# Relative Benefit Equilibrating Bargaining Solution and the Ordinal Interpretation of Gauthier's Arbitration Scheme*

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**Abstract**

In 1986 David Gauthier proposed an arbitration scheme for two player cardinal bargaining games based on interpersonal comparisons of players' relative concessions. In Gauthier's original arbitration scheme, players' relative concessions are defined in terms of Raiffa-normalized cardinal utility gains, and so it cannot be directly applied to ordinal bargaining problems. In this paper I propose a relative benefit equilibrating bargaining solution (*RBEBS*) for two and n-player ordinal and quasiconvex ordinal bargaining problems with finite sets of feasible basic agreements based on the measure of players' ordinal relative individual advantage gains. I provide an axiomatic characterization of this bargaining solution and discuss the conceptual relationship between *RBEBS* and ordinal egalitarian bargaining solution (*OEBS*) proposed by Conley and Wilkie (2012). I show the relationship between the measurement procedure for ordinal relative individual advantage gains and the measurement procedure for players' ordinal relative concessions, and argue that the proposed arbitration scheme for ordinal games can be interpreted as an ordinal version of Gauthier's arbitration scheme.

## 1 Introduction

Experiments with bargaining games suggest that information concerning the distribution of the bargaining gains among real-world decision-makers plays a role in their search for mutually agreeable solutions of bargaining problems. For example, a more recent extensive experimental study carried out by Herreiner and Puppe (2009) suggests that people are willing to trade strict Pareto efficiency for a more equitable distribution of the individual bargaining gains. It is fairly obvious that such "fairness" considerations inevitably involve *some kind* of interpersonal comparison of decision-makers' individual bargaining gains. Standard inequity aversion models, such as the one proposed by Fehr and Schmidt (1999),

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can be used to represent decision-maker's aversion to inequitable distributions of material bargaining gains via appropriate transformations of decision-makers' Von Neumann and Morgenstern expected utility functions (for details, see Fehr and Schmidt 1999). They cannot, however, be used to represent decision-makers' attitudes towards interpersonal comparisons of their utility gains.

The orthodox expected utility theory does not imply the interpersonal comparability of numbers representing decision-makers' Von Neumann and Morgenstern utility (for an extensive discussion of why this is so, see Luce and Raiffa 1957). It means that any interpretation of Von Neumann and Morgenstern utility function as a function representing interpersonally comparable levels of players' welfare gains goes beyond the uses of the utility representation which can be theoretically justified in terms of the orthodox expected utility theory (see Hammond 1991 and Binmore 2009). However, Raiffa (1953) has suggested a utility function normalization procedure which allows for abstract comparisons of decision-makers' *relative utility gains.* According to this normalization procedure, the relative utility gain from a bargaining outcome can be defined as the extent by which that outcome advances the decision-maker's personal utility from his/her reference point relative to the utopia point – the bargaining outcome associated with the maximum possible decision-maker's personal utility gain. Raiffa normalization is an abstract formal procedure which allows to establish a meaningful interpersonal comparison of decision-makers' individual utility gains in bargaining problems where comparisons of numbers representing decision-makers' utility gains are not initially meaningful (Raiffa 1953, Luce and Raiffa 1957). In other words, Raiffa normalization does not require decision-makers' utility numbers to be associated with any meaningfully interpersonally comparable notion of well-being or welfare (in fact, the numbers representing payoffs need not have any substantive meaning at all for the normalization procedure to yield meaningful information), and so its application is compatible with the expected utility theory (for extensive discussion, see Luce and Raiffa 1957, Hausman 1995).

In *Morals by Agreement* (1986), Gauthier suggested an arbitration scheme based on interpersonal comparisons of decison-makers' *relative concessions* for two player bargaining problems in which the set of feasible agreements is a closed convex hull[1]. A relative concession associated with a particular bargaining outcome was defined as the difference between decision-maker's maximum possible Raiffa-normalized utility gain (utopia payoff) and the actual Raiffa-normalized utility gain associated with that bargaining outcome. Gauthier's *minimax* bargaining solution is a bargaining agreement which minimizes the maximum relative concession among the interacting decision-makers.

However, Gauthier's solution can only be applied to cardinal bargaining games, and Gauthier himself has offered no suggestions of how an arbitration scheme based on comparisons of bargainers' relative concessions could be applied to ordinal bargaining games. In this paper I suggest an arbitration

[1]That is, the set of feasible agreements is defined over a set of lotteries over the set of possible allocations of a finite resource.

scheme based on comparisons of decision-makers' ordinal relative individual benefit gains, which can be applied in ordinal and quasiconvex ordinal bargaining games with a finite set of basic feasible agreements[2]. I propose an ordinal *relative benefit equilibrating bargaining solution* (later abbreviated as *RBEBS*) for ordinal and quasiconvex ordinal two and n-player bargaining games, and provide an axiomatic characterization of this bargaining solution. Finally, I show that the *RBEBS* of quasiconvex bargaining problems is formally equivalent to Conley and Wilkie ordinal egalitarian bargaining solution for finite sets of Pareto optimal points (*OEBS*), yet the equivalence relation between *RBEBS* and *OEBS* does not hold in purely ordinal bargaining games. I also show that the *RBEBS* solution can always be defined in terms of players' ordinal relative concessions, and so the *RBEBS* can be interpreted as an ordinal version of Gauthier's *minimax* bargaining solution.

# 2 The Ordinal *RBEBS*

## 2.1 The intuition behind the procedure for the measurement of ordinal relative individual benefit gains

In every bargaining situation a rational negotiator aims to maximize his/her individual benefit gains from a bargaining agreement. Each bargainer's individual benefit gains would be maximized if his/her most preferred feasible agreement were chosen to be implemented by the interacting parties. If bargainers have conflicting preferences over the feasible agreements (as it is the case in all the standard bargaining problems where bargainers negotiate over a set of feasible distributions of some (possibly imperfectly) divisible resource), an agreement can realistically be reached if at least one bargainer makes a *concession* – agrees with the implementation of a feasible agreement which is worse than his/her most preferred feasible agreement. In other words, in order for an agreement to be reached, at least one of the bargainers needs to 'give up' demanding the implementation of *at least one* feasible agreement (for an extensive discussion of the logic of concessional bargaining, see Zhang and Zhang 2008).

In standard ordinal bargaining problems, decision-makers are assumed not to have information about each other's preferences over the lotteries over the set of feasible agreements. This assumption implies that decision-makers do not know each other's Von Neumann and Morgenstern utility functions representing their cardinal preferences over the feasible agreements[3]. However, bargainers' ordinal preferences over feasible agreements are assumed to be common knowl-

[2]That is, to ordinal and quasiconvex ordinal bargaining games where the set of feasible allocations of some (possibly imperfectly) divisible resource is finite.

