Finite Jeffrey logic is not finitely axiomatizable

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Abstract

Bayes logics based on Bayes conditionalization as a probability updating mechanism have recently been introduced in [2]. It has been shown that the modal logic of Bayesian belief revision determined by probabilities on a finite set of elementary propositions or on a standard Borel space is not finitely axiomatizable [2, 5]. Apart from Bayes conditionalization there are other methods, extensions of the standard one, of updating a probability measure. One such important method is Jeffrey's conditionalization. In this paper we consider the modal logic $\mathbf{JL}_{<\omega}$ of probability updating based on Jeffrey's conditionalization where the underlying measurable space is finite. By relating this logic to the logic of absolute continuity and to Medvedev's logic of finite problems, we show that $\mathbf{JL}_{<\omega}$ is not finitely axiomatizable. The result is significant because it indicates that axiomatic approaches to belief revision might be severely limited.

Keywords: Bayesian inference, Bayes learning, Bayes logic, Medvedev frames, Jeffrey conditionalization, Jeffrey logic, Non finite axiomatizability.

1 Introduction and overview

In this paper we continue the investigations initiated in the recent paper [2] concerning logics of probabilistic updating. [2] introduced Bayes logics to study the modal logical properties of statistical inference based on Bayes conditionalization. The core idea was to look at Bayes conditionalization as a relation between probability measures: the probability measure q can be Bayes-accessed from the probability measure p if for some evidence (event) A we have $q(\cdot) = p(\cdot \mid A)$. Equivalently, we say in this situation that "q *can* be Bayes-learned from p". That "it is *possible* to obtain/learn q from p" is clearly a modal talk and calls for a logical modeling in terms of concepts of modal logic. This logical modeling has been done in [2] and that paper also hints that a similar analysis could be carried out when Bayes accessibility is replaced by the more general accessibility based on Jeffrey conditionalization. Indeed, Bayesian belief revision is just a particular type of belief revision: Various rules replacing the Bayes's rule have been considered in the context of belief change, one particular type is Jeffrey conditionalization (see [7] and [4]). Suppose $\{E_i\}_{i<n}$ is a finite partition of X with $p(E_i) \neq 0$ and we are given a probability measure $r : \mathcal{A} \to [0,1]$,

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called the *uncertain* evidence, on the subalgebra $\mathcal{A}$ of $\mathcal{B}$ generated by this partition. Given a prior probability p using the evidence r we infer to the measure q below by the "Jeffrey rule":

$$q(H) = \sum_{i<n} p(H \mid E_i) r(E_i) \tag{1}$$

Let $\langle X, \mathcal{B}\rangle$ be a measurable space and denote by $M(X, \mathcal{B})$ the set of all probability measures over $\langle X, \mathcal{B}\rangle$. Bayes accessibility relation has been defined in [2] as follows: For $v, w \in M(X, \mathcal{B})$ we say that w is *Bayes accessible* from v if there is an $A \in \mathcal{B}$ such that $w(\cdot) = v(\,\cdot \mid A)$. We denote the Bayes accessibility relation on $M(X, \mathcal{B})$ by $R(X, \mathcal{B})$. The notion of Bayes frames and Bayes logics have been introduced in [2] as follows.

Definition 1.1 (Bayes frames)**.** A Bayes frame is a Kripke frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\mathcal{F}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), R(X, \mathcal{B})\rangle$ for a measurable space $\langle X, \mathcal{B}\rangle$. □

Definition 1.2 (Bayes logics)**.** A family of normal modal logics have been defined in [2] based on finite or countable or countably infinite or all Bayes frames as follows.

$$\mathbf{BL}_{<\omega} = \{\phi :\ (\forall n \in \mathbb{N})\mathcal{F}(n, \wp(n)) \Vdash \phi\} \tag{2}$$

$$\mathbf{BL}_{\omega} = \{\phi :\ \mathcal{F}(\omega, \wp(\omega)) \Vdash \phi\} \tag{3}$$

$$\mathbf{BL}_{\leq\omega} = \mathbf{BL}_{<\omega} \cap \mathbf{BL}_{\omega} \tag{4}$$

$$\mathbf{BL}_{st} = \{\phi :\ (\forall \text{Standard Borel } \langle X, \mathcal{B}\rangle)\ \mathcal{F}(X, \mathcal{B}) \Vdash \phi\} \tag{5}$$

$$\mathbf{BL} = \{\phi :\ (\forall \text{ Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\} \tag{6}$$

We call $\mathbf{BL}_{<\omega}$ (resp. $\mathbf{BL}_{\leq\omega}$) the logic of finite (resp. countable) Bayes frames; however, observe that the set of possible worlds $M(X, \mathcal{B})$ of a Bayes frame $\mathcal{F}(X, \mathcal{B})$ is finite if and only if X is a one-element set, otherwise it is at least of cardinality continuum. $\mathbf{BL}_{st}$ is called the Standard Bayes logic. □

Bayes logics in Definition 1.2 capture the laws of Bayesian learning: $\mathbf{BL}_{<\omega}$ is the set of general laws of Bayesian learning based on all finite Bayes frames, while the *general laws of Bayesian learning* independent of the particular representation $\langle X, \mathcal{B}\rangle$ of the events is then the modal logic **BL**. The following theorem has been proved in [2, 5][1].

Theorem 1.3. $\mathbf{S4} \subseteq \mathbf{BL} \subseteq \mathbf{BL}_{st} \subseteq \mathbf{S4.1} \subsetneq \mathbf{BL}_{\omega} = \mathbf{BL}_{\leq\omega} \subsetneq \mathbf{S4.1} + \mathbf{Grz} \subsetneq \mathbf{BL}_{<\omega}$.

