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THE STRUCTURALIST THESIS RECONSIDERED

GEORG SCHIEMER AND JOHN WIGGLESWORTH

ABSTRACT. Øystein Linnebo and Richard Pettigrew ([2014]) have recently developed a version of non-eliminative mathematical structuralism based on Fregean abstraction principles. They argue that their theory of abstract structures proves a consistent version of the structuralist thesis that positions in abstract structures only have structural properties. They do this by defining a subset of the properties of positions in structures, so-called fundamental properties, and argue that all fundamental properties of positions are structural. In this paper, we argue that the structuralist thesis, even when restricted to fundamental properties, does not follow from the theory of structures that Linnebo and Pettigrew have developed. To make their account work, we propose a formal framework in terms of Kripke models that makes structural abstraction precise. The formal framework allows us to articulate a revised definition of fundamental properties, understood as intensional properties. Based on this revised definition, we show that the restricted version of the structuralist thesis holds.

1. INTRODUCTION

Structuralism in the philosophy of mathematics is the view that mathematical theories describe abstract structures and the structural properties of their objects. The position comes in various forms, depending on whether talk about abstract structures is taken literally or not. In particular, there exist a number of non-eliminative accounts of structuralism which posit structures as existing entities in an abstract realm.[1] Thus, according to non-eliminative structuralists, mathematicians' talk about structures, and positions in structures, should be taken at face value, as talk about abstract entities that stand in some form of connection with more concrete objects or systems of such.

What separates different versions of non-eliminative structuralism from each other is the understanding of how abstract mathematical structures and concrete systems are connected. The recent literature

Authors are listed in alphabetical order; the paper is fully collaborative. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No [715222]). We wish to thank Bahram Assadian, Hannes Leitgeb, Øystein Linnebo, Erich Reck, Gil Sagi and several anonymous referees for helpful comments and discussions of this material.

[1]See, for example, (Parsons [1990]; Resnik [1997]; Shapiro [1997]), as well as (Dedekind [1888]), though it is a matter of debate as to whether Dedekind was a non-eliminative structuralist (see Reck [2003]).

usually distinguishes here between *ante rem* structuralism and *in re* structuralism. According to the former, abstract structures and positions are 'bona fide' objects that exist independently of their concrete instantiations. In contrast, according to the latter account, structures are not ontologically independent of their instances. Mathematical structures and positions in them exist only if they are instantiated by some concrete mathematical system.[2]

In the present paper, our focus is on a particular account of *in re* structuralism. A natural way to think about the conceptual dependency between abstract mathematical structures and concrete instances is in terms of the notion of structural abstraction. Informally speaking, structures result through abstraction from concrete mathematical systems.[3] For instance, one could say that the natural number structure is what is acquired through the process of abstracting away all mathematically irrelevant properties of a given ω-sequence. Or consider the set-theoretic system of Dedekind cuts, which satisfies the axioms for a complete ordered field. If one abstracts away the irrelevant set-theoretic properties from this system, what is left is the real number structure.

This 'abstractionist' approach has a long history in modern mathematics and is expressed in work of some of the pioneers of a structural approach.[4] Observe, for instance, Dedekind's famous remark on arithmetical abstraction in *Was sind und was sollen die Zahlen* of 1888:

> If in the consideration of a simply infinite system N set in order by a mapping j, we entirely disregard the particular character of the elements, retaining merely their distinctness, and taking into account only the relations to one another in which they are placed by the order-setting mapping j, then are these elements called natural numbers or ordinal numbers or simply numbers, and the base-element 1 is called the base-number of the number-series N. With reference to this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind. (Dedekind [1888], p. 68, Def. 73)

This passage indicates that Dedekind's account of mathematical concept formation in his work on the foundations of arithmetic is based on a notion of structural abstraction. Moreover, it also contains an early formulation of a central thesis in modern non-eliminative structuralism. The thesis is sometimes called the 'incompleteness claim' (Linnebo [2008]) or simply the 'structuralist thesis' (Linnebo and Pettigrew [2014]): mathematical objects are merely placeholders or positions in structures, and as such they have no 'foreign' or nonstructural properties.

This paper will further clarify the structuralist thesis and contribute to the understanding of how an abstraction-based account of *in re* structuralism can be made formally precise. Our work builds on that of (Linnebo and Pettigrew [2014]) who have recently developed a version of mathematical structuralism based on Fregean abstraction principles.[5] They argue that this non-eliminative theory of abstract structures proves a consistent version of the structuralist thesis that positions in abstract structures only have structural properties. They do this by defining a subset of the properties of positions in abstract

[2]For a closer discussion of this distinction, see (Shapiro [1997]; Parsons [1990]).

[3]In the theory of *in re* structures that is the focus of this paper, concrete systems are understood model-theoretically. See section 3 for the details.

[4]See, in particular, (Mancosu [2015]) for a study of different notions of abstraction in nineteenth-century mathematics.

[5]Unless otherwise indicated, all references to Linnebo and Pettigrew are to their [2014] paper.

structures, so-called fundamental properties, and argue that all fundamental properties of positions are structural.

The present paper has two central aims. The first is to fill in some of the details where Linnebo's and Pettigrew's account requires further development. Regrettably, Linnebo and Pettigrew are not explicit about the general understanding of mathematical properties that is presupposed in their version of structuralism. They also remain silent as to how properties of positions in structures can be extended to hold of objects in other systems that exhibit the structure in question. Consequently, the structuralist thesis, even when restricted to fundamental properties, remains problematic for Linnebo's and Pettigrew's theory of abstract structures. In fact, the restricted version of the thesis that they propose, under a reasonable interpretation of their account that treats properties extensionally, is equivalent to the unrestricted version of the thesis, and the unrestricted version of the thesis is false.

The present paper will give a closer analysis of the potential weaknesses of the theory of structural abstraction as given in (Linnebo and Pettigrew [2014]) and then show how their restricted structuralist thesis can be vindicated. In this regard, the second aim of the paper is to capture the structural abstraction process in a formal framework based on Kripke models. Through this formal framework, we introduce a dynamic version of structural abstraction, showing how this notion is captured by the standard operation of Kripke model extension. According to this semantic approach, structural abstraction principles become model generating functions, building new Kripke models from old ones. Within this formal framework, we can represent mathematical properties intensionally, as maps from systems of objects to local extensions. We argue that this is a philosophically interesting and insightful account of mathematical properties, one that is well-suited to the general structuralist approach. Furthermore, the intensional account of mathematical properties can be put to good use in Linnebo's and Pettigrew's abstraction-based theory of mathematical structures. Treating mathematical properties intensionally, we give a revised definition of fundamental properties of positions in pure structures. Given this revised definition, we show that the restricted version of the structuralist thesis holds.

