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Freitas, P., Siegl, P., & Tretter, C. (2018). The damped wave equation with unbounded damping. Journal of Differential Equations, 264(12), 7023-7054. https://doi.org/10.1016/j.jde.2018.02.010

THE DAMPED WAVE EQUATION WITH UNBOUNDED DAMPING

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Abstract. We analyze new phenomena arising in linear damped wave equations on unbounded domains when the damping is allowed to become unbounded at infinity. We prove the generation of a contraction semigroup, study the relation between the spectra of the semigroup generator and the associated quadratic operator function, the convergence of non-real eigenvalues in the asymptotic regime of diverging damping on a subdomain, and we investigate the appearance of essential spectrum on the negative real axis. We further show that the presence of the latter prevents exponential estimates for the semigroup and turns out to be a robust effect that cannot be easily canceled by adding a positive potential. These analytic results are illustrated by examples.

1. Introduction

We consider the spectral problem associated with the linearly damped wave equation

$$u_{tt}(t,x) + 2a(x)u_t(t,x) = (\Delta - q(x))u(t,x), \quad t>0, \quad x \in \Omega, \tag{1.1}$$

with non-negative damping a and potential q on an open (typically unbounded) subset Ω of $\mathbb{R}^d$; when Ω is not all of $\mathbb{R}^d$ we shall impose Dirichlet boundary conditions on its boundary $\partial\Omega$. Here both the potential q and the damping a are allowed to be unbounded and/or singular.

The main goal of this paper is to analyze the new phenomena which arise when the damping term a is allowed to grow to infinity on an unbounded domain. To this end, we formally rewrite (1.1) as a first order system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}}_{G} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.2}$$

and realize the operator G in a suitable Hilbert space without assuming that the damping is dominated by $\Delta - q$. Our main results show that, even under these weak assumptions on the damping, G generates a contraction semi-group, but G may have essential spectrum that covers the entire semi-axis $(-\infty, 0]$. As a consequence, although the energy of solutions will still approach zero, this decay will now be polynomial and no longer exponential, *cf.* [11] and the discussion below. We further establish conditions for the latter and study the convergence of non-real eigenvalues in the asymptotic regime of diverging damping on a subdomain.

Date: January 31, 2018.

2010 *Mathematics Subject Classification.* 35L05, 35P05, 47A56, 47D06.

Key words and phrases. damped wave equation, unbounded damping, essential spectrum, quadratic operator function with unbounded coefficients, Schrödinger operators with complex potentials.

P.F. was partially supported by the Fundação para a Ciência e Tecnologia, Portugal, through project PTDC/MAT-CAL/4334/2014. The research of P.S. is supported by the *Swiss National Science Foundation* Ambizione grant No. PZ00P2_154786. C.T. gratefully acknowledges the support of the *Swiss National Science Foundation*, SNF, grant no. 169104.

In most of the literature on linearly damped wave equations on unbounded domains only bounded damping terms were considered. This is a natural condition to allow for the exponential decay of solutions, while large damping terms in fact tend to weaken the decay giving rise to the phenomenon known as over-damping. More precisely, increasing the damping term past a certain threshold will cause part of the spectrum to approach the imaginary axis, thus producing a slower decay. This phenomenon may already occur in finite-dimensional systems and in equations like (1.1) with bounded damping where its effect on individual eigenvalues is well-understood. Unbounded accretive or sectorial damping terms of equal strength as $-\Delta$ were considered as an application of semigroup generation results and of spectral estimates for second order abstract Cauchy problems in [13, 12] which allow to control the spectrum, in particular, near the imaginary axis.

To the best of our knowledge, the only article where the damping has been allowed to become unbounded at infinity is the recent preprint [11]. There the authors consider dampings on all of $\mathbb{R}^d$ ($d \geq 3$) and, using methods different from ours, they prove the existence and uniqueness of weak solutions whose energy decays at least with $(1+t)^{-2}$. In fact, our result on the essential spectrum will show that one of the characteristics of such systems is that the essential spectrum covers the whole semi-axis $(-\infty, 0]$, thus excluding exponential energy decay.

To illustrate this issue, consider the simple model case given by the generators G_n, $n \in \mathbb{N}$, of the wave equation on the real line with the family of damping terms

$$a_n(x) = x^{2n} + \mathfrak{a}_0, \quad x \in \Omega = \mathbb{R}, \quad n \in \mathbb{N}, \quad \mathfrak{a}_0 \geq 0, \tag{1.3}$$

and a constant potential $q(x) = \mathfrak{q}_0 \geq 0$, $x \in \mathbb{R}$. The formal limit as n goes to ∞ leads to a simple problem (in a different space) with

$$a_\infty(x) = \mathfrak{a}_0, \quad x \in \Omega_\infty = (-1, 1), \tag{1.4}$$

and Dirichlet boundary conditions at ± 1. The spectrum of the generator G_∞ is discrete and may be found explicitly. It consists of eigenvalues with all, but possibly finitely many, lying on the line $-\mathfrak{a}_0 + \mathrm{i}\mathbb{R}$, *cf.* Remark 6.2. Moreover, the energy of solutions of the corresponding wave equation is known to decay exponentially, *cf.* for instance [10]. A natural question is to what extent the properties of G_∞ are shared by G_n. The non-real eigenvalues of G_n, here given explicitly in terms of the eigenvalues of $-\mathrm{d}^2/\mathrm{d}x^2 + x^{2n}$ and located on rays of the form $\mathrm{e}^{\pm \mathrm{i} \frac{n+1}{2n+1}\pi}\mathbb{R}_+$, *cf.* Proposition 6.1, do indeed converge to those of G_∞. However, while the spectrum of G_∞ is discrete and does not contain 0, all G_n, $n \in \mathbb{N}$, have non-empty essential spectrum covering the entire negative semi-axis and thus 0 is in the spectrum of G_n. As a consequence, exponential decay of energy is lost, *cf.* for instance [6, Thm. 10.1.7].

The fundamental point here is that the essential spectrum can no longer be shifted away from 0 by adding a positive potential $q(x) \geq \mathfrak{q}_0 > 0$, as might be done for bounded damping, to ensure that exponential energy decay still holds, *cf.* for instance [25, 18]. In fact, even a potential q that is unbounded at infinity, but does not dominate the damping term, will not be enough to cancel this effect. On the other hand, a dominating potential q can be used to shift the essential spectrum from 0, *cf.* Remark 3.3.

As we will see, (1.3) is not an isolated example and our results cover the much more general setting with an open (typically unbounded) domain $\Omega \subset \mathbb{R}^d$, a potential q with low regularity and a damping a satisfying natural conditions allowing for a convenient separation property of the domain of the Schrödinger operator $-\Delta + q + \gamma a$ when γ belongs to $\mathbb{C} \setminus (-\infty, 0]$, *cf.* Assumption I, Remark 2.1.i) and Theorem 2.4.

We emphasize that the unbounded damping at infinity can by no means be viewed as "small" when compared to $-\Delta + q$ and our results, even those which are qualitative, do not follow by standard perturbation techniques, traditionally used to handle bounded or relatively bounded damping terms.

The proofs rely on a wider range of methods like elliptic estimates for Schrödinger operators with unbounded complex potentials, quadratic complements (quadratic operator functions associated with (1.2)), Fredholm theory, the use of suitable notions of essential spectra for non-self-adjoint operators, WKB expansions, convergence of sectorial forms acting in different spaces with $L^1_{\rm loc}$-coefficients, spectral convergence of holomorphic operator families, and properties of solutions of second order ODE's with polynomial potentials.

The crucial part of our analysis is the relation between the spectrum, and some of its subsets, of the generator G and the associated quadratic operator function T given by

$$T(\lambda) = -\Delta + q + 2\lambda a + \lambda^2, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0]. \tag{1.5}$$

While for bounded damping $T(\lambda)$ is defined for all $\lambda \in \mathbb{C}$ and the equivalence of $\lambda \in \sigma(G)$ and $0 \in \sigma(T(\lambda))$ is relatively straightforward, *cf.* for instance [23, Sec. 2.2, 2.3] for abstract results, the unboundedness of a is a major challenge that requires a new approach; in particular, first $T(\lambda)$ has to be introduced as a closed operator with non-empty resolvent set acting in $L^2(\Omega)$.

It is the precise description of $\mathrm{Dom}(T(\lambda))$, *cf.* Theorem 2.4, that enables us to prove both the generation of a contraction semigroup, *cf.* Theorem 2.2, and the spectral correspondence between G and $T(\lambda)$ for the restricted range $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, *cf.* Theorem 3.2. Clearly, there are crucial differences between $-\Delta + 2\lambda a$ for $\lambda > 0$ and $\lambda < 0$ since the quadratic form of the latter is not semi-bounded. Nevertheless, for a general $\lambda \in \mathbb{C}$, convenient properties of $T(\lambda)$ with $\lambda > 0$ remain valid also for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ since the possibly negative real part of $2\lambda a$ is compensated by the imaginary part of $2\lambda a$, *cf.* Section 2.1 for details.

When a is unbounded at infinity, we show that the set $\{\lambda \in \mathbb{C} \setminus (-\infty, 0] : 0 \in \sigma(T(\lambda))\}$, and hence the non-real spectrum of G, consists only of discrete eigenvalues of finite multiplicity which may only accumulate at the semi-axis $(-\infty, 0]$. Since the unboundedness of a is not required for the equivalences in Theorem 3.2, also the non-real essential spectrum of G can be analyzed by studying whether 0 belongs to the essential spectrum of $T(\lambda)$.

Because $T(\lambda)$ is not defined for $\lambda \in (-\infty, 0)$, the negative real spectrum of G is investigated directly for unbounded domains Ω. We show that if a grows to infinity in a channel in Ω whose radius may shrink at ∞ at a rate controlled by the growth of a, then 0 belongs to the essential spectrum of G. In fact, the whole real negative semi-axis belongs to the essential spectrum of G even when q is unbounded but does not dominate a, *cf.* Theorem 4.2.

In Section 5, motivated by examples (1.3), (1.4) above, we prove a convergence result for non-real eigenvalues and corresponding eigenfunctions of a sequence of quadratic functions $\{T_n(\lambda)\}_n$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, with dampings possibly diverging on a subset of Ω, *cf.* Theorem 5.1. We thus justify the formal limit considered in the examples above.

In Section 6 we analyze two examples, the first is on the whole real line based on (1.3) and (1.4), while the second is on a horizontal strip in $\mathbb{R}^2$ with damping $a(x, y) = x^2 + \mathfrak{a}_0$ and the corresponding discrete spectrum displaying the structure of a two–dimensional problem. Apart from showing what type of behavior one may now expect from isolated eigenvalues, more importantly both cases illustrate that having the discrete spectrum to the left of a line $\mathrm{Re}\,\lambda = -\alpha_0 < 0$ is, by itself, not enough to determine the type of decay of solutions in the presence of unbounded damping. Indeed, our results applied to both examples show that the essential spectrum covers the negative part of the real axis all the way up to 0, thus excluding the possibility of uniform exponential decay of solutions in general.

1.1. **Notation.** The following notations and conventions are used throughout the paper. The norm and inner product (linear in the first entry) in $L^2(\Omega)$ are denoted by $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(\Omega)}$, respectively. The domain of a multiplication operator by a measurable function m (here a and q) in $L^2(\Omega)$ is always taken to be maximal, *i.e.*

$$\mathrm{Dom}(m) := \{\psi \in L^2(\Omega) \,:\, m\psi \in L^2(\Omega)\}.$$

The Dirichlet Laplacian on Ω, introduced through the corresponding form, is denoted by Δ_{D}, *i.e.*

$$-\Delta_{\mathrm{D}}\psi = -\Delta\psi, \quad \mathrm{Dom}(\Delta_{\mathrm{D}}) = \{\psi \in W_0^{1,2}(\Omega) \,:\, \Delta\psi \in L^2(\Omega)\}.$$

When $-\Delta+q$ is viewed as an operator, the Dirichlet realization introduced through the form is meant, *i.e.*

$$\mathrm{Dom}(-\Delta+q) := \{\phi \in W_0^{1,2}(\Omega) \cap \mathrm{Dom}(q^{\frac{1}{2}}) \,:\, (-\Delta+q)\phi \in L^2(\Omega)\}. \tag{1.6}$$

For $\Omega_1 \subset \Omega$ we view $L^2(\Omega_1)$ as a subspace of $L^2(\Omega)$, $L^2(\Omega) = L^2(\Omega_1) \oplus L^2(\Omega \setminus \overline{\Omega}_1)$, *i.e.* we use zero extensions. On the other hand, for $f \in L^2(\Omega)$, $\|f\|_{L^2(\Omega_1)}$ means $\|f \restriction \Omega_1\|_{L^2(\Omega_1)}$. For consistency with earlier work, we denote the numerical range of a linear operator A acting in a Hilbert space $\mathcal{H}$ by

$$\mathrm{Num}(A) := \{\langle Af, f\rangle_{\mathcal{H}} \,:\, f \in \mathrm{Dom}(A),\ \|f\|_{\mathcal{H}} = 1\},$$

while $W(A)$ may be more common in the operator theoretic literature; the numerical range of a quadratic form is introduced analogously, *cf.* [15, Sec. VI].

The essential spectrum of a non-self-adjoint operator may be defined in several, different and in general not equivalent, ways. Here we use the definition via Weyl singular sequences, denoted by $\sigma_{\mathrm{e2}}(\cdot)$ in [7, Sec. IX],

$$\sigma_{\mathrm{e2}}(A) = \{\lambda \in \mathbb{C} : \exists\{\psi_n\} \subset \mathrm{Dom}(A),\ \|\psi_n\| = 1,\ \psi_n \xrightarrow{w} 0, (A-\lambda)\psi_n \to 0,\ n \to \infty\}.$$

2. Generation of a contraction semigroup

Throughout the paper, if not stated otherwise, we shall assume that the damping and the potential satisfy the following regularity conditions.