The finite Bayes frame case has been completely described in [2] and, in particular, it has been shown that $\mathbf{BL}_{<\omega}$ has the finite frame property and is *not* finitely axiomatizable (see Propositions 5.8, 5.9 in [2]). The standard case had been discussed in [5] it has been shown that $\mathbf{BL}_{st}$ is not finitely axiomatizable.

Given two measures $p, q \in M(X, \mathcal{B})$ one can define Jeffrey accessibility: q is Jeffrey accessible from p if there is a partition $\{E_i\}_{i<n}$ and uncertain evidence r such that eq. (1) holds. Denote the corresponding accessibility relation by $J(X, \mathcal{B})$.

[1] Basic terminology of modal logic, such as what **S4** is, is recalled at the end of the introduction.

Definition 1.4 (Jeffrey frames)**.** A Jeffrey frame is a Kripke frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\mathcal{J}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), J(X, \mathcal{B})\rangle$ for a measurable space $\langle X, \mathcal{B}\rangle$. $\square$

To capture the laws of Jeffrey learning we define the following normal modal logics.

Definition 1.5 (Jeffrey logics)**.** We define a family of normal modal logics based on finite or countable or countably infinite or all Jeffrey frames as follows.

$$
\begin{aligned}
\mathbf{JL}_n &= \{\phi :\ \mathcal{J}(n, \wp(n)) \Vdash \phi\} && (7)\\
\mathbf{JL}_{<\omega} &= \{\phi :\ (\forall n \in \mathbb{N})\mathcal{J}(n, \wp(n)) \Vdash \phi\} && (8)\\
\mathbf{JL}_{\omega} &= \{\phi :\ \mathcal{J}(\omega, \wp(\omega)) \Vdash \phi\} && (9)\\
\mathbf{JL}_{\leq\omega} &= \mathbf{JL}_{<\omega} \cap \mathbf{JL}_{\omega} && (10)\\
\mathbf{JL}_{st} &= \{\phi :\ (\forall \text{Standard Borel } \langle X, \mathcal{B}\rangle)\ \mathcal{J}(X, \mathcal{B}) \Vdash \phi\} && (11)\\
\mathbf{JL} &= \{\phi :\ (\forall \text{ Jeffrey frames } \mathcal{J})\ \mathcal{J} \Vdash \phi\} && (12)
\end{aligned}
$$

We call $\mathbf{JL}_{<\omega}$ (resp. $\mathbf{JL}_{\leq\omega}$) the logic of finite (resp. countable) Jeffrey frames; however, observe that the set of possible worlds $M(X, \mathcal{B})$ of a Jeffrey frame $\mathcal{J}(X, \mathcal{B})$ is finite if and only if X is a one-element set, otherwise it is at least of cardinality continuum. $\mathbf{JL}_{st}$ is called the Standard Jeffrey logic. $\square$

Our aim in this paper is to take the first steps in studying finite Jeffrey logic. In particular, we will prove that finite Jeffrey logic $\mathbf{JL}_{<\omega}$ is not finitely axiomatizable (Theorem 3.7). To gain such a result we follow the method presented in Shehtman [6] and we relate Jeffrey logic to Medvedev's logic of finite problems. (The necessary definitions and results will be recalled later on; for an overview about Medvedev's logic we refer to the book [3] and to Shehtman [6]).

Structure of the paper. In the remaining part of the introduction we recall useful facts from modal logic that we will make use of many times. In Section 2 it is shown that Jeffrey-accessibility and the accessibility based on absolute continuity coincide, provided the underlying measurable space is finite. Theorem 2.8 clarifies the containment relation between the logics of absolute continuity: the different logics are all comparable, and the larger the cardinality of X, the smaller the logic. The standard modal logical features of the Jeffrey logics are also determined in section 2. Section 3 is devoted to prove that the finite Jeffrey logic $\mathbf{JL}_{<\omega}$ is not finitely axiomatizable. Finally, in section 4 we close with some open problems.

Useful preliminaries. By a frame we always understand a Kripke frame, that is, a structure of the form $\mathcal{F} = \langle W, R\rangle$, where W is a non-empty set (of possible worlds) and $R \subseteq W \times W$ a binary relation (accessibility). Kripke models are tuples $\mathfrak{M} = \langle W, R, [\,\cdot\,]\rangle$ based on frames $\mathcal{F} = \langle W, R\rangle$, and $[\,\cdot\,] : \Phi \to \wp(W)$ is an evaluation of propositional letters. Truth of a formula φ at world w is defined in the usual way by induction:

- $\mathfrak{M}, w \Vdash p \iff w \in [\,p\,]$ for propositional letters $p \in \Phi$.
- $\mathfrak{M}, w \Vdash \varphi \wedge \psi \iff \mathfrak{M}, w \Vdash \varphi$ AND $\mathfrak{M}, w \Vdash \psi$.

- $\mathfrak{M}, w \Vdash \neg\varphi \iff \mathfrak{M}, w \nVdash \varphi$.
- $\mathfrak{M}, w \Vdash \Diamond\varphi \iff$ there is v such that wRv and $\mathfrak{M}, v \Vdash \varphi$.

Formula φ is valid over a frame $\mathcal{F}$ ($\mathcal{F} \Vdash \varphi$ in symbols) if and only if it is true at every point in every model based on the frame. For a class C of frames the modal logic of C is the set of all modal formulas that are valid on every frame in C:

$$\Lambda(\mathsf{C}) = \{\phi : \ (\forall \mathcal{F} \in \mathsf{C})\ \mathcal{F} \Vdash \phi\} \tag{13}$$

$\Lambda(\mathsf{C})$ is always a normal modal logic. Let us recall the most standard list of modal axioms (frame properties) that are often considered in the literature (cf. [1] and [3]).

