Maxwell gravitation

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Abstract

This paper gives an explicit presentation of Newtonian gravitation on the backdrop of Maxwell spacetime, giving a sense in which acceleration is relative in gravitational theory. However, caution is needed: assessing whether this is a robust or interesting sense of the relativity of acceleration depends upon some subtle technical issues, and upon substantive philosophical questions over how to identify the spacetime structure of a theory.

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1 Introduction

The following two observations are well-known to philosophers of physics:

1. Newtonian gravitation admits, in addition to the well-known velocity-boost and potential-shift symmetries, a "gravitational gauge symmetry" in which the gravitational field is altered.

2. Newtonian gravitation may be presented in a "geometrised" form known as Newton-Cartan theory,[1] in which the dynamically allowed trajectories are the geodesics of a non-flat connection.

Moreover, it is widely held that these two observations are intimately related. However, aspects of this relationship remain somewhat obscure. In particular, there is widespread disagreement over the sense in which the symmetry of observation 1 motivates the move from a non-geometrised formulation to the geometrised formulation of observation 2; and over the extent to which such motivation ought to be regarded as analogous to the use of the velocity-boost symmetry to motivate the move from Newtonian to Galilean spacetime, or to the use of the potential-shift symmetry to motivate the move from a formulation in terms of gravitational potentials to a formulation in terms of gravitational fields.

In this paper, I seek to clarify this relationship. First, I consider the symmetry from point 1 above, in the context of Newtonian gravitation set on Galilean spacetime. I then briefly review the geometrised formulation of the theory, and discuss some puzzling aspects concerning the relativity of acceleration. This motivates an exploration of Maxwell spacetime, and the presentation of a Newtonian theory of gravitation set on Maxwell spacetime. I then look at how this theory relates to Newton-Cartan theory, and explore how this illuminates the conceptual issues with which we began.

[1] Due originally to Trautman (1965).

2 Galilean gravitation

I will assume familiarity with the differential-geometric architecture standardly used to present classical gravitational theories.[2] All the theories we will consider postulate at least as much structure as that of Leibnizian spacetime, which comprises data $\langle M, t_a, h^{ab}\rangle$: here, M is a differential manifold which is diffeomorphic to $\mathbb{R}^4$; t_a is a smooth, curl-free 1-form; and h^{ab} is a smooth, symmetric rank-$(0,2)$ tensor, of signature $(0,+,+,+)$. t_a and h^{ab} are orthogonal, i.e., they satisfy

$$t_a h^{ab} = 0 \tag{1}$$

Given our topological assumptions, t_a induces a foliation of M into three-dimensional hypersurfaces; we require that each such hypersurface is diffeomorphic to $\mathbb{R}^3$. h^{ab} induces a three-dimensional metric on each hypersurface. We require that each hypersurface is complete relative to this induced metric, and that the induced metric is flat.[3] We will use L to denote a Leibnizian spacetime. If $L = \langle M, t_a, h^{ab}\rangle$ is a Leibnizian spacetime, then a connection ∇ on M is said to be *compatible* with L just in case it satisfies

$$\nabla_a t_b = 0 \tag{2a}$$

$$\nabla_a h^{bc} = 0 \tag{2b}$$

We will only consider compatible connections in this paper.

A *Galilean spacetime* is a Leibnizian spacetime equipped with a flat (compatible) connection. The first theory we will consider is that of Newtonian gravitation on Galilean spacetime—for short, "Galilean gravitation". Each model of such a theory comprises the following data:

- A Galilean spacetime $\langle L, \nabla\rangle$

[2]See Friedman (1983), Earman (1989), and—especially—Malament (2012).
[3]For more detail on the above, see (Malament, 2012, §4.1).

- A spacelike vector field G^a
- A rank-(2, 0) tensor field T^{ab}

satisfying the following equations:

$$\nabla_a G^a = -4\pi\rho \tag{3a}$$

$$\nabla^{[c} G^{a]} = 0 \tag{3b}$$

$$\nabla_n T^{na} = \rho G^a \tag{3c}$$

where $\rho = T^{ab} t_a t_b$.

The vector field G^a represents the gravitational field, and the tensor field T^{ab} represents the mass-momentum of whatever matter or fields are present (with the scalar field ρ representing the mass density). I have chosen to work with a gravitational field, related to the mass density by the source equation (3a), rather than with a gravitational potential. This is simply in order to remove the gauge symmetries of the potential, so that we can focus on those symmetries that alter the field itself. Equation (3b), the condition that the gravitational field is *twist-free*, ensures that this decision is harmless: given our assumptions about the topology of L, it holds of G^a if and only if there is a scalar field φ such that $G^a = -\nabla^a \varphi$.[4] Finally, equation (3c) encodes the dynamics of the matter (both gravitational and non-gravitaitonal).

To illuminate this last remark, note that wherever $\rho \neq 0$, we can decompose T^{ab} by defining[5]

$$\xi^a = \rho^{-1} T^{ab} t_b \tag{4a}$$

$$\sigma^{ab} = T^{ab} - \rho \xi^a \xi^b \tag{4b}$$

[4] See (Malament, 2012, Proposition 4.1.6). Note that this is analogous to the role played by the equation $\nabla \times \mathbf{E} = 0$ in electrostatics.

[5] The below follows (Malament, 2012, pp. 265–266).

so that

$$T^{ab} = \rho\xi^a\xi^b + \sigma^{ab} \tag{5}$$

ξ^a is a unit, future-directed timelike field (interpretable as the net motion of the matter) and σ^{ab} is a symmetric field spacelike in both indices (interpretable as the stress tensor for the matter). Equation (3c) then holds if and only if the equations

$$\rho\nabla_a\xi^a + \xi^a\nabla_a\rho = 0 \tag{6a}$$

$$\rho\xi^a\nabla_a\xi^b = \rho G^b - \nabla_a\sigma^{ab} \tag{6b}$$

hold. Thus, in the presence of mass, equation (3c) encodes both a continuity equation (6a) and an equation of motion (6b). Given a model of Galilean gravitation, we will refer to the integral curves of ξ^a as the *dynamical trajectories*: so the dynamical trajectories undergo an acceleration due to the gravitational field, and due to the non-gravitational forces encoded by the stress tensor. Obviously, in a realistic application one would impose further equations on T^{ab}, capturing the details of the non-gravitational dynamics. The theory (3) is only intended to provide a framework for analysing theories involving gravitation, at a reasonably high level of generality (whilst nevertheless including an explicit representation of the mass-momentum).

It will be helpful to have a term for a structure $\langle L, \nabla, G^a, T^{ab}\rangle$ which does not necessarily satisfy equations (3).[6] We will refer to such a structure as a *model-candidate* for Galilean gravitation. The metaphysically inlined may think of model-candidates as representing worlds which are metaphysically possible according to Galilean gravitation (they contain the right ontological ingredients), and of models as representing worlds which are physically possible according to Galilean gravitation (they contain the right ontological ingredients, arranged in the right way).

