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Psychology and the A Priori Sciences

The 'a priori sciences'[1] to be considered here are logic and arithmetic;[2] the 'psychology' includes experimental, especially developmental psychology, neurophysiology, and vision science. My goal is to examine the role these empirical theories can play in the philosophies of those sciences, or more precisely, the role I think they should play. Most of the psychological studies referred to here will be familiar to readers of this volume, though perhaps not the use to which I hope to put them.

I. Logic

Common sense tells us that much of the world comes packaged into middle-sized objects -- stones, coins, snails, apples, trees, the bodies of cats, apes, human beings -- and not without reason; these items are what we see and touch, encounter and engage with, in everyday life. Of course common sense doesn't always hold up under scrutiny, but meticulous science confirms that each of these is a

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[1] I use the term 'a priori science' as the customary label, not to endorse the view that these disciplines are in fact a priori in some sense or other.

[2] With a nod toward set theory in footnote 47.

rough collection of molecules, held together by various forces, resisting penetration due to other forces, moving as a bounded unit on a continuous spatiotemporal path.[3] Scientifically refined common sense also reveals that these objects have properties and stand in relations: stones come in a variety of sizes and shapes, apples in a various colors, domestic cats are generally smaller than adult humans.[4] This simple structuring validates a certain amount of rudimentary logic: if the apple is either red or green, and it's not red, then it must be green. This might all seem so obvious, so unavoidable, as to be true no matter what, true in 'all possible worlds', but in fact it breaks down at the quantum level: the particles don't behave as bounded units on continuous paths; the sense in which they enjoy properties (like position and momentum) is problematic; and some simple logical laws (e.g., the distributive law) appear to fail.[5] The inferences of this rudimentary logic are reliable as long as the requisite structure is in place, but not otherwise.

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[3] Some philosophers question this simple view on the grounds that the commonsense table is intuitively 'solid', while the scientific object is largely empty space, so the two cannot be the same. Even assuming that common sense does picture things as continuous matter (over and above being impenetrable), it seems more natural to say science has taught us that the objects of common sense are different than we first imagined, not that they don't exist. (For a bit on more such thinking from Eddington, Sellers, and Ladyman and Ross, see [2014], pp. 95-97, p. 99, footnote 9.) Other philosophers go farther, rejecting everything in science and common sense on radical skeptical grounds, but this challenge, too, I set aside for present purposes (for more, see [201?]).

[4] Dependencies between one situation and other are also important -- the coin is on the floor because the cat shoved it off the table -- as are universal properties, but I leave these aside for simplicity in this quick sketch of [2007], III.4 (also [2014]).

[5] In another skeptical move, it's sometimes suggested that science can't serve to ratify the objects of common sense, because any science that begins with those objects will inevitably end up ratifying them. In fact, a science (ours) that begins with them has ended up without them in the quantum world.

Psychology comes into this story as an investigation of how we come to those early commonsense beliefs about objects and their features. The ground-breaking developmental work of the 1980s and 90s[6] showed that young infants track cohesive, bounded, solid[7] individuals, despite occlusions, using spatiotemporal criteria like contiguity, common fate, and continuous motion; though they're aware of object's other properties, they typically don't use regularities of shape, color, texture or motion, or features or kinds to determine object boundaries or identity.[8] Studies of neonates and non-human animals suggest a distant evolutionary origin.

There's some disagreement over the precise interpretation of these experimental results. The contents of 'the object concept' vary slightly from writer to writer;[9] disagreements arise over whether the abilities catalogued are purely perceptual or somehow conceptual;[10] and

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[6] For a summary with references, see [2007], pp. 245-258. Carey [2009] (chapters 2 and 3) is a much-discussed survey and philosophical elaboration by one of the leading researchers in the area. This work swept away the earlier seminal theories of Piaget (featured in [1990], pp. 54-5), according to which the ability to represent objects comes later in development. Carey [2009], pp. 46-55, gives a fascinating re-analysis of Piaget's evidence.

[7] That is, impenetrable (a feature of both Eddington's tables in footnote 3).