Basic frame properties		
Name	Formula	Corresponding frame property
T	$\Box\phi \to \phi$	accessibility relation R is reflexive
4	$\Box\phi \to \Box\Box\phi$	accessibility relation R is transitive
M	$\Box\Diamond\phi \to \Diamond\Box\phi$	2nd order property not to be covered here
Grz	$\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi$	$\mathbf{T} + \mathbf{4} + \neg\exists P(\forall w \in P)(\exists v\ wRv)(v \neq w\ \wedge\ P(v))$
S4	$\mathbf{T} + \mathbf{4}$	preorder
S4.1	$\mathbf{T} + \mathbf{4} + \mathbf{M}$	preorder having endpoints

For two frames $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{G} = \langle W', R'\rangle$ we write $\mathcal{F} \trianglelefteq \mathcal{G}$ if $\mathcal{F}$ is (isomorphic as a frame to) a generated subframe of $\mathcal{G}$. We recall that if $\mathcal{F} \trianglelefteq \mathcal{G}$, then $\mathcal{G} \Vdash \phi$ implies $\mathcal{F} \Vdash \phi$, whence $\Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$ (see Theorem 3.14 in [1]). If $w \in W$, then we write $\mathcal{F}^w$ to denote the subframe of $\mathcal{F}$ generated by w, and we call such subframes point-generated subframes. Further, let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ denote a surjective, bounded morphism (sometimes called p-morphisms). Such morphisms preserve the accessibility relation and have the zig-zag property (see [1]). Recall that if $\mathcal{F} \twoheadrightarrow \mathcal{G}$, then $\mathcal{F} \Vdash \phi$ implies $\mathcal{G} \Vdash \phi$, hence $\Lambda(\mathcal{F}) \subseteq \Lambda(\mathcal{G})$ (see Theorem 3.14 in [1]).

2 The modal logic of absolute continuity

Recall that for $p, q \in M(X, \mathcal{B})$ we say that q is absolutely continuous with respect to p ($q \ll p$ in symbols) if $p(A) = 0$ implies $q(A) = 0$ for all $A \in \mathcal{B}$. Let now assume that $X = \{x_0, \ldots, x_{n-1}\}$ is finite (and hence $\mathcal{B} = \wp(X)$) and take any $p \in M(X, \wp(X))$. If $q \in M(X, \wp(X))$ is a probability measure such that $q \ll p$, then by taking the partition $E_i = \{x_i\}$ for $i < n$ and the probability $r(E_i) = q(E_i)$, we get

$$q(H) = \sum_{i<n} p(H \mid E_i) r(E_i) \tag{14}$$

This means that given any prior probability p and an other probability q that is absolutely continuous with respect to p, if the probability space is finite, then q can be obtained from p by the Jeffrey rule. In other words, absolute continuity and Jeffrey accessibility coincide in the finite case. This motivates us to introduce Kripke frames where the accessibility relation is defined by absolute continuity, as follows.

Definition 2.1. For a probability space $\langle X, \mathcal{B} \rangle$ we define the Kripke frame

$$\mathcal{A}(X, \mathcal{B}) = \left\langle M(X, \mathcal{B}),\ \gg \right\rangle \tag{15}$$

where $\gg$ stands for absolute continuity: For probability measures $p, q \in M(X, \mathcal{B})$ we write $p \gg q$ (or $q \ll p$) if $p(A) = 0$ implies $q(A) = 0$ for all $A \in \mathcal{B}$. $\square$

For a finite, or countably infinite set X we write $\mathcal{A}(X)$ in place of $\mathcal{A}(X, \wp(X))$.

Definition 2.2 (Logics of Absolute Continuity)**.** In a similar manner to Definitions 1.2 and 1.5 we define a family of normal modal logics based on absolute continuity. Let κ be a cardinal and $\ltimes \in \{=, <, \leq\}$.

$$\begin{aligned}
\mathbf{ACL}_{\ltimes\kappa} &= \{\phi :\ (\text{for all } \langle X, \mathcal{B}\rangle \text{ with } |X| \ltimes \kappa) \quad \mathcal{A}(X, \mathcal{B}) \Vdash \phi\} && (16)\\
\mathbf{ACL}_{st} &= \{\phi :\ (\forall \text{ standard Borel } \langle X, \mathcal{B}\rangle) \quad \mathcal{A}(X, \mathcal{B}) \Vdash \phi\} && (17)\\
\mathbf{ACL} &= \{\phi :\ (\forall \langle X, \mathcal{B}\rangle) \quad \mathcal{A}(X, \mathcal{B}) \Vdash \phi\} && (18)
\end{aligned}$$

$\square$

Observe that the set of possible worlds $M(X, \mathcal{B})$ of a frame $\mathcal{A}(X, \mathcal{B})$ is finite if and only if X is a one-element set. What does the frame $\mathcal{A}(X)$ look like? Suppose X is a countable set. Then for $p, q \in M(X, \wp(X))$ we have

$$p \gg q \quad \Longleftrightarrow \quad \operatorname{supp}(p) \supseteq \operatorname{supp}(q) \tag{19}$$

where $\operatorname{supp}(p) = \{x \in X :\ p(\{x\}) \neq 0\}$. Therefore, probability measures having the same support are all accessible from each other.