Assumption I (*Regularity assumptions on the damping a and the potential q*)**.** Let $a \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R})$, $q \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$ with $a, q \geq 0$. Suppose that a can be decomposed into a regular and singular part as

$$a = a_{\mathrm{r}} + a_{\mathrm{s}}, \quad a_{\mathrm{r}} \geq 0,$$

with $a_{\mathrm{r}} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega};\mathbb{R})$, $a_{\mathrm{s}} \in L^2_{\mathrm{loc}}(\Omega;\mathbb{R})$ and, for every $\varepsilon > 0$, there exists a constant $M_\nabla = M_\nabla(\varepsilon) > 0$ such that

$$|\nabla a_{\mathrm{r}}| \leq \varepsilon a_{\mathrm{r}}^{\frac{3}{2}} + M_\nabla(q^{\frac{1}{2}} + 1). \tag{2.1}$$

Further assume that, for every $\varepsilon > 0$, there exists a constant $M_{\mathrm{s}} = M_{\mathrm{s}}(\varepsilon) > 0$ such that, for all $\psi \in \mathrm{Dom}(a_{\mathrm{r}}) \cap \mathrm{Dom}(-\Delta+q)$, *cf.* (1.6),

$$\|a_{\mathrm{s}}\psi\| \leq \varepsilon(\|(-\Delta+q)\psi\| + \|a_{\mathrm{r}}\psi\|) + M_{\mathrm{s}}\|\psi\|. \tag{2.2}$$

Remark 2.1. The exponent $\frac{3}{2}$ in (2.1) is known to be optimal for the so-called separation property of the domain of $-\Delta + a$, *cf.* for instance [8], which will be proved (and used) here as well, *cf.* Theorem 2.4.

In some cases, we will assume, in addition, that a is unbounded at infinity which results in special spectral features like in Proposition 3.1 or Theorem 3.2.

Assumption II (*Unboundedness of damping a at infinity*)**.** Let a satisfy

$$\lim_{k\to\infty} \operatorname*{ess\,inf}_{x\in\Omega, |x|>k} a(x) = \infty.$$

We are mostly interested in the situation when a is not dominated by q, and a typical potential q being bounded (or even 0). The case where q dominates a is discussed in Remark 3.3.

In order to find a suitable operator realization of the formal operator matrix G, *cf.* (1.2), we denote by $\mathcal{W}(\Omega)$ the completion of the pre-Hilbert space

$$\Big(C_0^\infty(\Omega), \langle \nabla\cdot, \nabla\cdot\rangle + \langle q^{\frac{1}{2}}\cdot, q^{\frac{1}{2}}\cdot\rangle\Big)$$

the inner product of which is non-degenerate since ∇ is injective on $C_0^\infty(\Omega)$, and we define the product Hilbert space

$$\begin{aligned} \mathcal{H} &:= \mathcal{W}(\Omega) \times L^2(\Omega), \\ \langle(\phi_1,\phi_2),(\psi_1,\psi_2)\rangle_{\mathcal{H}} &:= \langle\nabla\phi_1,\nabla\psi_1\rangle + \langle q^{\frac{1}{2}}\phi_1, q^{\frac{1}{2}}\psi_1\rangle + \langle\phi_2,\psi_2\rangle. \end{aligned} \tag{2.3}$$

Here $\operatorname{Dom}((-\Delta+q)^{1/2}) = W_0^{1,2}(\Omega)\cap\operatorname{Dom}(q^{1/2}) \subset \mathcal{W}(\Omega)$ and equality holds if, for example, there is a positive constant $\mathfrak{q}_0$ such that $q \geq \mathfrak{q}_0 > 0$, *cf.* [4, Thm. 1.8.1], or if Ω has finite width and so Poincaré's inequality applies, *cf.* for instance [1, Thm. 6.30]; then $-\Delta+q$ is uniformly positive and the space $\mathcal{H}$ in (2.3) coincides with the usual choice of space for abstract operator matrices associated with quadratic operator functions in this case, *cf.* for instance [17], [12].

Moreover, by the first representation theorem [15, Thm. VI.2.1], $\operatorname{Dom}(-\Delta+q)$ and also its core $\mathcal{D}$ given by the restriction to functions with compact support, *cf.* (2.12), are dense in $\mathcal{W}(\Omega)$.

In $\mathcal{H}$ we introduce the densely defined operator

$$G_0 := \begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}, \qquad \operatorname{Dom}(G_0) := (\operatorname{Dom}(-\Delta+q)\cap\operatorname{Dom}(a))^2. \tag{2.4}$$

The following theorem states the fundamental property that

$$G := \overline{G_0} \tag{2.5}$$

generates a contraction semigroup; the proof is given at the end of Section 2.1 after all necessary ingredients have been derived.

Theorem 2.2. *Let a, q satisfy Assumption* I *and let G_0 be as in* (2.4)*. Then $-G_0$ is accretive and* $\operatorname{Ran}(G_0-1)$ *is dense in $\mathcal{H}$. Hence $-G_0$ is closable with m-accretive closure $-G = -\overline{G_0}$ and G generates a contraction semigroup in $\mathcal{H}$.*

2.1. **The associated quadratic operator function.** Employing sectorial forms, we introduce the family $T(\lambda)$, $\lambda\in\mathbb{C}\backslash(-\infty,0]$, *cf.* (1.5), of closed operators in $L^2(\Omega)$. Although the operator function T resembles one of the quadratic complements of G, *cf.* [23, Sec. 2.2], however, here $T(\lambda)$ is considered as an operator from $L^2(\Omega)$ to $L^2(\Omega)$ and not from $\mathcal{W}(\Omega)$ to $L^2(\Omega)$.

We shall introduce $T(\lambda)$ as

$$T(\lambda) := H_{2\lambda} + \lambda^2, \quad \operatorname{Dom}(T(\lambda)) := \operatorname{Dom}(H_{2\lambda}), \quad \lambda\in\mathbb{C}\setminus(-\infty,0], \tag{2.6}$$

via the one-parameter family of operators

$$H_\gamma = -\Delta + q + \gamma a, \quad \gamma\in\mathbb{C}\setminus(-\infty,0),$$

which will be defined rigorously below, using the first representation theorem for a *rotated* version of H_γ. In fact, the numerical range of H_γ is contained in a sector with semi-angle smaller then $\pi/2$ which need not lie in the right half-plane; however, after multiplication of H_γ by $\mathrm{e}^{-\mathrm{i}\arg(\gamma)/2}$, we obtain a sectorial operator

$\widetilde{H}_\gamma = \mathrm{e}^{-\mathrm{i}\arg(\gamma)/2} H_\gamma$. We mention that the operator family $\widetilde{H}_\gamma$, $\gamma \in \mathbb{C} \setminus (-\infty, 0]$, is not uniformly sectorial, *cf.* (2.7) below.

Note that here we have included $\gamma = 0$ on purpose, although the domains of H_γ for $\gamma \neq 0$ and for $\gamma = 0$ are very different. Clearly, for $\gamma = 0$, no rotation is needed since H_0 is self-adjoint and bounded from below. For convenience, we set $\arg(0) = 0$ in what follows.

In the definition of H_γ as well as in several auxiliary results, it suffices to assume less regularity of a than in Assumption I.

Lemma 2.3. *Let $a, q \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$ and $a, q \geq 0$. Then the following hold.*

i) *For fixed $\gamma \in \mathbb{C} \setminus (-\infty, 0)$, the form*

$$\widetilde{h}_\gamma := \mathrm{e}^{-\frac{\mathrm{i}}{2}\arg(\gamma)}(\|\nabla \cdot\|^2 + \|q^{\frac{1}{2}} \cdot\|^2) + \mathrm{e}^{\frac{\mathrm{i}}{2}\arg(\gamma)}\,|\gamma|\|a^{\frac{1}{2}} \cdot\|^2,$$
$$\mathrm{Dom}(\widetilde{h}_\gamma) := W^{1,2}_0(\Omega) \cap \mathrm{Dom}(\gamma a^{\frac{1}{2}}) \cap \mathrm{Dom}(q^{\frac{1}{2}}),$$

is closed in $L^2(\Omega)$ and sectorial with

$$\mathrm{Num}(\widetilde{h}_\gamma) \subset \left\{ z \in \mathbb{C} \,:\, |\arg z| \leq \frac{\arg(\gamma)}{2} \right\}. \tag{2.7}$$

ii) *$C^\infty_0(\Omega)$ is a core of $\widetilde{h}_\gamma$ and $\widetilde{h}_\gamma$ determines a unique m-sectorial operator $\widetilde{H}_\gamma$ in $L^2(\Omega)$.*

iii) *If Assumption* II *holds, then $\widetilde{H}_\gamma$, $\gamma \in \mathbb{C} \setminus (-\infty, 0]$, has compact resolvent.*

iv) *The operator family*

$$H_\gamma := \mathrm{e}^{\frac{\mathrm{i}}{2}\arg(\gamma)} \widetilde{H}_\gamma, \quad \gamma \in \mathbb{C} \setminus (-\infty, 0], \tag{2.8}$$

is a holomorphic family of closed operators.

Proof. i) We denote $\omega := \arg(\gamma)/2 \in (-\pi/2, \pi/2)$. Since

$$\begin{aligned} \mathrm{Re}\,\widetilde{h}_\gamma[\psi] &= \cos\omega \left(\|\nabla\psi\|^2 + \|q^{\frac{1}{2}}\psi\|^2 + |\gamma|\|a^{\frac{1}{2}}\psi\|^2 \right), \\ \mathrm{Im}\,\widetilde{h}_\gamma[\psi] &= -\sin\omega \left(\|\nabla\psi\|^2 + \|q^{\frac{1}{2}}\psi\|^2 - |\gamma|\|a^{\frac{1}{2}}\psi\|^2 \right), \end{aligned} \tag{2.9}$$

the sectoriality of $\widetilde{h}_\gamma$ follows from

$$\left|\mathrm{Im}\,\widetilde{h}_\gamma[\psi]\right| \leq |\sin\omega| \,(\|\nabla\psi\|^2 + \|q^{\frac{1}{2}}\psi\|^2 + |\gamma|\|a^{\frac{1}{2}}\psi\|^2) \leq |\tan\omega|\,\mathrm{Re}\,\widetilde{h}_\gamma[\psi]. \tag{2.10}$$

The form $\widetilde{h}_\gamma$ is closed since $\mathrm{Re}\,\widetilde{h}_\gamma$ is closed, *cf.* [4, Thm. 1.8.1]. The enclosure (2.7) of the numerical range of $\widetilde{h}_\gamma$ follows from (2.10).

ii) The core property of $C^\infty_0(\Omega)$ follows from [4, Thm. 1.8.1] and [15, Thm. VI.1.21]. The operator $\widetilde{H}_\gamma$ is determined by the first representation theorem [15, Thm. VI.2.1].

iii) The resolvent of $\widetilde{H}_\gamma$ is compact if and only if the resolvent of $\mathrm{Re}\,\widetilde{H}_\gamma$ is compact, *cf.* [15, Thm. VI.3.3.]. The operator $\mathrm{Re}\,\widetilde{H}_\gamma$, induced by the form $\mathrm{Re}\,\widetilde{h}_\gamma$, is self-adjoint and has compact resolvent if $\mathrm{Dom}(\widetilde{h}_\gamma)$ is compactly embedded in $L^2(\Omega)$, *cf.* [20, Thm. XIII.67]. If Ω is bounded, then $W^{1,2}_0(\Omega)$, and hence $\mathrm{Dom}(\widetilde{h}_\gamma)$, is compactly embedded in $L^2(\Omega)$ by the Rellich-Kondrachov Theorem, *cf.* [1, Thm. 6.3]. For unbounded Ω, let $\phi \in \mathrm{Dom}(\widetilde{h}_\gamma)$. The zero extension $\widetilde{\phi}$ of ϕ belongs to $W^{1,2}(\mathbb{R}^d)$, *cf.* [1, Lem. 3.27], and to $\mathrm{Dom}(a_{\mathrm{ext}}^{1/2})$ where

$$a_{\mathrm{ext}}(x) := \begin{cases} a(x), & x \in \Omega, \\ \operatorname*{ess\,inf}_{x\in\Omega, |x|>k} a(x), & x \notin \Omega,\ |x| = k. \end{cases}$$

Moreover, there exists a non-negative constant C, independent of ϕ, such that

$$\|\widetilde{\phi}\|^2_{W^{1,2}(\mathbb{R}^d)} + \|a_{\mathrm{ext}}^{\frac{1}{2}}\widetilde{\phi}\|^2_{L^2(\mathbb{R}^d)} \leq C(\mathrm{Re}\,\widetilde{h}_\gamma[\phi] + \|\phi\|^2_{L^2(\Omega)}).$$

The function $a_{\rm ext}$ satisfies Assumption II on $\mathbb{R}^d$ and thus, by Rellich's criterion, *cf.* [20, Thm. XIII.65], $\operatorname{Dom}(\widetilde{h}_\gamma)$ is compactly embedded in $L^2(\Omega)$ also for unbounded Ω.

iv) We verify that H_γ is holomorphic (in the sense of [15, Sec. VI.1.2]) in a neighborhood of any $\gamma_0 \in \mathbb{C} \setminus (-\infty, 0]$. The strategy is to use the analyticity of the associated quadratic form. Nonetheless, we first note that, in a neighborhood of γ_0, H_γ is equal to the operator

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\arg(\gamma_0)}\widehat{H}_\gamma := \mathrm{e}^{\frac{\mathrm{i}}{2}\arg(\gamma_0)}(\mathrm{e}^{-\frac{\mathrm{i}}{2}\arg(\gamma_0)}(-\Delta + q) + \mathrm{e}^{-\frac{\mathrm{i}}{2}\arg(\gamma_0)}\gamma a)$$

where $\widehat{H}_\gamma$ is the m-sectorial operator introduced through the sectorial form

$$\begin{aligned} \widehat{h}_\gamma &:= \mathrm{e}^{-\frac{\mathrm{i}}{2}\arg(\gamma_0)}(\|\nabla \cdot\|^2 + \|q^{\frac{1}{2}} \cdot\|^2) + \mathrm{e}^{-\frac{\mathrm{i}}{2}\arg(\gamma_0)}\gamma \|a^{\frac{1}{2}} \cdot\|^2, \\ \operatorname{Dom}(\widehat{h}_\gamma) &:= W_0^{1,2}(\Omega) \cap \operatorname{Dom}(a^{\frac{1}{2}}) \cap \operatorname{Dom}(q^{\frac{1}{2}}); \end{aligned}$$

the sectoriality and closedness of $\widehat{h}_\gamma$ can be verified as in the proof of i), *cf.* (2.9)–(2.10), and the equality of the operators H_γ and $\mathrm{e}^{\frac{\mathrm{i}}{2}\arg(\gamma_0)}\widehat{H}_\gamma$ follows from [15, Cor. VI.2.4]. Since the rotation in $\widehat{h}_\gamma$ is independent of γ, the form associated with $\widehat{H}_\gamma$ is obviously an analytic family of type (a) in a neighborhood of γ_0, *cf.* [15, Sec. VII.4.2]. Thus $\widehat{H}_\gamma$, and hence H_γ, are holomorphic in a neighborhood of γ_0. □

In case of higher regularity of a as required in Assumption I, we obtain the following separation property of $\operatorname{Dom}(T(\lambda)) = \operatorname{Dom}(H_{2\lambda})$ which ensures that $T(\lambda)$ is defined as a sum of unbounded operators. The strategy of the proof is similar to [16], but the different type of potentials used here requires new estimates.