[3]An ordinal representation of the bargaining problem does not imply that decision-makers' preferences over the feasible agreements cannot, at least in principle, be represented by Von Neumann and Morgenstern utility functions. It only implies that players negotiate in an *epistemic situation* where the cardinal information about preferences is not available to them, and so an arbitration scheme must be based on bargainers' ordinal preferences over the set of feasible agreements.

edge. This epistemic assumption implies that each bargainer knows every other bargainer's ordinal preferential ranking of feasible agreements, knows that every other bargainer knows this, knows that every bargainer knows that every other bargainer knows this, and so on *ad infinitum*[4]. If each decision-maker's ordinal ranking of feasible agreements is common knowledge, then each bargainer must know a subset of feasible agreements that each of the bargainers strictly prefers over any particular feasible agreement. If the set of feasible agreements is *finite*, then, for any feasible agreement $x$, every bargainer must also be able to determine the *number* of alternative feasible agreements that each of the bargainers strictly prefers over $x$. More technically, for any feasible agreement $x$, every bargainer must know every other bargainer's *cardinality of the preferred set of alternatives associated with* $x$ – the number of feasible agreements that every other bargainer strictly prefers over $x$. By letting $\mathcal{S}$ to denote a set of feasible agreements and $x, y \in \mathcal{S}$ to denote any two feasible agreements, we can define, for each bargainer, the cardinality of the preferred set of feasible agreements associated with some agreement $x \in \mathcal{S}$ in the following way:

$$c(x, \mathcal{S}) = \{|T|, \text{ where } y \in T \text{ iff } y \in \mathcal{S} \wedge y \succ x\}. \quad (1)$$

A rational bargainer always strictly prefers a feasible agreement $y$ associated with a smaller cardinality of the preferred set of alternatives to any feasible agreement $x$ associated with a larger cardinality of the preferred set of alternatives. A bargainer who aims to maximize his/her individual benefit gains thus seeks an implementation of an agreement which minimizes the cardinality of the preferred set of alternatives. Each bargainer's cardinality of the preferred set of alternatives associated with some feasible agreement $x$ can be interpreted as a *measure of the size of bargainer's concession*: It represents the number of preferred feasible agreements that the bargainer would forego if the feasible agreement $x$ were chosen to be implemented. Each bargainer's *ordinal relative concession* associated with some agreement $x$ can thus be defined as the ratio of the difference between bargainer's minimum possible concession and the concession associated with some agreement $x$ to the difference between bargainer's minimum possible concession and the maximum possible concession. By letting $c(x)$ to denote the cardinality of the preferred set of alternatives associated with agreement $x$, $c^{max}$ to denote the maximum possible cardinality of the preferred set of alternatives, and $c^{min}$ to denote the minimum possible cardinality of the preferred set of alternatives, we can define bargainer's *ordinal relative concession* associated with some feasible agreement $x$ in the following way:

$$r^o = \frac{c^{min} - c(x)}{c^{min} - c^{max}}. \quad (2)$$

Since the minimum possible cardinality of the preferred set of alternatives is associated with bargainer's most preferred feasible agreement, it is, by definition

[4]For an extensive technical discussion of how common knowledge assumptions can be formally represented by infinite belief hierarchies, see Perea 2012.

of cardinality of the preferred set of alternatives, always 0. Therefore, the ordinal relative concession definition can be simplified in the following way:

$$r^o = \frac{0 - c(x)}{0 - c^{max}} = \frac{-c(x)}{-c^{max}} = \frac{c(x)}{c^{max}}. \tag{3}$$

Since bargainer's concession associated with some agreement $x$ is the number of preferred feasible agreements that s/he would have to give up if the feasible agreement $x$ were chosen to be implemented, bargainer's ordinal individual benefit gain associated with $x$ can be defined as the number of feasible agreements that s/he would *not* have to give up if $x$ were chosen to be implemented. Bargainer's *ordinal relative individual benefit gain* associated with a feasible agreement $x$ can thus be defined as a ratio of the difference between bargainer's maximum possible concession and the concession associated with $x$ to the difference between bargainer's maximum and minimum possible concessions:

$$b^o = \frac{c^{max} - c(x)}{c^{max} - c^{min}} = \frac{c^{max} - c(x)}{c^{max} - 0} = \frac{c^{max} - c(x)}{c^{max}}. \tag{4}$$

Thus, a measurement of bargainer's ordinal concessions can be used to derive a measure of bargainers' individual benefit gains, and thus serve as a basis for an arbitration scheme based on comparisons of bargainers' ordinal individual advantage gains. *RBEBS* is based on an arbitration scheme which recommends an implementation of a feasible agreement which minimizes the difference between bargainers' ordinal relative individual benefit gains. *Since RBEBS* will, by definition, be a feasible agreement which minimizes the difference between bargainers' ordinal relative concessions, it can be viewed as an ordinal interpretation of Gauthier's *minimax* bargaining solution.

## 2.2 Formal characterization of the ordinal *RBEBS* for two player bargaining problems

Let $\mathcal{B}^o = \left(\{1,2\}, \{\mathcal{A}, d\}, \{\succeq_i\}_{i \in 1,2}\right)$ be a two player ordinal bargaining game where $\mathcal{A}$ is the set of possible agreements and $d = \{d_1, d_2\}$ is the disagreement point. Each player $i \in \{1,2\}$ has a complete, transitive and reflexive preference relation $\succeq_i$ over the set of possible agreements $\mathcal{A}$. Each agreement $x_i \in \mathcal{A}$ is a pair of demands $x_i = (g_1, g_2)$ over some amount of (possibly imperfectly) divisible resource[5], the total amount of which is $z$. Each demand $g_i$ of every

[5] Here I follow Sakovicks (2004) and Conley and Wilkie (2012) by defining the set of feasible basic agreements as a set of feasible physical allocations of some divisible resource. The proposed ordinal solution operates on assumption that a set of feasible basic allocations of resource is finite. This condition may be satisfied for two reasons. First, a resource may be imperfectly divisible, and so the players have to choose among a finite number of possible divisions of resource (e.g. a dollar can only be divided into a finite number of possible distributions). Second, the players may not be able to implement all the infinite agreements due to some external factors unrelated to the divisibility of the resource itself. The *RBEBS* solution can be applied to any ordinal game with a finite set of feasible agreements.

$i \in \{1, 2\}$ is selected from the interval $[d_i, ..., z]$. Let $\mathcal{S} \subseteq \mathcal{A}$ be a subset of feasible agreements of $\mathcal{B}^o$ which can be defined as follows:

$$\mathcal{S} = \{x_i = (g_1, g_2) \in \mathcal{S} : g_1 + g_2 = z \wedge x_i \succeq_i d \forall i \in \{1, 2\}\}. \tag{5}$$

From (5) it follows that $x_i \in \mathcal{S}$ if and only if a pair of demands $(g_1, g_2)$ is a Nash equilibrium.

For any pair $x_i = (g_1, g_2) \in \mathcal{S}$ and $x_{j \neq i} = (h_1, h_2) \in \mathcal{S}$, such that $h_i > g_i$, the preferences of $i \in \{1, 2\}$ are such that $h_i \succ_i g_i$, and so $x_j \succ_i x_i$. Since every $x_i \in \mathcal{S}$ is such that $g_1 + g_2 \in \mathcal{S}$, each player $i \in \{1, 2\}$ has a strict preference relation $\succ_i$ over $\mathcal{S} \subseteq \mathcal{A}$.

Suppose that $\mathcal{B}^o$ is such that $\mathcal{A} \neq \emptyset$ and $\mathcal{S} \subseteq \mathcal{A}$ is a finite set of agreements. In line with the standard interpretation of bargaining problems, the disagreement point will be treated as a feasible outcome, and so the set of considered feasible outcomes will be defined as $\{\mathcal{S}, d\} := \mathcal{S} \cup \{d\}$. For each $i \in \{1, 2\}$, the cardinality of the preferred set of agreements associated with some feasible agreement $x_i \in \mathcal{S}$ can be defined in the following way:

$$c_i(x_i, \{\mathcal{S}, d\}) = \{|T|, \text{ where } x_{j \neq i} \in T \Rightarrow x_j \in \{\mathcal{S}, d\} \wedge x_j \succ x_i\}. \tag{6}$$

From (6) it follows that $x_{j \neq i} \succ_i x_i$ if and only if $c_i(x_j, \{\mathcal{S}, d\}) < c_i(x_i, \{\mathcal{S}, d\})$.