The structure of the paper is as follows. Section 2 clarifies the unrestricted structuralist thesis that all properties of positions in abstract structures are structural. We rehearse the arguments against the unrestricted thesis and introduce Linnebo's and Pettigrew's restricted version. Section 3 sets up the non-eliminative theory of abstract structures that is the focus of the paper, a theory that is based on Fregean abstraction principles. We then focus in section 4 on Linnebo's and Pettigrew's claim that this theory proves their restricted version of the structuralist thesis, i.e., that all fundamental properties of positions in abstract structures are structural. We show that, on the extensional interpretation that is consistent with their definition of fundamental properties, all properties are fundamental. We then propose a formal framework for structural abstraction in section 5. The framework is given in terms of Kripke models, and it is able to formally capture a dynamic version of abstraction as well as an intensional account of mathematical properties. This intensional account of properties is crucial to the revised definition of fundamental properties given in section 6. In section 7, we finally prove that on this definition of fundamental properties, all fundamental properties turn out to be structural. Section 8 concludes.

2. THE STRUCTURALIST THESIS

A primary concern for the non-eliminative structuralist is to give an account of the relations that positions in abstract structures instantiate.[6] Importantly, these should not include relations that are specific to the objects in any particular system that exhibits the structure in question, relations that Dedekind ([1888]) described as 'foreign' because they are irrelevant to the pure mathematical structure of the system. This requirement has led some structuralists to endorse what has come to be called the 'incompleteness claim', which captures the idea that mathematical objects, as positions in pure structures, have no internal nature. Michael Resnik and Charles Parsons, for example, both endorse versions of the incompleteness claim.

> In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics ... are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure. (Resnik [1981], p. 530)

> The idea behind the structuralist view of mathematical objects is that such objects have no more of a 'nature' than is given by the basic relations of a structure to which they belong. (Parsons [2004], p. 57)

Because mathematical objects are merely positions in pure structures, any 'nature' that they may have is given entirely by the basic relations of the structure to which they belong. These basic relations are often called structural relations.

A natural question then arises as to how to make the notion of structural relation precise.[7] How one does this likely depends on the more general picture of pure structures that one has. As we describe in section 3, Linnebo and Pettigrew take pure structures to be abstracted from systems of objects. This approach allows for a straightforward way to make the notion of structural relation precise: structural relations are those relations that are invariant across systems that have the same structure. Systems are usually taken to have the same structure when they are isomorphic to one another.[8] The structural relations of objects in a system are then the relations that are invariant across all isomorphic systems. More specifically, a relation that holds of some particular object(s) in a particular system is structural if and only if the relation holds of the matching object(s) in every system that is isomorphic to the original. The structuralist thesis can then be articulated as follows: positions in pure structures only instantiate structural relations. Because the non-eliminative structuralist argues that mathematical objects are positions in pure structures, it follows that mathematical objects only instantiate structural relations.

Unfortunately, John Burgess ([1999]) has shown that the structuralist thesis appears to be inconsistent. Consider the property of instantiating only structural relations, which according to the structuralist thesis is a property that every position in a pure structure should instantiate. The problem is that this property is not structural. Arguably, this property isn't shared by any matching object in any distinct system that exhibits the relevant structure. Thus, in virtue of a position in a pure structure instantiating only structural properties, there is a property it instantiates that is not structural.

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[6] Here and in what follows, talk of relations includes one-place relations, i.e., monadic properties.

[7] See (Schiemer and Korbmacher [2017]) for a more detailed study of different ways to characterise the structural relations of mathematical objects.

[8] According to Linnebo and Pettigrew's theory of structures, systems are understood model-theoretically, which enables them to be isomorphic to one another. See section 3.

The most straightforward reading of the structuralist thesis, it seems, renders mathematical structuralism inconsistent. To resolve the inconsistency, Linnebo and Pettigrew ([2014]) articulate two versions of non-eliminative structuralism, each based on a different form of abstraction. We focus only on the version of structuralism that deploys so-called Fregean abstraction. This is a novel form of non-eliminative structuralism, of particular interest because it provides for the existence of abstract structures through Fregean abstraction principles. In this regard, it differs from familiar versions of non-eliminative structuralism, such as Stewart Shapiro's ([1997]) *ante rem* structuralism, in which the existence of structures is given axiomatically. For want of a better name, we refer to the version of non-eliminative structuralism based on Fregean abstraction principles as LP-structuralism, though it must be emphasised that Linnebo and Pettigrew do not necessarily endorse this version of structuralism.

In addition to being a viable form of non-eliminative structuralism, and a notable alternative to *ante rem* structuralism, LP-structuralism is also of interest because, as Linnebo and Pettigrew argue, it proves a consistent, restricted version of the structuralist thesis. By identifying a subclass of the relations that positions in pure structures instantiate, a subclass they call fundamental relations, they argue that LP-structuralism entails that all fundamental relations are structural. Furthermore, they identify this class of fundamental relations in a natural and non-trivial way, so that the restricted structuralist thesis is 'philosophically interesting and not merely a definitional truth' (Linnebo and Pettigrew [2014], pp. 271-2).

Despite LP-structuralism's success in proving the restricted structuralist thesis, Linnebo and Pettigrew are hesitant to endorse LP-structuralism, because, as they show, it is vulnerable to the well-known problem of non-rigid structures. This problem has long been recognised to threaten Shapiro's *ante rem* structuralism. It is significant that the problem persists across different versions of non-eliminative structuralism, signalling the importance of a solution. However, our goal is not to resolve the problem of non-rigid structures, a problem that has already been given its fair share of attention in the literature.[9] Rather, we discuss another problem for LP-structuralism, which concerns the class of fundamental relations that Linnebo and Pettigrew identify, and the fact that LP-structuralism can prove these relations are structural. This result is captured by Linnebo's and Pettigrew's *Purity* thesis. They also claim that LP-structuralism satisfies two further desirable theses, which they call *Instantiation* and *Uniqueness*. In order to state these three theses, some set up is required.

3. LP-STRUCTURALISM

The idea behind LP-structuralism is to apply Fregean abstraction principles to systems of objects in order to obtain the pure abstract structures of those systems. Fregean abstraction principles introduce identity conditions for certain types of objects by appealing to equivalence relations on other types of objects. For example, Frege ([1968], §64) uses an abstraction principle to give identity conditions for the directions of lines: the direction of line a is identical to the direction of line b if and only if a is parallel to b.

LP-structuralism appeals to abstraction principles to introduce identity conditions both for abstract structures and for positions in those structures.[10] Identity conditions for abstract structures are given

[9] See (Burgess [1999]; Keränen [2001], [2006]), and the responses in (Shapiro [2006], [2008]). Possible solutions to the problem for Shapiro's *ante rem* structuralism can be found in (Button [2006]; Ladyman [2005]; Leitgeb and Ladyman [2008]).

[10] The set up in this section follows closely the description of Fregean abstraction given in (Linnebo and Pettigrew [2014], pp. 274–5), though we have changed some notation to preserve uniformity.

by introducing an equivalence relation on systems of objects. Systems are essentially models, in the logical sense. A system $S = \langle D, R_1, ..., R_n \rangle$ is an ordered tuple comprising a domain and distinguished or primitive relations on the domain. One could also include primitive elements of the domain, and primitive functions on the domain. For simplicity, we restrict consideration to primitive relations, including one-place relations. The equivalence relation on systems required for the abstraction principle is the relation of being isomorphic, which intuitively captures the structuralist idea that certain systems have the same structure. Two systems $S = \langle D, R_1, ..., R_n \rangle$ and $S' = \langle D', R'_1, ..., R'_n \rangle$ are isomorphic ($S \cong S'$) if and only if there is a bijective function $f : D \to D'$, such that if R_i is a k-ary relation, then $(\forall x_1, ..., x_k \in D)[R_i(x_1, ...x_k) \leftrightarrow R'_i(f(x_1), ..., f(x_k))]$ (stipulating that for each i, R_i and R'_i have the same arity). If S is a system, let $[S]$ refer to the pure abstract structure of S. Identity conditions for pure structures are then given by the following abstraction principle.