Theorem 2.4. *Let a, q satisfy Assumption* I *and let* $\operatorname{Dom}(-\Delta + q)$ *be as in* (1.6)*. Then $T(\lambda)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, is a holomorphic family of type* (*A*) *with*

$$\operatorname{Dom}(T(\lambda)) = \operatorname{Dom}(-\Delta + q) \cap \operatorname{Dom}(a), \quad \lambda \in \mathbb{C} \setminus (-\infty, 0], \tag{2.11}$$

the set

$$\mathcal{D} := \{\psi \in \operatorname{Dom}(-\Delta + q) \,:\, \operatorname{supp} \psi \text{ is compact in } \mathbb{R}^d\} \subset \operatorname{Dom}(a_{\rm r}) \tag{2.12}$$

is a core of $T(\lambda)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, and

$$T(\lambda)^* = \mathcal{C} T(\lambda) \mathcal{C} = T(\overline{\lambda}), \quad \lambda \in \mathbb{C} \setminus (-\infty, 0], \tag{2.13}$$

where $\mathcal{C}$ is the (*antilinear*) *operator of complex conjugation in $L^2(\Omega)$.*

Proof. By (2.6), it suffices to analyze H_γ with $\gamma = 2\lambda \in \mathbb{C} \setminus (-\infty, 0]$. It follows from the first representation theorem, *cf.* [15, Thm. VI.2.1], that

$$\operatorname{Dom}(H_\gamma) \subset \{\psi \in \operatorname{Dom}(\widetilde{h}_\gamma) \,:\, (-\Delta + q + \gamma a)\psi \in L^2(\Omega)\}.$$

Similarly, for $H_\gamma^{\rm r} := -\Delta + q + \gamma a_{\rm r}$, introduced in the same way as H_γ through the form $\widetilde{h}_\gamma^{\rm r}$, we have

$$\operatorname{Dom}(H_\gamma^{\rm r}) \subset \{\psi \in \operatorname{Dom}(\widetilde{h}_\gamma^{\rm r}) \,:\, (-\Delta + q + \gamma a_{\rm r})\psi \in L^2(\Omega)\}.$$

Below we prove that $\mathcal{D}$ is a core of $H_\gamma^{\rm r}$ and that there exist positive constants k_1 and k_2 such that, for all $\psi \in \mathcal{D}$,

$$\begin{aligned} &k_1 \left(\|(-\Delta + q)\psi\|^2 + \|a_{\rm r}\psi\|^2 + \|\psi\|^2\right) \\ &\leq \|H_\gamma^{\rm r}\psi\|^2 + \|\psi\|^2 \leq k_2 \left(\|(-\Delta + q)\psi\|^2 + \|a_{\rm r}\psi\|^2 + \|\psi\|^2\right), \end{aligned} \tag{2.14}$$

from which it follows that $\operatorname{Dom}(H_\gamma^{\rm r}) = \operatorname{Dom}(-\Delta + q) \cap \operatorname{Dom}(a_{\rm r})$.

By (2.2) in Assumption I and (2.14), $\gamma a_{\rm s}$ is a relatively bounded perturbation of $H_\gamma^{\rm r}$ with relative bound 0, thus $\operatorname{Dom}(H_\gamma^{\rm r} + \gamma a_{\rm s}) = \operatorname{Dom}(H_\gamma^{\rm r}) = \operatorname{Dom}(-\Delta + q) \cap \operatorname{Dom}(a)$. Moreover, $H_\gamma^{\rm r} + \gamma a_{\rm s} \subset H_\gamma$ and a standard perturbation argument shows that, for sufficiently large positive z, we have $-\mathrm{e}^{\mathrm{i}\arg(\gamma)/2} z \in \rho(H_\gamma^{\mathrm{r}} + \gamma a_{\mathrm{s}})$. Hence $\mathrm{e}^{-\mathrm{i}\arg(\gamma)/2}(H_\gamma^{\mathrm{r}} + \gamma a_{\mathrm{s}})$ is m-sectorial, and $H_\gamma^{\mathrm{r}} + \gamma a_{\mathrm{s}} = H_\gamma$, *cf.* [15, Sec. V.3].

To prove (2.11), it therefore remains to be shown that $\mathcal{D}$ is a core of H_γ^{r} and that (2.14) holds. Take $\psi \in \mathrm{Dom}(H_\gamma^{\mathrm{r}})$ and notice that $a_{\mathrm{r}}\psi \in L^2_{\mathrm{loc}}(\overline{\Omega})$ by Assumption I, thus $(-\Delta+q)\psi \in L^2_{\mathrm{loc}}(\overline{\Omega})$ as well. We first prove the core property by a suitable cut-off, *cf.* [5, Proof of Thm. 8.2.1]. Let φ be a $C_0^\infty(\mathbb{R}^d)$ function taking on non-negative values such that $\varphi(x) = 1$ if $|x| < 1$ and $\varphi(x) = 0$ if $|x| > 2$. For $\psi \in \mathrm{Dom}(H_\gamma^{\mathrm{r}})$ define

$$\psi_n(x) := \psi(x)\varphi_n(x), \quad \varphi_n(x) := \varphi\left(\frac{x}{n}\right), \quad x \in \Omega, \quad n \in \mathbb{N}.$$

From the derived regularity of ψ and the compactness of $\operatorname{supp}\varphi_n$, we conclude that $\{\psi_n\} \subset \mathcal{D}$. Moreover, by the dominated convergence theorem, $\|\psi_n - \psi\| \to 0$ as $n \to \infty$, and

$$\|H_\gamma^{\mathrm{r}}(\psi - \psi_n)\| \le \|(1-\varphi_n)(-\Delta+q+\gamma a_{\mathrm{r}})\psi\| + \|2\nabla\psi.\nabla\varphi_n + \psi\Delta\varphi_n\| \longrightarrow 0, \quad n \to \infty,$$

since $\|\nabla\varphi_n\|_{L^\infty(\mathbb{R}^d)} = \frac{1}{n}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}$ and $\|\Delta\varphi_n\|_{L^\infty(\mathbb{R}^d)} = \frac{1}{n^2}\|\Delta\varphi\|_{L^\infty(\mathbb{R}^d)}$.

Next, we prove (2.14). The second inequality in (2.14) is obvious. To prove the first one, we consider the cases $\operatorname{Im}\gamma \neq 0$ and $\gamma \neq 0$ only, the symmetric case with $\gamma > 0$ being analogous and, in fact, simpler. For every $\psi \in \mathcal{D}$,

$$\begin{aligned}\|H_\gamma^{\mathrm{r}}\psi\|^2 &= \|(-\Delta+q)\psi\|^2 + |\gamma|^2\|a_{\mathrm{r}}\psi\|^2 + 2\operatorname{Re}\langle(-\Delta+q)\psi, \gamma a_{\mathrm{r}}\psi\rangle \\ &= \|(-\Delta+q)\psi\|^2 + |\gamma|^2\|a_{\mathrm{r}}\psi\|^2 + 2(\operatorname{Re}\gamma)\langle q^{\frac12}\psi, a_{\mathrm{r}} q^{\frac12}\psi\rangle \\ &\quad + 2\operatorname{Re}(\gamma\langle\nabla\psi, \nabla(a_{\mathrm{r}}\psi)\rangle);\end{aligned}$$

note that the second step is justified since it can be verified that $a_{\mathrm{r}}\psi \in \mathrm{Dom}(h_0)$. Straightforward manipulations with the last term yield that

$$\begin{aligned}2\operatorname{Re}(\gamma\langle\nabla\psi, \nabla(a_{\mathrm{r}}\psi)\rangle) &= 2(\operatorname{Re}\gamma)\langle\nabla\psi, a_{\mathrm{r}}\nabla\psi\rangle + 2\operatorname{Re}(\gamma\langle\nabla\psi, \psi\nabla a_{\mathrm{r}}\rangle) \\ &\ge 2(\operatorname{Re}\gamma)\langle\nabla\psi, \nabla(a_{\mathrm{r}}\psi)\rangle - 4|\gamma|\langle|\psi||\nabla a_{\mathrm{r}}|, |\nabla\psi|\rangle.\end{aligned}$$

Hence, for every $\varepsilon_1 \in (0,1)$,

$$\begin{aligned}\|H_\gamma^{\mathrm{r}}\psi\|^2 &\ge \|(-\Delta+q)\psi\|^2 + |\gamma|^2\|a_{\mathrm{r}}\psi\|^2 + 2(\operatorname{Re}\gamma)\langle(-\Delta+q)\psi, a_{\mathrm{r}}\psi\rangle \\ &\quad - 4|\gamma|\,\langle|\psi||\nabla a_{\mathrm{r}}|, |\nabla\psi|\rangle \\ &\ge \frac{\varepsilon_1}{1+\varepsilon_1}\|(-\Delta+q)\psi\|^2 + ((\operatorname{Im}\gamma)^2 - \varepsilon_1(\operatorname{Re}\gamma)^2)\|a_{\mathrm{r}}\psi\|^2 \\ &\quad - 4|\gamma|\,\langle|\psi||\nabla a_{\mathrm{r}}|, |\nabla\psi|\rangle\end{aligned}$$

where we used Young's inequality in the last step. Since $\psi \in \mathrm{Dom}(H_0)$ and a_{r} satisfies (2.1), we see that, for every $\varepsilon_2, \varepsilon_3 \in (0,1)$,

$$\begin{aligned}2\langle|\psi||\nabla a_{\mathrm{r}}|, |\nabla\psi|\rangle &\le 2\langle(\varepsilon_2 a_{\mathrm{r}}^{\frac32} + M_\nabla(q^{\frac12}+1))|\psi|, |\nabla\psi|\rangle \\ &\le \varepsilon_2(\|a_{\mathrm{r}}^{\frac12}\nabla\psi\|^2 + \|a_{\mathrm{r}}\psi\|^2) \\ &\quad + 2M_\nabla(\|\nabla\psi\|^2 + \|q^{\frac12}\psi\|^2 + \|\psi\|^2) \\ &\le \varepsilon_2(\|a_{\mathrm{r}}^{\frac12}\nabla\psi\|^2 + \|a_{\mathrm{r}}\psi\|^2) + \varepsilon_3\|(-\Delta+q)\psi\|^2 + C\|\psi\|^2\end{aligned} \tag{2.15}$$

where C is independent of ψ. Combining the estimates above, we obtain

$$\begin{aligned}\|H_\gamma^{\mathrm{r}}\psi\|^2 &\ge \left(\frac{\varepsilon_1}{1+\varepsilon_1} - 2|\gamma|\varepsilon_3\right)\|(-\Delta+q)\psi\|^2 \\ &\quad + ((\operatorname{Im}\gamma)^2 - \varepsilon_1(\operatorname{Re}\gamma)^2 - 2\varepsilon_2|\gamma|)\|a_{\mathrm{r}}\psi\|^2 \\ &\quad - 2\varepsilon_2|\gamma|\,\|a_{\mathrm{r}}^{\frac12}\nabla\psi\|^2 - 2|\gamma|C\|\psi\|^2.\end{aligned} \tag{2.16}$$

It remains to consider the term $\|a_{\mathrm{r}}^{\frac{1}{2}}\nabla\psi\|^2$ in (2.16). Clearly, we have

$$2\operatorname{Im}\langle H_\gamma^{\mathrm{r}}\psi, \operatorname{sgn}(\operatorname{Im}\gamma)(-\Delta+q)\psi\rangle \leq \frac{1}{\varepsilon_4}\|H_\gamma^{\mathrm{r}}\psi\|^2 + \varepsilon_4\|(-\Delta+q)\psi\|^2 \tag{2.17}$$

for any $\varepsilon_4 \in (0,1)$. On the other hand,

$$\begin{aligned}\operatorname{Im}\langle H_\gamma^{\mathrm{r}}\psi, \operatorname{sgn}(\operatorname{Im}\gamma)(-\Delta+q)\psi\rangle &= \operatorname{Im}(\gamma\operatorname{sgn}(\operatorname{Im}\gamma)\langle a_{\mathrm{r}}\psi,(-\Delta+q)\psi\rangle) \\ &\geq |\operatorname{Im}\gamma|\,\|a_{\mathrm{r}}^{\frac{1}{2}}\nabla\psi\|^2 - |\gamma|\,\langle|\nabla a_{\mathrm{r}}||\psi|, |\nabla\psi|\rangle.\end{aligned} \tag{2.18}$$

Thus using (2.17), (2.18) in (2.16) and (2.15), we arrive at

$$\begin{aligned}\left(1+\frac{1}{\varepsilon_4}\right)\|H_\gamma^{\mathrm{r}}\psi\|^2 \geq{}& \left(\frac{\varepsilon_1}{1+\varepsilon_1} - 3|\gamma|\varepsilon_3 - \varepsilon_4\right)\|(-\Delta+q)\psi\|^2 \\ &+ \left((\operatorname{Im}\gamma)^2 - \varepsilon_1(\operatorname{Re}\gamma)^2 - 3|\gamma|\varepsilon_2\right)\|a_{\mathrm{r}}\psi\|^2 \\ &+ (2|\operatorname{Im}\gamma| - 3|\gamma|\varepsilon_2)\|a_{\mathrm{r}}^{\frac{1}{2}}\nabla\psi\|^2 - 3|\gamma|C\|\psi\|^2.\end{aligned}$$

Hence, we can successively select $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in (0,1)$ such that the coefficients of the first three terms are positive. Then a standard argument shows the existence of $k_1 > 0$, *cf.* for instance [3, Proof of Lem. 2.9], as required in (2.14).