Let $c_i^{max}(\{\mathcal{S}, d\}) := arg\ max_{x_i \in \{\mathcal{S}, d\}}[c_i(x_i, \{\mathcal{S}, d\})]$ be the *maximum possible concession* of player $i \in \{1, 2\}$ associated with some outcome in the set $\{\mathcal{S}, d\}$. Since, for every $i \in \{1, 2\}$, it is the case that $x_i \succeq_i d_i \ \forall x_i \in \mathcal{S}$, it follows that $c_i^{max}(\{\mathcal{S}, d\}) = c_i(d)$ for every $i \in \{1, 2\}$. Let $c_i^{min}(\{\mathcal{S}, d\}) := arg\ min_{x_i \in \{\mathcal{S}, d\}}[c_i(x_i, \{\mathcal{S}, d\})]$ be the *minimum possible concession* of player $i \in \{1, 2\}$ associated with some outcome in the set $\{\mathcal{S}, d\}$. Since every $x_i = (g_1, g_2) \in \mathcal{S}$ is such that $g_1 + g_2 = z$, it follows that agreements associated with players' minimum concessions are $(z, d_2)$ and $(d_1, z)$, and so, from the structure of $\mathcal{B}^o$ and $\{\mathcal{S}, d\}$, it follows that $c_i^{min}(\{\mathcal{S}, d\}) = 0$ for every $i \in \{1, 2\}$.

Let $r_i^o(y_i, \{\mathcal{S}, d\}) \in [0, 1]$ denote an *ordinal relative concession* of player $i \in \{1, 2\}$ associated with some feasible outcome $y_i \in \{\mathcal{S}, d\}$. Player's ordinal relative concession associated with $y_i \in \{\mathcal{S}, d\}$ can be defined as follows:

$$r_i^o(y_i, \{\mathcal{S}, d\}) = \frac{c_i^{min}(\{\mathcal{S}, d\}) - c_i(y_i, \{\mathcal{S}, d\})}{c_i^{min}(\{\mathcal{S}, d\}) - c_i^{max}(\{\mathcal{S}, d\})}. \tag{7}$$

Since $c_i^{min}(\{\mathcal{S}, d\}) = 0$ and $c_i^{max}(\{\mathcal{S}, d\}) = c_i(d)$ for every $i \in \{1, 2\}$, (7) can be simplified as follows:

$$r_i^o(y_i, \{\mathcal{S}, d\}) = \frac{-c_i(y_i, \{\mathcal{S}, d\})}{-c_i(d)} = \frac{c_i(y_i, \{\mathcal{S}, d\})}{c_i(d)}. \tag{8}$$

Let $b_i^o(y_i, \{\mathcal{S}, d\}) \in [0, 1]$ denote an *ordinal relative individual benefit gain* of player $i \in \{1, 2\}$ associated with some feasible outcome $y_i \in \{\mathcal{S}, d\}$. Since the individual benefit gain represents the number of total possible concessions that $i \in \{1, 2\}$ does not make if outcome $y \in \{\mathcal{S}, d\}$ obtains, a relative ordinal individual benefit gain of player $i \in \{1,2\}$ associated with $y_i \in \{\mathcal{S}, d\}$ can be defined as follows:

$$b_i^o\left(y_i, \{\mathcal{S}, d\}\right) = \frac{c_i^{max}\left(\{S, d\}\right) - c_i\left(y, \{\mathcal{S}, d\}\right)}{c_i^{max}\left(\{\mathcal{S}, d\}\right) - c_i^{min}\left(\{\mathcal{S}, d\}\right)}. \tag{9}$$

Since $c_i^{min}\left(\{\mathcal{S}, d\}\right) = 0$ and $c_i^{max}\left(\{\mathcal{S}, d\}\right) = c_i\left(d\right)$ for every $i \in \{1,2\}$, (9) can be simplified as follows:

$$b_i^o\left(y_i, \{\mathcal{S}, d\}\right) = \frac{c_i\left(d\right) - c_i\left(y_i, \{S, d\}\right)}{c_i\left(d\right)}. \tag{10}$$

The ordinal *RBEBS* function $\varphi^o\left(\cdot\right)$ satisfies, for every $\{\mathcal{S}, d\}$,

$$\varphi^o\left(\{\mathcal{S}, d\}\right) \in arg\ min_{x_i \in \mathcal{S}}\left[\left|\left(b_i^o\left(x_i, \{\mathcal{S}, d\}\right) - b_{j\neq i}^o\left(x_i, \{\mathcal{S}, d\}\right)\right)\right|\right]. \tag{11}$$

From (10) it follows that, for every $i \in \{1,2\}$,

$$b_i^o\left(x_i, \{\mathcal{S}, d\}\right) = 1 - \frac{c_i\left(x_i, \{\mathcal{S}, d\}\right)}{c_i\left(d\right)} \forall x \in \mathcal{S}. \tag{12}$$

From (8) and (12) it follows that, for every $i \in \{1,2\}$,

$$b_i^o\left(x_i, \{\mathcal{S}, d\}\right) = 1 - r_i^o\left(x_i, \{\mathcal{S}, d\}\right) \forall x_i \in \mathcal{S}. \tag{13}$$

In terms of ordinal relative concessions, the *RBEBS* function $\varphi^o\left(\cdot\right)$ can be defined as follows:

$$\varphi^o\left(\{\mathcal{S}, d\}\right) = arg\ min_{x_i \in \mathcal{S}}\left[\left|\left(1 - r_i^o\left(x_i, \{\mathcal{S}, d\}\right)\right) - \left(1 - r_{j\neq i}^o\left(x_i, \{\mathcal{S}, d\}\right)\right)\right|\right], \tag{14}$$

which can be simplified to

$$\varphi^o\left(\{\mathcal{S}, d\}\right) = arg\ min_{x_i \in \mathcal{S}}\left[\left|r_i^o\left(x_i, \{\mathcal{S}, d\}\right) - r_{j\neq i}^o\left(x_i, \{\mathcal{S}, d\}\right)\right|\right]. \tag{15}$$

## 2.3 Axiomatic characterization

The ordinal *RBEBS* has a number of desirable properties:

**Existence in non-trivial cases:** *RBEBS* exists in every ordinal bargaining game with at least one feasible agreement.

If $x_i \in \mathcal{A}$ is a feasible agreement, it follows that $\mathcal{S} \neq \emptyset$. Since $\varphi^o\left(\{\mathcal{S}, d\}\right) : \{\mathcal{S}, d\} \to \mathcal{P}\left(\mathcal{S}\right)$ for any $\{\mathcal{A}, d\}$, it follows that $\varphi^o = \emptyset$ in every $\mathcal{B}^o$, such that $\mathcal{S} \neq \emptyset$.

**Pareto optimality:** *RBEBS* is always a Pareto optimal agreement.

Let $S^{po} \subseteq \mathcal{A}$ denote a set of Pareto optimal agreements which can be defined as follows:

$$\mathcal{S}^{wpo} = \left\{x_i \in \mathcal{S} : \forall i \in \{1,2\}, x_i \succeq_i x_{j\neq i} \forall x_j \in \mathcal{S}\right\}. \tag{16}$$

From (16) it follows that $x_i = (g_1, g_2) \in \mathcal{S} \Rightarrow g_1 + g_2 = z$. If $g_1 + g_2 = z$, it follows that $(g_1, g_2) \equiv (g_1, z - g_1) \equiv (z - g_2, g_2)$. It follows that $\nexists h_i \in [d_1, z] : h_i > g_i \wedge z - h_i \geq z - g_i$, which implies that $\nexists h_i \in [d_1, z] : h_i \succ_i g_i \wedge z - h_i \succeq_{j \neq i} z - g_i$, and so, for any $x_i = (g_1, g_2)$, if $g_1 + g_2 = z$, then $x_i \in S^{po}$. It follows that $x \in \mathcal{S} \Rightarrow x \in \mathcal{S}^{po}$. Since $\varphi^o(\{\mathcal{S}, d\}) : \{\mathcal{S}, d\} \to \mathcal{P}(\mathcal{S})$ for any $\{\mathcal{A}, d\}$, it follows that $\varphi^o \{\mathcal{S}, d\} \in \mathcal{P}(\mathcal{S}^{po})$ for any $\{\mathcal{A}, d\}$, such that $\mathcal{S} \neq \emptyset$.