Proposition 2.3. $\mathbf{JL}_n = \mathbf{ACL}_n$ and $\mathbf{JL}_{<\omega} = \mathbf{ACL}_{<\omega}$ for any $n \in \mathbb{N}$.

Proof. We remarked at the beginning of this section that for a finite X, a probability $q \in M(X, \wp(X))$ can be obtained from $p \in M(X, \wp(X))$ by means of Jeffrey conditionalizing if and only if $p \gg q$. This implies that the frames $\mathcal{A}(X)$ and $\mathcal{J}(X)$ are identical. Consequently $\mathbf{ACL}_n = \Lambda(\mathcal{A}(n)) = \Lambda(\mathcal{J}(n)) = \mathbf{JL}_n$, and $\mathbf{ACL}_{<\omega} = \bigcap_n \mathbf{ACL}_n = \bigcap_n \mathbf{JL}_n = \mathbf{JL}_{<\omega}$. $\blacksquare$

Next we recall the notion of Medvedev frames and Medvedev logic from [6].

Definition 2.4. For a non-empty set X we let $\mathcal{P}^0 = \langle \wp(X) \setminus \{\emptyset\}, \supseteq \rangle$. $\mathcal{P}^0(X)$ is called a Medvedev frame. For a cardinality κ and $\ltimes \in \{=, <, \leq\}$ we define

$$\mathbf{ML}_{\ltimes\kappa} = \left\{\phi :\ (\text{for all } |X| \ltimes \kappa) \quad \mathcal{P}^0(X) \Vdash \phi\right\} \tag{20}$$

$\mathbf{ML}_{\ltimes\kappa}$ is called the Medvedev logic based on sets $|X| \ltimes \kappa$. $\square$

$\mathcal{P}^0(X)$ can be visualized as a Boolean algebra with the least element $\emptyset$ cut out. Figure 2 shows $\mathcal{P}^0(\{1, 2, 3\})$. For a finite X, the frame $\mathcal{A}(X)$ can be obtained by blowing up each possible world $A \in \wp(X) - \{\emptyset\}$ of $\mathcal{P}^0(X)$ into the continuum sized complete graph having probability measures p with $\operatorname{supp}(p) = A$ as vertices; except for the singleton sets $\{x\} \in \wp(X) - \{\emptyset\}$: there is a single probability with support $\{x\}$, the Dirac measure $\delta_{\{x\}}$. The frame $\mathcal{A}(\{1, 2, 3\})$ is sketched in Figure 2.

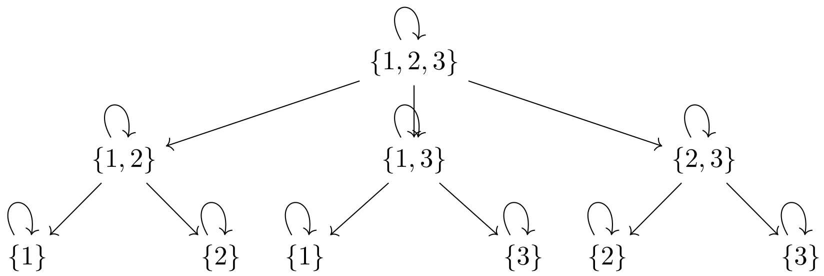

Figure 1: Medvedev frame $\mathcal{P}^0(\{1,2,3\})$. Arrows indicating transitivity are not drawn. We stress that the copies of the nodes $\{1\}$, $\{2\}$ and $\{3\}$ are identical, thus $\mathcal{P}^0(X)$ is *not* a tree but rather a Boolean algebra without the least element.

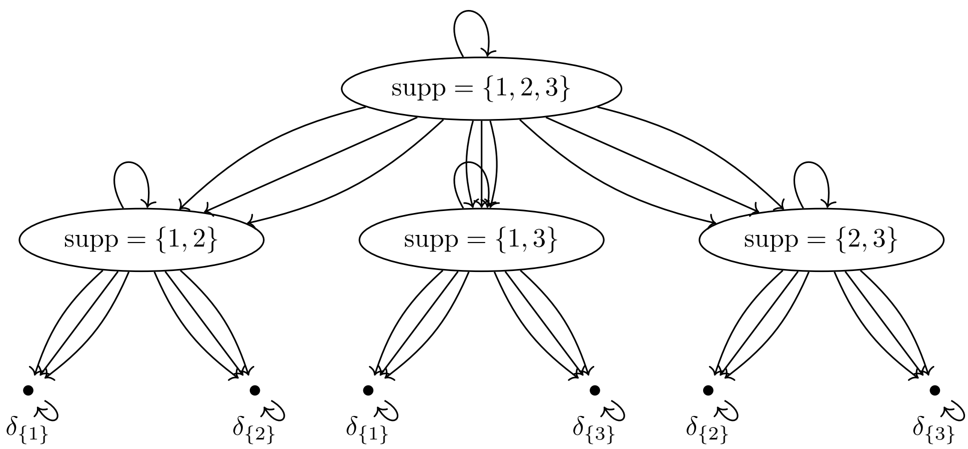

Figure 2: The frame $\mathcal{A}(\{1,2,3\})$. Arrows inside the bubbles and arrows indicating transitivity are not noted. We stress again that the copies of the nodes $\delta_{\{1\}}$, $\delta_{\{2\}}$ and $\delta_{\{3\}}$ are identical.

Lemma 2.5. For a countable X the mapping $f : \mathcal{A}(X) \twoheadrightarrow \mathcal{P}^0(X)$ defined by

$$f(p) = \operatorname{supp}(p) \tag{21}$$

is a surjective bounded morphism.