[8] Animate/inanimate, human/non-human appear to be exceptions. See Carey [2009], pp. 263-284, especially pp. 276-277, for more on this point.

[9] In one of the more dramatic examples, Burge departs from many psychologists (from Piaget on) in holding that the 'constitutive conditions' for representing bodies as such don't include tracking through occlusion: 'a capacity to perceptually track a body as a three-dimensionally bounded and cohesive volume shape while it remains in view … suffices' (Burge [2010], p. 460). In contrast, Hatfield [2009] requires tracking through occlusion, but holds that the developmental evidence doesn't conclusively show infants are representing objects as 'individual material objects (not as mere local collections of properties) … that … occupy … distinct … space-time worms … throughout their existence' (Hatfield [2009], p. 241).

[10] E.g., Burge [2010], pp. 438-450, in disagreement with Spelke [1998].

so on.[11] Fortunately, these niceties needn't trouble us here because

any of these options will be enough to serve as the building blocks,

the 'objects with properties', in the rudimentary logical structure

described above.

But 'object with properties' aren't all there is to that

rudimentary structure: a stone has a size and a shape, an apple can

be red or green, a coin can fail to be a quarter. More developmental

work of the 80s and 90s shows that young infants classify cats and

dogs so as to exclude birds, and even cats so as to exclude

superficially similar dogs. They're also sensitive to correlations of

features: infants[12] aware of three possibilities for each of the

features A, B, C, D, and E (that is, A1, A2, and A3, and so on),

habituated to items with correlations between these features -- for

example, items with (A1∧B1∧C1)∧(D1∨D2)∧E1∨E2) and items with

(A2∧B2∧C2)∧(D1∨D2)∧(E1∨E2) -- find a new correlated combination (like

A2-B2-C2-D1-E2) familiar but an uncorrelated combination (like A1-B2-

C1-D1-E2) just as novel as one that's entirely new (like A3-B3-C3-D3-

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[11] It could be that some of these disagreements run deeper than the sort of
thing scouted in footnote 9. Sticking with Burge and Hatfield as our
examples, notice that Burge takes the goal of the project to be determining
what's 'constitutive' of objecthood -- 'Our question concerns necessary
minimal constitutive conditions for having the capacity to attribute the kind
body in perception' (Burge [2010], p. 465) -- where this presupposes a fact
of the matter to be discovered (perhaps by rational intuition, perhaps with a
hint of essentialism in the appeal to 'natural kinds'). In contrast,
Hatfield [2009], p. 241) only claims that 'we as adult perceivers typically
see [things] as individual objects' with the features listed in footnote 9,
and that the developmental evidence doesn't establish that infants do this,
too. He describes this situation by referring to 'the adult concept', but
there's no indication of an underlying Burge-like metaphysics; he could just
be using the phrase to highlight the possibility of a significant cognitive
shift.

[12] This pattern and the next emerge in 10-month-olds, still pre-linguistic.

E3)! In addition to these conjunctions, infants also appear sensitive

to disjunctions -- habituated to cats or horses, they find a dog novel

-- and to relations -- for example, 'above', 'below', 'between'.

Results like these strongly suggest that we humans are sensitive to

rudimentary logical structures from an early age.[13]

   Still, as is well-known, it's entirely possible to respond to a

feature of the world without representing it: the frog's visual

system might allow it to detect (then catch and eat) flies without

representing them as flies. On this point, I'm less confident than

Tyler Burge that 'representation' is a psychological natural kind,[14]

and even more doubtful that its contours can be discovered, as he

suggests, by uniquely philosophical means.[15] In contrast, Gary

Hatfield [1988] undertakes a more modest task, firmly grounded in

contemporary vision science.[16] An ongoing debate pits those who

believe that the visual system employs symbolic representations in an

internal symbolic 'language' -- that rules are encoded and applied,

hypotheses formed and tested (in the tradition of Helmholtz) -- and

those who insist that the visual system is not representational, that

it's simply tuned to directly register the rich and complex

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[13] For a summary, with references, see [2007], pp. 258-262.

[14] See Burge [2010], p.291: 'Psychological explanations have a distinct
explanatory paradigm. Psychology … discovers its own kinds. One of them is
the kind representation'.