**Individual rationality:** *RBEBS* is always individually rational.

Suppose that $f^o(\cdot) : \{\mathcal{A}, d\} \to \mathcal{P}(\mathcal{A})$ is some ordinal solution function. It will satisfy the individual rationality axiom iff, for any $\{\mathcal{A}, d\} := \mathcal{A} \cup \{d\}$,

$$f^o : \{\mathcal{A}, d\} \to \mathcal{P}(\mathcal{A}) : x_i \succeq_i d_i \forall x_i \in \mathcal{P}(\mathcal{S}). \tag{17}$$

From (5) it follows that $x_i \in \mathcal{S}$ iff $x_i \succeq_i d_i \forall i \in \{1, 2\}$. Since $\varphi^o : \{\mathcal{S}, d\} \to \mathcal{P}(\mathcal{S})$ for every $\{\mathcal{A}, d\}$, it follows that, for every $\{\mathcal{A}, d\}$, $\varphi^o : \{\mathcal{S}, d\} \to \mathcal{P}(\mathcal{S}) : \forall i \in \{1, 2\}, x_i \succeq_i d_i \forall x_i \in \mathcal{P}(\mathcal{S})$.

**Invariance under additions of Pareto irrelevant agreements:** For any two ordinal bargaining problems $\mathcal{B}^o$ and $\mathcal{B}'^o$, such that $\mathcal{A}^{po} = \mathcal{A}'^{po}$, it is always the case that $\varphi^o \{\mathcal{S}, d\} = \varphi^o(\mathcal{S}', d)$.

Note that $x_i \notin \mathcal{A}^{po} \Rightarrow \exists x_{j \neq i} \in \mathcal{A} : x_j \succ_i x_i \wedge x_j \succeq_{j \neq i} x_i$. It follows that $x_i = (g_1, g_2) \in \mathcal{A}^{po} \Rightarrow g_1 + g_2 = z$ , $x_i = (g_1, g_1) \in \mathcal{S} \Rightarrow g_1 + g_2 = z$ and $x_i \in \mathcal{S}^{po} \Rightarrow x_i \in \mathcal{S} \wedge x_i \in \mathcal{A}^{po}$. It follows that $\nexists x_i \in \mathcal{A}^{po} : x_i \in \mathcal{S} \wedge x_i \notin \mathcal{S}^{po}$ and $\nexists x_i \in \mathcal{A}^{po} : x_i \notin \mathcal{S} \wedge x_i \in \mathcal{S}^{po}$, and so $\mathcal{A}^{po} = \mathcal{S}^{po} = \mathcal{S}$.

Let $\mathcal{B}^o$ and $\mathcal{B}'^o$ be two ordinal bargaining problems, such that $\mathcal{A}' \supseteq \mathcal{A}$ and $\mathcal{A}^{po} = \mathcal{A}'^{po}$. It follows that $\mathcal{S} = \mathcal{S}'$ and $\mathcal{S}^{po} = \mathcal{S}'^{po}$. Notice that $\varphi^o : \{\mathcal{S}, d\} \Rightarrow \mathcal{P}(\mathcal{S})$. Since $\mathcal{S} = \mathcal{S}^{po}$ it follows that $\varphi : \{\mathcal{S}, d\} \Rightarrow \mathcal{P}(\mathcal{S}^{po})$. Since $\mathcal{A}^{po} = \mathcal{S} = \mathcal{S}^{po} = \mathcal{A}'^{po} = \mathcal{S}' = \mathcal{S}'^{po}$, it follows that $\varphi^o \{\mathcal{S}, d\} = \varphi^o \{\mathcal{S}', d\}$.

## 2.4 Application to quasiconvex two player ordinal bargaining problems

In a two player bargaining problem where the number of feasible agreements is odd and both players have a strict preferential ordering of outcomes in $\{\mathcal{S}, d\}$, the *RBEBS is a unique* feasible agreement $x \in \mathcal{S}$, such that

$$b_1^o(x_i, \{\mathcal{S}, d\}) = b_2^o(x_i, \{\mathcal{S}, d\}). \tag{18}$$

In terms of relative ordinal concessions, $x \in \mathcal{S}$ must be such that

$$r_1^o(x_i, \{\mathcal{S}, d\}) = r_2^o(x_i, \{\mathcal{S}, d\}), \tag{19}$$

which will only be the case if

$$c_1(x_i, \{\mathcal{S}, d\}) = c_2(x_i, \{\mathcal{S}, d\}). \tag{20}$$

The *RBEBS* solution will therefore be formally equivalent to the ordinal egalitarian bargaining solution for finite sets of Pareto optimal points suggested by

| Agreement | $b_1(x_i, \{\mathcal{S}, d\})$ | $b_2(x_i, \{\mathcal{S}, d\})$ |
|---|---|---|
| $(5, 0)$ | 1 | 0 |
| $(4, 1)$ | 0.8 | 0.2 |
| $(3, 2)$ | 0.6 | 0.4 |
| $(2, 3)$ | 0.4 | 0.6 |
| $(1, 4)$ | 0.2 | 0.8 |
| $(0, 5)$ | 0 | 1 |
| $(0, 0)$ | 0 | 0 |

Table 1: Discrete Divide the Cake game with an even number of feasible agreements.

Conley and Wilkie (2012). However, the *RBEBS* solution will not be unique in bargaining problem with an even number of feasible agreements. For example, consider the discrete Divide the Cake game depicted in Table 1, in which players have to agree on how to share 5 pieces of "pie". Agreement $(0, 0)$ represents the outcome associated with a disagreement point, as well as outcome associated with a Pareto inefficient Nash equilibrium, in which both players demand 5 pieces of "pie". This bargaining problem has two feasible agreements which minimize the difference between players' ordinal relative individual benefit gains – $(3, 2)$ and $(2, 3)$. This is a case of a bargaining problem in which the available information about players' ordinal preferences over a finite set of feasible physical allocations of resource is not sufficient to define a unique solution of the bargaining problem. According to the arbitration scheme suggested by Conley and Wilkie, the players could resolve the problem with a fair lottery over the two agreements which minimize the difference between players' cardinalities of the preferred sets of alternatives (i.e. the difference between players' ordinal concessions) (for details, see Conley and Wilkie 2012). However, such an arbitration scheme requires additional information about players' attitudes towards lotteries over feasible physical allocations of resource. If the required additional information is not available to the interacting parties, an arbitration scheme involving a non-degenerate lottery (i.e. a lottery over more than one feasible physical allocation of resource) cannot be conceptually justified[6].

The ordinal *RBEBS*, however, can be applied to quasiconvex ordinal bargaining problems if players' attitudes to lotteries satisfy a number of basic axioms.