Proof. Surjectivity of f is straightforward. f is a homomorphism (preserves accessibility) because for $p, q \in M(X, \wp(X))$ we have $p \gg q$ if and only if $\operatorname{supp}(p) \supseteq \operatorname{supp}(q)$ (see (19)). To verify the zig-zag property, suppose $\operatorname{supp}(p) \supseteq A$. We need $q \in M(X, \wp(X))$ such that $p \gg q$ and $\operatorname{supp}(q) = A$. Finding such a q is easy, take for example the conditional probability $q(\cdot) = p(\cdot \mid A)$. ∎

Corollary 2.6. $\mathbf{ACL}_{\ltimes\kappa} \subseteq \mathbf{ML}_{\ltimes\kappa}$ holds for $\ltimes \in \{=, <, \leq\}$ and κ countable.

Proof. Immediate from Lemma 2.5. ∎

Corollary 2.7. $\mathbf{JL}_n \subseteq \mathbf{ML}_n$ and $\mathbf{JL}_{<\omega} \subseteq \mathbf{ML}_{<\omega}$ for all $n \in \mathbb{N}$.

Proof. Combine Corollary 2.6 and Proposition 2.3. ∎

Theorem 2.8. *The following containments hold.*

$$\mathbf{S4} \subseteq \mathbf{ACL} \subsetneq \mathbf{S4.1} \subseteq \mathbf{ACL}_\omega = \mathbf{ACL}_{\leq\omega} \subseteq \mathbf{ACL}_{<\omega} \subseteq \mathbf{ACL}_{n+k} \subseteq \mathbf{ACL}_n.$$

Proof. From the very definition the following containments are straightforward:

$$\mathbf{ACL} \subseteq \mathbf{ACL}_{\leq\omega} \subseteq \mathbf{ACL}_{<\omega} \subseteq \mathbf{ACL}_n \quad \text{and} \quad \mathbf{ACL} \subseteq \mathbf{ACL}_{\leq\omega} \subseteq \mathbf{ACL}_\omega \tag{22}$$

Next we show $\mathbf{ACL}_m \subseteq \mathbf{ACL}_n$ for $m > n$ and $\mathbf{ACL}_\omega \subseteq \mathbf{ACL}_{<\omega}$. The proof relies on the next lemma. If $\langle X, \mathcal{B}\rangle$ and $\langle Y, \mathcal{S}\rangle$ are measurable spaces, then we say that $\langle X, \mathcal{B}\rangle$ can be embedded into $\langle Y, \mathcal{S}\rangle$ ($\langle X, \mathcal{B}\rangle \hookrightarrow \langle Y, \mathcal{S}\rangle$ in symbols) if there is a surjective measurable function $f : Y \to X$ such that $f^{-1} : \mathcal{B} \to \mathcal{S}$ is a σ-algebra homomorphism.

Lemma 2.9. If $\langle X, \mathcal{B}\rangle \hookrightarrow \langle Y, \mathcal{S}\rangle$, then $\mathcal{A}(Y, \mathcal{S}) \twoheadrightarrow \mathcal{A}(X, \mathcal{B})$

Proof. Let $f : Y \to X$ be a surjective measurable function ($f^{-1} : \mathcal{B} \to \mathcal{S}$ is a σ-algebra homomorphism). For a probability measure $p \in M(Y, \mathcal{S})$ let us assign the probability measure $F(p) \in M(X, \mathcal{B})$ defined by the equation

$$F(p)(A) = p\big(f^{-1}(A)\big) \quad (A \in \mathcal{B})$$

Then $F : \mathcal{A}(Y, \mathcal{S}) \twoheadrightarrow \mathcal{A}(X, \mathcal{B})$ is a surjective bounded morphism. ∎

Now, for $m > n$ we have $\mathcal{A}(m) \twoheadrightarrow \mathcal{A}(n)$ and $\mathcal{A}(\mathbb{N}) \twoheadrightarrow \mathcal{A}(n)$. Hence, the containments $\mathbf{ACL}_m \subseteq \mathbf{ACL}_n$ for $m > n$ and $\mathbf{ACL}_\omega \subseteq \mathbf{ACL}_{<\omega}$ follow. We also obtain $\mathbf{ACL}_\omega = \mathbf{ACL}_{\leq\omega}$ as $\mathbf{ACL}_{\leq\omega} = \mathbf{ACL}_\omega \cap \mathbf{ACL}_{<\omega}$.

To see $\mathbf{S4} \subseteq \mathbf{ACL}$ note that absolute continuity is reflexive and transitive (but not antisymmetric), so every frame $\mathcal{A}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), \gg\rangle$ validates $\mathbf{S4} = \mathbf{T} + \mathbf{4}$. If a frame validates $\mathbf{S4}$, then it validates $\mathbf{M}$ (and thus $\mathbf{S4.1}$) if and only if the accessibility relation has endpoints in the following sense:

$$\forall w \exists u (w \gg u \ \wedge\ \forall v (u \gg v \to u = v)) \tag{23}$$

If X is countable, then the Dirac measures $\delta_{\{x\}}$ for $x \in X$ are endpoints, therefore $\mathbf{S4.1} \subseteq \mathbf{ACL}_{\leq\omega}$. To see that $\mathbf{M} \not\subseteq \mathbf{ACL}$ it is enough to give an example for an $\mathcal{A}(X, \mathcal{B})$ in which there are paths without endpoints. Consider the frame $\mathcal{A} = \langle M([0,1], \mathcal{B}), \gg\rangle$ where $[0,1]$ is the unit interval and $\mathcal{B}$ is the Borel σ-algebra. Then, for the Lebesgue measure w we have

$$\mathcal{A} \not\models \exists u (w \gg u \ \wedge\ \forall v (u \gg v \to u = v)) \tag{24}$$

∎

We note that none of the logics $\mathbf{ACL}_n$ (for $n > 1$) validate the Grzegorczyk axiom $\mathbf{Grz}$ as $\mathcal{A}(X)$ always contain a complete subgraph of cardinality continuum.