> *Frege Abstraction for Pure Structures*: Given systems S and S',
>
> $$[S] = [S'] \leftrightarrow S \cong S'.$$

Two systems have the same pure structure if and only if they are isomorphic.[11] For the mathematical structuralist, pure mathematics is primarily concerned with these abstract structures. Individual mathematical objects are given as positions in these abstract structures. Accordingly, Linnebo and Pettigrew also introduce an abstraction principle to give identity conditions for positions in pure structures.

> *Frege Abstraction for Positions in Pure Structures*: Given systems S and S' and elements x of S and x' of S',
>
> $$[x]_S = [x']_{S'} \leftrightarrow \exists f(f : S \cong S' \wedge f(x) = x').$$

If x is an element in the domain D of a system S, then $[x]_S$ is the matching position in the domain $[D]_S = \{[x]_S : x \in D\}$ of the pure structure $[S]$ of system S. Two positions are identical if and only if there is an isomorphism between their respective systems that maps one corresponding object to the other. Finally, Linnebo and Pettigrew present a third principle to give an account of the relations that hold between positions in a pure structure.

> *Pure Relations on Pure Domains*: Suppose S is a system and R is an n-ary relation on the domain D of S. Then: $[R]_S(x_1, ..., x_n)$ if and only if there are elements $u_1, ..., u_n$ of D such that, for each i, $[u_i]_S = x_i$, and $R(u_1, ..., u_n)$.

The definition shows how to abstract from concrete relations between the objects of a given system in order to yield a pure relation on the matching positions in a structure. This third principle allows us to think of pure structures as entities that usually come equipped with some internal relational structure or structural composition, similar to the systems instantiating them.

LP-structuralism, as given by these three principles, provides us with a formally precise account of how the relation between abstract mathematical structures and concrete systems instantiating them (such as concrete groups, graphs, or number systems) can be captured in terms of the notion of structural abstraction. However, Linnebo and Pettigrew leave open precisely how these structures, and the positions in them, should be understood. One can view the principles as axiomatic conditions that specify the

[11] As Linnebo and Pettigrew note, if one is not careful, this abstraction principle can lead to the Burali-Forti paradox. To avoid this consequence, they suggest that one could take systems to be sets and abstractions to be *sui generis* mathematical objects.

behaviour of the abstraction operators expressed by their bracket notation.[12] Nevertheless, their account does not further specify how these functions from systems to structures (and from elements in systems to pure positions) actually work. More specifically, it remains unclear how to think of the respective co-domains of the operators described by the three abstraction principles. We will return to this issue in section 5.

The idea that pure mathematical structures are the result of abstracting away mathematically irrelevant properties from systems of objects is extremely natural and intuitive. LP-structuralism is a good first attempt at formally capturing this idea. As we briefly mentioned, these particular abstraction principles do not get it right for non-rigid structures. But for a wide range of structures of interest to mathematicians and philosophers of mathematics, like the natural numbers as given by the standard model of arithmetic, or the real numbers as given by any complete ordered field, these abstraction principles comprise a theory of pure structures that is simple, elegant, and highly successful.

In particular, Linnebo and Pettigrew argue that, when restricted to rigid systems, LP-structuralism satisfies three desirable theses:

> *Instantiation*: $S \cong [S]$ (systems are isomorphic to their pure structures).
> *Purity*: Suppose a is a position in $[S]$ and R is a property. If R is a fundamental property of a, then for each system S', such that $f : [S] \cong S', R$ is a property of $f(a)$.
> *Uniqueness*: $[S]$ is unique in satisfying *Instantiation* and *Purity*.

In section 4, we examine Linnebo's and Pettigrew's claim that LP-structuralism proves the Purity thesis. Such a claim depends on the definition of fundamental properties and relations. Linnebo and Pettigrew propose a definition that naturally emerges from their description of LP-structuralism. But as we will see, whether this definition delivers a proof of the Purity thesis depends crucially on the more general theory of properties and relations that one subscribes to.

4. Purity

Linnebo and Pettigrew argue that the structuralist must identify a special class of relations, which they call fundamental relations. The structuralist must then be able to distinguish between fundamental and non-fundamental relations in a principled way so as to make it an interesting truth that all fundamental relations are structural. The thesis that all fundamental relations are structural is captured by Linnebo's and Pettigrew's Purity thesis. According to them, 'Purity is our consistent reformulation of the structuralists' claim that positions in pure structures have no non-structural properties' (p. 272). In order to prove that the Purity thesis is a consequence of LP-structuralism, Linnebo and Pettigrew offer a sufficient condition for a relation to be fundamental (p. 276).

> *Fundamental Relations amongst Positions*: Suppose R is a relation on the positions of $[S]$. Then R is *fundamental* if there is a relation Q on the domain of S, such that $[Q]_S = R$.

Fundamental relations are those obtained, through abstraction, from the relations on systems. Given this definition, Linnebo and Pettigrew claim that LP-structuralism proves the Purity thesis for rigid systems, in the form of their Proposition 5.2 (p. 277):

> **Proposition 5.2**: If S is rigid and $x_1, ..., x_n$ are the elements of S, then

$$[Q]_S([x_1]_S, ..., [x_n]_S) \leftrightarrow Q(x_1, ..., x_n)$$

[12]Compare (Leach-Krouse [2017]) for a related axiomatic theory of structures based on Fregean abstraction principles.

Let $[Q]_S$ be a property of a position $[x]_S$ in a pure structure $[S]$, which is abstracted from some property Q of an object x in the system S. The Purity thesis says that if $[Q]_S$ is a fundamental property, then it holds of each object corresponding to $[x]_S$ in every system isomorphic to $[S]$ (i.e., every system that has the pure structure $[S]$). Strictly speaking, however, this cannot happen on Linnebo's and Pettigrew's account. According to their definition of pure relations, $[Q]_S$ can only hold of pure positions in pure structures. Linnebo and Pettigrew even make this restriction explicit: 'Where Q is a monadic property, $[Q]_S$ is the property that holds of an object iff that object is a pure position in the pure structure $[S]$ and the occupant of this position in the system S has the property Q' ([2014], p. 275). It follows that $[Q]_S$ is not invariant across isomorphic systems.[13]

At this point we can see that something has gone wrong with Linnebo's and Pettigrew's account. Their definition of fundamental relations appears to be incompatible with the Purity thesis. In order to make their account work, either the definition of fundamental relations must be revised, or the Purity thesis must be dropped. In section 6, we propose a revised definition of fundamental relations, allowing us to endorse the Purity thesis, and maintain the spirit of the structuralist approach.

While Linnebo's and Pettigrew's definition of fundamental relations is the immediate cause of the problem for their account, we argue that this problem is rooted in a more general conceptual issue. The conceptual issue concerns how mathematical relations should be understood on a structuralist approach to mathematical objects. Regrettably, Linnebo and Pettigrew are not explicit about the general theory of relations that they have in mind.