Let $L(\mathcal{S})$ denote a set of lotteries over the finite set $\mathcal{S} \subseteq \mathcal{A}$ of feasible alternative agreements (i.e. a finite set of feasible basic allocations of resource) containing $m > 0$ elements. Let $\mathcal{L} = (p_1, ..., p_m; x_1, ..., x_m)$ denote a particular lottery in $L(\mathcal{S})$, where $(p_1, ..., p_m)$ is a probability distribution over the set $\mathcal{S} = \{x_1, ..., x_m\}$ and $x_{i \in \{1,...,m\}} \in \mathcal{S}$ is some basic feasible agreement in $\mathcal{S}$. The support of each $\mathcal{L} \in L(\mathcal{S})$ can be defined in the following way:

$$Supp(\mathcal{L}) = \{x_i \in Supp(\mathcal{L}) : p(x_i) > 0\}. \tag{21}$$

[6] According to the standard interpretation of axiomatic bargaining solutions, an arbitration scheme is considered to be "fair" only if it operates on the basis of information available to the interacting parties. For discussion, see Myerson 1991, Luce and Raiffa 1957.

The goal is to provide a set of axioms representing a set of preference domain restrictions that are necessary for a construction of an ordinal representation of players' preferences over the set of basic lotteries over the finite set of feasible allocations of resource. The set of axioms suggested here is a variation of axioms which for the same purposes have been suggested by Karni and Schmeidler (1991), Grant and Kajii (1995), Dhillon and Mertens (1999) and Conley and Wilkie (2012).

**Archimedean Axiom ($\boldsymbol{AA}$):** For any triple $x_i, x_{j\neq i}, x_{k\neq j\neq i} \in \mathcal{S}$, such that $x_i \succ_i x_j \succ_i x_k$, there always exists a unique $p \in [0,1]$, such that $px_i + (1-p)\,x_k \sim_i x_j$ for every $i \in \{1,2\}$.

**Preservation of Certainty Ordering ($\boldsymbol{PCO}$):** For any pair $x_i, x_{j\neq i} \in \mathcal{S}$, such that $x_i \succ_i x_j$, and any $p \in (0,1)$, it is the case that $x_i \succ_i px_i + (1-p)\,x'_j \succ_i x_j$ for every $i \in \{1,2\}$.

**Preference Continuity ($\boldsymbol{PC}$):** For any pair $x_i, x_{j\neq i} \in \mathcal{S}$, such that $x_i \succ_i x_j$, and any pair of lotteries $\mathcal{L} = px_i + (1-p)\,x_j$ and $\mathcal{L}' = qx_i + (1-q)\,x_j$, such that $p > q$, it must be the case that $\mathcal{L} \succ_i \mathcal{L}'$ for every $i \in \{1,2\}$.

**First Order Stochastic Dominance ($\boldsymbol{FOSD}$):** For any pair $x_i, x_j \in \mathcal{S}$, such that $x_i \succ_i x_j$, and any $\mathcal{L} \in L(\mathcal{S})$, if $1 \geq p > q \geq 0$ and $\mathcal{L}' = px_i + (1-p)\,\mathcal{L}$ and $\mathcal{L}'' = qx_j + (1-q)\,\mathcal{L}$ , then $\mathcal{L}' \succ_i \mathcal{L}''$ for every $i \in \{1,2\}$.

**Quasiconvexity ($\boldsymbol{QC}$):** For any pair $\mathcal{L}, \mathcal{L}' \in \mathcal{L}(\mathcal{S})$, such that $\mathcal{L} \succ_i \mathcal{L}'$, for any $p \in [0,1]$, it is the case that $\mathcal{L} \succ_i p\mathcal{L} + (1-p)\,\mathcal{L}'$ for every $i \in \{1,2\}$.

**Strict Dominance of Basic Alternatives ($\boldsymbol{SDBA}$):** For any pair $x_i, x_{j\neq i} \in \mathcal{S}$ and any pair $\mathcal{L}, \mathcal{L}' \in L(\mathcal{S})$, such that $x_i \sim_i \mathcal{L}$ and $x_j \sim_i \mathcal{L}'$, then, for any $p \in [0,1]$, it is the case that $px_i + (1-p)\,x_j \succ_i p\mathcal{L} + (1-p)\,\mathcal{L}'$ for every $i \in \{1,2\}$.

Suppose that players' preferences satisfy $\boldsymbol{AA}$, $\boldsymbol{PCO}$, $\boldsymbol{PC}$, $\boldsymbol{FOSD}$, $\boldsymbol{QC}$ and $\boldsymbol{SDBA}$ axioms and this is common knowledge among the interacting parties. A quasiconvex ordinal bargaining problem can be defined as a triple

$$\mathcal{B}^{qo} = \left\{L(\mathcal{S}), d, \{\succeq_i\}_{i\in\{1,2\}}\right\}, \tag{22}$$

where $L(\mathcal{S})$ is the set of lotteries over $\mathcal{S} \subseteq \mathcal{A}$, $d = \{d_1, d_2\}$ is the disagreement point and $\succeq_i$ is player $i$'s preference relation over $L(\mathcal{S})$. As in the purely ordinal case, it will be assumed that the preferences of every player $i \in \{1,2\}$ are such that $\mathcal{L} \succeq_i d_i \forall \mathcal{L} \in L(\mathcal{S})$.

The set of considered outcomes will be defined as $\{L(\mathcal{S}), d\} := L(\mathcal{S}) \cup \{d\}$.

Let $x_i, x_{j\neq i} \in \mathcal{S}$ be a pair of feasible agreements, such that $x \succ_i x_j$ for player $i \in \{1,2\}$. Recall that in purely ordinal case the preferences of player $i \in \{1,2\}$ over $\mathcal{S} \subseteq \mathcal{A}$ are such that, for any pair $x_i, x_{j\neq i} \in \mathcal{S}$, $x_i \succ_i x_{j\neq i}$ if and only if $c_i(x_i, \{\mathcal{S}, d\}) < c_i(x_j, \{\mathcal{S}, d\})$.

Let $\mathcal{L} = px_i + (1-p)\,x_j$ be a basic lottery over pair $x_i, x_j \in \mathcal{S}$, where $p \in (0,1)$. From $PCO$ axiom it follows that a preference relation of $i \in \{1,2\}$

must be such that $x_i \succ_i px_i + (1-p)\,x_j \succ_i x_j$. Let $\mathcal{L}, \mathcal{L}' \in L(\mathcal{S})$ be a pair of lotteries over $x_i, x_{j \neq i} \in \mathcal{S}$ , such that $\mathcal{L} = px_i + (1-p)\,x_j$, $\mathcal{L}' = qx_i + (1-q)\,x_j$ and $1 \geq p > q \geq 0$. From $PC$ axiom it follows that $\mathcal{L} \succ_i \mathcal{L}'$ for every $i \in \{1, 2\}$. It is easy to check that this preference relation over a pair of basic lotteries will be preserved if, in terms of cardinalities of the preferred sets of alternatives, players' preferences over basic lotteries will be defined in the following way:

$$c_i\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) = pc_i\left(x_i, \{\mathcal{S}, d\}\right) + (1-p)\,c_i\left(x_j, \{\mathcal{S}, d\}\right); \tag{23}$$

$$c_i\left(\mathcal{L}', \{L(S), d\}\right) = qc_i\left(x_i, \{\mathcal{S}, d\}\right) + (1-q)\,c_i\left(x_j, \{\mathcal{S}, d\}\right). \tag{24}$$