[15] See Burge [2010], p. xviii: 'philosophy has … a set of methodological and
conceptual tools that position it uniquely to make important contributions to
understanding the world. … Many of its topics remain of broadest human
concern. Where, constitutively, representational mind begins is such a
topic'.

[16] Obviously this contrast (elaborated in footnotes 14 and 15) is reminiscent
of the one in footnotes 9 and 11.

information available in the ever-changing array of ambient light (Gibson and his followers). Both sides acknowledge that processing takes place between the retina and the visual experience. The debate between them hinges on the question: is this processing purely physiological or does it break down into psychologically significant components, and in particular, into components with a characteristically representational role?[17] Hatfield threads the needle between the two schools of thought, arguing that there are representational components, but that they needn't involve a symbolic system.

To see how this goes, consider using a slide rule to multiply n times m: locate n on the A scale; slide 1 on the B scale beneath n on the A scale; find the number on the A scale that's above m on the B scale. The procedure works because the scales are laid out logarithmically and $n \ x \ m = \ln^{-1}(\ln(n) + \ln(m))$. That same equation could be programmed into a digital computer and multiplication carried out in that way, in which case the logarithmic algorithm itself would be encoded, represented, in the computer's program, but this isn't true for the slide rule: there, the algorithm is effectively followed, but it isn't literally represented.[18] The lengths on the slide rule represent numbers because of what the device is designed to

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[17] See Ullman [1980], p. 374, Hatfield [1988], §1.

[18] As Burge ([2010], p. 504) points out, an odometer's computation of the distance travelled depends on the circumference of the tires (it records a tick for each rotation), but the circumference is nowhere represented. Hatfield ([1988], p. 75) makes a similar point about a 'tension adder': n and m are represented by small weights placed on a pan and their sum registered by a pointer on the front of the device, but no algorithm is encoded.

do (multiply, among other things) and how it was designed to do it (relying on, but not representing, the properties of logarithms). The computer is also designed to multiply (among other things), but it's designed to do so quite differently, by applying explicit rules in an internal symbolic system. So, what a device does or doesn't represent depends on how it does what it's designed to do.

If we now replace the slide rule designer or the computer programmer with the evolutionary pressures on our species,[19] the representational status of some element of the visual system can be assessed in the same way: it depends on the function of the visual system in the evolved human organism, the function of that element within the visual system, and the method it uses to perform that function.[20] Hatfield ([1988], pp. 63-65) gives the example of seeing a circle at a slant rather than an ellipse. This function could be achieved by registering the retinal ellipse, registering slant information from shading, and computationally combining these two; or it might be achieved by a single registration of shading across the retinal ellipse. Obviously it's an empirical matter which of these algorithms is actually implemented; it can be investigated by psychological experiments with carefully timed disruptions or by physiological investigation of the neuroanatomy. In these ways, we could determine whether or not, say, the projective retinal shape by

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[19] In practice, determining what aspects of the visual system are adaptations and which are spandrels is a very difficult undertaking. See, e.g., Warren [2012], Anderson [2015].

[20] Burge [2010], chapter 8, soundly rejects accounts of representation based in biological function, for reasons I don't fully understand and won't attempt to explicate.

itself is represented. But either way, it's not at all obvious that this sort of representation would involve a symbolic system.[21]

So, to return to our theme, we know that infants respond to conjunctions, disjunctions, negations, and so on, but do they actually represent them as such? As Burge notes ([2010], p. 406), the fact that we infer in accord with a logical rule doesn't imply that the rule is somehow encoded in our psychology, presumably in some language of thought. Our concern here, though, isn't with inference, but with simple logical structuring, and (following Hatfield) the representation needn't be symbolic. The question is whether the infant represents the stone as small and round, the apple as red or green, the coin as not-a-quarter. Assuming that sensory sensitivity to these worldly features is adaptive,[22] a Hatfield-style answer to this question hinges on how that sensory sensitivity is achieved: does the scientific story of that ability break down into psychologically significant parts, into representational components, like the separate representations of projective shape and of shading