Standard philosophical treatments of relations include extensional accounts and intensional accounts. The most natural reading of Linnebo's and Pettigrew's account treats mathematical relations extensionally. Extensional relations are identical when they have the same extension in a mathematical system. For example, in the natural number structure, the properties 'being an even prime' and 'being the successor of 1' have the same extension. On an extensional treatment, they are the same (one-place) relation. Extensional relations offer a 'local' account of relations, as they are understood relative to a particular system. For instance, on an extensional approach, the ordering relation, $<$, on the natural numbers is a different from the ordering relation on the finite von Neumann ordinals, because the numbers and ordinals comprise different mathematical systems.

It is unlikely, however, that an extensional account of relations can do the work that the LP-structuralist requires. If one adopts the characterisation of structural relations as given by Linnebo and Pettigrew, it's very hard to see how an extensional relation could be structural, because the structural relations of objects in a system hold of all of the corresponding objects in every other system that is isomorphic.

Even if extensional relations could somehow be 'applied' in other systems, there would still be several negative consequences of combing LP-structuralism with an extensional account. In particular, one can show that all extensional relations of positions in pure structures satisfy Linnebo's and Pettigrew's definition of fundamental relations. If Purity holds for LP-structuralism, it then follows that all relations of positions in pure structures are structural. That is, every relation that holds of positions in a pure structure holds of the corresponding objects in every system that exhibits the structure in question.

To see that LP-structuralism makes all extensional relations of positions in pure structures fundamental, pick some system S that exhibits the structure $[S]$, and let R be any relation that holds of some positions in $[S]$. For simplicity, let R be a property, holding of each of the positions $a_1, a_2, ...$ in $[S]$,

[13] Many thanks to an anonymous referee for discussion of this point.

though what follows holds for relations of any arity, and the positions need not be enumerable. By the Instantiation thesis, there exists an isomorphism $f : [S] \cong S$. The isomorphism gives us another property Q that holds of each of the matching objects $x_1, x_2, \ldots$ in S, where for all i, $[x_i]_S = a_i$. By the principle of *Pure Relations on Pure Domains*, there is a property $[Q]_S$ that holds of each of $a_1, a_2, \ldots$. And by the principle of *Fundamental Relations amongst Positions*, it follows that $[Q]_S$ is a fundamental property. As the properties $[Q]_S$ and R have the same extension, and we are treating properties extensionally, it follows that $[Q]_S = R$. So the property R is fundamental. As this property was chosen arbitrarily, all properties on positions in pure structures are fundamental. The *Purity* thesis would then entail that all properties on pure structures are structural. The same result holds for relations of any arity, and so all relations on pure structures are structural.

An extensional account of relations would therefore make LP-structuralism vulnerable to two familiar objections. First, because all extensional relations are structural on this account, LP-structuralism would incorrectly classify nonstructural properties, like being John's favourite number, as structural. Second, as Burgess has shown, every position in a pure structure would have the property of having no nonstructural properties. This property is not a structural property. There would then be a non-structural property that positions in pure structures instantiate, which is in tension with the result that LP-structuralism makes all relations on pure structures structural.

Given these considerations, an intensional account of relations is more appropriate. In the context of LP-structuralism, intensional relations can be understood as functions from systems to collections of ordered tuples of objects. For each system, this collection of ordered tuples is the local extension of the relation in the system in question. Understood intensionally, it is relatively straightforward to see that relations need not be bound to a particular system. Relations can therefore be instantiated by matching objects in isomorphic systems, and thus have a chance at being structural. Arguably, an intensional account of relations is what Linnebo and Pettigrew envision for LP-structuralism, though they do not make the details explicit. The next section fills in the details by proposing a formal framework that can capture both structural abstraction and intensional relations.

5. A FORMAL FRAMEWORK FOR STRUCTURAL ABSTRACTION

In order to vindicate the restricted structuralist thesis, we present a new formal framework for structural abstraction. The formal framework, which is given in terms of Kripke models, appeals to a dynamic and predicative version of abstraction and allows for an understanding of mathematical relations as intensional, rather than extensional. Given this formal framework, we present a revised notion of fundamental relations, one that is not vulnerable to the objections that threaten Linnebo's and Pettigrew's approach. With the more refined notion of fundamental relations in hand, the formal framework that we introduce allows us to prove the restricted structuralist thesis that every fundamental relation of positions in pure structures is structural.

To introduce the formal framework, recall Linnebo's and Pettigrew's model-theoretic presentation of mathematical systems. Let S be a relational system of the form $\langle D, R_1, \ldots, R_n \rangle$, which is a tuple consisting of a domain and a collection of primitive relations (each of a given arity) on the domain.[14] Now consider a variable domain Kripke model, where each world in the model is a relational system. A

[14]For simplicity, as before, we consider only purely relational systems, though what follows also holds for systems that include primitive functions and elements.

variable domain Kripke model is a quadruple $\mathcal{M} = \langle D, W, \sim, v\rangle$, such that D is a non-empty universal domain of quantification, W is a non-empty set of worlds, and $\sim$ is a binary accessibility relation on W. The function v interprets relations in the usual way, except that it also assigns to each $w \in W$ a set $D_w \subseteq D$, which is the local domain of quantification for w. For our purposes, given a particular Kripke model $\mathcal{M}$, each $w \in W$ is taken to be a relational system in a given language $\mathcal{L}$, and the accessibility relation $\sim$ holds between two systems if and only if the systems are isomorphic. On this account, worlds in a Kripke model are just $\mathcal{L}$-systems as Linnebo and Pettigrew understand them. The accessibility relation can be more formally defined, using the standard definition of isomorphism as a bijection between systems that preserves relations. In order to present the full details, we must first describe the intensional account of mathematical relations that we propose.

Intensional accounts of relations typically define relations as functions from possible worlds to extensions. An intensional account of relations is particularly suited to this formal framework, as the worlds can be taken to be relational systems in a Kripke model $\mathcal{M} = \langle D, W, \sim, v\rangle$. Mathematical relations can then be understood intensionally, as functions from systems in W to local extensions at those systems. Given a particular relational system $S = \langle D, R_1, \ldots, R_n\rangle$, the relations $R_1, \ldots, R_n$ should then be understood as local to S. They are like the extensional relations described in section 4, applying only to the objects in system S. More accurately, they are the local extensions of intensional relations. Intensional relations are global relations that can be applied across different systems in a Kripke model. For example, consider the ordering relations on the natural numbers and the finite von Neumann ordinals. On an extensional account, these are different relations. But on an intensional account, they are the local extensions of an intensional ordering relation that applies to all ω-sequences.

Formally an intensional n-ary relation R is a function with domain W and co-domain $\mathcal{P}(D^n)$ — the set of all subsets of n-tuples of the domain of the model. Each n-ary relation R has a (possibly empty) local extension R_w at each system $w \in W$, such that $R_w = \{\langle x_1, \ldots, x_n\rangle \in D_w^n \mid R(x_1, \ldots, x_n)\}$. Two relations R_1 and R_2 are identical if and only if they are co-extensional at each system. That is, they are identical if and only if for all $w \in W, R_{1w} = R_{2w}$ (the local extension of R_1 at w is identical to the local extension of R_2 at w). This intensional account of mathematical relations essentially agrees with David Lewis's ([1986]) possible worlds approach to relations, where possible worlds are replaced with mathematical systems.

