Davies, 2003], Ch. 1) treat this as indicative of the impossibility of silence, but see [Sorensen, 2008], p. 287 for an effective rebuttal.

[21]This example is from [Sorensen, 2008], p. 269.

however, is the following similarity between zero and silence; zero perceptions are precisely characterised by a total lack of expected stimulation, and so there are no *particular* qualities associated with zero other than its 'zero-ness' (for example, zero is not coloured in the same way as darkness, which is black in colour). This resonates with Sorensen's account of silence expressed in the quotation above.

Drawing on this character of zero, one can then view perceptions of zero as the kind of absence considered by [Farennikova, 2013]. There she considers a model of absence perception on which absences are understood through *mismatches*. For example, suppose I am looking for and failing to find my keys in the usual expected places. On her picture, I develop a rough visual template of my keys in my working memory and attempt to project it onto my visual surroundings. When the template fails to project accurately onto my surroundings, I am aware of a mismatch between an expectation arising from the working memory projection and the world, yielding a sensation of absence.[22]

Farennikova' Mismatch Model can be adapted to the current case to provide an elegant interpretation of perception of zero numerosity. In a situation of numerosity perception we are projecting possible numbers of objects to be (possibly approximately) matched. In a token occurrence of zero perception, we have an expectation arising from projection of positive numerosity in a cardinal context mismatched with a lack of any number of things. Thus, in addition to understanding zero as a *numerosity* property, we should also understand it as an *absence* property.

The suggestion that we should understand zero an absence perception arising from mismatches meshes well with the fact that subitizing is often explained via the use of an *object-tracking* mechanism. Thus, an absence of positive numerosity (i.e. a token perception of zero numerosity) precisely indicates a mismatch between an *object* tracking mechanism and a *lack* of objects.

Two points are in order here. First, despite the fact we have characterised zero perception as *expectation* violation (in the sense of mismatches between the world and expectations arising from projections), one can still *expect* (in a different sense) a perception of zero numerosity. Suppose I tell a younger sibling to always leave at least one sweet in the jar (out of politeness), but lacking self-control they usually eat all of them. I come down in the morning, expecting to find the jar containing zero sweets, and predictably it does. How can I have a perception of zero numerosity via mismatch if I actually expect zero sweets?

The answer is already dealt with in [Farennikova, 2013] and we adapt that response to the current context. Despite the fact that I *doxastically* expect there to be no sweets, I nonetheless *project* the positive numerosity of sweets onto my surroundings in an attempt to discern the mismatch and see the absence. In other words, I assess my doxastic expectation that there won't be any sweets by comparing the world against a perceptual expectation. The objection that I can predict the relative absence conflates two notions of expectation; the expectation that arises from beliefs and the cognitive expectation that arises from projecting in the process of a search.

A second interesting point here is that the case of zero proposed, if satisfactory, reveals a *generality* to Farennikova's model. [Farennikova, 2013] is primarily concerned with *visual* absences, but later conjectures that the account can be generalised to other modalities. The current proposal suggests that this model may be *very* general. We know that numerosity is perceivable in a variety of modalities (e.g. sound as

[22] Interestingly, Farennikova holds that there may be challenges in applying her account to silence ([Farennikova, 2013], p. 452), citing the example of no longer hearing the footsteps of a retreating lover. She argues that there may be difficulties in saying that one hears the silence when no mismatch occurs. We are happy to accept the conclusion that her agent fails to hear the silence when no footsteps are heard without mismatching; a mere failure to hear footsteps does not entail hearing silence.

well as vision). Given the current account, there is no obstacle to *mixed* modalities of absence perception. Consider a case where we are asked to perceive the total number of tones heard and circles displayed on a screen. No circles are displayed and no tones sound. It seems reasonable to suggest, if we accept the current proposal, that this is a *multi-modal* absence of both visual and auditory stimulus *at the same time*.

So, to sum up the positive account, we have proposed that zero should be understood as follows:

1. A *numerosity* property corresponding to a type of *absence perception*.
2. This is naturally understood through a model of absence perception as a *mismatch* between projections and the world.
3. The property corresponding to zero is not tied to any one modality, and indeed can be instantiated *multi-modally*.