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[21] Does this mean that the frog is representing flies? Opponents of biological function views suggest that evolutionary considerations aren't enough to show that the frog is representing flies as opposed to moving black dots, or even flies as opposed to nothing at all, given that a frog's detector will occasionally go off on its own. Regarding the first point as subject to further investigation, granting it now only for the sake of argument, Hatfield contends that nevertheless biological function 'can serve as the basis for ascribing to states of the frog's visual system the content *target fly/moving dot*, or some such coarse-grained content', that '[a]mong the functions of the frog's visual system is to represent small moving things as being there when they are, and not to represent them as being there when they aren't (Hatfield [1991], pp. 122-123). In other words, the biological function account has room to regard the frog's detector as having misfired when it goes off on its own.

[22] See footnote 19.

information in the multi-stage algorithm for seeing the circle at a slant? For our case: given that the infant can represent stones, smallness and roundness, is her representation of a small, round stone related to other representations in a way that merits describing it as a conjunctive representation? This needn't involve encoding in some language of thought any more that the circle-at-a-slant case does, but it is a straightforward empirical question for experimental psychology and neuroscience.

If a definitive answer to this question is known, it isn't known to me, but the study of visual working memory offers a hint of how a small part of it might go. Evidence suggests that we're able to store information about a limited number of objects (around four) and their features over short periods of time. This raises the question how several features of one object are bound together: what distinguishes a scene with a vertical red bar and a horizontal green bar from one with a vertical green bar and a horizontal red one? One proposal is synchronized neural firing: a particular neuron fires repeatedly to encode a single feature; when the repeated firings of the neurons for two separate features are synchronized, they form a unit, 'cell assembly'.[23] In Hatfield's terms, the initial firings represent red, green, horizontal and vertical bars; when the 'vertical' and 'red' neurons fire in unison, the resulting assembly represents a vertical red bar. The position I'm proposing, on pure speculation, requires

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[23] See Vogel, Woodman, and Luck [2001] for discussion and references. Also Olson and Jiang [2002].

that this isn't exceptional, that rudimentary logical structuring is widely represented, one way or another.

If all this is granted, what role is psychology playing in this philosophy of logic? The ground of logical truth, what makes it true (where it is true), is the objective logical structuring in the world, so there's no trace of psychologism. Psychology's role, then, might be thought to be epistemological. For example, a sufficiently externalist epistemologist, one who thinks the evolutionary pressures responsible for our logical cognition produce a reliable process,[24] might conclude that we know (at least some of) the world's logical structure a priori. I prefer to leave the policing of 'know' and 'a priori' to the specialists and to say only this much: we come to believe what we do about the logical structures in the world on the basis of primitive cognitive mechanisms, many of which we share with other animals, but our evidence for the correctness of those beliefs comes from common sense and its subsequent (partial) ratification by scientific means.

So far, this is a fairly slight philosophical impact for psychology, but I think there's an important moral concerning our philosophical preconceptions about logical truth. Because our logical beliefs rest on such primitive cognitive mechanisms, it's hard for us to see how things could be otherwise, how a world failing to instantiate those rudimentary logical forms is even possible. When

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[24] This needn't be a fallacious argument of the form: 'this evolved, therefore it's reliable'. Instead, it might run roughly along the lines traced here: first science establishes that much of the world is logically structured; then psychology defends an evolutionary story of how we come to detect and represent that structure.

quantum mechanics shows us not only that a world can fail do this, but that our very own micro-world so fails, often the result is that we find quantum mechanics deeply problematic, not that we take logic itself to be contingent.[25] It seems to me that the psychology here is showing us *why* we're so easily inclined to believe that logical truth is necessary, a priori, certain -- a stubborn preconception that vastly distorts our theorizing about it.[26] It's hard to imagine a more valuable lesson for the philosophy of logic!

II. Arithmetic

 Obviously any patch of the world with logical structuring into objects with properties, standing in relations, will also have number properties: so many objects, so many with this particular feature, so many standing in this relation to this particular individual, and so on. When it comes to our cognitive access to those number properties, though, it's well known that the first four or so have special status: infants' expectations about how many objects will appear behind a screen after individual objects have been added or removed are accurate up to 3; adults can hold 3-4 objects in working memory[27] and

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[25] I suspect many of us have heard our fellow philosophers assert with great confidence that quantum mechanics must be false, on a priori grounds.