The $\mathcal{C}$-self-adjointness of H_γ is straightforward by the representation theorem [15, Thm. VI.2.1] and thus also (2.13) follows. □

Remark 2.5. If q satisfies certain regularity assumptions similar to those for a, then also

$$\operatorname{Dom}(-\Delta+q) = \operatorname{Dom}(\Delta_{\mathrm{D}}) \cap \operatorname{Dom}(q).$$

The latter holds *e.g.* if there is a decomposition $q = q_{\mathrm{r}} + q_{\mathrm{s}}$ with $q_{\mathrm{r}} \geq 0$, $q_{\mathrm{r}} \in W^{1,\infty}_{\mathrm{loc}}(\overline{\Omega})$, $q_{\mathrm{s}} \in L^2_{\mathrm{loc}}(\Omega)$ and, for each $\varepsilon > 0$, there are constants $M_{\nabla,q} = M_{\nabla,q}(\varepsilon) \geq 0$ and $M_{\mathrm{s},q} = M_{\mathrm{s},q}(\varepsilon) \geq 0$ such that

$$|\nabla q_{\mathrm{r}}| \leq \varepsilon q_{\mathrm{r}}^{\frac{3}{2}} + M_{\nabla,q}$$

and, for all $\psi \in \operatorname{Dom}(\Delta_{\mathrm{D}}) \cap \operatorname{Dom}(q_{\mathrm{r}})$,

$$\|q_{\mathrm{s}}\psi\| \leq \varepsilon(\|\Delta_{\mathrm{D}}\psi\| + \|q_{\mathrm{r}}\psi\|) + M_{\mathrm{s},q}\|\psi\|.$$

The proof is a simpler version of the proof of Theorem 2.4.

Proof of Theorem 2.2. Using integration by parts, it is straightforward to check that, for all $\Phi := (\phi_1, \phi_2) \in \operatorname{Dom}(G_0)$,

$$\langle G_0\Phi, \Phi\rangle_{\mathcal{H}} = 2\mathrm{i}\operatorname{Im}\left(\langle\nabla\phi_2, \nabla\phi_1\rangle + \langle q^{\frac{1}{2}}\phi_2, q^{\frac{1}{2}}\phi_1\rangle\right) - 2\|a^{\frac{1}{2}}\phi_2\|^2.$$

Thus $\operatorname{Num}(-G_0) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ and so G_0 is closable by [15, Thm. V.3.4].

Let $\mathcal{D}$ be the core of $(-\Delta+q)$ defined in (2.12). We prove that $\mathcal{D} \times C_0^\infty(\Omega) \subset \operatorname{Ran}(G_0 - 1)$. To this end, we take an arbitrary $\Psi := (\psi_1, \psi_2) \in \mathcal{D} \times C_0^\infty(\Omega)$ and find a solution $\Phi := (\phi_1, \phi_2) \in \operatorname{Dom}(G_0)$ of $(G_0 - 1)\Phi = \Psi$, *i.e.* of the system

$$\begin{aligned}-\phi_1 + \phi_2 &= \psi_1, \\ (\Delta - q)\phi_1 - (2a+1)\phi_2 &= \psi_2.\end{aligned}$$

Solving the first equation for ϕ_2 and inserting this into the second equation, we get

$$(-\Delta + q + 2a + 1)\phi_1 = -(\psi_2 + (2a+1)\psi_1).$$

Note that the left hand side equals $T(1)\phi_1$ with $T(\lambda)$ defined in Section 2.1, *cf.* (2.6). Moreover, for $\lambda = 1$, $\operatorname{Dom}(T(1)) = \operatorname{Dom}(-\Delta+q) \cap \operatorname{Dom}(a)$, *cf.* Theorem 2.4, and $0 \notin \sigma(T(1))$ since $T(1)$ is uniformly positive. Thus $T(1)^{-1}$ is a bounded operator in $L^2(\Omega)$ and hence we obtain the solution $\Phi = (\phi_1, \phi_2)$,

$$\phi_1 = -T(1)^{-1}(\psi_2 + (2a+1)\psi_1), \qquad \phi_2 = \psi_1 + \phi_1.$$

Since $\psi_1 \in \mathcal{D} \subset \operatorname{Dom}(-\Delta+q)$ and $\operatorname{supp}\psi_1$ is compact, we have $a\psi_1 \in L^2(\Omega)$ due to (2.2) and $a_{\rm r} \in L^\infty_{\rm loc}(\overline{\Omega})$. By Theorem 2.4 and because $\psi_2 \in C_0^\infty(\Omega)$, we see that $\phi_1 \in \operatorname{Dom}(T(1))$ and thus $\phi_2 \in \operatorname{Dom}(T(1))$ since $\psi_1 \in \mathcal{D} \subset \operatorname{Dom}(T(1))$. Altogether this proves $\Phi \in \operatorname{Dom}(G_0)$. □

3. Spectral equivalence for the generator G and the associated quadratic function T

In this section we prove spectral equivalence for the generator G and the quadratic operator function T. To this end, we first derive some basic spectral properties of the operator family $T(\lambda)$, $\lambda \in \mathbb{C}\setminus(-\infty,0]$.

Proposition 3.1. *Let $a, q \in L^1_{\rm loc}(\Omega;\mathbb{R})$ and $a, q \geq 0$, let $T(\lambda)$, $\lambda \in \mathbb{C}\setminus(-\infty,0]$, be as in* (2.6) *and let $a_{\inf} := \operatorname{ess\,inf}(a)$. Then, for every $\lambda \in \mathbb{C}\setminus(-\infty,0]$,*

i) $0 \in \sigma(T(\lambda)) \iff 0 \in \sigma(T(\overline{\lambda}))$,
ii) $0 \in \sigma(T(\lambda)) \implies \operatorname{Re}\lambda \leq -a_{\inf}$ *and* $|\lambda|^2 \geq \inf\sigma(-\Delta+q)$,
iii) *if, in addition, Assumption* II *is satisfied, then, for all $\lambda \in \mathbb{C}\setminus(-\infty,0]$,*

$$0 \in \sigma(T(\lambda)) \iff 0 \in \sigma_{\rm disc}(T(\lambda))$$

and the set $\{\lambda \in \mathbb{C}\setminus(-\infty,0] : 0 \in \sigma(T(\lambda))\}$ consists only of isolated points which may accumulate at most at $(-\infty,0]$.

Proof. i) The claim is immediate from $T(\lambda)^* = T(\overline{\lambda})$, $\lambda \in \mathbb{C}\setminus(-\infty,0]$, *cf.* (2.13).

ii) We rely on a numerical range argument. Recall that $T(\lambda)$ is defined through the sectorial form $\widetilde{h}_{2\lambda}$ and so for $0 \in \sigma(T(\lambda))$ it is necessary that $0 \in \overline{\operatorname{Num}(T(\lambda))} = \mathrm{e}^{\mathrm{i}\arg(\lambda)/2}\operatorname{Num}(\widetilde{h}_{2\lambda}) + \lambda^2$, *cf.* Lemma 2.3. But this is impossible if $\operatorname{Re}\lambda > 0$ by the enclosure (2.7) with $\gamma = 2\lambda$. We proceed further by contradiction. Let $\lambda \in \mathbb{C}\setminus(-\infty,0]$ with $\operatorname{Re}\lambda \leq 0$ be such that $0 \in \sigma(T(\lambda))$ and $a_{\inf} + \operatorname{Re}\lambda > \varepsilon > 0$ or $\inf\sigma(-\Delta+q) - |\lambda|^2 > \varepsilon > 0$. By the numerical range argument above, there is a sequence $\{z_n\} \subset \mathrm{e}^{\mathrm{i}\arg(\lambda)/2}\operatorname{Num}(\widetilde{h}_{2\lambda}) + \lambda^2$ such that $z_n \to 0$. Then there is a sequence $\{\psi_n\} \subset \operatorname{Dom}(\widetilde{h}_{2\lambda})$, $\|\psi_n\| = 1$, such that

$$\|\nabla\psi_n\|^2 + \|q^{\frac12}\psi_n\|^2 + 2\lambda\|a^{\frac12}\psi_n\|^2 + \lambda^2\|\psi_n\|^2 = z_n. \tag{3.1}$$

Taking the real and imaginary part of (3.1), we find

$$\|\nabla\psi_n\|^2 + \|q^{\frac12}\psi_n\|^2 + 2\operatorname{Re}\lambda\|a^{\frac12}\psi_n\|^2 + (\operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 = \operatorname{Re} z_n, \tag{3.2}$$

$$2\operatorname{Im}\lambda\big(\|a^{\frac12}\psi_n\|^2 + \operatorname{Re}\lambda\big) = \operatorname{Im} z_n. \tag{3.3}$$

Recall that $\operatorname{Im}\lambda \neq 0$ since $\lambda \in \mathbb{C}\setminus(-\infty,0]$ with $\operatorname{Re}\lambda \leq 0$.

First consider the case when $a_{\inf} + \operatorname{Re}\lambda > \varepsilon > 0$. Then (3.3) yields

$$\frac{|\operatorname{Im} z_n|}{2|\operatorname{Im}\lambda|} = \|a^{\frac12}\psi_n\|^2 + \operatorname{Re}\lambda > \varepsilon > 0,$$

a contradiction to $z_n \to 0$.

In the second case when $\inf\sigma(-\Delta+q) - |\lambda|^2 > \varepsilon > 0$, we solve (3.3) for $\|a^{1/2}\psi_n\|^2$ and insert this into (3.2) to obtain

$$\|\nabla\psi_n\|^2 + \|q^{\frac12}\psi_n\|^2 - |\lambda|^2 = \operatorname{Re} z_n - \frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\operatorname{Im} z_n. \tag{3.4}$$

Since the minimum of the spectrum of a self-adjoint operator coincides with the infimum of its numerical range and by the assumption on $|\lambda|$, we have

$$\|\nabla\psi_n\|^2 + \|q^{\frac12}\psi_n\|^2 - |\lambda|^2 \geq \inf_{\psi\in\operatorname{Dom}(\widetilde{h}_{2\lambda}),\|\psi\|=1}(\|\nabla\psi\|^2 + \|q^{\frac12}\psi\|^2) - |\lambda|^2 > \varepsilon. \tag{3.5}$$

Inserting (3.5) into (3.4), we again arrive at a contradiction to $z_n \to 0$.

iii) The claim follows from [15, Thm. VII.1.10] if we show that $T(\lambda)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, is a holomorphic family of closed operators in $L^2(\Omega)$ with compact resolvent and that there is a $\lambda_0 \in \mathbb{C} \setminus (-\infty, 0]$ for which $T(\lambda_0)^{-1}$ exists and is bounded in $L^2(\Omega)$. The compactness of the resolvents is proved in Lemma 2.3 iii) and $T(\lambda)$ is holomorphic since $H_{2\lambda}$ is holomorphic, *cf.* Lemma 2.3 iv), and λ^2, viewed as a multiplication operator, is a bounded holomorphic family, *cf.* [14, Prob. VII.1.2]. Since, for any $\lambda_0 > 0$, $T(\lambda_0)$ is a uniformly positive operator, we can choose $\lambda_0 \in (0, \infty)$ arbitrarily. □

In the case where the spectrum is discrete and there are no real eigenvalues, it is possible to extend Proposition 3.1.ii) and derive further estimates on the absolute values of eigenvalues for quadratic pencils, *cf.* [9] for the matrix case and also for wave equations on bounded domains with bounded damping via discretization.

Theorem 3.2. *Let a, q satisfy Assumption* I, *and let G, $T(\lambda)$ be as in* (2.5), (2.6), *respectively. Then, for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$,*

$$\begin{aligned} \lambda \in \sigma(G) &\iff 0 \in \sigma(T(\lambda)), \\ \lambda \in \sigma_{\mathrm{p}}(G) &\iff 0 \in \sigma_{\mathrm{p}}(T(\lambda)), \\ \lambda \in \sigma_{\mathrm{e2}}(G) &\iff 0 \in \sigma_{\mathrm{e2}}(T(\lambda)), \end{aligned} \tag{3.6}$$

and

$$\psi \in \operatorname{Ker}(T(\lambda)) \iff (\psi, \lambda\psi) \in \operatorname{Ker}(G - \lambda).$$

If, in addition, a satisfies Assumption II, *then $\sigma(G) \cap \mathbb{C} \setminus (-\infty, 0]$ consists only of eigenvalues of finite multiplicity which may only accumulate at* $(-\infty, 0]$.

Proof. Let $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ be fixed. We split the proof into several steps.

• Claim i): $\Psi = (\psi_1, \psi_2) \in \operatorname{Ran}(G - \lambda)^\perp \implies \overline{\lambda}\psi_1 + \psi_2 = 0$ and $\psi_2 \in \operatorname{Ker}(T(\lambda)^*)$.

To see this, take $\Phi = (\phi_1, \phi_2) \in \operatorname{Dom}(G_0)$ with G_0 as in (2.4). Then we have $\langle (G_0 - \lambda)\Phi, \Psi \rangle_{\mathcal{H}} = 0$ or, equivalently, *cf.* (2.3),

$$\langle \nabla(\phi_2 - \lambda\phi_1), \nabla\psi_1 \rangle + \langle q^{\frac{1}{2}}(\phi_2 - \lambda\phi_1), q^{\frac{1}{2}}\psi_1 \rangle + \langle (\Delta - q)\phi_1 - (2a + \lambda)\phi_2, \psi_2 \rangle = 0. \tag{3.7}$$

If we set $\phi_2 = \lambda\phi_1$, we get $\langle T(\lambda)\phi_1, \psi_2 \rangle = 0$ for all $\phi_1 \in \operatorname{Dom}(T(\lambda))$. Hence, $\psi_2 \in \operatorname{Dom}(T(\lambda)^*) = \operatorname{Dom}(T(\lambda))$ and $T(\lambda)^*\psi_2 = 0$. On the other hand, if we set $\phi_2 = 0$, then (3.7) and $\psi_2 \in \operatorname{Dom}(T(\lambda)^*) \subset \operatorname{Dom}(h_0)$, *cf.* Theorem 2.4, imply that

$$\langle \nabla\phi_1, \nabla(\overline{\lambda}\psi_1 + \psi_2) \rangle + \langle q^{\frac{1}{2}}\phi_1, q^{\frac{1}{2}}(\overline{\lambda}\psi_1 + \psi_2) \rangle = 0.$$

Since $\operatorname{Dom}(-\Delta + q) \cap \operatorname{Dom}(a)$ is dense in $\mathcal{W}(\Omega)$, we obtain $\overline{\lambda}\psi_1 + \psi_2 = 0$.

• Claim ii): $\Psi = (\psi_1, \psi_2) \in \operatorname{Ker}(G - \lambda) \iff \lambda\psi_1 - \psi_2 = 0$ and $\psi_2 \in \operatorname{Ker}(T(\lambda))$.