Our concern in this paper is with a certain transformation one can make

[6] That is, what in e.g. Belot (2007) is referred to as a “kinematical possibility”.

of the models of this theory—specifically, one obtained by altering the connection and gravitational field as follows:

$$\nabla \mapsto \nabla' = (\nabla, \eta^a t_b t_c) \tag{7a}$$

$$G^a \mapsto G'^a = G^a - \eta^a \tag{7b}$$

where η^a is any spacelike vector field such that $\nabla^a \eta^b = 0$. The notation $(\nabla, \eta^a t_b t_c)$ follows Malament: (Malament, 2012, Proposition 1.7.3) shows that given any connection ∇ on a manifold M, any other connection ∇' may be expressed in the form (∇, C^a_{bc}) (for some symmetric tensor field C^a_{bc}), meaning that for any tensor field $T^{a_1 \ldots a_r}_{b_1 \ldots b_s}$ on M:

$$\begin{aligned} \nabla'_c T^{a_1 \ldots a_r}_{b_1 \ldots b_s} = & \nabla_c T^{a_1 \ldots a_r}_{b_1 \ldots b_s} \\ & - C^{a_1}_{cn} T^{n a_2 \ldots a_r}_{b_1 \ldots b_s} - \cdots - C^{a_r}_{cn} T^{a_1 \ldots a_{r-1} n}_{b_1 \ldots b_s} \\ & + C^n_{cb_1} T^{a_1 \ldots a_r}_{n b_2 \ldots b_s} + \cdots + C^n_{cb_s} T^{a_1 \ldots a_r}_{b_1 \ldots b_{s-1} n} \end{aligned} \tag{8}$$

It is straightforward to show that the transformation (7) is a symmetry of Galilean gravitation, in the following sense: if $\nabla' = (\nabla, \eta^a t_b t_c)$ and $G'^a = G^a - \eta^a$ are substituted into the equations (3), we get the same equations out again (and if ∇ is flat, then so is ∇'). Consequently, any model-candidate $\langle L, \nabla, G^a, T^{ab} \rangle$ is a model of Galilean gravitation if and only if $\langle L, \nabla', G'^a, T^{ab} \rangle$ is also a model of Galilean gravitation.

Now, if we read the theory literally, then these two models would appear to represent distinct possibilities (since the two models are not isomorphic to one another). That is, if all the mathematical structures present in the models are taken to represent physical structure, then the two models disagree over what the world is like: they disagree over the magnitude of the gravitational field, for instance, and over the acceleration of matter. Yet this is a problematic judgment, since it seems that two such possibilities would be epistemically indistinguishable from one another: all seemingly observationally accessible quantities, such as relative distances, are the same in the two models. Such epistemic underdetermination gives us some reason to think that we should seek another theory which, read literally, does not give rise to such a problem (whilst still capturing the "good" content of Galilean gravitation, i.e., the content that is invariant under (7)).[7]

3 Newton-Cartan gravitation

The standard view is that such a theory is provided by *Newton-Cartan gravitation*. Let us say that a *Newton-Cartan connection*, for a given Leibnizian spacetime, is a (compatible) connection $\widetilde{\nabla}$ whose curvature tensor $\widetilde{R}^a{}_{bcd}$ obeys the *homogeneous Trautman conditions*:

$$\widetilde{R}^{ab}{}_{cd} = 0 \tag{9a}$$

$$\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b} \tag{9b}$$

and that a *Newton-Cartan spacetime* consists of a Leibnizian spacetime L together with a Newton-Cartan connection for L. Note that all flat connections obey the conditions (9), and so are Newton-Cartan connections; as such, Galilean spacetime is a Newton-Cartan spacetime. A model of Newton-Cartan gravitation then comprises

- A Newton-Cartan spacetime $\langle L, \widetilde{\nabla} \rangle$
- A tensor field T^{ab}

[7]The above kind of argument is an instance of a more general one: the claim that that the differences between symmetry-related models of a theory are (in some sense) not differences that should be taken seriously, and which should motivate us either to interpret the theory in such a way that it is not committed to that structure, or to replace the theory by a more parsimonious one (for discussion, see Møller-Nielsen (2016)). However, it is controversial both how exactly the notion of "symmetry" should be defined, and how (or whether) this general interpretational maxim should apply (see Saunders (2003), Brading and Castellani (2003), Baker (2010), Dewar (2015), Caulton (2015), Dasgupta (2016), and references therein). Since the general debate is tangential to our purposes, I pass over it here.

such that the following equations hold:

$$\widetilde{R}_{bc} = 4\pi\rho t_b t_c \tag{10a}$$

$$\widetilde{\nabla}_n T^{na} = 0 \tag{10b}$$

where the Ricci tensor $\widetilde{R}_{bc} = \widetilde{R}^a{}_{bca}$ and (as before) $\rho = T^{ab}t_a t_b$.

Thus, the source equation (10a) relates the mass density to the curvature of spacetime, rather than to the gravitational field. If we have $\rho \neq 0$, then we can decompose T^{ab} as in equation (5) to obtain

$$\rho\widetilde{\nabla}_a\xi^a + \xi^a\widetilde{\nabla}_a\rho = 0 \tag{11a}$$

$$\rho\xi^a\widetilde{\nabla}_a\xi^b = -\widetilde{\nabla}_a\sigma^{ab} \tag{11b}$$

So the continuity equation (11a) is unchanged, but the equation of motion (11b) only features acceleration due to non-gravitational forces: the gravitational acceleration has been "absorbed" into the curved Newton-Cartan connection.

The relationship between Galilean gravitation and Newton-Cartan gravitation is captured in what are known as the *geometrisation* and *recovery* theorems.[8] The former states that from any model of Galilean gravitation, one can obtain a unique model of Newton-Cartan gravitation: namely, that given by taking $\widetilde{\nabla} = (\nabla, G^a t_b t_c)$. Note that two models of Galilean gravitation which are related by the transformation (7) will generate the same model of Newton-Cartan gravitation. The latter asserts that given a model of Newton-Cartan gravitation, there is a model of Galilean gravitation related to it by $\widetilde{\nabla} = (\nabla, G^a t_b t_c)$ for some twist-free spacelike field G^a; several models, in fact, corresponding to different choices of G^a (and all related to one another by transformations of the form (7)). It is in this sense that Newton-Cartan gravitation captures the invariant content of Galilean gravitation: there is a systematic one-to-one correspondence between models of

[8]See (Malament, 2012, Propositions 4.2.1, 4.2.5), Trautman (1965).

Newton-Cartan gravitation and equivalence classes of (7)-related models of Galilean gravitation.