3 The logic of finite Jeffrey frames is not finitely axiomatizable

The logic of finite Jeffrey frames $\mathbf{JL}_{<\omega}$ is proved to be equal to $\mathbf{ACL}_{<\omega}$ (see Proposition 2.3). We aim at proving $\mathbf{ACL}_{<\omega}$ is not finitely axiomatizable. We show first that $\mathbf{ACL}_{<\omega}$ is a logic of *finite* frames (thus it has the finite frame property).

For each $k, n \in \mathbb{N}$ we define the finite frame $\mathcal{A}_k(n)$ as follows. Take the frame $\mathcal{A}(n)$. For each non-singleton set $A \subseteq n$ the frame $\mathcal{A}(n)$ contains a complete subgraph of cardinality continuum (measures p with support $\mathrm{supp}(p) = A$). Replace this infinite complete graph with the complete graph on k vertices and keep everything else fixed. A more precise definition is the following.

Definition 3.1. Let $n, k > 0$ be natural numbers. For each non-singleton set $a \in \wp(n) - \{\emptyset\}$ take new distinct points $[a]_1, \ldots, [a]_k$, and for each singleton $a \in \wp(n)$ take $[a]_1 = \cdots = [a]_k$ to be a single new point. The set of possible worlds of the frame $\mathcal{A}_k(n)$ is the set

$$A_k(n) = \big\{[a]_1, \ldots, [a]_k : \ a \in \wp(n) - \{\emptyset\}\big\} \tag{25}$$

For two points $[a]_i, [b]_j \in A_k(n)$ we define the accessibility relation $\to$ as

$$[a]_i \to [b]_j \quad \text{if and only if} \quad a \supseteq b \tag{26}$$

□

Figure 3 illustrates the frame $\mathcal{A}_3(3)$ (arrows indicating transitivity are omitted).

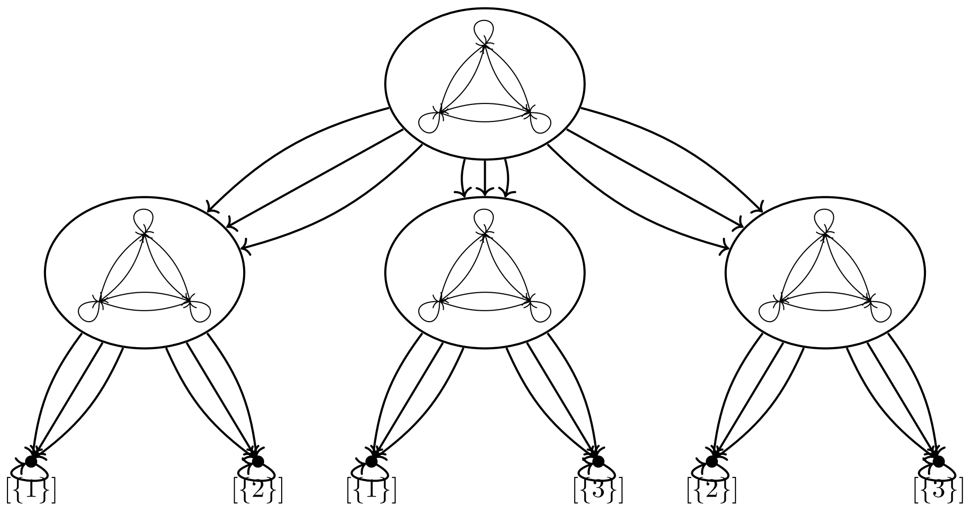

Figure 3: The frame $\mathcal{A}_3(3)$. Arrows indicating transitivity are not drawn. We stress that vertices $[\{1\}]$, $[\{2\}]$ and $[\{3\}]$ at the bottom has been drawn twice though the copies are identical.

Lemma 3.2. For all n and k we have $\mathcal{A}(n) \twoheadrightarrow \mathcal{A}_k(n)$.

Proof. For a measure $p \in M(n)$ the support $\mathrm{supp}(p)$ is a non-empty subset of n, therefore $[\mathrm{supp}(p)]_1, \ldots, [\mathrm{supp}(p)]_k$ are elements of $A_k(n)$. Take any mapping $f : M(n) \to A_k(n)$ such that

$$f(p) \;=\; [\mathrm{supp}(p)]_i \qquad \text{for some } i \in \{1, \ldots, k\} \tag{27}$$

and f is a surjection. Such a mapping clearly exists as for each $a \in \wp(n) - \{\emptyset\}$ we have

$$|\{p: \ \mathrm{supp}(p) = a\}| = 2^{\aleph_0} > k \tag{28}$$

We claim that f is a surjective bounded morphism:

Homomorphism. Take $p, q \in M(n)$ and suppose $f(p) = [\mathrm{supp}(p)]_i$, $f(q) = [\mathrm{supp}(q)]_j$. Then $p \gg q$ if and only if $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ if and only if $[\mathrm{supp}(p)]_i \to [\mathrm{supp}(q)]_j$.