[26] See [2014a], chapter 6, for a comparison of this conclusion with the late Wittgenstein's take on logic.

[27] Feigenson [2011] describes how visual working memory can encode more than four slots worth of information by 'chunking', as when we remember a phone number by dividing it into three blocks of digits, or one of her infant subjects remembers two cats and two cars, but not four individual cats. (See also Carey [2009], pp. 149-150.) The 'chunk' is often referred to as a 'set', as exhibiting the higher ranks that differentiate sets from mere aggregates. I once appealed to analogous considerations (e.g., in [1990], p. 165), but for what it's worth, I'm no longer convinced anything essentially

track 3-4 objects through complex motions, but these abilities break

down quickly for higher quantities. Non-human animals share these

abilities and limitations, indicating another primitive cognitive

system.[28]

The mechanism underlying these abilities -- the object-tracking

or parallel individuation system -- apparently includes so-called

'object files' of mid-level vision,[29] which follow objects

spatiotemporally and encode features as they go ('it's a bird, it's a

plane, it's Superman'), and visual working memory, which keeps visual

information accessible over short periods.[30] The two are closely

intertwined, with some evidence of complementary emphases, on tracking

over motion and retention of object properties, respectively.[31] Though

the infant expectation experiments are often described in arithmetic

terms -- 1+1=2, 3-2=1 -- it's widely agreed that these representations

are not truly numerical: not '3', but the simply logical 'a thing,

another thing, and yet another thing', most likely the opening of

three successive object files.[32] In cases of 'subitizing' -- immediate

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'higher-order' is involved in such cases. Seeing two cats and two cars could
just be a particular way of seeing the cats and the cars, not a way of seeing
something else (a set of cats, a set of cars).

[28] See [2007], pp. 319-326, for more on the story in this and the following
three paragraphs, with references. See Carey [2009], chapter 8, for her
elaboration.

[29] Kahnemann et al [1992]. See [2007], pp. 255-257, 319-320, for a brief
discussion with references.

[30] See, e.g., the references in footnote 23.

[31] See Hollingworth and Rasmussen [2010]. The two are often lumped together
without comment, or even identified.

[32] Burge points out that '[t]here need be no use of conjunction or negation
in the perceptual representation (as in: this is a body and this is a body

recognition (without counting) of up to 3 or 4 objects -- perhaps

visual working memory is engaged, but again, mostly likely through the

opening of three distinct information slots,[33] rather than an explicit

numerical representation.

Yet another primitive system we share with other animals is

sensitive to approximate quantities: it can distinguish 1 dot from 3

more easily than 2 dots from 3 (the 'distance effect'); it can

distinguish 2 dots from 3, but not 8 dots from 9 (the 'magnitude

effect').[34] The mechanism for this is so far unknown (at least to me),

but neurological studies on monkeys suggest a two-step process that

begins with a group of neurons that encode locations of objects,

ignoring other features, then feeds into an array of neurons whose

responses are bell-shaped curves each peaking at a certain number.[35]

This model would explain the distance effect -- the ranges of firing

for '1-neurons' and '3-neurons' overlap less than those for '2-

neurons' and '3-neurons' -- and the magnitude effect -- the bell

curves for large numbers are broader. In any case, this is clearly a

more quantitative system than the object tracker, but it can't truly

be said to represent cardinality. Burge ([2010], p. 482) suggests a

return to the ancient notion of 'pure magnitude', neither continuous

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and this is not that)' (Burge [2010], p. 486). He's right: it's unlikely
that anything like this is encoded in a language of thought. But in
Hatfield's terms, the opening of three successive object files could
represent the corresponding logical feature of the scene.

[33] See Chesney and Haladjian [2011] for evidence that subitizing and object
tracking rely on a shared visual mechanism.

[34] See, e.g., Carey [2009], pp. 118-137, Dehaene [2011].