It is straightforward to check the implication "$\Longleftarrow$" since the assumptions imply that $\Psi \in \operatorname{Dom}(G_0)$, *cf.* Theorem 2.4 and (2.4). To prove the implication "$\Longrightarrow$", we first integrate by parts to conclude that the operator

$$G_0^{\mathrm{c}} := \begin{pmatrix} 0 & -I \\ -\Delta + q & -2a \end{pmatrix}, \quad \operatorname{Dom}(G_0^{\mathrm{c}}) := \mathcal{D} \times \mathcal{D},$$

with $\mathcal{D}$ defined as in (2.12) is a densely defined restriction of $G^* = G_0^*$. Then $\Psi \in \operatorname{Ker}(G - \lambda)$ implies that, for all $\Phi = (\phi_1, \phi_2) \in \operatorname{Dom}(G_0^{\mathrm{c}})$,

$$0 = \langle (G - \lambda)\Psi, \Phi \rangle_{\mathcal{H}} = \langle \Psi, (G_0^{\mathrm{c}} - \overline{\lambda})\Phi \rangle_{\mathcal{H}}$$

or, equivalently,

$$-\langle \nabla\psi_1, \nabla(\overline{\lambda}\phi_1 + \phi_2) \rangle - \langle q^{\frac{1}{2}}\psi_1, q^{\frac{1}{2}}(\overline{\lambda}\phi_1 + \phi_2) \rangle - \langle \psi_2, (\Delta - q)\phi_1 + (2a + \overline{\lambda})\phi_2 \rangle = 0. \tag{3.8}$$

Setting $\phi_2 = -\overline{\lambda}\phi_1$, we obtain $\langle \psi_2, T(\lambda)^*\phi_1 \rangle = 0$ for all $\phi_1 \in \mathcal{D}$. Since $\mathcal{D}$ is a core of $T(\lambda)^*$, we have $\psi_2 \in \operatorname{Dom}(T(\lambda))$ and $T(\lambda)\psi_2 = 0$. Finally, setting $\phi_2 = 0$ and using (3.8), we find that, for all $\phi_1 \in \mathcal{D}$,

$$\langle \nabla(\lambda\psi_1 - \psi_2), \nabla\phi_1 \rangle + \langle q^{\frac{1}{2}}(\lambda\psi_1 - \psi_2), q^{\frac{1}{2}}\phi_1 \rangle = 0,$$

hence $\lambda\psi_1 - \psi_2 = 0$ because $\mathcal{D}$ is dense in $\mathcal{W}(\Omega)$.

• Claim iii): $0 \in \sigma_{e2}(T(\lambda)) \iff \lambda \in \sigma_{e2}(G)$.

Let $0 \in \sigma_{e2}(T(\lambda))$ and let $\{\psi_n\} \subset \operatorname{Dom}(T(\lambda))$ be a corresponding singular sequence, *i.e.* $\|\psi_n\| = 1$, $\psi_n \xrightarrow{w} 0$ and $T(\lambda)\psi_n \to 0$ in $L^2(\Omega)$ as $n \to \infty$. Then $\Psi_n := (\psi_n, \lambda\psi_n) \in \operatorname{Dom}(G_0)$, $n \in \mathbb{N}$, and

$$\frac{\|(G_0 - \lambda)\Psi_n\|_{\mathcal{H}}}{\|\Psi_n\|_{\mathcal{H}}} \le \frac{\|T(\lambda)\psi_n\|}{|\lambda|} \to 0, \quad n \to \infty.$$

Thus it remains to be verified that $\widetilde{\Psi}_n := \Psi_n / \|\Psi_n\|_{\mathcal{H}} \xrightarrow{w} 0$ as $n \to \infty$ in $\mathcal{H}$. Since $\|\widetilde{\Psi}_n\|_{\mathcal{H}} = 1$, it suffices to check weak convergence on $\mathcal{D} \times \mathcal{D}$ which is dense in $\mathcal{H}$. Indeed, for $\Phi = (\phi_1, \phi_2) \in \mathcal{D} \times \mathcal{D}$,

$$|\langle \widetilde{\Psi}_n, \Phi \rangle_{\mathcal{H}}| \le \frac{|\langle \psi_n, (-\Delta + q)\phi_1 \rangle| + |\lambda||\langle \psi_n, \phi_2 \rangle|}{|\lambda|} \to 0, \quad n \to \infty,$$

since $\psi_n \xrightarrow{w} 0$ in $L^2(\Omega)$ as $n \to \infty$. Hence the implication "$\Longrightarrow$" is proved.

To prove the reverse implication "$\Longleftarrow$", assume that $0 \notin \sigma_{e2}(T(\lambda))$. In order to show that $\lambda \notin \sigma_{e2}(G)$, we construct a (bounded) left approximate inverse, *cf.* [7, Def. I.3.8], of $G - \lambda$. Then it follows from [7, Thm. I.3.13] that $G - \lambda$ is semi-Fredholm. Moreover, we have $\dim \operatorname{Ker}(G - \lambda) < \infty$ by claim ii) proved above.

It remains to construct a left approximate inverse of $G - \lambda$. Since $T(\lambda)$ is J-self-adjoint, we have $\dim \operatorname{Ker}(T(\lambda)) = \dim \operatorname{Ker}(T(\lambda)^*)$, *cf.* [7, Lem. III.5.4], thus $T(\lambda)$ is Fredholm. Hence there exists a generalized inverse $T(\lambda)^{\#}$, *cf.* [19, Sec. 5], *i.e.*

$$\begin{aligned} T(\lambda)T(\lambda)^{\#}\psi &= \psi - Q\psi, \quad \psi \in L^2(\Omega), \\ T(\lambda)^{\#}T(\lambda)\psi &= \psi - P\psi, \quad \psi \in \operatorname{Dom}(T(\lambda)), \end{aligned} \tag{3.9}$$

where P, Q are the orthogonal projections on $\operatorname{Ker}(T(\lambda))$, $\operatorname{Ker}(T^*(\lambda))$, respectively.

Let $\Phi = (\phi_1, \phi_2) \in \operatorname{Dom}(G_0)$ and $\Psi = (\psi_1, \psi_2) \in \mathcal{H}$ be so that $(G_0 - \lambda)\Phi = \Psi$, *i.e.*

$$\begin{aligned} \phi_2 - \lambda\phi_1 &= \psi_1, \\ (\Delta - q)\phi_1 - (2a + \lambda)\phi_2 &= \psi_2; \end{aligned}$$

notice that $\psi_1 \in \operatorname{Dom}(-\Delta + q)$ by the first equation and since $\Phi \in \operatorname{Dom}(G_0)$. Solving the first equation for ϕ_1, *i.e.* $\phi_1 = \lambda^{-1}(\phi_2 - \psi_1)$, and inserting this expression into the second equation, we obtain, after multiplication by λ,

$$T(\lambda)\phi_2 = (-\Delta + q)\psi_1 - \lambda\psi_2.$$

Applying the generalized inverse $T(\lambda)^{\#}$, we find

$$\phi_2 = T(\lambda)^{\#}(-\Delta + q)\psi_1 - \lambda T(\lambda)^{\#}\psi_2 + P\phi_2,$$

and thus, recalling that $\phi_1 = \lambda^{-1}(\phi_2 - \psi_1)$, we arrive at

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\lambda}\left(T(\lambda)^{\#}(-\Delta + q) - I\right) & -T(\lambda)^{\#} \\ T(\lambda)^{\#}(-\Delta + q) & -\lambda T(\lambda)^{\#} \end{pmatrix}}_{=:\widehat{R}_\lambda} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \frac{1}{\lambda}P \\ 0 & P \end{pmatrix}}_{=:K_\lambda} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3.10}$$

Hence, for all $\Phi \in \operatorname{Dom}(G_0)$, $\widehat{R}_\lambda(G_0 - \lambda)\Phi = \Phi - K_\lambda\Phi$ and K_λ is compact since P has finite rank and is everywhere defined, both as an operator in $L^2(\Omega)$ and as an operator from $L^2(\Omega)$ to $\mathcal{W}(\Omega)$ because $\operatorname{Ker}(T(\lambda)) \subset \operatorname{Dom}(T(\lambda)) \subset \mathcal{W}(\Omega)$.

Next we show that $\widehat{R}_\lambda$ has a bounded extension R_λ onto $\mathcal{H}$, which is a left approximate inverse for the closure $G-\lambda$ of $G_0-\lambda$, *i.e.* ,

$$R_\lambda(G-\lambda)\Phi = \Phi - K_\lambda\Phi, \quad \Phi \in \mathrm{Dom}(G).$$

To this end, in the representation of $\widehat{R}_\lambda$, *cf.* (3.10), we replace $T(\lambda)^{\#}$ first by $(T(\lambda)+\lambda_0)^{-1}$ with some $\lambda_0 \in \rho(T(\lambda)) \neq \emptyset$ and then the latter by the self-adjoint operator $T(1)^{-1}$. More precisely, with the help of (3.9), we derive the resolvent-type identities

$$\begin{aligned} T(\lambda)^{\#} &= T(\lambda)^{\#}(T(\lambda)+\lambda_0)(T(\lambda)+\lambda_0)^{-1} \\ &= (I-P)(T(\lambda)+\lambda_0)^{-1} + \lambda_0 T(\lambda)^{\#}(T(\lambda)+\lambda_0)^{-1}, \\ T(\lambda)^{\#} &= (T(\lambda)+\lambda_0)^{-1}(T(\lambda)+\lambda_0)T(\lambda)^{\#} = (T(\lambda)+\lambda_0)^{-1}(I-Q+\lambda_0 T(\lambda)^{\#}), \end{aligned}$$

hence

$$T(\lambda)^{\#} = (I-P)(T(\lambda)+\lambda_0)^{-1} + \lambda_0(T(\lambda)+\lambda_0)^{-1}(I-Q+\lambda_0T(\lambda)^{\#})(T(\lambda)+\lambda_0)^{-1}.$$

Similarly,

$$\begin{aligned} (T(\lambda)+\lambda_0)^{-1} &= T(1)^{-1} - (T(\lambda)+\lambda_0)^{-1}(2(\lambda-1)a+\lambda^2-1+\lambda_0)T(1)^{-1}, \\ (T(\lambda)+\lambda_0)^{-1} &= T(1)^{-1} - T(1)^{-1}(2(\lambda-1)a+\lambda^2-1+\lambda_0)(T(\lambda)+\lambda_0)^{-1}. \end{aligned}$$

Since $\mathrm{Dom}(T(\lambda)) = \mathrm{Dom}(T(\lambda)^*) = \mathrm{Dom}(-\Delta+q)\cap\mathrm{Dom}(a)$ for all $\lambda\in\mathbb{C}\setminus(-\infty,0]$, the composition $a(T(\lambda)+\lambda_0)^{-1}$ is bounded on $L^2(\Omega)$; since $(T(\lambda)+\lambda_0)^{-1}a \subset (a((T(\lambda)+\lambda_0)^{-1})^*)^*$, the operator $(T(\lambda)+\lambda_0)^{-1}a$ has a bounded extension onto $L^2(\Omega)$.

A careful inspection of the individual terms in $\widehat{R}_\lambda$ using the identities derived for $T(\lambda)^{\#}$ shows that the most problematic term is $T(1)^{-1}(-\Delta+q)$; we will show that it has an extension to a bounded operator from $\mathcal{W}(\Omega)$ to $\mathcal{W}(\Omega)$. The remaining terms can be handled in a similar (simpler) way; notice also that the terms containing P or Q are of finite rank and everywhere defined since $\mathrm{Dom}(T(\lambda)) = \mathrm{Dom}(T(\lambda)^*) \subset \mathcal{W}(\Omega)$.

Now let $\phi \in \mathrm{Dom}(-\Delta+q)$. Then, using the second representation theorem [15, Thm. VI.2.23] for $-\Delta+q$ and denoting $\psi := (-\Delta+q)^{1/2}\phi$, we obtain

$$\begin{aligned} &\frac{\|\nabla T(1)^{-1}(-\Delta+q)\phi\|^2 + \|q^{\frac12}T(1)^{-1}(-\Delta+q)\phi\|^2}{\|\nabla\phi\|^2+\|q^{\frac12}\phi\|^2} \\ &\qquad = \frac{\|(-\Delta+q)^{\frac12}T(1)^{-1}(-\Delta+q)\phi\|^2}{\|(-\Delta+q)^{\frac12}\phi\|^2} \\ &\qquad = \frac{\|(-\Delta+q)^{\frac12}(-\Delta+q+2a+1)^{-1}(-\Delta+q)^{\frac12}\psi\|^2}{\|\psi\|^2}. \end{aligned}$$

Since

$$\begin{aligned} &(-\Delta+q)^{\frac12}(-\Delta+q+2a+1)^{-1}(-\Delta+q)^{\frac12} \\ &\quad \subset (-\Delta+q)^{\frac12}(-\Delta+q+2a+1)^{-\frac12}((-\Delta+q)^{\frac12}(-\Delta+q+2a+1)^{-\frac12})^* \end{aligned} \tag{3.11}$$

and the operator on the right-hand side of (3.11) is bounded on $L^2(\Omega)$, we have

$$\frac{\|(-\Delta+q)^{\frac12}(-\Delta+q+2a+1)^{-1}(-\Delta+q)^{\frac12}\psi\|^2}{\|\psi\|^2} \le M < \infty.$$

Hence $T(1)^{-1}(-\Delta+q)$ is bounded on a dense subset of $\mathcal{W}(\Omega)$, so it has a bounded extension on $\mathcal{W}(\Omega)$.

• Claim iv): $0\in\rho(T(\lambda)) \iff \lambda\in\rho(G)$.

The implication "$\Longrightarrow$" follows immediately from claims iii), i) and ii) since $\mathrm{Ran}(G-\lambda)$ is closed and $\dim\mathrm{Ker}(G-\lambda) = \dim\mathrm{Ran}(G-\lambda)^\perp = 0$. To show

the other direction, notice that if $0 \in \sigma(T(\lambda))$, then $0 \in \sigma_{\rm p}(T(\lambda))$ or $0 \in \sigma_{\rm e2}(T(\lambda))$ since $T(\lambda)$ is $\mathcal{C}$-self-adjoint, *cf.* (2.13). Hence by claims ii) and iii), respectively, we have shown that then $\lambda \in \sigma_{\rm p}(G)$ or $\lambda \in \sigma_{\rm e2}(G)$.
• Finally, if a additionally satisfies Assumption II, the last claim follows from the established equivalences (3.6) and Proposition 3.1.iii). □

The following is a straightforward extension of the claim of Theorem 2.4 to $\mathbb{C} \setminus (-\infty, -\alpha_q]$ for some $a_q > 0$; the details are left to the reader.