With this intensional understanding of relations, we can give the formal definition of the accessibility relation $\sim$ in terms of isomorphism between systems. A system w consists of a domain D_w together with local extensions of intensional relations on the domain. In other words, w is a system in the model-theoretic sense that Linnebo and Pettigrew use. Given two systems, w and $v, w \sim v$ if and only if there exists an isomorphism between w and v, which holds if and only if there is a bijective function $f : D_w \to D_v$, such that if R_w is the local extension of an n-ary relation on D_w, then $(\forall x_1, ..., x_n \in D_w)[R_w(x_1, ..., x_n) \leftrightarrow R_v(f(x_1), ..., f(x_n))]$, with $f(x_1), ..., f(x_n) \in D_v$.

The formal framework provided by taking systems as worlds in a Kripke model provides insight into the nature of mathematical abstraction. The abstraction process takes one from a single system S to the system's pure structure $[S]$. The process can be generalised to apply to a collection of systems $S_1, S_2, ...$, by which we arrive at the collection of pure structures $[S_1], [S_2], ...$. One way to understand abstraction is as a 'dynamic' process that begins with a collection of systems and extends this collection by adding all of the pure structures of the original systems. According to this dynamic process, one moves from an initial collection of systems to a new, extended collection that also includes a pure structure for each system in the original collection.

We may recall that mathematical abstraction is often associated with the neo-logicist programme in the philosophy of mathematics. However, the dynamic approach to abstraction differs from how abstraction has traditionally been understood in the neo-logicist literature. According to the traditional neo-logicist, abstraction is a 'static' process that provides identity conditions for first-order objects using impredicative abstraction principles. The principles are impredicative because the first-order objects whose identity conditions are given already belong to the first-order domain of quantification. However, the impredicativity of abstraction principles like Frege's Basic Law V contributes, in part, to their resulting in inconsistency. In response, some have argued for a predicative and dynamic approach to abstraction principles. Linnebo ([2009]) has had particular success applying predicative and dynamic abstraction to resolve the bad company problem that has dominated recent debates in the neo-logicist literature.[15] He has since developed a more general theory of dynamic and predicative abstraction principles.

> I argue that predicative abstraction principles can be laid down with *no presuppositions whatsoever*. ... The restriction to predicative abstraction results in an entirely natural class of abstraction principles, which has no unacceptable members (or so-called 'bad companions'). My account therefore avoids the 'bad company problem'. Instead, I face a complementary challenge. Although predicative abstraction principles are uniquely unproblematic and free of presuppositions, they are mathematically weak. My response to this challenge takes the form of a novel account of 'dynamic abstraction' ... Since abstraction often results in a larger domain, we can use this extended domain to provide criteria of identity for yet further objects, which can be obtained by further steps of abstraction. ... The successive 'formation' of sets described by the influential iterative conception of sets is just one instance of the more general phenomenon of dynamic abstraction. Legitimate abstraction steps are iterated indefinitely to build up ever larger domains of abstract objects. (Linnebo [forthcoming], p. x, emphasis in original)

We do not wish to engage too deeply with any purely ontological consequences that may hinge on the difference between predicativity and impredicativity, for example, regarding the debate between mathematical realists and anti-realists. Mathematical abstraction, understood as a predicative and dynamic process, simply allows one to consider larger and larger domains of mathematical entities, independently of the question of their objective existence. In the case of LP-structuralism, the relevant abstraction principles introduce pure structures into the domain of consideration by giving their identity conditions, as well as the identity conditions for the pure positions that belong to those structures.[16]

One of the benefits of the Kripke model framework is that it can easily capture dynamic structural abstraction through the standard operation of Kripke model extension. Linnebo's and Pettigrew's principles of structural abstraction, if understood as predicative and dynamic abstraction principles, induce

[15] James Studd ([2016]) has also used dynamic abstraction to resolve the bad company problem.

[16] According to some 'thin' versions of realism (for example, Linnebo [2012]), all that is required for the existence of an object is the provision of consistent identity conditions for the object. Without committing to this form of realism, we recognise that it may be a useful context in which to understand dynamic abstraction, both generally and in its application in LP-structuralism.

the extension of a Kripke model $\mathcal{M}$ to a model $\mathcal{M}' = \langle D', W', \sim', v' \rangle$, with $D \subseteq D', W \subseteq W'$, $\sim \subseteq \sim'$, and $v \subseteq v'$. Intuitively, extending a Kripke model involves extending the set W of worlds to include the members of a set W_S of pure structures, and extending the universal domain D to include the members of a set D_P of pure positions. These new sets are given by two abstraction operators, which essentially serve as model generating functions, because they generate an extension of the initial Kripke model.

The first abstraction operator is a pure structure operator $\S : W \to W_S$ (with $W \cap W_S = \emptyset$) such that for all $w_1, w_2 \in W$:

$$\S(w_1) = \S(w_2) \text{ iff } w_1 \sim w_2$$

The set of pure structures is then defined as $W_S = \{\S(w) \mid w \in W\}$, and the extended set of worlds in the new model is $W' = W \cup W_S$. The second abstraction operator is a pure positions operator $\sigma : D \to D_P$ (with $D \cap D_P = \emptyset$) such that for all $a \in D_{w_1}, b \in D_{w_2}$:

$$\sigma(a) = \sigma(b) \text{ iff there is an isomorphism } f \text{ between } w_1 \text{ and } w_2 \text{ (i.e., } w_1 \sim w_2\text{) and } f(a) = b$$

The set of pure positions is then defined as $D_P = \{\sigma(a) \mid a \in D\}$, and the extended universal domain of the new model is $D' = D \cup D_P$.

The accessibility relation must also be extended to hold between systems and their pure structures. To do this, the accessibility relation of the extended model can be defined in terms of the accessibility relation of the initial model and the pure structure abstraction operator $\S$, such that for all $w_1, w_2 \in W'$:

$$w_1 \sim' w_2 \text{ iff } w_1 \sim w_2 \vee w_1 = \S(w_2) \vee w_2 = \S(w_1)$$

It may be helpful to visualise the idea of extending a Kripke model $\mathcal{M}$ to a model $\mathcal{M}'$ that includes the pure structures of the systems in the initial model $\mathcal{M}$.