Since

$$c_i\left(x_i, \{\mathcal{S}, d\}\right) < c_i\left(x_j, \{\mathcal{S}, d\}\right) \wedge p > q, \tag{25}$$

it follows that

$$c_i\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) < c_i\left(\mathcal{L}', \{L(\mathcal{S}), d\}\right), \tag{26}$$

and so

$$\mathcal{L} \succ_i \mathcal{L}' \forall i \in \{1, 2\}. \tag{27}$$

Thus, for each player $i \in \{1, 2\}$, the ordinal ranking of any lottery $\mathcal{L} = \{p_1, ..., p_m; x_1, ..., x_m\}$, such that $\mathcal{L} \in L(\mathcal{S})$, can be defined in terms of cardinalities of the preferred sets of alternative agreements associated with each element of the finite set $\mathcal{S} = (x_1, ..., x_m)$ (i.e. with each feasible basic allocation of a divisible resource):

$$c_i\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) = \sum_{x_i \in \{x_1, \ldots, x_m\}} p_i c_i\left(x_i, \{\mathcal{S}, d\}\right). \tag{28}$$

Let $b_i^{qo}\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) \in [0, 1]$ denote the ordinal relative individual benefit gain of player $i \in \{1, 2\}$ associated with some lottery $\mathcal{L} \in L(\mathcal{S})$ which, in terms of cardinalities of the preferred sets of alternatives, can be defined in the following way:

$$b_i^{qo}\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) = \frac{c_i(d) - c_i\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right)}{c_i(d)}. \tag{29}$$

Let $L(\mathcal{S})^{po} \subseteq L(\mathcal{S})$ denote a set of Pareto optimal lotteries over $\mathcal{S} \subseteq \mathcal{A}$ which can be defined in the following way:

$$L(\mathcal{S})^{po} = \{\mathcal{L} \in L(\mathcal{S}) : \forall i \in \{1, 2\}, \mathcal{L}(\mathcal{S}) \succeq_i \mathcal{L}'(\mathcal{S}) \forall \mathcal{L}' \in L(\mathcal{S})\}, \tag{30}$$

which, in terms of cardinalities of the preferred sets of alternative agreements, is equivalent to

$$L(\mathcal{S})^{po} = \{\mathcal{L} \in L(\mathcal{S}) : \forall i \in \{1, 2\}, c_i\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) \leq c_i\left(\mathcal{L}, \{L(\mathcal{S}), d\}\right) \forall \mathcal{L}' \in L(\mathcal{S})\} \tag{31}$$

which implies that, in terms of players' ordinal relative individual advantage gains,

$$L(\mathcal{S})^{po} = \{\mathcal{L} \in L(\mathcal{S}) : \forall i \in \{1,2\}, b_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) \geq b_i^{qo}(\mathcal{L}', \{L(\mathcal{S}), d\}) \forall \mathcal{L}' \in L(\mathcal{S})\}. \quad (32)$$

In quasiconvex ordinal bargaining problems, the *RBEBS* function $\varphi^{qo}(\cdot)$ satisfies, for every $\{L(\mathcal{S}), d\}$,

$$\varphi^{qo} \in arg\ min_{\mathcal{L} \in L(\mathcal{S})^{po}} \left[\left| b_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) - b_{j \neq i}^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\})\right|\right]. \quad (33)$$

Let $r_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) \in [0,1]$ denote the ordinal relative concession of player $i \in \{1,2\}$ associated with some lottery $L \in L(S)$. From (29) it follows that

$$r_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) = 1 - b_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) \forall \mathcal{L} \in L(\mathcal{S}). \quad (34)$$

In terms of ordinal relative concessions, the *RBEBS* function $\varphi^{qo}(\cdot)$ can be defined as follows:

$$\varphi^{qo}(\{L(\mathcal{S}), d\}) \in arg\ min_{\mathcal{L} \in L(\mathcal{S})^{po}} \left[\left| r_i(\mathcal{L}, \{L(\mathcal{S}), d\}) - r_{j \neq i}(\mathcal{L}, \{L(\mathcal{S}), d\})\right|\right]. \quad (35)$$

It is straightforward to check that *RBEBS* function for quasiconvex bargaining problems satisfies all the axioms that are satisfied by the *RBEBS* of purely ordinal bargaining problems. However, in quasiconvex ordinal bargaining problems, *RBEBS* also satisfies a specific version of symmetry axiom:

**Ex ante symmetry:** In any quasiconvex ordinal bargaining problem, *RBEBS* is a lottery, such that, *ex ante*,

$$c_i(\varphi^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\})) = c_{j \neq i}(\varphi^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\})). \quad (36)$$

Let $\wp_i : \{\mathcal{S}, d\} \to Z^+$ denote an ordinal ranking function of player $i \in \{1,2\}$ which maps every $y_i \in \{\mathcal{S}, d\}$ into a set of positive integers in the following way:

$$\wp_i(y_i, \{\mathcal{S}, d\}) = c_i(y_i, \{\mathcal{S}, d\}) + 1. \quad (37)$$

An ordinal bargaining problem $\mathcal{B}^o$ is symmetric if and only if, for every pair $h_i, g_i \in \mathcal{S}$, such that $\wp_i(h_i, \{\mathcal{S}, d\}) = \wp_{j \neq i}(g_i, \{\mathcal{S}, d\})$, it is always the case that $c_i(h_i, \{\mathcal{S}, d\}) = c_j(g_i, \{\mathcal{S}, d\})$. If an ordinal bargaining problem is symmetric, it follows that $c_i(d, \{\mathcal{S}, d\}) = c_{j \neq i}(d, \{\mathcal{S}, d\})$. Suppose that the set $\mathcal{S} = \{x_1, ..., x_m\}$ is finite. If the bargaining problem is symmetric, then the following must be the case:

$$\sum_{x_i \in \mathcal{S}} c_i(x_i, \{\mathcal{S}, d\}) = \sum_{x_i \in \mathcal{S}} c_{j \neq i}(x_i, \{\mathcal{S}, d\}) = C. \quad (38)$$

Notice that the value of function $\varphi^{qo}(\{\mathcal{S},d\})$ will be minimized when

$$b_i^{qo}(\mathcal{L},\{L(\mathcal{S}),d\}) = b_{j\neq i}^{qo}(\mathcal{L},\{L(\mathcal{S}),d\}). \tag{39}$$

This will be the case if

$$\frac{c_i(d) - \left(\sum_{x_i\in\mathcal{S}} p_i c_i(x_i,\{\mathcal{S},d\})\right)}{c_i(d)} = \frac{c_{j\neq i}(d) - \left(\sum_{x_i\in\mathcal{S}} p_i c_j(x_i,\{\mathcal{S},d\})\right)}{c_j(d)}. \tag{40}$$

Since $c_i(d) = c_{j\neq i}(d)$, it follows that an *RBEBS* must be a lottery $\mathcal{L}\in L(\mathcal{S})$, such that

$$\sum_{x_i\in\mathcal{S}} p_i c_i(x_i,\{\mathcal{S},d\}) = \sum_{x_i\in\mathcal{S}} p_i c_{j\neq i}(x_i,\{\mathcal{S},d\}) = \frac{C}{2}. \tag{41}$$

Let $c(\mathcal{S})$ denote the cardinality of $\mathcal{S}\subseteq\mathcal{A}$ which can be defined as follows:

$$c(\mathcal{S}) = \{|T|\,,\text{ where } x_i\in T\Rightarrow x_i\in\mathcal{S}\ \}. \tag{42}$$

The *RBEBS* must be a lottery $\mathcal{L}\in L(\mathcal{S})$, such that

$$\varphi^{qo}(\{\mathcal{S},d\}) \quad = \quad \begin{cases} x_i\in S:\forall i\in\{1,2\}, \\ c_i(x_i,\{\mathcal{S},d\}) = \dfrac{c(\mathcal{S})}{2}, \text{if } c(\mathcal{S}) \text{ is odd.} \\ \mathcal{L}\in L(\mathcal{S}):\forall i\in\{1,2\}, \\ c_i(\mathcal{L},\{L(\mathcal{S}),d\}) = \dfrac{c(\mathcal{S})}{2}, \text{if } c(\mathcal{S}) \text{ is even.} \end{cases} \tag{43}$$