Zag property. Assume $f(p) \to [a]_i$ for some $a \in \wp(n) - \{\emptyset\}$. This can be the case if and only if $\mathrm{supp}(p) \supseteq a$. By surjectivity of f there is q such that $f(q) = [a]_i$, whence $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ which means $p \gg q$. ∎

Lemma 3.3. For each modal formula φ there is $k \in \mathbb{N}$ such that $\mathcal{A}(n) \not\Vdash \varphi$ implies $\mathcal{A}_k(n) \not\Vdash \varphi$.

Proof. We prove that if φ uses the propositional letters $p_1, \ldots, p_k$ only, then $\mathcal{A}(n) \not\Vdash \varphi$ implies $\mathcal{A}_{2^k}(n) \not\Vdash \varphi$. If $\mathcal{A}(n) \not\Vdash \varphi$, then there is an evaluation V such that the model $\langle \mathcal{A}(n), V \rangle \not\Vdash \varphi$. The truth of a formula in a model depends only on the evaluation of the propositional letters the formula uses, therefore we may assume that V is restricted to p_1, …, p_k.

For $x \in \mathcal{A}(n)$ we define a 0–1 sequence of length k according to whether $x \in V(p_i)$ holds for $1 \leq i \leq k$:

$$P_x(i) = \begin{cases} 1 & \text{if } x \in V(p_i) \\ 0 & \text{otherwise.} \end{cases} \qquad (1 \leq i \leq k) \tag{29}$$

As there are 2^k different 0–1 sequences of length k, the number of possible P_x's is at most 2^k.

Take any *surjective* mapping $f : \mathcal{A}(n) \to \mathcal{A}_{2^k}(n)$ such that

$$f(x) = [\mathrm{supp}(x)]_i \qquad \text{for some } i \in \{1, \ldots, k\} \tag{30}$$

and for $x, y \in \mathcal{A}(n)$ with $\mathrm{supp}(x) = \mathrm{supp}(y)$ we have

$$P_x = P_y \quad \text{implies} \quad f(x) = f(y) \tag{31}$$

Such a mapping f must exist as for each non-singleton $a \in \wp(n) - \{\emptyset\}$ we have 2^k elements $[a]_1$, …, $[a]_{2^k}$ in $\mathcal{A}_{2^k}(n)$, and this is the number of the possible P_x's. Let us now define the evaluation V' over $\mathcal{A}_{2^k}(n)$ by

$$V'(p_i) = \{f(x) : \ x \in V(p_i)\} \tag{32}$$

for $1 \leq i \leq k$. Condition (31) ensures that if x and y agree on $p_1, \ldots, p_k$, then so do the images $f(x)$ and $f(y)$. Thus, V' is well-defined. Following the proof of 3.2 one obtains that

$$f : \langle \mathcal{A}(n), V \rangle \twoheadrightarrow \langle \mathcal{A}_{2^k}(n), V' \rangle \tag{33}$$

is a surjective bounded morphism. As $\langle \mathcal{A}(n), V \rangle \Vdash \neg\varphi$ we arrive at $\langle \mathcal{A}_{2^k}(n), V' \rangle \Vdash \neg\varphi$. This means $\mathcal{A}_{2^k}(n) \not\Vdash \varphi$. ∎

Proposition 3.4. $\mathbf{ACL}_{<\omega} = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \Lambda(\mathcal{A}_k(n))$.

Proof. By combining Lemmas 3.2 and 3.3 the equality

$$\bigcap_{n=1}^{\infty} \Lambda\left(\mathcal{A}(n)\right) \;=\; \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \Lambda\left(\mathcal{A}_k(n)\right) \tag{34}$$

follows immediately. The right-hand side of the equation is the definition of $\mathbf{ACL}_{<\omega}$. ∎

Let us recall a theorem of Jankov and de Jongh

Lemma 3.5 (cf. Proposition 4 in [6])**.** Let $\mathcal{F}$ be a point-generated finite **S4**-frame. Then there is a modal formula $\chi(\mathcal{F})$ with the following properties:

(A) For any **S4**-frame $\mathcal{G}$ we have $\mathcal{G} \nVdash \chi(\mathcal{F})$ if and only if $\exists u\; \mathcal{G}^u \twoheadrightarrow \mathcal{F}$.

(B) For any logic $\mathbf{L} \supseteq \mathbf{S4}$ we have $\mathbf{L} \subseteq \Lambda(\mathcal{F})$ if and only if $\chi(\mathcal{F}) \notin \mathbf{L}$.

Corollary 3.6. Let K be a class of finite, transitive frames, closed under point-generated subframes. For every finite, transitive, point-generated frame $\mathcal{F}$ we have

$$\mathcal{F} \Vdash \Lambda(\mathsf{K}) \;\text{ if and only if }\; \exists(\mathcal{G} \in \mathsf{K})\; \mathcal{G} \twoheadrightarrow \mathcal{F}.$$

Proof. ($\Leftarrow$) If there is $\mathcal{G} \in \mathbf{K}$ such that $\mathcal{G} \twoheadrightarrow \mathcal{F}$, then $\Lambda(\mathsf{K}) \subseteq \Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$.

($\Rightarrow$) By way of contradiction suppose $\mathcal{G} \not\twoheadrightarrow \mathcal{F}$ for all $\mathcal{G} \in \mathbf{K}$. Then by Lemma 3.5 we have $\mathcal{G} \Vdash \chi(\mathcal{F})$ for all $\mathcal{G} \in \mathsf{K}$, in particular, $\chi(\mathcal{F}) \in \Lambda(\mathsf{K})$. It is straightforward to see that $\mathcal{F} \nVdash \chi(\mathcal{F})$, thus $\mathcal{F} \nVdash \Lambda(\mathsf{K})$. ∎

Theorem 3.7. $\mathbf{ACL}_{<\omega}$ *is not finitely axiomatizable.*

Proof. A logic $\mathbf{L}$ is not finitely axiomatizable if and only if for any formula $\phi \in \mathbf{L}$ there is a frame $\mathcal{F}_\phi$ such that $\mathcal{F}_\phi \nVdash \mathbf{L}$ but $\mathcal{F}_\phi \Vdash \phi$.