[35] See Dehaene [2011], pp. 247-254, Neider [2011].

nor discrete, but nevertheless stands in ratios. However that may be, what matters for our purposes is that features of the world's logical structure are being represented, albeit only approximately.[36]

So far, we're in step with the non-human animals, still far short of human arithmetic. The leading theory is that what sets us apart is the child's ability to combine the proto-numerical fruits of the object-tracking system and the approximate system via her command of the counting sequence.[37] 'One, two, three …' is first learned as a verbal nonsense scheme -- like 'eeny, meeny, miney, moe …' -- and the act of reciting it while pointing to each of a group of objects in turn is just play, of no numerical significance. Young children do realize that use of the word 'one' correlates with the presence of a single object, with a single opened object file or a single item in visual working memory, but the sense of larger number words comes only gradually, between 2½ and 3½: first 'two' is associated with the presence of an object and another; a few months later 'three', and maybe even 'four', gains meaning from the object-tracking system.[38] This far the non-human animals can follow, but what happens next is

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[36] Oddly enough, on small numbers, where the two systems overlap, the infant's object tracking system appears to override the approximate system. E.g., they prefer a box where 3 treats have been placed to a box where 1 or 2 treats have been placed, but when the numbers are 2 and 5, beyond the object tracker's capacity, they perform at chance -- despite the fact that the ratio is big enough for the approximate system to detect easily. See Carey [2009], pp. 84-85, 139-141, 153-155.

[37] Here again, Piaget was in disagreement. See Dehaene [2011], pp. 30-36, for an amusing account of how the empirical results were misinterpreted.

[38] Some hold that the object-tracking system isn't involved, that the underlying mechanism here is the approximate number system (see, e.g., Piazza [2011]), which is most precise for small numbers. Dehaene ([2011], pp. 256-259), who once entrusted small numbers to the approximate number system (what he calls 'the number sense'), explains what changed his mind.

uniquely human: apparently the child notices that an extra object in the scene corresponds to the next number in the counting sequence, and suddenly the true meaning of counting becomes clear: the last number recited in the procedure is the number of objects in the scene.

It's sometimes assumed that this is the end of the story of how humans come to a full understanding of arithmetic, but it isn't, for at least two reasons. First, consider a child who knows how to count, and knows there are 'just as many' of these as those when the same number word results from counting these as those. That is, she knows that if she counts *n* children and *n* cookies, she'll be able to give each child exactly one cookie with no cookies left over. Richard Heck makes the case that a child can know all this without having the notion of a one-to-one correspondence, which is, after all, 'very sophisticated' (Heck ([2000], p. 170). Of course when she counts, she forms what we understand to be a one-to-one correspondence, but she needn't understand it as such; she's just implementing the counting procedure.[39] So this is one respect in which the child still hasn't grasped a notion some consider essential to the concept of 'cardinal number'.

Another tempting assumption is that a child who understands that one more object corresponds to the next number word must also

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[39] Heck also notes that the child can understand 'just as many' without understanding counting: there are just as many cookies as children if she can make sure everyone has exactly one cookie with none left over. He then shows how the Peano Axioms can be derived with 'just as many' in place of Frege's 'one-to-one correspondence'.

understand that there's no largest number.[40] The only empirical study

touching on this question that I know of, Harnett and Gelman [1998],

actually aims to show that it's relatively easy for children to learn

that the number sequence has no end -- easy compared to learning

fractions! -- so its design includes more coaching than would be ideal

for present purposes. Still, children in kindergarten and first

grade[41] did quite poorly on questions like 'is there a biggest number

of all numbers?' and 'is there a last number?'. They did somewhat

better but still far from perfectly on leading questions like 'if we

count and count and count, will we ever get to the end of the

numbers?' and 'can we always add one more, or is there a number so big

we'd have to stop?', despite having been primed with exercises in

counting larger and larger numbers.[42]

Explaining their answers, the six-year-olds might suggest that we

have to stop counting 'cause you need to eat breakfast and dinner' or

'because we need sleep', or that we couldn't then start up again where

we left off because 'you forget where you stopped'. There's even a

hint at mortality: if we try to add one more after counting to a very

big number, 'I guess you'll be old, very old'. Though answers like

these were classified as 'unacceptable', there is an straightforward

sense in which the children have it right: there *are* practical

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[40] For a bit more on the line of thought in the remainder of this section,
see [2014b].