### 3.4 Satisfying the constraints

We now have a positive proposal on the table. Let us see how it fares with respect to the earlier outlined constraints.

The Ontological Constraint is very easily satisfied. Zero is understood as a property corresponding to numerosity just like the other natural numbers. Conceived as a property, it is in exactly the same ontological standing as the other natural numbers, which may vary according to philosophical preference. As a result, we can leave our correspondence-style theory of truth exactly as it is.

Moreover, the Phenomenological Constraint is also clearly satisfied. Though we have acknowledged that zero is a numerosity property, it is the *only* number characterised by an *absence* property (and the accompanying mismatch model explanation of absence), which accounts for its distinctive null-like quality in our phenomenological experience.

For similar reasons, the Epistemological Constraint is satisfied. We can integrate zero into the usual story of numerosity-perception and counting procedures as given in the cognitive science literature. Again, it is the only number within this story which has an additional component of absence perception provided by the mismatch model, giving it a distinctive epistemological flavour.

The Technical Constraint is a little more complex. Our account *does* deal with the role of zero in a natural number context, possibly supplemented by feeding it into the usual accounts of numerosity cognition and natural number epistemology.[23] However, we have thus far only explained its role as a *natural number* and not the role it plays in other technical contexts (such as in the integers, real numbers, or in the context of set theory).

However, we contend that our account provides the foundation for a generalisation to these contexts. In the case of the integers for example, we can generate an account of why the integer sequence should look as it does rather than going straight from $1$ to $-1$ in considering an inverse for the successor function. In this way, integrating our account of zero into what was *good* about the attempt to explain zero in terms of counting procedures, we see that it meshes well with such procedures, even if it is not fully characterised by them. Within the integer ring, we then have that zero can play the role that it does as (for example) an absorbing element for multiplication but for the deeper conceptual reason that no matter how many times one repeats an absence, one will always return an absence. Thinking of the real numbers

[23] For example, [Dehaene, 1997] and [Pantsar, 2014].

as giving mathematical structure to the notion of line, we can think of the origin as providing the point from which the length of particular lines can be measured. Zero itself can then be thought of as an *absence* of positive length (since the distance from the point $0$ to the point $0$ is zero). This explains the relationship between the natural number zero and the real number zero; both are characterised by absences of a sort, the former in numerosity and the latter in length. But it is this absence-like quality that implies the similar role of zero as an identity element for addition in the natural, integer, and real number contexts.

Moreover, conceiving of the real number zero in this way shows that while it may be arbitrary which point we pick on a line to serve as the real number zero, once that point has been picked it has a privileged status; it is the unique point on that line that has an absence of distance from that pre-picked point. So while our Structuralist from earlier said something right about the zero point; selecting it *is* arbitrary, we can say that there is nonetheless more to zero in the context of the real numbers (more in line with our Realist's position). Similarly in set theory, where the role of $0$ is normally played by the empty set, we can see that the empty set is the unique object that has cardinal numerosity zero. Applying this to the Fregean account; we can see that the class abstract $\{x|x \neq x\}$ will always denote a concept with numerosity zero instances, yielding information about how our epistemology of numerosity perception interacts with class abstraction operators. Moreover, the account we have provided here can supplement and complement existing explanations of $\aleph_0$. We might take numerosity zero as the beginning for the processes-as-objects account of $\aleph_0$ in [Pantsar, 2015], or as an initial point in the conceptual blending account provided in [Núñez, 2005][24].

Our answer to the Technical Constraint is somewhat partial in that a full account of how zero figures in many mathematical contexts would require a satisfactory epistemology for each of these contexts, each of which would require lengthy treatments in themselves. However, we have shown that our account is *flexible* in that it can be integrated into a wide variety of different accounts, and that for many of these it performs well in explaining the role of zero.

# 4 Objections

We seem to now have an attractive epistemological and metaphysical account of zero for the Realist, responding well to the constraints and indicating ways in which the account can be worked into further study. There are, however, some salient objections. This section explains and addresses these objections, and shows how they help to clarify the account.