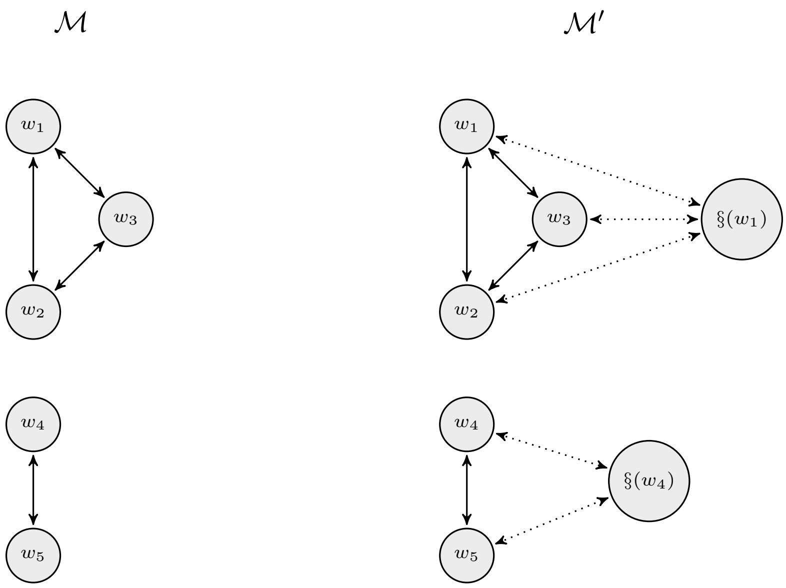

FIGURE 1. Extending a Kripke model

The arrows indicate the isomorphism relation between systems, as captured by the accessibility relation between worlds. The dotted arrows indicate how the accessibility relation is extended in moving from

$\mathcal{M}$ to $\mathcal{M}'$. The additional worlds in $\mathcal{M}'$, labeled $\S(w_1)$ and $\S(w_4)$, represent the pure structures of the systems in the initial model. It should be noted that $\S(w_1) = \S(w_2) = \S(w_3)$ and $\S(w_4) = \S(w_5)$.

We still need to say what relations are instantiated by the pure positions in D_P. Recall that on the proposed framework, relations are intensional entities: functions on W, mapping members of W to local extensions. So we need to give an account of how to extend relations so that they become functions on W'. This account can be given by specifying the local extensions of relations at the worlds in W_S.

Before presenting our specific account of extended relations in the next section, we can already illustrate one advantage that comes from appealing to intensional relations in the Kripke model framework. The Kripke model framework, when combined with an intensional notion of relations, blocks the problematic consequence that all relations on pure positions are fundamental. Recall that the objection proceeded by being able to identify an arbitrary relation on positions in a pure structure with one that is abstracted from the matching relation on the matching objects in an isomorphic system. On an extensional account, it follows immediately that these relations are identical, as each relation is bound to a particular system or structure, and these relations are co-extensional at the relevant structure. But on an intensional account of relations, this conclusion does not immediately follow, because the identity of two relations is more fine-grained: two relations are identical when they have the same extension at every system or structure. While the relations in question have the same local extension at a particular pure structure, the argument does not show that they match in extension at every system. It therefore does not immediately follow on the intensional account that every pure relation is fundamental.

In addition to blocking the argument that all fundamental relations are structural, there are other advantages of the Kripke model framework as presented so far. Most immediately, it gives a formal characterisation of the dynamic abstraction process in terms of Kripke models. Because the behaviour of Kripke models has been extensively explored, applying this formal framework provides clarity and further insight into the idea of structural abstraction.

Another advantage of the Kripke model framework is that it says precisely what pure structures are by giving a uniform account of systems and structures. Linnebo's and Pettigrew's theory of pure structures can be understood as an axiomatic approach to structuralism, as given by the abstraction principles for structures, positions, and relations presented in section 3. Though the axiomatic approach tells us something about how they work, one is left with questions as to exactly what kind of entities structures, positions, and relations are. In the particular case of pure structures, some specific clarification is required. For Linnebo and Pettigrew, pure structures are supposed to be like systems, in that they comprise a domain and relations on that domain. However, Linnebo and Pettigrew take systems to be (represented by) sets, but require that structures are not sets. Rather, they are *sui generis* mathematical objects (p. 274). The requirement is imposed to avoid the Burali-Forti paradox.[17] On the semantic approach that we propose, as given in terms of Kripke models, pure structures can be exactly the same kind of thing as the systems that they are abstracted from. If systems are sets, pure structures can be sets as well. This uniform treatment of systems and structures avoids paradox because the pure structures are not members of the set of worlds in the initial Kripke model, but are introduced through a dynamic abstraction process as captured by extending the initial model.

A further advantage of the Kripke model framework is that it supplies us with a natural understanding of pure or abstracted relations, as is shown in the next section.

[17]See fn 11.

6. FUNDAMENTAL RELATIONS

The Kripke model framework easily blocks the consequence that all relations are fundamental, and therefore structural. But it does not tell us what pure or abstracted relations are. Linnebo and Pettigrew provide some details, but not many. For example, they specify that for each relation on a system, there exists a corresponding relation on the pure structure of that system. But they have not said what this pure relation is, or how it is connected, if at all, to the relation on the system that it corresponds to. One also wants an account of how relations that hold between objects in systems can be applied to the positions in pure structures that are introduced through applications of the abstraction operators. Linnebo and Pettigrew are silent on these questions. By filling in the details of how relations are characterised when moving from an initial Kripke model to an extended model, we can provide some answers.

These details require the notion of one relation extending another. In the Kripke model framework, abstraction is understood as a dynamic process. This process is captured by the familiar notion of a Kripke model extension. Beginning with an intitial Kripke model $\mathcal{M} = \langle W, D, \sim, v\rangle$, the abstraction operators for pure structures and pure positions serve as model generating functions that extend $\mathcal{M}$ to a new model $\mathcal{M}' = \langle W', D', \sim', v'\rangle$. We saw earlier how W', D', and $\sim'$ are given. What remains to be seen is how the valuation function v' assigns local extensions to relations at the new pure structures. These extensions are given, in part, by the following definition.

Definition 1 (Extended Relation)**.** *An n-ary relation R is an extension of an n-ary relation Q iff*

(1) $Q_w = R_w$, for all $w \in W$;

(2) for all $u \in W_S$ and all $d_1, \dots, d_n \in D_u$: $\langle d_1, \dots, d_n\rangle \in R_u$ iff there exists a $v \in W$ with $b_1, \dots, b_n \in D_v$ such that (i) $d_i = \sigma(b_i)$ (for all $i \in n$) and (ii) $\langle b_1, \dots, b_n\rangle \in Q_v$.

There are two things to note about this definition. First, in the move from $\mathcal{M}$ to $\mathcal{M}'$, as a relation Q is extended to a relation R, the local extensions of Q at worlds belonging to W remain fixed. In effect, R is just like Q, but with additional local extensions at the new worlds, which are the pure structures added in the Kripke model extension. Second, Definition 1 does not give a full characterisation of every relation on positions in pure structures. There may be relations on positions in pure structures that are not extensions of relations on objects in systems.

The notion of one relation extending another offers a natural way to connect pure abstracted relations to the relations that they are abstracted from. We can simply say that an abstracted relation is an extended relation. It also suggests a first attempt at refining the notion of fundamental relation, with the goal of proving the restricted structuralist thesis that all fundamental relations are structural. For we could define a relation R on positions of a pure structure to be fundamental if and only if there is a relation Q on the objects of an isomorphic system, and R is an extension of Q. This definition is a natural way to capture the intuitive notion of a structural relation. We would like the structural relations of positions in a structure to be those that are connected to the relations on systems that exhibit that structure. The idea of extending these relations on systems shows precisely how they are connected by extending them from local extensions on systems to local extensions on structures.