From (37), (41), (42) and (43) it follows that

$$\varphi^{qo}(\{\mathcal{S},d\}) \quad = \quad \begin{cases} x_i\in S:\forall i\in\{1,2\}, \\ \wp_i(x_i,\{\mathcal{S},d\}) = \dfrac{c(\mathcal{S})}{2} + \dfrac{1}{2}, \text{if } c(\mathcal{S}) \text{ is odd.} \\ \mathcal{L} = \left(\dfrac{1}{2},\dfrac{1}{2},x_i,x_{j\neq i}\right)\in L(\mathcal{S}): \\ \wp_i(x_i,\{S,d\}) = \wp_{j\neq i}(x_{j\neq i}\{\mathcal{S},d\}) = \\ = \dfrac{c(\mathcal{S})}{2}, \text{if } c(\mathcal{S}) \text{ is even.} \end{cases} \tag{44}$$

Thus, in quasiconvex ordinal bargaining games the *RBEBS* is technically equivalent to *OEBS* suggested by Conley and Wilkie (for details, see Conley and Wilkie 2012).

## 2.5 Application to n-player ordinal and quasiconvex n-player ordinal bargaining problems

Let $\mathcal{B}^o = \left(I,\{\mathcal{A},d\},\{\succeq_i\}_{i\in I}\right)$ be an ordinal bargaining problem where $I = \{1,...,n\}$ is the set of $n\geq 3$ players, $\mathcal{A}$ is the set of possible agreements and $d = \{d_1,...,d_n\}$ is the disagreement point. Each player $i\in I$ has a complete, transitive and reflexive preference relation $\succeq_i$ over the set $\mathcal{A}$. Each agreement $x_i \in \mathcal{A}$ is an n-tuple of demands $x_i = (g_1, ..., g_n)$ over some (possibly imperfectly) divisible resource, the total sum of which is $z$. Each demand $g_i$ of every $i \in I$ is selected the interval $[d_i, ..., z]$. Let $\mathcal{S} \subseteq \mathcal{A}$ be a subset of feasible agreements which is defined as follows:

$$\mathcal{S} = \left\{ x = (g_1, ..., g_n) \in \mathcal{S} : \sum_{i \in I} g_i = z \wedge x_i \succeq_i d \forall i \in \{1, 2\} \right\}. \tag{45}$$

From (45) it follows that $x \in \mathcal{S}$ iff $x$ is a Nash equilibrium.

Suppose that $\mathcal{B}^o$ is such that $\mathcal{A} \neq \emptyset$ and $\mathcal{S} \subseteq \mathcal{A}$ is finite. Let $\{S, d\} := \mathcal{S} \cup \{d\}$ be the set of considered outcomes. Each player's preference relation over the set of possible outcomes is identical to the one defined in a two player case. A measure of cardinality of the preferred set of alternatives and each player's ordinal relative individual advantage gain are defined in the same way as in the two player case.

Let $\sum_{i \in I} b_i^o (x_i, \{\mathcal{S}, d\})$ be the sum of ordinal relative individual advantage gains associated with some feasible agreement $x_i \in \mathcal{S}$ of all the players in the set $I$. Suppose that $e = (e_1, ..., e_n) \in \mathcal{S}$ is a strictly egalitarian *RBEBS.* In that case, $e \in \mathcal{S}$ should have the following property:

$$b_1^o (e, \{\mathcal{S}, d\}) = b_2^o (e, \{\mathcal{S}, d\}) = ... = b_n^o (e, \{\mathcal{S}, d\}). \tag{46}$$

From (46) it follows that the following must be the case for every $i \in \{1, ..., n\}$:

$$\frac{b_1^o (e, \{\mathcal{S}, d\})}{\sum_{i \in I} b_i^o (e, \{\mathcal{S}, d\})} = \frac{b_2^o (e, \{\mathcal{S}, d\})}{\sum_{i \in I} b_i^o (e, \{\mathcal{S}, d\})} = ... = \frac{b_n^o (e, \{\mathcal{S}, d\})}{\sum_{i \in I} b_i^o (e, \{\mathcal{S}, d\})} = \frac{1}{n}. \tag{47}$$

Notice that $\sum_{i \in I} b_i^o (x_i, \{\mathcal{S}, d\}) = 1$ for any $x_i \in \mathcal{S}$, and so (47) can be simplified as follows:

$$b_1^o (e, \{\mathcal{S}, d\}) = ... = b_n^o (e, \{\mathcal{S}, d\}) = \frac{1}{n}. \tag{48}$$

It follows that $e \in \mathcal{S}$ is strictly ordinally egalitarian iff $b_i^o (e, \{\mathcal{S}, d\}) = \frac{1}{n} \; \forall i \in I$.

For any two feasible agreements $x_i \in \mathcal{S}$ and $x_{j \neq i} \in \mathcal{S}$, agreement $x_i = (g_1, ..., g_n)$ is associated with a more equitable distribution of ordinal relative individual advantage than agreement $x_j = (h_1, ..., h_n)$ iff

$$\left| b_i^o (x_i, \{\mathcal{S}, d\}) - \frac{1}{n} \right| < \left| b_i^o (x_j, \{\mathcal{S}, d\}) - \frac{1}{n} \right| \forall i \in I. \tag{49}$$

The ordinal *RBEBS* function for n-player ordinal bargaining problems $\varphi^o (\cdot)$ satisfies, for every $\{\mathcal{S}, d\}$,

$$\varphi^o (\{\mathcal{S}, d\}) \in arg\ min_{x_i \in \mathcal{S}} \left[ \left| \left( b_i^o (x_i, \{\mathcal{S}, d\}) - \frac{1}{n} \right) \right| \forall i \in I \right]. \tag{50}$$

Recall that $b_i^o (x_i, \{\mathcal{S}, d\}) = 1 - r_i^o (x_i, \{\mathcal{S}, d\}) \forall x_i \in \mathcal{S}$ for every $i \in I$. It follows that $e \in \mathcal{S}$ is a strictly egalitarian solution iff

$$1 - r_1^o (e, \{\mathcal{S}, d\}) = ... = 1 - r_n^o (e, \{\mathcal{S}, d\}), \tag{51}$$

which can be simplified to

$$r_1^o\left(e,\{\mathcal{S},d\}\right)=...=r_n^o\left(e,\{\mathcal{S},d\}\right). \tag{52}$$

Let $\sum_{i\in I} r_i^o\left(x_i,\{\mathcal{S},d\}\right)$ denote the sum of ordinal relative concessions associated with some feasible agreement $x_i\in\mathcal{S}$ of all the players in the set $I=\{1,...,n\}$. If $e\in\mathcal{S}$ is strictly egalitarian, it must be the case that

$$\frac{r_1^o\left(e,\{\mathcal{S},d\}\right)}{\sum_{i\in I} r_i^o\left(e,\{\mathcal{S},d\}\right)}=...=\frac{r_n^o\left(e,\{\mathcal{S},d\}\right)}{\sum_{i\in I} r_i^o\left(e,\{\mathcal{S},d\}\right)}=\frac{1}{n}. \tag{53}$$

Since $\sum_{i\in I} r_i^o\left(x_i,\{\mathcal{S},d\}\right)=1$ for every $x_i\in\mathcal{S}$, (53) can be simplified to

$$r_1^o\left(e,\{\mathcal{S},d\}\right)=...=r_n^o\left(e,\{\mathcal{S},d\}\right)=\frac{1}{n}. \tag{54}$$