We will use the proof that Medvedev's modal logic of finite problems, $\mathbf{ML}_{<\omega}$, is not finitely axiomatizable. We refer to [6] where it has been proved that for each modal formula $\phi \in \mathbf{ML}_{<\omega}$ there is a finite, transitive, point-generated frame $\mathcal{G}_\phi$ such that $\mathcal{G}_\phi \Vdash \phi$ while $\mathcal{G}_\phi \nVdash \mathbf{ML}_{<\omega}$. The construction therein is such that $\mathcal{G}_\phi$, as a graph, has no directed cycles apart from the loops.

We intend to show that $\mathcal{G}_\phi \nVdash \mathbf{ACL}_{<\omega}$. This is enough because $\mathbf{ACL}_{<\omega} \subset \mathbf{ML}_{<\omega}$. By Proposition 3.4 $\mathbf{ACL}_{<\omega}$ is the logic of the class $\mathsf{K} = \{\mathcal{A}_k(n) :\; n, k \in \mathbb{N}\}$ of finite, transitive frames, closed under point-generated subframes. Therefore, to show $\mathcal{G}_\phi \nVdash \mathbf{ACL}_{<\omega}$, by Corollary 3.6 it is enough to prove that $\mathcal{G}_\phi$ is not a bounded morphic image of any $\mathcal{A}_k(n)$. Suppose, seeking a contradiction, that there exists a bounded morphism $f : \mathcal{A}_k(n) \twoheadrightarrow \mathcal{G}_\phi$. Then for each $a \in \wp(n) - \{\emptyset\}$ the elements $[a]_1, \ldots, [a]_k$ should be mapped into the same point x_a in $\mathcal{G}_\phi$. This is because the points $[a]_i$ are all accessible from each other, while in $\mathcal{G}_\phi$ there are no non-singleton sets in which points are mutually accessible. It follows that f induces a bounded morphism $f^* : \mathcal{P}^0(n) \to \mathcal{G}_\phi$ from the Medvedev frame $\mathcal{P}^0(n)$ into $\mathcal{G}_\phi$ by letting $f^*(a) = x_a$ for $a \in \wp(n) - \{\emptyset\}$. But this is impossible as $\mathcal{G}_\phi \nVdash \mathbf{ML}_{<\omega}$. ∎

4 Closing words and open problems

The recent paper [2] introduced Bayes logics based on Bayes conditionalization as a probability updating mechanism. Apart from Bayes conditionalization there are other methods, extensions of the standard one, of updating a probability measure. Jeffrey's conditionalization might be among the most important and well studied ones. In this paper we considered the modal logic of updating based on Jeffrey's conditionalization where the underlying measurable space is finite. We have seen that in such a case Jeffrey's conditionalization and the accessibility based on absolute continuity give the same class of Kripke frames and logics: $\mathbf{JL}_{<\omega} = \mathbf{ACL}_{<\omega}$ (Proposition 2.3). This logic can be related to the well-known Medvedev logic of finite problems $\mathbf{ML}_{<\omega}$, in fact $\mathbf{JL}_{<\omega} \subseteq \mathbf{ML}_{<\omega}$ (Corollary 2.7).

It has been shown in [2] that $\mathbf{BL}_{<\omega}$ has the finite frame property and is *not* finitely axiomatizable (see Propositions 5.8, 5.9 in [2]), and not finite axiomatizability of the standard Bayes logic $\mathbf{BL}_{st}$ has been proved in [5]. These results are clearly significant because they indicate that axiomatic approaches to belief revision might be severely limited. In this paper we proved that non finite axiomatizability is not a feature of just Bayes-learning: by Theorem 3.7 finite Jeffrey logic $\mathbf{JL}_{<\omega}$ is not finitely axiomatizable (and has the finite frame property). This result puts a further limit to axiomatic approaches to belief revision.

We do not yet have results about the infinite case. Proposition 2.3 cannot directly be extended to infinite measurable spaces, thus there is no straightforward proof of $\mathbf{JL}_{\omega} = \mathbf{ACL}_{\omega}$, for example. Note that in equation (1) we relied on a finite partition. One might define Jeffrey conditionalization allowing countable partitions in a similar manner: Given a prior probability p and a countable partition $\{E_i\}_{i\in\mathbb{N}}$ of X with $p(E_i) \neq 0$ we can infer to the probability measure q if the following equation hold:

$$q(H) = \sum_{i\in\mathbb{N}} p(H \mid E_i)q(E_i) \tag{35}$$

Let us call this updating *infinite Jeffrey's conditionalization.* It is easy to see that absolute continuity and accessibility based on infinite Jeffrey's conditionalization coincide even in the countably infinite case. (However, not in general). We close the paper with some open problems.

Problem 4.1. What the exact relations between Bayes and Jeffrey logics are?

Problem 4.2. Is $\mathbf{JL}_{\omega}$, $\mathbf{JL}_{st}$ or $\mathbf{JL}$ finitely axiomatizable? What about $\mathbf{ACL}_{\omega}$, $\mathbf{ACL}_{st}$ or $\mathbf{ACL}$? Does it make a difference if we allow infinite Jeffrey conditionalization?

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