At the same time, however, there is something potentially puzzling about this case. As mentioned above, the acceleration of the matter represented by ξ^a is not invariant under the transformations (7). If models related by such a transformation correspond to the same physical situation, then the natural reading would seem to be that accelerations are not a real, or objective, or absolute feature of the world (according to Newtonian gravitational theory). This notion is supported by reflection on the transition from setting Newtonian gravitation on Newtonian spacetime (wherein there is a standard of absolute rest) to setting it on Galilean spacetime. Here, we observe that applying a "boost" transformation is a symmetry of the dynamics. In Newtonian spacetime, trajectories have (absolute) velocities, relative to absolute space; but those velocities are not invariant under boosts. This is generally taken to licence the claim that such velocities are not real, or objective, or absolute features of the world (according to the best interpretation of the theory). This claim is supported by the fact that we can set the theory instead on Galilean spacetime, in which there is not the structure required to impute absolute velocities to trajectories. So if this transition involves the repudiation of absolute velocities (since they are not invariant under boosts), analogous reasoning would suggest that the move from Galilean gravitation to Newton-Cartan gravitation should involve the repudation of absolute accelerations (since they are not invariant under (7)).

However, the orthodox view is that this is decisively *not* the case. The reason for this is straightforward: any model of Newton-Cartan gravitation *does* have enough structure to make pronouncements on the accelerations of trajectories, since it contains a privileged connection $\tilde{\nabla}$. As such, in transitioning from Galilean to Newton-Cartan gravitation,

> We eliminate the notions of absolute acceleration and rotation relative to ∇, but we replace them with new notions of absolute

> acceleration and rotation relative to $\widetilde{\nabla}$. Hence, the move from [Galilean gravitation] to [Newton-Cartan gravitation] does not involve a relativization of acceleration parallel to the relativization of velocity [...][9]

Here is another way of expressing the idea that Newton-Cartan spacetime is just as committed to absolute acceleration as Galilean spacetime was: the Newton-Cartan connection is not invariant under a transformation of the form (7a).[10] So let us consider what kind of structure is so invariant.

4 Maxwell gravitation

Given a Galilean spacetime $\langle L, \nabla \rangle$, the structure that is invariant under a transformation of the form (7a) goes by the moniker of *Maxwell spacetime*.[11] Intuitively, the idea is that a Maxwell spacetime contains a "standard of rotation", but no "standard of acceleration". More precisely,[12] we say that a pair of connections ∇ and ∇' compatible with a given Leibnizian spacetime L are *rotationally equivalent* if, for any unit timelike field θ^a on L, $\nabla^{[a}\theta^{b]} = 0$ iff $\nabla'^{[a}\theta^{b]} = 0$. Then, a *Maxwell spacetime* comprises

- A Leibnizian spacetime L
- A *standard of rotation* $[\nabla]$: an equivalence class of rotationally equivalent flat affine connections (compatible with L)

The following proposition demonstrates the invariance of Maxwell spacetime under (7a):

[9] (Friedman, 1983, p. 122); I have modified Friedman's notation to fit with that used in this paper.

[10] The question of whether it is invariant under a transformation of the form (7) is rather more subtle.

[11] (Earman, 1989, chap. 2)

[12] This definition follows Weatherall (2015).

Proposition 1. Let $\langle L, [\nabla]\rangle$ be a Maxwell spacetime, and consider any $\nabla \in [\nabla]$. For any other flat connection ∇', $\nabla' \in [\nabla]$ (i.e. ∇' is rotationally equivalent to ∇) iff $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^a \eta^b = 0$.

Proof. The "if" direction is straightforward: if $\nabla' = (\nabla, \eta^a t_b t_c)$, then

$$\begin{aligned}\nabla'^{[a}\theta^{b]} &= \nabla^{[a}\theta^{b]} - t_n t_k \theta^k h^{n[a}\eta^{b]} \\ &= \nabla^{[a}\theta^{b]}\end{aligned}$$

and so ∇ and ∇' are rotationally equivalent.

The "only if" direction follows immediately from the proof of Proposition 3 in Weatherall (2015). □

So given a pair of models of Galilean gravitation related by (7), the structure shared by their Galilean spacetimes $\langle L, \nabla\rangle$ and $\langle L, \nabla'\rangle$ is that of their common Maxwell spacetime $\langle L, [\nabla]\rangle$.

Recently, Saunders (2013) has queried whether we really should regard Newton-Cartan theory as the spacetime theory that properly encodes the lessons of the symmetry canvassed above: he argues that we can "interpret [Newton's] laws [. . .] directly as concerning the relative motions of particle pairs",[13] and hence, as describing a theory set on Maxwell spacetime rather than Galilean spacetime.[14] Saunders' analysis concerns the point-particle formulation of Newtonian gravitation, but he continues:

> There remain important questions, above all, moving over to a manifold formulation: What is the relation between a theory of gravity (and other forces) formulated in Maxwell space-time and one based on Newton-Cartan space-time?[15]

[13](Saunders, 2013, p. 41)

[14]Strictly, against the backdrop of a spacetime structure equivalent to it, which Saunders refers to as "Newton-Huygens spacetime".

[15](Saunders, 2013, p. 46)

Obviously, assessing that relationship requires us to first present such a theory set on Maxwell spacetime.

Without further ado, then, a model of *Maxwell gravitation* comprises

- A Maxwell spacetime $\langle L, [\nabla] \rangle$
- A tensor field T^{ab}

such that the following equations hold wherever $\rho \neq 0$:

$$t_a \nabla_n T^{na} = 0 \tag{12a}$$

$$\nabla_a(\rho^{-1} \nabla_n T^{na}) = -4\pi\rho \tag{12b}$$

$$\nabla^c(\rho^{-1} \nabla_n T^{na}) - \nabla^a(\rho^{-1} \nabla_n T^{nc}) = 0 \tag{12c}$$

where ∇ is an arbitrary element of $[\nabla]$. Moreover, we also require that if there are regions of L in which $\rho = 0$, then the quantity $\rho^{-1}\nabla_n T^{na}$ converges as such a region is approached.

This is only well-specified if the choice of ∇ is indeed arbitrary. The following proposition shows that this is, indeed, the case.

Proposition 2. Let $\langle L, [\nabla], T^{ab} \rangle$ be a model-candidate for Maxwell gravitation, and consider any $\nabla, \nabla' \in [\nabla]$. Then the equations (12) hold with respect to ∇ iff they hold with respect to ∇'.

Proof. By Proposition 1, $\nabla' = (\nabla, \eta^a t_b t_c)$, for some spacelike field η^a such that $\nabla^a \eta^b = 0$. It follows that

$$\nabla'_n T^{na} = \nabla_n T^{na} - \rho\eta^a \tag{13}$$

First, from equation (13)

$$t_a \nabla'_n T^{na} = t_a \nabla_n T^{na} \tag{14}$$

so equation (12a) holds with respect to ∇ iff it holds with respect to ∇'.