One line of attack is to put pressure on the account of numbers as understood via subitizing and counting procedure, and with numbers integrated into a mental number line. This has been pursued in the cognitive sciences by Núñez[25] who argues that cultural factors are important in determining our epistemology of number (rather than it being 'hardwired' by numerosity perception and then made precise by an understanding of counting procedure). In the philosophical literature, [Buijsman, F] argues that the 'approximate' feature of the object-tracking subitizing procedure fails to deliver the precision required for properly arithmetic understanding and that subitizing fails to explain the ontogenetic delay in infants progressive

[24] Though Núñez would not accept this in virtue of other theoretical commitments, such as his rejection of the role of numerosity perception in providing a cognitive theory of number.

[25] See, for example, [Núñez, 2009] and [Núñez, 2011].

understanding of the meanings of the numbers 1–3.

While we find the account of natural number as understood through counting procedure and the approximate number perception system to be plausible, let us suppose that Núñez and Buijsman are in fact correct and that number epistemology should be understood through different means. Even accepting this, we contend that our account can still play an important role in the explanation of the epistemology of natural number.

Even if, as argued by Núñez, our arithmetical practices are substantially culturally grounded, it *remains* the case that such cultural practices containing a use for $0$ in their mathematics (in any reasonable sense of what $0$ means) will use it in such a way that it can be used to represent an absence of positive numerosity. Therefore, while we may accept that our story of zero in this case is not *constitutive* of the concept of zero, it nonetheless has an important role to play in *explaining* our epistemology of zero. We would still be use zero as a numerosity property to track absences of objects, absence perception and the approximate number would still be features of our cognitive make-up, it is just that additional epistemological steps would need to be made in connecting the epistemology to the perception. Similar remarks apply to [Buijsman, F]'s account of the natural numbers as understood through Hume-style principles. Even if the epistemology is not *exactly* as we have described it, the underlying account of zero as a numerosity property related to absence perception can be transferred to these cases.

Two objections relate to the cultural and infant specific ontogeny of zero. The first point is the following: Historically speaking, the use of zero was a comparatively late technological advance. Notation for zero first appears in around $400$BC in Babylonian mathematics, but it was not until the 7$^{\text{th}}$ century AD that we have record of it being used as a legitimate number in computation in the work of Brahmagupta.[26]

The historical and cultural story is mirrored by the ontogenetic development of children. For example, children of between three and four years of age can commonly count backwards to zero, however fail to integrate this adequately with their other numerical knowledge, often responding that one is smaller than zero.[27]

This raises the following objection: If zero is, as we've suggested, to be understood in terms of numerical absence perception, and this numerical absence perception is, roughly 'hardwired' in the human brain, then why is zero not a culturally and historically ubiquitous phenomenon, presenting contemporaneously with other numbers? Similarly in the case of child development, why is it, if zero is linked to the same perceptual faculties that give us knowledge of the very smallest cardinal numbers, that manifestation of its knowledge in the ordering of cardinal numbers presents much later (at around 6 years[28])?

We think that these points can be answered. The key point is that just because something is a perceivable quality, does not necessarily mean that it is either easy or necessary to integrate into a theory of how we navigate the world. It might just

[26]For an excellent survey of the history here, and an account of zero in the cognitive sciences, see [Nieder, 2016].

[27]Again, see [Nieder, 2016] for a concise survey of some of the cognitive science literature.

[28]See here [Wellman and Miller, 1986] for an early study into ontogenetic development of zero and [Merritt et al., 2009], p. 3:

> "Together, research with babies and children suggests that both in symbolic and in nonsymbolic form, children's concept of empty sets lags behind that of nonzero whole numbers. In addition, children's understanding of zero does not happen all at once, but rather, children gradually acquire the important elemental properties of zero before those elements are fully integrated."

be that in virtue of it being a kind of absence, zero is more difficult to integrate into conceptual systems. Indeed, we might hypothesise that the Mismatch Model would account for this; a failed search given a projection might well be more computationally and conceptually taxing than an immediate successful matching. This might then account for why children (and in fact mature adults) find it more difficult to compute with zero, and why a full understanding presents later.

Moreover, in the historical context it is unclear that we *need* to differentiate the perception of an absence of *some objects or stuff* from a the absence of *a positive numerosity* of that particular object or stuff. Suppose we are in some sort of 'state of nature' context where nutrition is scarce and key to survival. Is the distinction between whether I have an *absence of bananas* or *an absence of a positive numerosity of bananas* an important one? Roughly they come down to the same nutritionally relative state of affairs; I have no bananas to eat.