In terms of ordinal relative concessions, the *RBEBS* function $\varphi^o(\cdot)$ can be defined as follows:

$$\varphi^o\left(\{\mathcal{S},d\}\right)=arg\ min_{x_i\in\mathcal{S}}\left[\left|r_i^o\left(x_i,\{\mathcal{S},d\}\right)-\frac{1}{n}\right|\forall i\in I\right]. \tag{55}$$

The *RBEBS* can be easily extended to quasiconvex ordinal bargaining problems with $n\geq 3$ players. Let $\mathcal{B}^{qo}=\left(I,L\left(\mathcal{S}\right),d,\{\succeq_i\}_{i\in I}\right)$ denote a quasiconvex ordinal bargaining problem where $I=\{1,...,n\}$ is the set of players, $L\left(\mathcal{S}\right)$ is the set of lotteries over $\mathcal{S}\subseteq\mathcal{A}$, $d=(d_1,...,d_n)$ is the disagreement point and $\succeq_i$ is the preference relation of $i\in I$ over the set $L\left(\mathcal{S}\right)$. As in the purely ordinal case, it is assumed that the preferences of each $i\in I$ are such that $\mathcal{L}\succeq_i d\forall\mathcal{L}\in L\left(\mathcal{S}\right)$. The set of considered outcomes will be defined as $\{L\left(\mathcal{S}\right),d\}:=L\left(\mathcal{S}\right)\cup\{d\}$.

Suppose that the preferences of every $i\in I$ satisfy ***AA***, ***PCO***, ***PC***, ***FOSD***, ***QC*** and ***SDBA*** axioms. It follows that for each player the preference relation over the set $L\left(\mathcal{S}\right)$ can be defined in terms of cardinalities of the preferred sets of alternatives associated with elements in $\mathcal{S}\subseteq\mathcal{A}$:

$$c_i\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)=\sum_{x_i\in\{x_1,...,x_m\}} p_i c_i\left(x_i,\{\mathcal{S},d\}\right). \tag{56}$$

The ordinal relative individual advantage gain of every $i\in I$ can be defined in the same way as in the two player case:

$$b_i^{qo}\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)=\frac{c_i\left(d\right)-c_i\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)}{c_i\left(d\right)}. \tag{57}$$

Let $\sum_{i\in I} b_i^{qo}\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)$ be the sum of ordinal relative individual advantage gains associated with some lottery $\mathcal{L}\in L\left(\mathcal{S}\right)$ of all the players in the set $I$. Notice that $\sum_{i\in I} b_i^{qo}\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)=1$ for every $\mathcal{L}\in L\left(\mathcal{S}\right)$. If $\mathcal{L}\in L\left(\mathcal{S}\right)$ is a strictly egalitarian solution, it must satisfy the following property:

$$b_1^{qo}\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)=...=b_n^{qo}\left(\mathcal{L},\{L\left(\mathcal{S}\right),d\}\right)=\frac{1}{n}. \tag{58}$$

It follows that $\mathcal{L} \in L(\mathcal{S})$ is strictly egalitarian iff

$$b_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) = \frac{1}{n}. \tag{59}$$

Let $L(\mathcal{S})^{po} \subseteq L(\mathcal{S})$ denote a set of Pareto optimal lotteries over $\mathcal{S} \subseteq \mathcal{A}$, which will be defined as follows:

$$L(\mathcal{S})^{po} = \{\mathcal{L} \in L(\mathcal{S}) : \forall i \in I, \mathcal{L} \succeq \mathcal{L}' \forall \mathcal{L}' \in L(\mathcal{S})\}. \tag{60}$$

The *RBEBS* function for quasiconvex n-player ordinal bargaining problems $\varphi^{qo}(\cdot)$ satisfies, for every $\{L(\mathcal{S}), d\}$,

$$\varphi^{qo}(\{L(\mathcal{S}), d\})\, arg\ min_{\mathcal{L} \in L(\mathcal{S})^{po}} \left[ \left| b_i^{qo}(\mathcal{L}, \{L(\mathcal{S}), d\}) - \frac{1}{n} \right| \forall i \in I \right]. \tag{61}$$

# 3 Conclusion

Gauthier's *minimax* bargaining solution is based on assumption that bargainers care not only about their absolute individual benefit gains associated with a particular agreement, but also about how that agreement is reached, and how the benefits of a bargaining agreement are distributed among them. In real-world bargaining problems an agreement can realistically be reached only if at least one of the bargainers makes a concession – agrees to play a part in implementing an agreement which is worse than his/her most preferred agreement (i.e. utopia agreement). Each of the feasible agreements is associated with a specific combination of concessions that bargainers would have to make in order to reach an agreement to implement it. Therefore, even if every feasible agreement is mutually advantageous, the interacting decision-makers may not see them as equally "fair": Some of the agreements may require one interacting party to make a much larger concession than other interacting parties, while other agreements may be associated with a relatively equitable distribution of concessions among the interacting parties. If the interacting parties care about the distribution of concessions, they may evaluate the "fairness" of the agreement on the basis of interpersonal comparisons of concessions. Gauthier's *minimax* bargaining solution is based on assumption that interacting parties accept an arbitration scheme which recommends a minimization of the difference between the relative concessions of interacting parties. Each player's relative concession associated with a particular agreement is defined as the difference between player's Raiffa-normalized maximum possible payoff (utopia payoff) and the Raiffa-normalized payoff associated with the agreement. A *minimax bargaining solution minimizes* the maximum relative concession among the interacting parties, thus selecting an agreement which requires equal concessions from the interacting decision-makers.

In ordinal bargaining games, interacting decision-makers have limited information about each other's preferences over feasible agreements. As a result, a development of a compelling arbitration scheme for ordinal bargaining problems poses significant conceptual challenges. However, as has been argued in this paper, the ordinal information about decision-makers' preferences is sufficient to define a very basic measure of relative concessions, based on a simple counting of players' preferred agreements. Therefore, purely ordinal information about preferences is sufficient for an arbitration scheme which recommends the minimization of the difference of players' ordinal relative concessions, and so an ordinal interpretation of Gauthier's arbitration scheme is possible.

Gauthier's arbitration scheme seems to be psychologically compelling: In the absence of any information about players' cardinal preferences over outcomes, a measure of players' ordinal concessions seems to be one of the few procedures that could be used to determine the "fairness" of the achieved agreement. Therefore, it seems natural to expect the decision-makers aiming to find an agreement with the most equitable distribution of concessions to accept an arbitration scheme based on the principles roughly similar to the ones outlined in this paper.

As has been shown in this paper, the *RBEBS* satisfies a number of axioms which, at least intuitively, should be satisfied by any credible ordinal bargaining solution: Pareto optimality, individual rationality and invariance under additions of Pareto irrelevant alternatives. In quasiconvex symtric ordinal bargaining problems, the *RBEBS* also satisfies the *ex ante* symmetry axiom. Therefore, in quasiconvex symmetric bargaining problems, the *RBEBS* is, in terms of the symmetry property, equivalent to the *OEBS* bargaining solution for finite sets of Pareto optimal points suggested by Conley and Wilkie (2012). However, unlike *OEBS*, the *RBEBS* does not satisfy the symmetry axioms in purely ordinal games, since the arbitration scheme suggested in this paper does not justify the use of lotteries in situations where information about decision-makers' attitudes towards lotteries is not available to the interacting parties. Further study of the relationships between *RBEBS* and other ordinal bargaining solutions, as well as a study of how the proposed arbitration scheme could be applied to ordinal bargaining games with infinite sets of feasible agreements seem like viable directions of future research.

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