Second, we find that

$$\begin{aligned}\nabla'_a(\rho^{-1}\nabla'_n T^{na}) &= \nabla'_a(\rho^{-1}\nabla_n T^{na} - \eta^a)\\ &= \nabla_a(\rho^{-1}\nabla_n T^{na} - \eta^a) - \eta^a t_a t_r(\rho^{-1}\nabla_n T^{nr} - \eta^r)\\ &= \nabla_a(\rho^{-1}\nabla_n T^{na}) - \nabla_a \eta^a\end{aligned}$$

Since $\nabla^a \eta^b = 0$, $\nabla_a \eta^b = t_a \theta^n \nabla_n \eta^b$, where θ^n is any future-directed unit timelike field; it follows that $\nabla_a \eta^a = 0$.[16] So (12b) holds with respect to ∇ iff it holds with respect to ∇'.

Finally,

$$\begin{aligned}\nabla'^c(\rho^{-1}\nabla'_n T^{na}) &= \nabla'^c(\rho^{-1}\nabla_n T^{na} - \eta^a)\\ &= \nabla^c(\rho^{-1}\nabla_n T^{na} - \eta^a) - h^{dc}\eta^a t_d t_e(\rho^{-1}\nabla_n T^{ne} - \eta^e)\\ &= \nabla^c(\rho^{-1}\nabla_n T^{na})\end{aligned}$$

And so equation (12c) also holds with respect to ∇ iff it holds with respect to ∇'. □

As with the two previous theories, wherever $\rho \neq 0$ we can decompose T^{ab} using (5). It is then straightforward to show that (12a) holds iff

$$\rho \nabla_a \xi^a + \xi^a \nabla_a \rho = 0 \qquad (15)$$

does—i.e., the continuity equation carries over.

There is not a straightforward analogue of (6b) or (11b) for Maxwell gravitation (which is to be expected, given that Maxwell spacetime lacks an absolute standard of acceleration). However, we can show that Maxwell gravitation determines the *relative* acceleration of the dynamical trajectories. That is, given a unit timelike vector field θ^a on a Maxwell spacetime $\langle L, [\nabla]\rangle$, let λ^a be a *connecting field* for θ^a: a spacelike vector field such that $\mathcal{L}_\theta \lambda^a = 0$

[16]This observation is adapted from (Malament, 2012, p. 277).

(where $\mathcal{L}_\theta$ denotes the Lie derivative along θ^a). Intuitively, we think of λ^a as joining integral curves of θ^a to "neighbouring" integral curves. The relative acceleration of such neighbouring curves is then given by

$$\theta^n \nabla_n (\theta^m \nabla_m \lambda^a) \tag{16}$$

and has radial component (magnitude in the direction of λ^a)

$$\lambda_a \theta^n \nabla_n (\theta^m \nabla_m \lambda^a) \tag{17}$$

where $\lambda_a = \hat{h}_{ab}\lambda^b$, for $\hat{h}_{ab}$ the spatial metric associated to θ^a.[17] These expressions are easily shown to be independent of the choice of $\nabla \in [\nabla]$, but they do depend on λ^a. If, however, we introduce three connecting fields $\overset{1}{\lambda}{}^a, \overset{2}{\lambda}{}^a, \overset{3}{\lambda}{}^a$ which are orthonormal to one another, then we can define the *average radial acceleration* of θ^a as the average of the three radial components,

$$A_\theta := \frac{1}{3} \sum_{i=1}^{3} \overset{i}{\lambda}_a \theta^n \nabla_n (\theta^m \nabla_m \overset{i}{\lambda}{}^a) \tag{18}$$

It can then be shown that the average radial acceleration is independent of the choice of connecting fields $\overset{i}{\lambda}{}^a$; indeed, we have

Proposition 3. Let θ^a be a unit timelike field on some Maxwell spacetime $\langle L, [\nabla] \rangle$, and suppose that $\{\overset{i}{\lambda}{}^a\}_i$ are three orthonormal spacelike fields such that $\mathcal{L}_\theta \overset{i}{\lambda}{}^a = 0$. Then for any $\nabla \in [\nabla]$,

$$A_\theta = \frac{1}{3} \nabla_a (\theta^n \nabla_n \theta^a) \tag{19}$$

Proof. First, some straightforward algebra shows that for any connecting

[17] In fact, given that λ^a is spacelike, we could have used the spatial metric associated to any unit timelike field; but since we have a particular such field knocking around, it is helpful to fix on it.

field λ^a,[18]

$$\theta^n \nabla_n (\theta^m \nabla_m \lambda^a) = \lambda^m \nabla_m (\theta^n \nabla_n \theta^a) \tag{20}$$

Since the connecting fields are orthonormal,[19]

$$\sum_i \overset{i}{\lambda}_a \overset{i}{\lambda}{}^c = \delta_a{}^c - t_a \theta^c \tag{21}$$

Therefore,

$$\begin{aligned}
A_\theta &= \frac{1}{3} \sum_{i=1}^{3} \overset{i}{\lambda}_a \theta^n \nabla_n (\theta^m \nabla_m \overset{i}{\lambda}{}^a) \\
&= \frac{1}{3} \sum_i \overset{i}{\lambda}_a \overset{i}{\lambda}{}^c \nabla_c (\theta^n \nabla_n \theta^a) \\
&= \frac{1}{3} (\delta_a{}^c - t_a \theta^c) \nabla_c (\theta^n \nabla_n \theta^a) \\
&= \frac{1}{3} \nabla_a (\theta^n \nabla_n \theta^a)
\end{aligned}$$

□

Now observe that, given equation (15),

$$\nabla_n (\rho \xi^n \xi^a + \sigma^{na}) = \rho \xi^n \nabla_n \xi^a + \nabla_n \sigma^{na} \tag{22}$$

It follows that if T^{ab} obeys equation (12b), then

$$A_\xi = -\frac{4}{3} \pi \rho - \frac{1}{3} \nabla_a (\rho^{-1} \nabla_n \sigma^{na}) \tag{23}$$

In other words, Maxwell gravitation specifies the relative acceleration of trajectories (and characterises them as having both a gravitational and non-gravitational component).

[18] The calculation is just an adaptation of the proof of (Malament, 2012, Proposition 1.8.5) to the case where θ^a is not a geodesic and ∇ is flat.

[19] (Malament, 2012, Equation 4.1.12)

5 Comparing Maxwell gravitation and Newton-Cartan gravitation

We now consider the relationship between Maxwell gravitation and Newton-Cartan gravitation. First, we say that a connection is *compatible* with a given Maxwell spacetime if it is compatible with the Leibnizian substructure of the Maxwell spacetime, and rotationally equivalent to the members of $[\nabla]$. We now prove an intermediate proposition, giving the relationship between different Newton-Cartan connections compatible with a given standard of rotation.