Culturally speaking, the importance of distinguishing an absence of positive numerosity zero as opposed to absence *simpliciter* only becomes acute when we require a relatively sophisticated ideas. Issues like the lending of money, positive and negative charge, and scientific theories requiring algebraic structures utilising zero are the applications that require zero rather than mere absence. Zero then enters the picture in order to mediate the distinction between the positive and the negative, and provide an appropriate numerical property that can fill the required algebraic roles. But until such needs arise, there is no especial pressure to assert that a lack of numerical competence with a particular arithmetical concept is indicative of a lack of perception of the property corresponding to that concept.

Consider, as an analogy, the Pirahã people who have an approximate proto-arithmetic involving the words "one", "two", and "many". In their proto-arithmetic one plus one is two, two plus one is many, and many plus one remains many.[29] Does this threaten our epistemology of natural number as given by an integration of numerosity procedure with counting practices? It seems that the Pirahã have all the pieces (albeit a different counting procedure). Yet they fail to have a determinate concept for three, despite the fact that they are (presumably) regularly experiencing numerosity three perceptions.

We contend that this does not threaten the account of natural numbers as understood through numerosity perception and counting procedures. The reason is the same as in the zero case; while they may be having perceptions of a certain kind, this does not entail that they have sophisticated concepts that can apply to them. We do not need to maintain that counting and numerosity perception *has* to entail a satisfactory epistemology of number, just that it *can* for the particular arithmetical concepts we employ.[30] Similarly for zero, it may very well be that one can have zero numerosity perceptions without possessing a concept and associated language to describe them.

# 5 Conclusions and open questions

We have argued that we can come to a reasonably satisfactory account of zero on a Realist philosophy of mathematics by holding that zero is a kind of numerosity property corresponding to an absence of positive numerosity. This account is able to explain both the ontological contiguity of zero with the other numbers, whilst

[29] See [Gordon, 2004] for the details.

[30] [Pantsar, 2014] makes a related point (although he is less confident that the practice of the Pirahã can be described as proto-arithmetical) that the use of the perceptual approximate number system might underdetermine the eventual arithmetic adopted.

providing an account of why zero should play the role it does in our phenomenology and mathematical theories, as well as how its epistemology is distinctive. As we've just argued, salient criticisms can be responded to.

There are, however, some open questions regarding zero both from a philosophical and cognition perspective. On the cognitive science side, whilst we have argued that zero can fulfil particular philosophical and mathematical functions if viewed as a kind of absence property, this part of the story has not yet been corroborated by empirical evidence, including neuronal activation. Part of the problem here is that it is unclear if there is *specific* neuronal activation corresponding to absence perception *in general*, or whether absence perception is rather a higher-level philosophical concept. Nonetheless, examining how token perceptions of zero and other kinds of absence interact would be an interesting comparison in providing a full picture of how philosophical concepts and neurological facts interact. We therefore ask the following:

**Question 1.** How does zero behave empirically with respect to other kinds of absence perception, including comparison of neuronal activation?

Our next question concerns the fact that, as hinted to in the text, there seems no barrier to considering numerosities instantiated multi-modally. We know that human (and animal) perception of numerosity can be discriminated in different modalities, but this is not the same thing as analysing them in different modalities *simultaneously*. As far as we know, this has not been studied in detail, and so we ask the following for the specific case of zero:

**Question 2.** What empirical data can we glean from studies involving the multimodal instantiation of numerosity zero?

This also suggests a related question for the philosophical literature:

**Question 3.** How does the possibility of perceiving a multi-modal absence inform our understanding of absence perception?

Finally, we leave open a question concerning how the account provided generalises to other cases. Earlier, we remarked that our account of zero can be integrated into other accounts of mathematical structures that make use of zero (e.g. the real numbers, set theory). However, we did not fill in the details of the account. The following question is therefore important:

**Question 4.** What are the *details* concerning how we integrate this account of zero into the wider practice of mathematics, and how does this account inform the philosophy of those areas?

We are hopeful that an answer to these questions will yield an account of mathematics on which the fundamental importance of zero as both a technical device and philosophically distinctive entity is developed. For now, we take ourselves to have made an initial step in this direction.

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