[41] Averaging just under 6 and 7 years old, respectively.

[42] One group of subjects in one of the studies was questioned about the
largest number, etc., before the counting exercises. Their performance was
even worse than the group who did the counting exercises first.

limitations on how far we're inclined to count, and even physical

limits on how far we could count.[43] The young children aren't wrong

exactly, they're just failing to grasp the spirit of the question.

What's being asked is whether there's any limit to how far we could

count, *in principle*.

In contrast to the kindergarteners and first graders, the second

graders[44] in this study generally answered the questions as they were

intended: there is no largest number, period. Closer analysis of the

experimental results led Harnett and Gelman to the observation that

the children in a position 'to benefit from a conversation that offers

cues' (p. 361) were those who could count beyond 100:

> Once children master the sequence from 1 to 20 and the list of
> decade words, they have most but not all of the vocabulary they
> need to apply the recursive procedures by which larger and larger
> numbers are generated. As they count beyond 100, they come to
> learn that not only the digits, but also the decade terms, are
> recycled over and over. [Younger] [c]hildren are still at work
> memorizing the teens and decade terms and are less able to
> appreciate that the count sequences is systematic. (Harnett and
> Gelman [1998], p. 361)[45]

This suggests, as the psychologist Paul Bloom proposes, that

> [T]he generative nature of human numerical cognition develops
> only as a result of children acquiring the linguistic counting
> system of their culture. Many, but not all, human groups have
> invented a way of using language to talk about number, through
> use of a recursive symbolic grammar. (Bloom [2000], p. 236)

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[43] Russell once remarked that running through an infinite decimal expansion
is '*medically* impossible' (Russell [1935/6], p. 143).

[44] Averaging just under 8 years old.

[45] Though Harnett and Gelman speak of 'recursive procedures by which larger
and larger numbers are generated', obviously they're talking about linguistic
procedures that generate numerical expressions. (Understanding that adding
one results in a larger number was another predictor for successful response
to the cues.)

This would mean that children's belief in the infinity of the numbers

derives from their belief in the infinity of numerical expressions,

not vice versa:

> [I]t is not that somehow children know that there is an infinity
> of numbers and infer that you can always produce a larger number
> word. Instead, they learn that one can always produce a larger
> number word and infer that there must therefore be an infinity of
> numbers. (Ibid., p. 238)

In this way, our question -- how do we come to believe there's no

largest number? -- is pushed back one step to: how do we come to

believe that there's no largest numerical expression?

Harnett and Gelman's studies show that it's quite easy for

children to come to this view once they've appreciated the intricacies

of the systematic generation of numerical expressions. What's

striking is that they don't seem bothered by concerns about the

practical or physical limitations on, for example, the length of those

numerical expressions or the breathe needed to utter them or the need

to stop for lunch -- all that apparently matters is coming to grasp

the recursive character of the rules of formation. Why is the

intended 'in principle' reading of the question more natural here,

when it's posed for numerical expressions, than it was when posed for

the numbers themselves? To engage once again in rank speculation, I

suggest that this traces to the recursive element of the innate

linguistic faculty, whatever it is in our genetically endowed

cognitive machinery that underlies our ability to understand and

produce indefinitely varied and complex linguistic items:

> All approaches agree that a core property of [the linguistic
> faculty] is recursion … [The linguistic faculty] takes a finite
> set of elements and yields a potentially infinite array of
> discreet expressions. This capacity … yields a discrete infinity

        (a property that also characterizes the natural numbers).
        (Hauser, Chomsky and Fitch [2002], p. 1571)

The suggestion is that this linguistic capacity is what produces our

intuitive grasp of the 'in principle' question.