Proposition 4. Let $\langle L, [\nabla]\rangle$ be a Maxwell spacetime, and let $\widetilde{\nabla}$ be any Newton-Cartan connection compatible with $[\nabla]$. Then for any other connection $\widetilde{\nabla}'$, $\widetilde{\nabla}'$ is a Newton-Cartan connection compatible with $[\nabla]$ if and only if $\widetilde{\nabla}' = (\widetilde{\nabla}, \zeta^a t_b t_c)$, for some spacelike field ζ^a such that $\widetilde{\nabla}^{[a}\zeta^{b]} = 0$.

Proof. First, suppose that $\widetilde{\nabla}' = (\widetilde{\nabla}, \zeta^a t_b t_c)$ for such a field ζ^a. Then for any timelike θ^a,

$$\begin{aligned}\widetilde{\nabla}'^{[a}\theta^{b]} &= h^{n[a}\widetilde{\nabla}'_n\theta^{b]}\\ &= h^{n[a}\widetilde{\nabla}_n\theta^{b]} - h^{n[a}\zeta^{b]}\theta^m t_m t_n\\ &= \widetilde{\nabla}^{[a}\theta^{b]}\end{aligned}$$

So clearly, $\widetilde{\nabla}'^{[a}\theta^{b]} = 0$ iff $\widetilde{\nabla}^{[a}\theta^{b]} = 0$, i.e., $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are rotationally equivalent. It remains to show that $\widetilde{\nabla}'$ satisfies the homogeneous Trautman conditions (9). Applying the standard condition relating two Riemann tensors,[20] we obtain

$$\widetilde{R}'^a{}_{bcd} = \widetilde{R}^a{}_{bcd} + 2t_b t_{[d}\widetilde{\nabla}_{c]}\zeta^a \tag{24}$$

[20](Malament, 2012, Equation 1.8.2)

It is then a straightforward computation to show that

$$\widetilde{R}'^{ab}{}_{cd} = \widetilde{R}^{ab}{}_{cd} \tag{25}$$

So clearly, $\widetilde{R}'^{ab}{}_{cd} = 0$ iff $\widetilde{R}^{ab}{}_{cd} = 0$.

Next, suppose that $\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b}$. Again, a straightforward computation (together with the twist-freedom of ζ^a) yields

$$\widetilde{R}'^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}'^{c}{}_{d}{}^{a}{}_{b} \tag{26}$$

where the third equality uses our supposition, and the twist-freedom of ζ^a. Showing that if $\widetilde{R}'^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}'^{c}{}_{d}{}^{a}{}_{b}$ then $\widetilde{R}^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{c}{}_{d}{}^{a}{}_{b}$ proceeds similarly.

The converse half of the proof is adapted from Weatherall (2015). Suppose that $\widetilde{\nabla}'$ is a Newton-Cartan connection compatible with $[\nabla]$. Since $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are both compatible with L, there is some antisymmetric tensor field κ_{ab} such that $\widetilde{\nabla}' = (\widetilde{\nabla}, 2h^{an}t_{(b}\kappa_{c)n})$.[21] Now let θ^a be some unit timelike field such that $\widetilde{\nabla}^{[a}\theta^{b]} = 0$ (some such field is guaranteed to exist, since $\widetilde{\nabla}$ obeys the homogeneous Trautman conditions).[22] Using the fact that $\widetilde{\nabla}'^{[a}\theta^{b]} = 0$, we can show that $\widetilde{\nabla}' = (\widetilde{\nabla}, \zeta^a t_b t_c)$ for some spacelike field ζ^a (see (Weatherall, 2015, p. 91) for details of the computation).

It remains to show that ζ^a is twist-free. By using equation (24), we obtain

$$\widetilde{R}'^{a}{}_{b}{}^{c}{}_{d} = \widetilde{R}^{a}{}_{b}{}^{c}{}_{d} + 2t_b t_d \widetilde{\nabla}^c \zeta^a \tag{27}$$

So by exchange of indices, and applying the second homogeneous Trautman condition,

$$t_b t_d \widetilde{\nabla}^c \zeta^a = t_b t_d \widetilde{\nabla}^a \zeta^c \tag{28}$$

Since $t_a \neq 0$, $\widetilde{\nabla}^{[c}\zeta^{a]} = 0$. □

We can now explore the relationship between Maxwell gravitation and

[21](Malament, 2012, Proposition 4.1.3)
[22]See (Malament, 2012, Proposition 4.3.7).

Newton-Cartan gravitation. The relationship is limited in an important way: we can establish a correspondence between the models of Maxwell gravitation and those of Newton-Cartan gravitation only in the case of an everywhere nonvanishing mass density. However, each such model of Newton-Cartan gravitation is naturally associated with a unique such model of Maxwell gravitation, and vice versa. This provides a sense in which the two theories might be regarded as equivalent over the nonvanishing-mass sector, since the mutual pair of associations might be regarded as showing how the two theories are intertranslatable with one another.[23]

Proposition 5. Let $\langle L, \widetilde{\nabla}, T^{ab}\rangle$ be a model of Newton-Cartan gravitation such that at all points in L, $\rho \neq 0$. Then there is a unique standard of rotation $[\nabla]$ such that $\widetilde{\nabla}$ is compatible with $[\nabla]$; and $\langle L, [\nabla], T^{ab}\rangle$ is a model of Maxwell gravitation.

Proof. First, define $[\nabla]$ as consisting of all and only those connections which are flat, and which are rotationally equivalent to $\widetilde{\nabla}$. By the Trautman recovery theorem, there is at least one such connection, so $[\nabla]$ is nonempty. Hence, it is indeed a standard of rotation with which $\widetilde{\nabla}$ is compatible—and it is manifestly unique in this regard.

It remains to show that $\langle L, [\nabla], T^{ab}\rangle$ is a model of Maxwell gravitation. Let ∇ be an arbitrary element of $[\nabla]$. ∇ is a Newton-Cartan connection,[24] and is evidently compatible with $[\nabla]$; so by Proposition 4, $\widetilde{\nabla} = (\nabla, \zeta^a t_b t_c)$ for a spacelike ζ^a such that $\widetilde{\nabla}^{[a}\zeta^{b]} = 0$. Since $\widetilde{\nabla}$ and ∇ are rotationally equivalent, we also have that $\nabla^{[a}\zeta^{b]} = 0$. By equation (10b),

$$\rho\zeta^a = \nabla_n T^{na} \tag{29}$$

So first, the fact that ζ^a is spacelike entails that (12a) is satisfied.

[23] cf. Glymour (1970), Glymour (1977), Barrett and Halvorson (2015).

[24] As remarked earlier, any flat connection trivially satisfies the homogeneous Trautman conditions.

Second, from (10a) and the standard equation relating curvature tensors for different connections, we obtain

$$\begin{aligned} 4\pi\rho t_b t_c &= \widetilde{R}_{bc} \\ &= 2t_b t_{[a}\nabla_{c]}\zeta^a \\ &= -t_b t_c \nabla_a(\rho^{-1}\nabla_n T^{na}) \end{aligned}$$

Since $t_a \neq 0$, it follows that equation (12b) is satisfied.