However, though the proposed definition is natural, it requires further conditions on the notion of fundamental if we want to endorse the restricted structuralist thesis that all fundamental relations are structural. These further conditions are necessary, as there are relations that satisfy the proposed definition that are clearly not structural. To see this, consider an initial Kripke model with two systems, one of which is the system, v, of finite von Neumann ordinals, and the other is the system, z, of finite Zermelo ordinals, each with its usual ordering. Now take an arbitrary set-theoretic property P, for example, having exactly two members. In this case the extension of P at v has one member containing the third von Neumann ordinal: $P_v = \{\{\emptyset, \{\emptyset\}\}\}$. The extension of P at z is empty, as all finite Zermelo ordinals have exactly one member, except for the first ordinal, which has none. These systems both exhibit the structure of the natural numbers, so $\S(v) = \S(z) = \mathbb{N}$. We can extend the property P to P^*, whose local extensions at v and z are identical to those of P, and whose local extension at the pure structure $P^*_{\mathbb{N}} = \{2\}$, where 2 is the position in $\mathbb{N}$ that matches the third von Neumann and Zermelo ordinals. The extended property P^* is fundamental according to the proposed definition because it extends a property that holds of the third von Neumann ordinal. But clearly the property is not structural, as it fails to hold of the third Zermelo ordinal, and so does not hold of each matching object in every matching system. What has gone wrong is that we extended the wrong kind of property, as we simply chose some property arbitrarily. The question is: what's the right kind of property to extend? In other words, are there further conditions that we can add to the definition of fundamental property that accurately captures the structural properties?

To find a satisfactory condition, it is helpful to consider the actual examples of mathematical relations that Linnebo and Pettigrew take to be 'intuitively fundamental'. Being the additive identity in a complete ordered field is one such property (of the zero position). Being an annihilating element for multiplication in such a field is another. The list could be extended for other types of structures: being an even number or being the second successor of the zero position are fundamental properties of certain places in the natural number structure. Being a node with a certain degree, that is, having a certain number of edges incident to it, is a fundamental property of nodes in a graph structure. And so on.

What is characteristic of these properties of positions in abstract structures is that they can all be induced by abstraction from a special type of 'concrete' property of objects in the exemplifying systems. The 'concrete' properties from which one abstracts are special in the sense that they concern only the structural composition or the relational features of the systems in question. Put differently, fundamental relations always seem to be abstracted from relations dealing with (or about) the internal structure of the systems in question. The special class of relations admissible for this kind of abstraction can be characterised more precisely with Linnebo's and Pettigrew's model-theoretic presentation of mathematical systems. Recall that a relational system $S = \langle D, R_1, \ldots, R_n \rangle$ is a tuple consisting of a domain and a collection of relations on the domain. In mathematical logic, systems of this sort are models, and they are usually described by first-order (or second-order) languages with a given signature. The signature corresponding to S consists of a number of relation symbols or predicates, each of a specified arity, that can be interpreted by the primitive relations in the system. We say that a system together with a matching signature determine a formal language in which the system, as well as systems of the same logical form, can be described. Let $\mathcal{L}$ be the first-order language determined by S. In this case, we say that S is an $\mathcal{L}$-system. The language in question here consists of the first-order formulae whose relation symbols belong to the signature of S. We further assume here that the satisfaction of such formulae of $\mathcal{L}$ in a system S is defined in the usual way. In particular, if $\varphi(x_1, \ldots, x_n)$ is a formula with free variables $x_1, \ldots, x_n$ and $d_1, \ldots, d_n$ are objects in the domain of S, $S \models \varphi(d_1, \ldots, d_n)$ is taken to express the fact that $\varphi(x_1, \ldots, x_n)$ is satisfied in S relative to an assignment of d_i to x_i, for each $i \in n$.

In the formal framework we propose, a system is a world $w = \langle D_w, R_{1w}, ..., R_{jw} \rangle$ in a Kripke model, comprising a local domain D_w and local extensions $R_{1w}, ..., R_{jw}$ of the intensional relations $R_1, ..., R_j$ at w. The primitive relations, and the relations that are logically constructible from them, can be given a uniform characterisation in terms of the notion of model-theoretic definability:

Definition 2 (Definable Relation)**.** *Let $\mathcal{L}$ be a language and $w = \langle D_w, R_{1w}, ..., R_{jw} \rangle$ be an $\mathcal{L}$-system. We say that an n-ary relation R_i is definable iff there exists an $\mathcal{L}$-formula $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ and for all w there are elements $b_1, ..., b_m \in D_w$ such that for all $d_1, ..., d_n \in D_w$*

$$\langle d_1, \ldots, d_n \rangle \in R_{iw} \Leftrightarrow w \vDash \varphi(d_1, \ldots, d_n, b_1, ..., b_m).$$

The relations between elements of a given system characterised informally above are precisely the ones definable in the corresponding language $\mathcal{L}$.[18] In particular, each primitive relation R in the system is defined by its corresponding predicate in the signature of $\mathcal{L}$. Moreover, given the standard recursive definition of terms and formulas in $\mathcal{L}$, it follows that every relation on D_w that is logically constructible from these primitive relations is also definable by a formula of the language.

We can thus take the restricted class of relations from which fundamental relations should be abstracted to be the class of definable relations in the system.

Definition 3 (Fundamental Relation). *An n-ary relation R on the positions of a pure structure $\S(w)$ is fundamental iff there exists an n-ary relation Q on the objects of an $\mathcal{L}$-system w and a formula φ in $\mathcal{L}$ such that:*

(i) *Q is defined by φ, and*
(ii) *R is an extension of Q.*

Stated less formally, fundamental relations are understood here as relations of pure structures that extend definable relations of a system that exemplifies the structure in question.

As a direct consequence of this definition, the specification of fundamental properties and relations becomes strongly dependent on the logical language in use. Depending on whether one chooses a first-order or second-order language to describe the mathematical systems in question, different sets of properties will turn out to be definable in these systems. Certain properties of objects in a given system will likely not be definable in a first-order language, but only in a second-order (or even higher-order) language.

This language-relativity (or 'structural relativity') in the specification of the properties of places in a structure has been discussed in detail in (Resnik [1997]).[19] In the present context, we will not further address the issue whether this language-relativity poses a general problem for the non-eliminative structuralist. We will simply assume here that fundamental relations of a structure are always to be characterised relative to a language of a specified logical strength. We further suggest that relations are always discussed here relative to a given class of mathematical systems (or structures) of a specified signature. Thus, we say that relations are specified for systems that can be described in a given formal language $\mathcal{L}$ in the sense specified above. For instance, we think of arithmetical relations between natural numbers, such as the 'less than' relation, as specified for the context of natural number systems

[18]It should be noted that this definition concerns the definability of properties and relations of objects in systems, not the definability of Kripke frame properties, for example, the property of having a reflexive accessibility relation.

[19]In particular, for Resnik 'structural relations' of a given mathematical pattern are precisely those relations definable in a given logical language ([1997], pp.250-254).

described by the standard first-order language of Peano arithmetic. Relations so conceived can thus hold between the objects of a particular system of that class.

Our new definition of fundamental relations has three important consequences. First, it ensures that a fundamental relation R on positions in a structure is pure in Linnebo's and Pettigrew's sense and can be induced by abstraction from a relation Q on objects in a system. More precisely, condition (ii) of Definition 3 requires that the local extension of R in the structure $\S(w)$ can be abstracted from the local extension of Q in system w.