        Assuming this sketch of the psychology is roughly right -- a big

assumption, subject to empirical test -- the consequences for the

philosophy of arithmetic are fundamental. Simple arithmetical claims

like 2+2=4 and 12<191 are ordinary facts about worldly logical

structures (where they're present), but the subject matter of

mathematical arithmetic -- the standard model, what we now think of as

an omega-sequence -- doesn't depend on any contingent features of the

actual world, which may or may not be finite. Insofar as arithmetic

is 'about' anything, it's about an intuitive picture of a recursive

sequence of potentially infinite extent, an intuitive picture we

humans share thanks to the evolved linguistic faculty common to our

species.

        Now we all tend to believe that the structure of the standard

model of arithmetic, that simple omega-sequence, is coherent, unique

and determinate. But if it's really just a matter of an intuitive

picture, what reason do we have to believe these things? As

Wittgenstein once asked, 'what if the picture began to flicker in the

far distance?' (*RFM*, V.10). Our innate cognitive structuring may well

give rise to these firm convictions, but if the story told here is

correct, our capacity for mathematical arithmetic could be a mere

spandrel, generated just by the way as we evolved toward language, and

even if it is an adaption in itself, that's no guarantee of

reliability.[46] Under the circumstances, we reflective beings should

want more support for our faith in the cogency of an omega-sequence

than just our brute inclination to believe it. I think there are

facts we can appeal to, but they're hardly conclusive: our biological

similarity as humans is reason to think your intuitive picture is

more-or-less the same as mine; the apparent coherence of the picture,

plus long experience of the species with mathematical arithmetic,

provides some evidence for its consistency; the lack of any important

independent statement comparable to the Continuum Hypothesis suggests

it may be fully determinate.[47] But our sense that arithmetic is more

secure than that may be an illusion -- another valuable lesson from

psychology!

III. Conclusions

Though psychologists sometimes take their work to support a brand

of anti-realism about mathematics -- Stanislas Dehaene's influential

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[46] See the fallacy described in footnote 24.

[47] There's an analogous question for set theory, where the relevant intuitive
picture -- the iterative hierarchy -- seems to rest on three elements:
recursion (presumably based in the same cognitive faculty as the standard
model of arithmetic); the combinatorial notion of an arbitrary subset, not
beholden to any rule, definition, or construction (perhaps related to Heck's
'very sophisticated' one-to-one correspondence?); and Cantor's gusty bet on
the completed infinite (see [1988], I.5). This picture isn't definitive of
the field in the way the standard model is for arithmetic: it wasn't present
when set theory was founded by Cantor and others, and it could be altered or
replaced in the future (e.g., by the multiverse conception, though for now
I'm skeptical about that (see [201?a], III)). In any case, given the added
vagaries of the two additional elements, any case for cogency is
correspondingly weaker: determinacy is undercut by independent statements
like the CH, and our biological similarity gives less support for uniqueness.
Perhaps the apparent coherence of the conception delivers some evidence of
consistency, but considerably less than in the case of arithmetic. Still,
this would be a form of so-called 'intrinsic' support distinct from the
merely instrumental role described in [2011].

Number Sense, for example, bears the subtitle *How the Mind Creates Mathematics* -- in fact their skepticism doesn't extend to the contingent logical/numerical structure I've been attributing to the world or our cognitive access to it:

> [A]rithmetic … draw[s] upon a store of fundamental knowledge accumulated over millions of years of evolution *in a physical world which, at the scale we live it, is … numerically structured*. (Dehaene in Dehaene and Brannon [2011], p. 187, emphasis added)

This type of straightforward realism breaks down, I've suggested, with the potential infinite, the standard model of arithmetic, where attention to the psychological facts reveals that our cognitive architecture does, in a sense, 'create' the subject matter under investigation. In addition to this positive semantic or metaphysical conclusion, empirical work in psychology also uncovers the less-than-firm underpinings of some of our firmest philosophical preconceptions: that logic is necessary, that arithmetic is obviously cogent (coherent, unique, determinate). This valuable therapeutic helps free the philosophies of these subjects from traditional baggage and set them on a more vital course. In these ways, psychological inquiry stands to play a central, highly beneficial role in our philosophizing about the a priori disciplines.[48]

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