Remark 3.3. Let the assumptions of Theorem 3.2 hold and let, in addition, q satisfy the conditions in Remark 2.5 ensuring that $\mathrm{Dom}(-\Delta + q) = \mathrm{Dom}(\Delta_{\rm D}) \cap \mathrm{Dom}(q)$. If there are constants $k_1 \geq 0$, $k_2 \in \mathbb{R}$ such that

$$a \leq k_1 q + k_2. \tag{3.12}$$

and M_q denotes the q-bound of a, *i.e.* the infimum of k_1 for which (3.12) holds, then the spectral equivalence (3.6) holds for $\lambda \in \mathbb{C} \setminus (-\infty, -\alpha_q]$ with

$$\alpha_q := \frac{1}{2M_q} \in (0, +\infty].$$

If, in addition, a satisfies Assumption II, then $\sigma(G) \setminus (-\infty, \alpha_q]$ consists only of eigenvalues with finite multiplicity which may accumulate only at points in $(-\infty, -\alpha_q]$.

4. Real essential spectrum of the generator G

In this section we investigate the essential spectrum of G lying on the negative real semi-axis which is not accessible via the quadratic operator function $T(\lambda)$ since the latter is not defined for $\lambda \in (-\infty, 0]$. Informally, if the underlying domain Ω contains a sufficiently large neighborhood of a ray where the damping a diverges as $|x| \to \infty$ and the potential q does not dominate a, then $(-\infty, 0] \subset \sigma_{\rm e2}(G)$. We emphasize that we do not require the potential q to be bounded.

In the sequel we decompose $x \in \mathbb{R}^d$ as $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{d-1}$. If $d > 1$ and $\Omega \neq \mathbb{R}^d$, we suppose that Ω contains a ray $\Gamma := \{(x_1, 0) : x_1 > 0\}$ and a "tubular" neighborhood U_ω of Γ given by

$$U_\omega := \left\{(x_1, x') \in \mathbb{R}^d \,:\, x_1 > 0,\ |x'| < \omega(x_1)^{-1}\right\} \tag{4.1}$$

where $\omega : (0, \infty) \to (0, \infty)$ is a continuous function satisfying certain assumptions to be specified in Theorem 4.2 below. The radius $1/\omega(x_1)$ may shrink to 0 at ∞, and the possible shrinking rate is controlled by the growth of the damping a. Note that for $d = 1$, we may let $U_\omega = \Gamma$ and no function ω is needed.

We start with the simple observation that $0 \in \sigma_{\rm e2}(G)$ if Ω contains a cone and q decays therein as $|x| \to \infty$.

Proposition 4.1. *Let a, q satisfy Assumption* I *and assume, in addition, that $q \in L^2_{\rm loc}(\Omega)$ and that G is given by* (2.5)*. If Ω contains a cone*

$$C_\delta := \{(x_1, x') \in \mathbb{R}^d \,:\, x_1 > 0,\ |x'| < \delta x_1\} \tag{4.2}$$

for some $\delta > 0$ and if

$$\lim_{k \to \infty} \ \operatorname*{ess\,sup}_{x \in C_\delta, |x| > k} \ q(x) = 0, \tag{4.3}$$

then $0 \in \sigma_{\rm e2}(G)$.

Proof. It suffices to find a sequence $\{\Phi_n\} \subset \mathrm{Dom}(G_0)$, $\Phi_n \neq 0$, $n \in \mathbb{N}$, such that $\Phi_n / \|\Phi_n\|_{\mathcal{H}} \xrightarrow{w} 0$ in $\mathcal{H}$ and $G_0 \Phi_n / \|\Phi_n\|_{\mathcal{H}} \to 0$ in $\mathcal{H}$ as $n \to \infty$.

For $d>1$, we work in spherical coordinates $x=(|x|,\Theta)$ with $\Theta\in S^{d-1}$; the simplifications for $d=1$ are obvious. Let $0\neq\varphi\in C_0^\infty((0,1))$, $0\neq\chi\in C_0^\infty(S^{d-1}\cap C_\delta)$, and define

$$\phi_n(|x|,\Theta):=|x|^{-\frac{d-1}{2}}\varphi_n(|x|)\chi(\Theta),\quad \varphi_n(|x|):=\rho_n^{\frac14}\varphi(\rho_n^{\frac12}|x|-n),\quad n\in\mathbb{N},$$

where

$$\rho_n:=\operatorname*{ess\,sup}_{x\in C_\delta,|x|>n} q(x).$$

Straightforward, but lengthy, calculations yield that, as $n\to\infty$,

$$\|\varphi_n\|_{L^2(\mathbb{R})}=\mathcal{O}(1),\qquad \|\varphi_n'\|_{L^2(\mathbb{R})}=\mathcal{O}(\rho_n^{\frac12}),\qquad \|\varphi_n''\|_{L^2(\mathbb{R})}=\mathcal{O}(\rho_n),$$
$$\|\nabla\phi_n\|^{-1}=\mathcal{O}(\rho_n^{-\frac12}),\qquad \|\Delta\phi_n\|=\mathcal{O}(\rho_n),\qquad \|q\phi_n\|=\mathcal{O}(\rho_n).$$

If we define $\Phi_n:=(\phi_n,0)$, then $\Phi_n/\|\Phi_n\|_{\mathcal{H}}\xrightarrow{w}0$ as $n\to\infty$ since $\operatorname{supp}\phi_n$ moves to infinity. Using assumption (4.3), we obtain

$$\frac{\|G_0\Phi_n\|_{\mathcal{H}}^2}{\|\Phi_n\|_{\mathcal{H}}^2}=\frac{\|G_0\Phi_n\|_{\mathcal{H}}^2}{\|\nabla\phi_n\|^2+\|q^{\frac12}\phi_n\|^2}\leq 2\frac{\|\Delta\phi_n\|^2+\|q\phi_n\|^2}{\|\nabla\phi_n\|^2}=\mathcal{O}(\rho_n)=\mathrm{o}(1),$$

as $n\to\infty$. □

The following Theorem 4.2 provides conditions under which a fixed $\lambda\in(-\infty,0)$ belongs to $\sigma_{\mathrm{e2}}(G)$. We remark that in the case where the damping a dominates the potential q in a suitable U_ω, *i.e.* $q(x)=\mathrm{o}(a(x))$ as $|x|\to\infty$ in U_ω (and the remaining regularity and growth conditions, then independent of λ, are satisfied), *every* $\lambda\in(-\infty,0)$ belongs to $\sigma_{\mathrm{e2}}(G)$, hence $0\in\sigma(G)$ as well. This effect is clearly visible in the examples, *cf.* Section 6. We also mention that for the simplest choice $\omega(x_1)=x_1^\alpha$, $x_1>0$, $\alpha\in\mathbb{R}$, the first two conditions in (4.6) are satisfied since, for $k\in\mathbb{N}$, $|\omega^{(k)}(x_1)|=\mathcal{O}(1/x_1^k)\omega(x_1)$ as $x_1\to+\infty$.

Theorem 4.2. *Let a,q satisfy Assumption* I *and assume, in addition, that $q\in L^2_{\mathrm{loc}}(\Omega)$, and that G is defined as in* (2.5). *If, for $\lambda\in(-\infty,0)$, Ω contains a tubular neighborhood U_ω of a ray Γ such that:*

i) *there is a decomposition*

$$q(x)+2\lambda a(x)+\lambda^2=-A(x_1)+B(x),\quad x\in U_\omega,$$

where $A\in C^1(\mathbb{R}_+)$,

$$\lim_{u\to+\infty}A(u)=+\infty,\qquad \lim_{u\to+\infty}\frac{|A'(u)|}{A(u)}=0,\tag{4.4}$$

and

$$\lim_{n\to\infty}\operatorname*{ess\,sup}_{x\in U_\omega,|x|>n}\frac{|B(x)|^2}{A(x_1)}=0\tag{4.5}$$

ii) *if $d>1$, then $\omega\in C^2(\mathbb{R}_+)$ and*

$$\frac{\omega'(u)}{\omega(u)}=\mathrm{o}(1),\qquad \frac{\omega''(u)}{\omega(u)^3}=\mathcal{O}(1),\qquad \frac{\omega(u)^4}{A(u)}=\mathrm{o}(1),\quad u\to\infty,\tag{4.6}$$

then $\lambda\in\sigma_{\mathrm{e2}}(G)$.

Proof. Using a one-dimensional WKB expansion, we construct a singular sequence of the form $\{\Phi_n\}=\{(\phi_n,\lambda\phi_n)\}\subset\operatorname{Dom}(G_0)=\operatorname{Dom}(T(\lambda))\times\operatorname{Dom}(T(\lambda))$ with $\operatorname{supp}\phi_n\subset U_\omega$ compact, $n\in\mathbb{N}$. We give a detailed proof for $d>1$; the simplifications for $d=1$ are obvious. The first components ϕ_n of Φ_n will be constructed such that

$$\frac{\|(-\Delta-A)\phi_n\|+\|B\phi_n\|}{\|\nabla\phi_n\|}\to 0,\quad n\to\infty,$$

and $\operatorname{supp}\phi_n$ moves to infinity in U_ω; this implies

$$\|(G-\lambda)\Phi_n\|_{\mathcal{H}} = \frac{\|T(\lambda)\phi_n\|}{\|\Phi_n\|_{\mathcal{H}}} \leq \frac{\|(-\Delta - A)\phi_n\| + \|B\phi_n\|}{\|\nabla\phi_n\|} \to 0, \quad n\to\infty \tag{4.7}$$

and $\Phi_n/\|\Phi_n\|_{\mathcal{H}} \xrightarrow{w} 0$ as $n\to\infty$, respectively.

By (4.4), we have $A(u)>0$ for all $u\in(\alpha,\infty)$ with some $\alpha>0$ and

$$\rho_n := \sup_{t>n} \frac{|A'(t)|}{A(t)} \to 0, \quad n>\alpha, \quad n\to\infty.$$

We write $x=(x_1,x')\in\mathbb{R}^d$ and denote by $\mathcal{B}'$ the open $(d-1)$-dimensional unit ball. For $n\in\mathbb{N}$, we choose $\phi_n(x_1,x') := \varphi_n(x_1)\psi_\lambda(x_1)\chi(x)$ where

$$\begin{aligned}
\psi_\lambda(x_1) &:= \exp\left(\mathrm{i}\int_\alpha^{x_1} A(t)^{\frac12}\,\mathrm{d}t\right), \\
\chi(x) &:= \widetilde{\chi}(\omega(x_1)x'), \qquad \widetilde{\chi}\in C_0^\infty(\mathcal{B}'), \qquad \|\widetilde{\chi}\|_{L^2(\mathbb{R}^{d-1})}=1, \\
\varphi_n(x_1) &:= \omega(x_1)^{\frac{d-1}{2}}\rho_n^{\frac14}\varphi\left(\rho_n^{\frac12}x_1 - n\right), \quad \varphi\in C_0^\infty((0,1)), \quad \|\varphi\|_{L^2(\mathbb{R})} = \left(\int_{\mathcal{B}'}\mathrm{d}y'\right)^{-1}.
\end{aligned}$$

Then $\operatorname{supp}\phi_n\subset\operatorname{supp}\chi\subset U_\omega$ and by the change of variables $(x_1,x')=(y_1,\omega^{-1}(y_1)y')$

$$\int_{U_\omega}|\varphi_n(x_1)|^2\,\mathrm{d}x = \int_0^\infty |\varphi(y_1)|^2\,\mathrm{d}y_1\cdot\int_{\mathcal{B}'}\mathrm{d}y' = 1.$$

Moreover, using the notation $\|f\|_{\infty,n} := \operatorname{ess\,sup}_{x\in\operatorname{supp}\varphi_n}|f(x)|$, $n\in\mathbb{N}$, and the assumptions (4.6) we obtain that, as $n\to\infty$,

$$\begin{aligned}
&\int_{U_\omega}|\varphi_n'(x_1)|^2\mathrm{d}x = \mathcal{O}\left(\|\omega'\omega^{-1}\|_{\infty,n}^2 + \rho_n\|\varphi'\|_{L^2(\mathbb{R})}^2\right) = \mathrm{o}(1), \\
&\int_{U_\omega}|\varphi_n''(x_1)|^2\mathrm{d}x = \mathcal{O}\left(\rho_n\|\omega'\omega^{-1}\|_{\infty,n}^2 + \|\omega'\omega^{-1}\|_{\infty,n}^4\|\omega''\omega^{-1}\|_{\infty,n}^2 + \rho_n^2\right) = \mathrm{o}(\|A\|_{\infty,n}).
\end{aligned}$$

We also note that (4.4) implies that

$$\sup_{u,v\in\operatorname{supp}\varphi_n}\left|\log\frac{A(u)}{A(v)}\right| \leq \int_{\operatorname{supp}\varphi_n}\frac{|A'(t)|}{A(t)}\,\mathrm{d}t = \mathcal{O}(\rho_n^{\frac12}), \quad n\to\infty,$$

hence

$$\frac{\|A\|_{\infty,n}}{\inf\limits_{u\in\operatorname{supp}\varphi_n} A(u)} = \mathcal{O}(1), \quad n\to\infty. \tag{4.8}$$

Clearly, we have

$$\psi_\lambda'(x_1) = \mathrm{i}A(x_1)^{\frac12}\psi_\lambda(x_1), \qquad \psi_\lambda''(x_1) = -A(x_1)\psi_\lambda(x_1) + \frac{\mathrm{i}}{2}\frac{A'(x_1)}{A(x_1)^{\frac12}}\psi_\lambda(x_1)$$

and, since $|\psi_\lambda|=1$,

$$\|\nabla\phi_n\| \geq \|\partial_1\phi_n\| \geq \|\varphi_n\psi_\lambda'\chi\| - \|\varphi_n'\chi\| - \|\varphi_n\partial_1\chi\|.$$

By straightforward calculations and using that $|x'|<\omega(x_1)^{-1}$ for $x\in U_\omega$ as well as (4.6) and (4.8), we arrive at

$$\|\varphi_n\psi_\lambda'\chi\|^{-2} = \mathcal{O}(\|A\|_{\infty,n}), \quad \|\varphi_n'\chi\| + \|\varphi_n\partial_1\chi\| = \mathrm{o}(1), \quad n\to\infty,$$

whence

$$\|\nabla\phi_n\|^{-2} = \mathcal{O}(\|A\|_{\infty,n}), \quad n\to\infty.$$

On the other hand, tedious but straightforward, and hence omitted, calculations and estimates yield that

$$\|(-\Delta - A)\phi_n\|^2 = \mathrm{o}(\|A\|_{\infty,n}), \quad n\to\infty.$$

Finally, assumption (4.5) implies that $\|B\phi_n\|^2 = \mathrm{o}(\|A\|_{n,\infty})$ as $n \to \infty$ and so (4.7) follows. □

Remark 4.3. If Ω contains a cone C_δ, *cf.* (4.2), with some $\delta > 0$ and a, q are radial functions (or perturbations thereof of the type (4.5) in C_δ), the above construction of a singular sequence can be adapted accordingly. In this case, for $\lambda \in (-\infty, 0)$ we have $\lambda \in \sigma_{\mathrm{e}2}(G)$ if there exists a decomposition

$$q(x) + 2\lambda a(x) + \lambda^2 = \widetilde{A}(|x|) + B(x), \quad x \in C_\delta,$$

such that $A(u) := \widetilde{A}(|x|)$ and B satisfy conditions (4.4) and (4.5).