Finally,

$$\begin{aligned} \nabla^c(\rho^{-1}\nabla_n T^{na}) - \nabla^a(\rho^{-1}\nabla_n T^{nc}) &= \nabla^{[c}\zeta^{a]} \\ &= 0 \end{aligned}$$

So equation (12c) is satisfied. □

Proposition 6. Let $\langle L, [\nabla], T^{ab}\rangle$ be a model of Maxwell gravitation such that at all points in L, $\rho \neq 0$. Then there is a unique Newton-Cartan connection $\widetilde{\nabla}$ compatible with $[\nabla]$ such that $\langle L, \widetilde{\nabla}, T^{ab}\rangle$ is a model of Newton-Cartan gravitation.

Proof. First, we show existence. Let ∇ be an arbitrary element of $[\nabla]$, and define

$$\widetilde{\nabla} = (\nabla, t_b t_c \rho^{-1}\nabla_n T^{na}) \tag{30}$$

$\widetilde{\nabla}$ is a Newton-Cartan connection compatible with $[\nabla]$. For, given Proposition 4, it suffices to observe that $\rho^{-1}\nabla_n T^{na}$ is a spacelike field which is twist-free (by equations (12a) and (12c)).

Further, $\langle L, \widetilde{\nabla}, T^{ab}\rangle$ is a model of Newton-Cartan gravitation. First, from equation (12b),

$$\begin{aligned} \widetilde{R}_{bc} &= -t_b t_c \nabla_a(\rho^{-1}\nabla_n T^{na}) \\ &= 4\pi\rho t_b t_d \end{aligned}$$

So equation (10a) is satisfied. Second,

$$\begin{aligned}\widetilde{\nabla}_n T^{na} &= \nabla_n T^{na} - t_n t_k(\rho^{-1}\nabla_m T^{mn}) - t_n t_k(\rho^{-1}\nabla_m T^{ma})T^{nk} \\ &= \nabla_n T^{na} - \nabla_m T^{ma} \\ &= 0\end{aligned}$$

where we have used equation (12a). So equation (10b) is satisfied.

We now prove uniqueness. Suppose that $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are two Newton-Cartan connections, compatible with $[\nabla]$, such that $\widetilde{\nabla}_n T^{na} = \widetilde{\nabla}'_n T^{na} = 0$. By Proposition (4), $\widetilde{\nabla}' = (\widetilde{\nabla}, \zeta^a t_b t_c)$, where $\widetilde{\nabla}^{[a}\zeta^{b]} = 0$. But then by equation (13), $\widetilde{\nabla}'_n T^{na} = \widetilde{\nabla}_n T^{na} - \zeta^a$. So by supposition (and the fact that $\rho \neq 0$), $\zeta^a = 0$, and so $\widetilde{\nabla}' = \widetilde{\nabla}$. □

6 Constructing spacetime

Let's take stock. On the face of it, a model of Maxwell gravitation $\langle L, [\nabla], T^{ab}\rangle$ might be imagined to have strictly less structure than a model of Newton-Cartan gravitation $\langle L, \widetilde{\nabla}, T^{ab}\rangle$: the latter has all the same stuff which the former has, but also includes a standard of acceleration. What Proposition 6 shows is that—in the case that ρ is nowhere-vanishing—there is a sense in which this appearance is misleading, since the "extra" structure (the standard of acceleration) can be defined from the other structure in the model: the standard of acceleration is defined as that according to which the net gravitational acceleration of the matter encoded by T^{ab} is zero.

Note that we do need to represent the matter by a mass-momentum tensor (rather than just a mass density) if this reconstruction is to work: a mere mass density does not carry enough information to fix a standard of acceleration, i.e., to determine a unique Newton-Cartan connection.[25] For

[25] Wallace (2016b) and Weatherall (2015) both make the same observation: the underlying point is just that Poisson's equation admits of homogeneous solutions which correspond to nontrivial gravitational fields—and since it is linear, superimposing such a solution onto

example,[26] let $\langle L, \widetilde{\nabla}, T^{ab}\rangle$ be some model of Newton-Cartan gravitation, and consider the structures $\langle L, \widetilde{\nabla}, \rho\rangle$ and $\langle L, (\widetilde{\nabla}, (\widetilde{\nabla}^a\phi)t_b t_c), \rho\rangle$ (with $\rho = T^{ab}t_a t_b$) where in some coordinate system (t, x, y, z) adapted to L,

$$\phi = e^x e^y \sin(\sqrt{2}z) \tag{31}$$

One can show that both structures satisfy equation (10a) (the satisfaction of equation (10b) does not arise)—and clearly, both structures give rise to the same standard of rotation, and so both correspond to the same Maxwell-spacetime-based structure $\langle L, [\nabla], \rho\rangle$.

Now, compare the possibility of reconstructing a model of Newton-Cartan gravitation from a model of Maxwell gravitation with an observation made by Pooley.[27] He notes that the presentation by Earman and Friedman of Newtonian spacetime as $\langle L, \nabla, A^a\rangle$ (where A^a is the timelike vector field representing absolute space) has a certain redundancy: $\langle L, A^a\rangle$ has the same structure, in the sense that the derivative operator ∇ may be defined from the structure of L and A^a. One way of thinking about Proposition 6 is as showing that a Newton-Cartan model $\langle L, \widetilde{\nabla}, T^{ab}\rangle$ (in which $\rho \neq 0$ everywhere) carries a similar form of redundancy: provided we know the standard of rotation associated to $\widetilde{\nabla}$, and provided we know the character of T^{ab}, we can "fill in the blanks" to reconstruct $\widetilde{\nabla}$ itself.

That said, there are two important differences between this case and the case raised by Pooley. The first is that in the example of Newtonian spacetime, we note that a piece of spatiotemporal structure (the connection) may be defined in terms of other pieces of spatiotemporal structure (the Leibnizian spacetime structure, plus the structure of absolute space). By contrast, here

a given solution for a fixed mass density ρ will yield another solution for that same mass density ρ. Note that imposing boundary conditions will typically restore uniqueness of solutions.

[26] I take this example from Jim Weatherall; for further discussion, see Dewar and Weatherall (2017).