A second consequence, closely related to the first, is that the fundamental relation R, which is an extension of the relation Q, is definable by φ, the formula that also defines Q. To see this, let f be an isomorphism between w and $\S(w)$ and let $f(Q_w) = \{\langle f(x_1) \ldots, f(x_n)\rangle \in D^n_{\S(w)} \mid \langle x_1, \ldots, x_n\rangle \in Q_w\}$ be the isomorphic image of Q_w in $\S(w)$. The local extension Q_w is definable by a formula φ of language $\mathcal{L}$. Given this, one can show with a simple proof by induction on the complexity of formulas that for all $d_1, \ldots, d_n \in D_w : \varphi(d_1, \ldots, d_n)$ holds at w if and only if $\varphi(f(d_1), \ldots, f(d_n))$ holds at $\S(w)$. Hence, the relation $f(Q_w)$ can be defined by φ in $\S(w)$. Since R is the extension of Q to the pure structure $\S(w)$, it follows that $R_{\S(w)} = f(Q_w)$. The fact that definability is preserved under extensions of relations is important for the proof of the restricted structuralist thesis given in section 7.

A third consequence is that all of the intuitively fundamental relations mentioned by Linnebo and Pettigrew turn out to be fundamental according to this revised definition. Consider, for instance, properties of the positions in the natural number structure discussed above. Each of these can be induced by abstraction from a concrete property of elements in a natural number system that is definable in terms of the primitive non-logical vocabulary of the language of Peano arithmetic. The property of being an even number, for example, is clearly fundamental in this sense, since it can be abstracted from a property of numbers in a given concrete natural number system that is definable by a first-order formula '$\exists y(y + y = x)$'. At the same time, it is no longer the case that all relations between places in pure structures trivially turn out to be fundamental. Properties such as being John's favourite number fail to be fundamental since there are no definable properties from which they can be abstracted. Moreover, given our intensional understanding, such properties are no longer identified with locally co-extensional properties of positions that are fundamental according to the new definition. Given our new definition, the class of fundamental relations is thus effectively restricted to those abstract relations whose concrete counterparts express some facts about the structural composition or the internal structural content of the system from which one abstracts.

7. The structuralist thesis vindicated

We saw that Linnebo and Pettigrew ([2014]) aim to defend a restricted version of the structuralist thesis that all relations between positions in pure structures are structural. Their Purity thesis—that all fundamental relations are structural—is taken as an explication of this position. Given our new definition of fundamental relations, this thesis turns out to be not a simple 'definitional truth' but an interesting and substantial result.

Recall that a property of a position in a pure structure is structural if it holds of the isomorphic copies of this position in every system that exhibits the structure in question. Let $\S(w)$ be a structure of systems of a given mathematical type.

Definition 4 (Structural properties)**.** *A property P is a structural property of position a in the domain of $\S(w)$ iff for all systems w and for all isomorphisms f between w and $\S(w)$, we have:*

$$a \in P_{\S(w)} \Rightarrow f(a) \in P_w$$

A property of a given position in a structure turns out to be structural if it is preserved under isomorphisms in the following sense: in case the position is in the local extension of the property in the pure structure then its isomorphic copies will always belong to the property's local extensions in the instantiating systems.[20] Notice that this definition of structural properties strongly favours an intensional account of mathematical properties in the sense specified above. A property can only qualify as structural if it is understood as being applicable to different systems of a given mathematical type.

Given this account of structural properties, we can finally prove the restricted structuralist thesis that all fundamental properties of positions in pure structures are structural. We consider, for simplicity, the special case of monadic properties (though the proof of the more general result involving relations of any arity is also straightforward):

Proposition 1 (Structuralist thesis)**.** *Suppose a is a position in structure $\S(w)$. If R is a fundamental property of a in $\S(w)$, then R is a structural property of a in $\S(w)$.*

Proof. Let R be a fundamental property of position a in the domain of the pure structure $\S(w)$, where $\S(w)$ is an $\mathcal{L}$-structure in the model theoretic sense. It follows from Definition 3 that there exists a formula $\varphi(x) \in \mathcal{L}$ that defines the extension of R in $\S(w)$, and therefore $\varphi(a)$ holds at $\S(w)$. Let f be an isomorphism between the structure $\S(w)$ and a system w. One can show by induction on the complexity of formulas that $\varphi(a)$ holds at $\S(w)$ iff $\varphi(f(a))$ holds at w. Since the formula φ is taken to define the local extension of property R in each $\mathcal{L}$-system, it follows that $f(a) \in R_w$. Hence, R is invariant under isomorphisms between $\mathcal{L}$-systems and thus by Definition 4, structural. $\square$

This result shows that the restricted structuralist thesis formulated in (Linnebo and Pettigrew [2014]), with the appropriate definitions of fundamental and structural relations, can be vindicated after all: all fundamental relations between positions in pure structures turn out to be structural on our modified account.

8. Conclusion

In this paper, we outlined a consistent version of the structuralist thesis in non-eliminative mathematical structuralism. The account given here is closely based on the attempt to present non-eliminative mathematical structuralism in terms of Fregean abstraction principles by (Linnebo and Pettigrew [2014]). Specifically, we focused on their restricted version of the 'structuralist thesis', namely, their claim that all fundamental relations between the positions of an abstract structure—specifiable in terms of their Fregean abstraction principles—are structural, or invariant under isomorphisms. Linnebo and Pettigrew argue that, at least in the case of rigid mathematical structures, such as the natural and real number structures, this version of the structuralist thesis holds. To show this, they present a formal explication of the thesis, in the form of their Purity thesis for abstract structures.

[20]See also (Schiemer and Korbmacher [2017]) for a more detailed analysis of this invariance-based definition of structural properties.

Our aim in the present paper was twofold. Based on a brief discussion of their account of structural abstraction, we first showed that, even in the context of rigid structures, the Purity thesis fails under a purely extensional understanding of mathematical properties. In particular, given their definition of fundamental relations, any extensional relation between positions of a given abstract structure turns out to be fundamental. This renders the Purity thesis false given that there are properties of such positions that we would intuitively take to be nonstructural. Still worse, their Purity thesis may be inconsistent if one accepts the argument from (Burgess [1999]) against the general structuralist thesis that all properties of positions are structural.

The second aim in the paper was to show how Linnebo's and Pettigrew's Purity thesis can be made to work given their account of structural abstraction. We did so by proposing a formal framework, in terms of Kripke models, that captures a dynamic version of structural abstraction. This formal framework accommodates an intensional account of mathematical relations, which allows for an alternative definition of the notion of fundamental relations between the positions in a structure. According to this new definition, a relation between positions in a structure is fundamental if (i) it can be induced by abstraction from a definable relation between objects in a system exemplifying that structure, and (ii) is an extension of the relation from which it is abstracted. As was shown, this definition is adequate in the sense of capturing the kind of 'intuitively fundamental' properties discussed by Linnebo and Pettigrew while ruling out unintended properties and relations. Moreover, given the modified account of fundamental relations, it was shown that the restricted structuralist thesis turns out to be substantive thesis in non-eliminative and abstraction-based structuralism.

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