We mention that, in spherical coordinates $x = (|x|, \Theta) \in (0,\infty) \times S^{d-1}$, where S^{d-1} is the $(d-1)$-dimensional unit sphere, a suitable singular sequence has the‘form

$$\phi_n(|x|, \Theta) := |x|^{-\frac{d-1}{2}} \varphi_n(|x|) \psi_\lambda(|x|) \chi(\Theta), \quad n \in \mathbb{N},$$

where

$$\psi_\lambda(|x|) := \exp\left(\mathrm{i} \int_\alpha^{|x|} \widetilde{A}(t)^{\frac{1}{2}} \mathrm{d}t \right), \quad 0 \neq \chi \in C_0^\infty(S^{d-1} \cap C_\delta),$$

$$\varphi_n(|x|) := \rho_n^{\frac{1}{4}} \varphi(\rho_n^{\frac{1}{2}} |x| - n), \qquad 0 \neq \varphi \in C_0^\infty((0,1)).$$

5. Convergence of non-real eigenvalues

In this section we consider a sequence of dampings $\{a_n\}$ that are unbounded at infinity in the sense of Assumption II and which converge in a suitable sense to a limit function a_∞ on some open subset $\Omega_\infty \subset \Omega \subset \mathbb{R}^d$.

To this end, we study the spectral convergence for the quadratic operator functions

$$T_n(\lambda) := -\Delta + q + 2\lambda a_n + \lambda^2, \quad n \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}, \quad \lambda \in \mathbb{C} \setminus (-\infty, 0], \qquad (5.1)$$

in $L^2(\Omega)$ for $n \in \mathbb{N}$ and in $L^2(\Omega_\infty) \subset L^2(\Omega)$ for $n = \infty$. While we allow for the case $\Omega_\infty = \Omega$, the example (1.3), (1.4) discussed in the introduction illustrates the need to consider dampings a_n that diverge on the non-empty interior of $\Omega \setminus \Omega_\infty$, and hence for $T_n(\lambda)$ and $T_\infty(\lambda)$ acting in possibly different spaces $L^2(\Omega)$ and $L^2(\Omega_\infty)$. In fact, the dampings a_n are only supposed to converge to a_∞ in $L^2_{\mathrm{loc}}(\Omega_\infty)$. Recall that, for $\{b_n\} \subset L^2_{\mathrm{loc}}(\Omega')$, $b \in L^2_{\mathrm{loc}}(\Omega')$ and $\Omega' \subset \mathbb{R}^d$ open, we have $b_n \to b$ in $L^2_{\mathrm{loc}}(\Omega')$ as $n \to \infty$, if for all compact sets $K \subset \Omega'$,

$$\int_K |b_n - b|^2 \to 0, \quad n \to \infty.$$

We shall also need the so-called *segment condition* for Ω_∞ which means that the domain Ω_∞ does not lie on both sides of part of its boundary or, more precisely, that every $x \in \partial\Omega$ has a neighborhood U_x and a non-zero vector $y_x \in \mathbb{R}^d$ such that if $z \in \overline{\Omega} \cap U_x$, then $z + t y_x \in \Omega$ for $0 < t < 1$, *cf.* [1, Sec. 3].

Our convergence result in Theorem 5.1 below is formulated for quadratic operator functions T_n, $n \in \mathbb{N}$, requiring a_n and q to be only in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$. If even Assumption I is satisfied, then spectral convergence for the corresponding generators G_n, $n \in \mathbb{N}$, follows from this result by Theorem 3.2.

Assumption III. Let $\emptyset \neq \Omega_\infty \subset \Omega \subset \mathbb{R}^d$ be open and assume that Ω_∞ satisfies the segment condition. Suppose that

(III.i) $q \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$, $\{a_n\}_{n \in \mathbb{N}_0} \subset L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$ and $a_\infty \in L^1_{\mathrm{loc}}(\Omega_\infty; \mathbb{R})$,

(III.ii) for all $n \in \mathbb{N}_0$, $a_n \geq 0$ and

$$\lim_{k \to \infty} \operatorname*{ess\,inf}_{x \in \Omega, |x| > k} a_n(x) = \infty,$$

(III.iii) for all $n \in \mathbb{N}$, $a_n \geq a_0$ in Ω and $a_\infty \geq a_0$ in Ω_∞,
(III.iv) $a_n^{\frac{1}{2}} \restriction \Omega_\infty \to a_\infty^{\frac{1}{2}}$ in $L^2_{\rm loc}(\Omega_\infty)$, $n \to \infty$,
(III.v) for all $n \in \mathbb{N}$, $a_n^{-\frac{1}{2}} \restriction \Omega_0 \in L^2_{\rm loc}(\Omega_0)$ and $a_n^{-\frac{1}{2}} \to 0$ in $L^2_{\rm loc}(\Omega_0)$, $n \to \infty$, where $\Omega_0 := (\Omega \setminus \Omega_\infty)^{\rm o}$.

Note that Assumption (III.v) is relevant only when $(\Omega \setminus \Omega_\infty)^{\rm o} \neq \emptyset$, which is not excluded here. The quadratic operator functions $T_n(\lambda) = H_{2\lambda,n} + \lambda^2$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, are defined as in Section 2.1, via the Schrödinger operators

$$H_{\gamma,n} := -\Delta + q + \gamma a_n, \quad n \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}, \quad \gamma \in \mathbb{C} \setminus (-\infty, 0], \tag{5.2}$$

and Assumption (III.ii) ensures that Assumption II is satisfied. Thus the non-real spectrum of T_n consists only of eigenvalues by Proposition 3.1.

The main result of this section is the following *spectral exactness* theorem for $\{T_n\}$, $n \in \mathbb{N}$. The latter means that all eigenvalues of the limiting operator function T_∞ are approximated by eigenvalues of T_n and all finite accumulation points of eigenvalues of T_n outside $(-\infty, 0]$ are eigenvalues of T_∞, *i.e.* no spectral pollution occurs. An illustration of this result may be found in example (1.3), (1.4) in Section 6.1.

Theorem 5.1. *Let Assumption* III *be satisfied and let* $\{T_n(\lambda)\}_{n \in \mathbb{N}^*}$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, *be as in* (5.1). *Then the following hold.*

i) *If* $\lambda \in \sigma_{\rm p}(T_\infty)$, *then there exists a sequence* $\{\lambda_n\}_{n \in \mathbb{N}}$, $\lambda_n \in \sigma_{\rm p}(T_n)$, *such that* $\lambda_n \to \lambda$, $n \to \infty$. *Conversely, if* $\{\lambda_n\}_{n \in \mathbb{N}}$, $\lambda_n \in \sigma_{\rm p}(T_n) \subset \mathbb{C} \setminus (-\infty, 0]$, *has a subsequence* $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ *such that* $\lambda_{n_k} \to \lambda \in \mathbb{C} \setminus (-\infty, 0]$, $k \to \infty$, *then* $\lambda \in \sigma_{\rm p}(T_\infty)$.

ii) *If* $\lambda_n \to \lambda$, $n \to \infty$, *where* $\lambda_n \in \sigma_{\rm p}(T_n)$, $\lambda \in \sigma_{\rm p}(T_\infty)$ *and* $\{f_n\}$ *is a sequence of normalized eigenfunctions of* T_n *at* λ_n, *then the sequence* $\{f_n\}$ *is compact in* $L^2(\Omega)$ *and its accumulation points* (*which belong to* $L^2(\Omega_\infty)$) *are normalized eigenvectors of* T_∞ *at* λ.

In the first step of the proof of Theorem 5.1, we establish generalized strong resolvent convergence of $H_{\gamma,n}$ to $H_{\gamma,\infty}$ as $n \to \infty$ for all $\gamma = 2\lambda \in \mathbb{C} \setminus (-\infty, 0]$, *cf.* Proposition 5.2; here "generalized" refers to the fact that the operators act in possibly different spaces; this is reflected by the presence of the characteristic function χ_∞ of Ω_∞ in (5.3) below. In the second step, we employ abstract spectral convergence results for analytic Fredholm operator functions [24, Satz 4.1.(18)] for $T_n(\lambda) = H_{2\lambda,n} + \lambda^2$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $n \in \mathbb{N}$.

Proposition 5.2. *Let Assumption* III *be satisfied and let* $\{H_{\gamma,n}\}_{n \in \mathbb{N}^*}$, $\gamma \in \mathbb{C} \setminus (-\infty, 0]$ *be as in* (5.2). *Then, for all* $\gamma \in \mathbb{C} \setminus (-\infty, 0]$ *and every* $f \in L^2(\Omega)$,

$$\|(H_{\gamma,n} + I)^{-1} f - (H_{\gamma,\infty} + I)^{-1} \chi_\infty f\| \to 0, \quad n \to \infty, \tag{5.3}$$

where χ_∞ *is the characteristic function of* Ω_∞.

Proof. To simplify the notation within this proof, we drop the subscript γ in the sequel and denote $\omega := \arg(\gamma)/2$. First we notice that

$$-1 \in \bigcap_{n \in \mathbb{N}^*} \rho(H_n)$$

by the numerical range enclosure, *cf.* (2.7), and the fact that $\widetilde{H}_n = {\rm e}^{-{\rm i}\omega} H_n$, $n \in \mathbb{N}^*$, is m-sectorial, *cf.* (2.8). Clearly, (5.3) is equivalent to

$$\|(\widetilde{H}_n + {\rm e}^{-{\rm i}\omega})^{-1} f - (\widetilde{H}_\infty + {\rm e}^{-{\rm i}\omega})^{-1} \chi_\infty f\| \to 0, \quad n \to \infty,$$

which we prove by contradiction in the following.

Suppose that there exists a function f in $L^2(\Omega)$ and a $\delta > 0$ such that

$$\|(\widetilde{H}_n + \mathrm{e}^{-\mathrm{i}\omega})^{-1} f - (\widetilde{H}_\infty + \mathrm{e}^{-\mathrm{i}\omega})^{-1} \chi_\infty f\| \geq \delta > 0 \tag{5.4}$$

for all $n \in J$ for some infinite subset $J \subset \mathbb{N}$. Then, for $\psi_n := (\widetilde{H}_n + \mathrm{e}^{-\mathrm{i}\omega})^{-1} f$, $n \in J$,

$$\widetilde{h}_n[\psi_n] + \mathrm{e}^{-\mathrm{i}\omega}\|\psi_n\|^2 = \langle f, \psi_n \rangle, \quad n \in J, \tag{5.5}$$

and the enclosure of the numerical range (2.7) implies that

$$\|\psi_n\| \leq \|(\widetilde{H}_n + \mathrm{e}^{-\mathrm{i}\omega})^{-1}\| \|f\| \leq \frac{\|f\|}{\operatorname{dist}(-\mathrm{e}^{-\mathrm{i}\omega}, \operatorname{Num}(\widetilde{h}_n))} \leq \frac{\|f\|}{\cos\omega}, \quad n \in J. \tag{5.6}$$

Taking real parts in (5.5) and using (5.6), we get

$$\|\nabla\psi_n\|^2 + \|q^{\frac{1}{2}}\psi_n\|^2 + |\gamma| \|a_n^{\frac{1}{2}}\psi_n\|^2 + \|\psi_n\|^2 \leq \frac{\|f\|^2}{(\cos\omega)^2}, \quad n \in J. \tag{5.7}$$

This shows that $\{\psi_n\}_{n\in J}$ is bounded in the Hilbert space $(\mathcal{H}_0, \langle\cdot,\cdot\rangle_{\mathcal{H}_0})$ defined by

$$\begin{aligned} \mathcal{H}_0 &:= W_0^{1,2}(\Omega) \cap \operatorname{Dom}(q^{\frac{1}{2}}) \cap \operatorname{Dom}(a_0^{\frac{1}{2}}), \\ \langle\cdot,\cdot\rangle_{\mathcal{H}_0} &:= \langle\cdot,\cdot\rangle_{W^{1,2}} + \langle q^{\frac{1}{2}}\cdot, q^{\frac{1}{2}}\cdot\rangle + \langle a_0^{\frac{1}{2}}\cdot, a_0^{\frac{1}{2}}\cdot\rangle. \end{aligned} \tag{5.8}$$

Thus $\{\psi_n\}_{n\in J}$ has a weakly convergent subsequence $\{\psi_n\}_{n\in J'}$ where J' is an infinite subset of J, in $\mathcal{H}_0$. Since the embedding $\mathcal{H}_0 \hookrightarrow L^2(\Omega)$ is compact due to Assumption (III.iii), *cf.* the proof of Lemma 2.3.iii), $\{\psi_n\}_{n\in J'}$ converges in $L^2(\Omega)$. Moreover, (5.7) shows that $\{a_n^{1/2}\psi_n\}_{n\in J}$ is bounded in $L^2(\Omega)$, thus we can assume that $\{a_n^{1/2}\psi_n\}_{n\in J'}$ converges weakly in $L^2(\Omega)$. Altogether, there exist $\psi \in \mathcal{H}_0$ and $g \in L^2(\Omega)$ such that, for $n \in J'$ and as $n \to \infty$,

$$\forall\, \eta \in \mathcal{H}_0 \quad \langle\psi_n, \eta\rangle_{\mathcal{H}_0} \longrightarrow \langle\psi, \eta\rangle_{\mathcal{H}_0}, \tag{5.9}$$

$$\|\psi_n - \psi\| \longrightarrow 0, \tag{5.10}$$

$$\forall\, \zeta \in L^2(\Omega) \quad \langle a_n^{\frac{1}{2}}\psi_n, \zeta\rangle \longrightarrow \langle g, \zeta\rangle. \tag{5.11}$$

If $\Omega_0 = (\Omega \setminus \Omega_\infty)^{\mathrm{o}} \neq \emptyset$, we choose arbitrary $\varphi \in C_0^\infty(\Omega_0)$. Then, by Assumption (III.v) and the boundedness of $\|a_n^{1/2}\psi_n\|$, *cf.* (5.7), we obtain

$$|\langle\psi_n, \varphi\rangle| = |\langle a_n^{\frac{1}{2}}\psi_n, a_n^{-\frac{1}{2}}\varphi\rangle| \leq \|a_n^{\frac{1}{2}}\psi_n\| \|a_n^{-\frac{1}{2}}\varphi\| \longrightarrow 0, \quad n \in J', \ n \to \infty.$$

Hence $\langle\psi, \varphi\rangle = \lim_{n\in J', n\to\infty} \langle\psi_n, \varphi\rangle = 0$, and so $\psi = 0$ a.e. in Ω_0. Since $\psi \in \mathcal{H}_0 \subset W_0^{1,2}(\Omega)$, the latter implies that $\psi \restriction \Omega_\infty \in W_0^{1,2}(\Omega_\infty)$, *cf.* [1, Lem. 3.27, Thm. 5.29].