[27] (Pooley, 2013, §4.5)

we have a piece of spatiotemporal structure (the standard of acceleration) being defined in terms of spatiotemporal structure (the Maxwellian spacetime structure) *and* non-spatiotemporal structure (the mass-momentum tensor). This gives us a better handle on the question of whether acceleration is absolute or relative in the context of Newtonian gravitation. To claim that acceleration is relative in Maxwell gravitation would mean taking the spacetime structure in a model $\langle L, [\nabla], T^{ab} \rangle$ to be given by the Maxwell spacetime $\langle L, [\nabla] \rangle$, rather than by the Newton-Cartan structure $\langle L, \widetilde{\nabla} \rangle$ definable within the model. In favour of this interpretation, note that L and $[\nabla]$ are the only primitive geometrical structures in any model of Maxwell gravitation; so on a view which identifies spacetime structure as just the primitive geometrical structure of a theory, it would be very natural to read this theory as a theory with merely relative acceleration.[28] On the other hand, if one has a different conception of spacetime structure, then it may well be that the Newton-Cartan connection is properly identified as spatiotemporal structure—the fact that it is derived from material dynamical structures (i.e., T^{ab}) notwithstanding. In particular, Knox's "spacetime functionalism"[29] holds that the spacetime structure in a theory is whatever structure encodes the relevant notion of inertial frame in that theory. There are good grounds for thinking that this role is played by the Newton-Cartan connection—and hence, for the spacetime functionalist to maintain that acceleration in Maxwell gravitation is absolute. Thus, this case provides a useful (although admittedly partial) illustration of the so-called "dynamical approach to spacetime geometry",[30] in which one seeks to characterise spacetime geometry as a codification of the behaviour of dynamical structures.[31]

The second (perhaps related) distinction is that such a unique reconstruc-

[28]For example, Maudlin (2012) and Dorr (2011) are both plausibly read as employing a methodology of this kind.

[29]Knox (2014)

[30]Brown (2005), Stevens (2015)

[31]Wallace (2016a) discusses these issues in more depth.

tion is always available in the Newtonian spacetime case,[32] whereas unique reconstruction is here only guaranteed by requiring the nonvanishing of the matter: in effect, by requiring that there be sufficient material structure to everywhere "probe" the spatiotemporal structure.

What happens when the matter distribution does vanish in some regions, then? In such a case, we are still able to construct a Newton-Cartan connection—but, in general, the connection will not be unique. For example, consider the case where $T^{ab} = \mathbf{0}$. Trivially, $\langle L, [\nabla], \mathbf{0} \rangle$ is a model of Maxwell gravitation; but we can show that $\langle L, \nabla, \mathbf{0} \rangle$ and $\langle L, (\nabla, (\nabla^a \phi) t_b t_c), \mathbf{0} \rangle$, where $\nabla \in [\nabla]$ and ϕ is as in equation (31), are both models of Newton-Cartan gravitation for which $T^{ab} = 0$. However, these models are distinct (non-isomorphic): the connection $(\nabla, (\nabla^a \phi) t_b t_c)$ is not flat (but merely has a vanishing Ricci tensor).

Bearing this in mind, consider the following remarks of Saunders:

> What of possible worlds, and distinctions among them drawn in [Newton-Cartan gravitation], invisible to ours? Take possible worlds each with only a single structureless particle. Depending on the connection, there will be infinitely many distinct trajectories, infinitely many distinct worlds of this kind. But in [Maxwell-gravitation] terms, [...] there is only one such world—a trivial one in which there are no meaningful predications of the motion of the particle at all. Only for worlds with two or more particles can distinctions among motions be drawn.[33]

We have now seen how to extend this observation to a field-theoretic formulation of Newtonian gravitation: in general, there are distinct but "materially identical" models of Newton-Cartan gravitation (such as $\langle L, \nabla, \mathbf{0} \rangle$ and $\langle L, (\nabla, (\nabla^a \phi) t_b t_c), \mathbf{0} \rangle$), which will correspond to a single model of Maxwell

[32] Admittedly, "always" is a slightly odd term to use here, since there is effectively only one case: Newtonian spacetime is unique up to isomorphism.

[33] (Saunders, 2013, pp. 46–47)

gravitation.

The natural next question is whether Saunders is correct that the extra structure of Newton-Cartan gravitation compared to Maxwell gravitation is "surplus". Consider a pair of such materially identical models M, M' of Newton-Cartan gravitation. The only difference between M and M' concerns the nature of spacetime in empty regions. So, at issue is whether such a difference constitutes an *empirical* difference. It turns out, however, that this is not a clear-cut question, for one can find (intuitively plausible) criteria of empirical equivalence that generate different answers. On the one hand, M and M' agree with respect to all material structure: thus, the full collection of every piece of observational data regarding M is identical to that regarding M'. On the other, it is not straightforwardly the case that M and M' agree on the content of all possible observations. For although there is not (in fact) any matter in the empty regions, there could have been—and were such matter to have been introduced, the motions that it would have made would suffice to empirically discriminate between M and M' (or to rule them both out in favour of some third alternative). More generally, the distinction at issue is whether unactualised dispositions may properly be considered as empirically respectable properties.[34]

Finally, I turn to comparing the analysis given here with the (related) account of Weatherall (2015). One difference is with regards to the framework: Weatherall's analysis represents the source matter via a mass density ρ, and considers what kinds of trajectories for test particles would be permissible for such a mass density. By contrast, the analysis above uses the mass-momentum tensor T^{ab} to represent matter which is simultaneously source and test: in the Newton-Cartan theory, for instance, equation (10a) encodes T^{ab}'s role as source matter, and equation (10b) encodes its role as test matter. Moreover, the only dynamics in play in Weatherall's paper is that of

[34]For an illuminating discussion of Newton's attitude towards such dispositions (in the gravitational context), see Stein (1970).

gravitation.

Within this framework, Weatherall characterises the dynamically permissible trajectories (for a given mass density ρ on Maxwell space-time) as follows. First, observe that given a Maxwell spacetime equipped with a mass density, $\langle L, [\nabla], \rho \rangle$, for any $\nabla \in [\nabla]$, there exists a spacelike vector field G^a such that $\langle L, \nabla, \rho, G^a \rangle$ satisfies equations (3a) and (3b). Given such a G^a, the allowed trajectories are then all and only those curves whose tangents satisfy

$$\xi^n \nabla_n \xi^a = G^a \tag{32}$$

Note that the choice of G^a (for a given ∇) is not unique, and not just in the manner captured by the gravitational gauge symmetry (7): for instance, given a scalar field ϕ of the form (31), then $\langle L, [\nabla], \rho, G^a + \nabla^a \phi \rangle$ will also satisfy (3a) and (3b)—but will pick out a different set of allowed trajectories, where the two sets of trajectories do not even agree on the relative accelerations of bodies (and hence, correspond to distinct Newton-Cartan connections).

The models of gravitation on Maxwell spacetime are then identified as follows: $\langle L, [\nabla], \rho, \{\gamma\} \rangle$ (where $\{\gamma\}$ is a set of timelike curves on L) is a model if and only if (i) for any $\nabla \in [\nabla]$, there is some spacelike field G^a_∇ such that $\langle L, \nabla, G^a_\nabla, \rho, \{\gamma\} \rangle$ satisfies equations (3a), (3b) and (32); and (ii) $\{\gamma\}$ is appropriately maximal, i.e., if γ' is a curve such that $\xi'^n \nabla_n \xi'^a = G^a_\nabla$ (with respect to any $\nabla \in [\nabla]$), then $\gamma' \in \{\gamma\}$. Note that these conditions don't quite line up with Maxwell gravitation as I've defined it, even allowing for the difference in framework: Weatherall's approach doesn't encode a continuity equation. More significantly, each model is equipped with all the allowed trajectories for test particles, even in empty regions (i.e. regions in which $\rho = 0$).

Weatherall's key result is then the following (where I have modified his notation, to match that used in this paper):

> Let $\{\gamma\}_\rho$ be the collection of allowed trajectories for a given

> mass distribution ρ in Maxwell-Huygens [i.e., Maxwell] spacetime $\langle L, [\nabla] \rangle$ [...]. Then there exists a unique derivative operator $\widetilde{\nabla}$ such that (1) $\{\gamma\}_\rho$ consists of the timelike geodesics of $\widetilde{\nabla}$ and (2) $\langle L, \widetilde{\nabla} \rangle$ is a model of Newton-Cartan theory for mass density ρ.[35]

One word of warning: speaking of *the* collection of allowed trajectories for a given mass distribution (in a Maxwell spacetime) is a little infelicitous, since—as discussed above—a mass density on Maxwell spacetime does not fix a *unique* collection of allowed trajectories for test particles. So it would be better to speak of *a* collection of allowed trajectories.[36]

Now, to facilitate the comparison between this and Proposition 6, recall that (in the contexts where $\rho \neq 0$, i.e., the contexts in which Proposition 6 applies) we can decompose the mass-momentum tensor into a vector field ξ^a and a stress tensor σ^{ab}—and if σ^{ab} vanishes (i.e. in the absence of non-gravitational interactions) the reconstructed connection is that according to which the integral curves of ξ^a are geodesics. So whereas Weatherall's observation is that a full collection of dynamically allowed trajectories is sufficient to pick out a unique Newton-Cartan connection, Proposition 6 shows that a single congruence of such trajectories is sufficient. This makes Weatherall's result slightly less strong than Proposition 6, at least in the context of non-vanishing ρ: it is a generic feature of differential geometry that a connection is uniquely identified by its geodesics, whereas it is not typically the case that a single congruence of geodesics is sufficient.[37] (It suffices in the context of Proposition 6 only because of the further requirement that the Newton-Cartan connection be compatible with the background Maxwell spacetime.)

[35] (Weatherall, 2015, Proposition 4)

[36] To be clear, it's evident that Weatherall appreciates this—I'm just aiming to forestall potential confusions that might arise from quoting him out of context.

[37] Which is not to say that the observation is trivial: it is a nontrivial fact that one can identify a collection of allowed trajectories in such a manner that they will be apt to be the geodesics of some connection. (For a discussion of how to determine whether a class of curves may be interpreted as the geodesics of some connection, see Matveev (2012).)

That said, because Weatherall's approach also includes the trajectories for test particles in empty regions, a model of Newton-Cartan gravitation can *always* be reconstructed from a model of Weatherall gravitation, even if there are empty regions.

Weatherall argues that this result shows that Saunders has made an error here:

> [The proposition above]—at least as I interpret it here—reveals a certain inadequacy in Saunders's account. Saunders insists that there is no privileged standard of acceleration in Maxwell-Huygens space-time. [...] Nonetheless, it turns out that once one takes the dynamically allowed trajectories into account, one can define a standard of acceleration, namely, the unique one relative to which the allowed trajectories are geodesics.[38]

Of course, Weatherall's technical claim here is quite correct; but I suggest that the technical claim doesn't quite capture what Saunders has in mind. From Saunders' remarks, it seems clear that he is *not* including all dynamically allowed trajectories as part of the empirical content of the theory; rather, he is including only the *actual* trajectories, the *actual* motions of matter. In other words, the disagreement between Saunders and Weatherall is essentially that already discussed, over what the most appropriate criterion of empirical equivalence between models of Newton-Cartan gravitation is. Saunders appeals to the former criterion (where empirical equivalence means agreement with respect to material structure), and so concludes that Newton-Cartan gravitation draws distinctions without differences; Weatherall appeals to the latter criterion (where empirical equivalence requires agreement about the counterfactual motions of hypothetical test particles),[39] and so denies

[38](Weatherall, 2015, pp. 89–90)

[39]For instance, "given some distribution of matter in space-time, it is these curves [the allowed trajectories] that form the empirical content of Newtonian gravitational theory." (Weatherall, 2015, p. 89)

that Newton-Cartan gravitation draws distinctions without differences. Insofar as Maxwell gravitation *does* collapse those distinctions, it—rather than Weatherall's theory—represents the natural extension of Saunders' remarks to the field-theoretic context.

Finally, even besides these differences over which class of models are picked out, there is also (I claim) a value to having equations which more simply and directly pick out the models. In particular, it helps us see a little more clearly the reason why the theory may be set on Maxwell spacetime, but not on anything weaker. If the game is just that of picking out a certain class of models, then we can set a gravitational theory on *Leibniz* spacetime just as easily as upon Maxwell spacetime. For consider the following theory, of "Leibniz gravitation": a triple $\langle L, \rho, \{\gamma\}\rangle$ is a model of Leibniz gravitation if and only if for some ∇ compatible with L, there is some spacelike field G^a such that $\langle L, \nabla, G^a, \rho, \{\gamma\}\rangle$ is a model of Galilean gravitation; and (ii) $\{\gamma\}$ is appropriately maximal. We can prove a reconstruction theorem for Leibniz gravitation of just the same sort as Weatherall gravitation: given any model of Leibniz gravitation $\langle L, \rho, \{\gamma\}\rangle$, there is a unique derivative operator $\widetilde{\widetilde{\nabla}}$ such that $\langle L, \widetilde{\widetilde{\nabla}}, \rho, \{\gamma\}\rangle$ is a model of Newton-Cartan gravitation.[40]

Yet Leibniz gravitation is a blatant pseudo-theory—"arrant knavery", as Belot rightly derides it.[41] Why is it knavery? I say: because we cannot give any set of equations, formulated in terms which refer only to the structure of Leibnizian spacetime, which picks out those models. This is not to say that there isn't a distinction between the forms of Leibniz gravitation and Weatherall gravitation: in Leibniz gravitation, rather than *universally* quantifying over connections compatible with the background structure, we *existentially* quantified over them. My claim is just that the fact that Maxwell gravitation is a legitimate theory, whereas Leibniz gravitation is not, can be

[40]We can only do this because of the presence of *all* members of $\{\gamma\}$, though. Unlike Maxwell spacetime, Leibniz spacetime has insufficient structure to enable one to infer a unique connection from a single vector field.

[41]Belot (2000)

hard to see when both are presented merely as classes of models. By contrast, if we insist that the class of models be picked out by a set of equations, then we can more easily keep ourselves honest.[42]

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[42]This raises some questions for the semantic view of theories: if a theory may be any class of models, picked out by any means whatsoever, then it is hard to see how the distinction appealed to here might be drawn.

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