Now let $\phi \in C_0^\infty(\Omega_\infty)$. By Assumption (III.iv), $a_n^{1/2}\phi \to a_\infty^{1/2}\phi$ in $L^2(\Omega_\infty)$ as $n \to \infty$ and thus $\sup_{n\in J'} \|a_n^{1/2}\phi\| < \infty$. Hence (5.10) implies that

$$\langle a_n^{\frac{1}{2}}\psi_n, \phi\rangle = \langle\psi_n - \psi, a_n^{\frac{1}{2}}\phi\rangle + \langle\psi, a_n^{\frac{1}{2}}\phi\rangle \longrightarrow \langle\psi, a_\infty^{\frac{1}{2}}\phi\rangle, \quad n \in J', \ n \to \infty.$$

On the other hand, $\langle a_n^{1/2}\psi_n, \phi\rangle \to \langle g, \phi\rangle$, *cf.* (5.11). Since $\phi \in C_0^\infty(\Omega_\infty)$ was arbitrary, $g \restriction \Omega_\infty = a_\infty^{1/2}\psi \restriction \Omega_\infty$ a.e. in Ω_∞. Therefore $\{a_n^{1/2}\psi_n \restriction \Omega_\infty\}_{n\in J'}$ converges weakly to $a_\infty^{1/2}\psi \restriction \Omega_\infty$ in $L^2(\Omega_\infty)$. Using $\sup_{n\in J'} \|a_n^{1/2}\psi_n\| < \infty$, *cf.* (5.7), and Assumption (III.iv), we finally obtain, for $n \in J'$ and as $n \to \infty$,

$$\langle a_n^{\frac{1}{2}}\psi_n, a_n^{\frac{1}{2}}\phi\rangle = \langle a_n^{\frac{1}{2}}\psi_n, (a_n^{\frac{1}{2}} - a_\infty^{\frac{1}{2}})\phi\rangle + \langle a_n^{\frac{1}{2}}\psi_n, a_\infty^{\frac{1}{2}}\phi\rangle \longrightarrow \langle a_\infty^{\frac{1}{2}}\psi, a_\infty^{\frac{1}{2}}\phi\rangle. \tag{5.12}$$

In summary, (5.9), (5.10) and (5.12) show that, for any $\phi \in C_0^\infty(\Omega_\infty)$ and for $n \in J'$, $n \to \infty$,

$$\begin{aligned}\langle f, \phi\rangle_{L^2(\Omega_\infty)} = \langle f, \phi\rangle &= \widetilde{h}_n(\psi_n, \phi) + \mathrm{e}^{-\mathrm{i}\omega}\langle \psi_n, \phi\rangle \\ &\to \mathrm{e}^{-\mathrm{i}\omega}\Big(\langle \nabla\psi, \nabla\phi\rangle_{L^2(\Omega_\infty)} + \langle q^{\frac12}\psi, q^{\frac12}\phi\rangle_{L^2(\Omega_\infty)}\Big) \\ &\quad + \mathrm{e}^{\mathrm{i}\omega}|\gamma|\langle a_\infty^{\frac12}\psi, a_\infty^{\frac12}\phi\rangle_{L^2(\Omega_\infty)} + \mathrm{e}^{-\mathrm{i}\omega}\langle \psi, \phi\rangle_{L^2(\Omega_\infty)} \\ &= \widetilde{h}_\infty(\psi, \phi) + \mathrm{e}^{-\mathrm{i}\omega}\langle \psi, \phi\rangle_{L^2(\Omega_\infty)},\end{aligned} \tag{5.13}$$

and hence the first and the last term in (5.13) must be equal. This and the representation theorem [15, Thm. VI.2.1] imply that $\psi \restriction \Omega_\infty \in \mathrm{Dom}(\widetilde{H}_\infty)$ and $(\widetilde{H}_\infty + \mathrm{e}^{-\mathrm{i}\omega})(\psi \restriction \Omega_\infty) = f \restriction \Omega_\infty$, *i.e.* $(\widetilde{H}_\infty + \mathrm{e}^{-\mathrm{i}\omega})^{-1}(f \restriction \Omega_\infty) = \psi \restriction \Omega_\infty$. The latter and (5.10) yield that $\|(H_n + \mathrm{e}^{-\mathrm{i}\omega})^{-1} f - (H_\infty + \mathrm{e}^{-\mathrm{i}\omega})^{-1}\chi_\infty f\| \to 0$ as $n \to \infty$ with $n \in J' \subset J$, a contradiction to (5.4). □

Proof of Theorem 5.1. We define operator functions A_n and A_∞ whose values are bounded linear operators in $L^2(\Omega)$ and $L^2(\Omega_\infty)$, respectively, by

$$A_n(\lambda) := (H_{2\lambda,n} + \lambda^2)(H_{2\lambda,n} + I)^{-1} = I + (\lambda^2 - 1)\,(H_{2\lambda,n} + I)^{-1},$$

$n \in \mathbb{N}^*$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$; these functions are well-defined since $-1 \in \bigcap_{n\in\mathbb{N}^*} \rho(H_{2\lambda,n})$ due to (2.7).

It is easy to verify that, for every $n \in \mathbb{N}^*$, if a nonzero ψ in $\mathrm{Dom}(H_{2\lambda,n})$ satisfies $(H_{2\lambda,n} + \lambda^2)\psi = 0$, then $A_n(\lambda)\psi = 0$ and, conversely, if a nonzero ψ in $L^2(\Omega)$ or in $L^2(\Omega_\infty)$ for $n = \infty$ satisfies $A_n(\lambda)\psi = 0$, then $\psi \in \mathrm{Dom}(H_{2\lambda,n})$ and $(H_{2\lambda,n} + \lambda^2)\psi = 0$. Hence, for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $n \in \mathbb{N}^*$,

$$0 \in \sigma_{\mathrm{p}}(H_{2\lambda,n} + \lambda^2) \iff 0 \in \sigma_{\mathrm{p}}(A_n(\lambda)).$$

The claims of Theorem 5.1 will follow from convergence results for holomorphic operator functions, *cf.* [24, Satz 4.1.(18)] or the summary in [2, Sec. 1.1.2], if we verify the following assumptions therein:

a) $\lambda \mapsto A_n(\lambda)$, $n \in \mathbb{N}^*$, is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$,
b) $A_n(\lambda)$, $n \in \mathbb{N}^*$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, is Fredholm with index 0,
c) for all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $\{A_n(\lambda)\}$ converges regularly to $A_\infty(\lambda)$,
d) there exists $\lambda_0 \in \mathbb{C} \setminus (-\infty, 0]$ such that $\lambda_0 \in \rho(A_\infty)$,
e) for every $K \subset \mathbb{C} \setminus (-\infty, 0]$, K compact,
$$\sup_{n\in\mathbb{N}} \max_{\lambda\in K} \|A_n(\lambda)\| < \infty.$$

We have already shown that a) holds for $n \in \mathbb{N}^*$, *cf.* Lemma 2.3 iv). The validity of condition b) follows from [7, Thm. IX.2.1] since $A_n(\lambda) - I$, $n \in \mathbb{N}^*$, is a compact operator, *cf.* Lemma 2.3.iii). For d), we observe that $A_\infty(1) = I$ so we can choose $\lambda_0 = 1$. The bound in e) follows immediately from the compactness of K and from

$$\|(H_{2\lambda,n} + I)^{-1}\| \le \mathrm{dist}(-\mathrm{e}^{-\mathrm{i}\frac{\arg(\lambda)}{2}}, \mathrm{Num}(\widetilde{h}_n))^{-1} \le \frac{1}{\cos\frac{\arg(\lambda)}{2}}, \quad n \in \mathbb{N}^*, \tag{5.14}$$

cf. (2.7) and (2.8).

The only remaining point is c), the *regular convergence*, *cf.* [24] or the summary in [2, Sec. 1.1.2]. In detail, we need to show that

i) for any $\{\psi_n\} \subset L^2(\Omega)$ and $\psi \in L^2(\Omega_\infty)$ such that $\|\psi - \psi_n\| \to 0$, $n \to \infty$, we have $\|A_n(\lambda)\psi_n - A_\infty(\lambda)\psi\| \to 0$, $n \to \infty$, and

ii) for any bounded $\{\psi_n\} \subset L^2(\Omega)$ such that every infinite subsequence of $\{A_n(\lambda)\psi_n\}$ contains a convergent subsequence, every infinite subsequence of $\{\psi_n\}$ also contains a convergent subsequence.

The validity of condition i) follows from (5.14) and the resolvent convergence proved in Proposition 5.2. In fact, since $\psi = \chi_\infty \psi$ where χ_∞ is the characteristic function of Ω_∞,

$$\begin{aligned}&\|A_n(\lambda)\psi_n - A(\lambda)\psi\|\\ &\leq \|\psi_n - \psi\| + (|\lambda|^2+1)\|(H_{2\lambda,n}+I)^{-1}\psi_n - (H_{2\lambda,\infty}+I)^{-1}\psi\|\\ &\leq \big(1 + (|\lambda|^2+1)\|(H_{2\lambda,n}+I)^{-1}\|\big)\,\|\psi_n - \psi\|\\ &\quad + (|\lambda|^2+1)\|(H_{2\lambda,n}+I)^{-1}\psi - (H_{2\lambda,\infty}+I)^{-1}\psi\| \longrightarrow 0, \quad n\to\infty.\end{aligned}$$

To verify condition ii), due to the relations,

$$\psi_n = A_n(\lambda)\psi_n - (\lambda^2-1)(H_{2\lambda,n}+I)^{-1}\psi_n, \quad n\in\mathbb{N},$$

it suffices to show that $\{(H_{2\lambda,n}+I)^{-1}\psi_n\}_{n\in J}$ with infinite $J\subset\mathbb{N}$ has a convergent subsequence. This can be shown in a similar way as in (5.6)–(5.8). In detail, $\{(H_{2\lambda,n}+I)^{-1}\psi_n\}$, and hence $\{\widetilde{H}_{2\lambda,n} + \mathrm{e}^{-\mathrm{i}\frac{\arg(\lambda)}{2}}\}$, is bounded due to the boundedness of $\{\psi_n\}$ and (5.14). Thus, as in (5.5), we obtain

$$\widetilde{h}_{2\lambda,n}[\psi_n] + \mathrm{e}^{-\mathrm{i}\frac{\arg(\lambda)}{2}}\|\psi_n\|^2 = \langle(\widetilde{H}_{2\lambda,n} + \mathrm{e}^{-\mathrm{i}\frac{\arg(\lambda)}{2}})\psi_n,\psi_n\rangle, \quad n\in\mathbb{N}.$$

and we proceed as in the paragraphs below (5.8) to finish the proof of ii) and hence of the theorem. □

6. Examples

As an illustration of our abstract results, we fully characterize the spectrum of the generator G for several examples of damping terms. For $\lambda\in(-\infty,0)$ and for $\lambda=0$, we employ the result on the essential spectrum of

$$G = \begin{pmatrix} 0 & I \\ \Delta - q & -2a\end{pmatrix},$$

cf. Theorem 4.2, while for $\lambda\in\mathbb{C}\setminus(-\infty,0]$ we use the spectral correspondence $\lambda\in\sigma(G)\iff 0\in\sigma(T(\lambda))$ between G and the quadratic operator function

$$T(\lambda) = -\Delta + q + 2\lambda a + \lambda^2, \qquad \lambda\in\mathbb{C}\setminus(-\infty,0],$$

cf. Theorem 3.2. As mentioned in the introduction, these examples show that the growing damping term will prevent uniform exponential decay of solutions by the creation of essential spectrum covering the entire negative semiaxis $(-\infty,0]$, independently of the existence of eigenvalues with real parts converging to 0. That this effect is quite general and not restricted to some particular examples may be seen from our abstract results, see Theorem 4.2.

6.1. Examples for $d=1$.

We start with the family of examples (1.3) with $\Omega=\mathbb{R}$ described in the introduction where the dampings are given by

$$a_n(x) = x^{2n} + \mathfrak{a}_0, \quad x\in\Omega=\mathbb{R}, \quad n\in\mathbb{N}, \quad \mathfrak{a}_0\geq 0,$$

and we consider a constant potential $q(x)\equiv\mathfrak{q}_0\geq 0$. The non-real eigenvalues of the corresponding generators G_n can be expressed in terms of the eigenvalues $\{\mu_k(n)\}_{k\in\mathbb{N}_0}\subset(0,\infty)$ of the self-adjoint anharmonic oscillators

$$S_n = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2n}, \quad \mathrm{Dom}(S_n) = W^{2,2}(\mathbb{R})\cap\mathrm{Dom}(x^{2n}), \tag{6.1}$$

in $L^2(\mathbb{R})$. These eigenvalues are known to satisfy

$$\mu_k(n) = \begin{cases} 2k+1, \quad k\in\mathbb{N}_0, & n=1,\\ \left(\dfrac{\pi}{\Sigma_{2n}}\right)^{\frac{2n}{n+1}} k^{\frac{2n}{n+1}}\left(1+\mathrm{o}_k(1)\right), \quad k\to\infty, & n\geq 2,\end{cases}$$

where $\Sigma_{2n} := \int_{-1}^{1}(1-x^{2n})^{\frac{1}{2}}\,\mathrm{d}x$, *cf.* for instance [22].

Proposition 6.1. *Let $\Omega = \mathbb{R}$, let G_n be as in (2.5) with $q(x) \equiv \mathfrak{q}_0 \geq 0$ and $a_n(x) = x^{2n} + \mathfrak{a}_0$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $\mathfrak{a}_0 \geq 0$, and let $\{\mu_k(n)\}_{k\in\mathbb{N}_0}$ be the eigenvalues of S_n defined by (6.1). Then*

$$\sigma(G_n) = (-\infty, 0]\,\dot\cup \bigcup_{k\in\mathbb{N}_0}\left\{\lambda_k(n,\mathfrak{a}_0,\mathfrak{q}_0), \overline{\lambda_k(n,\mathfrak{a}_0,\mathfrak{q}_0)}\right\} \subset \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \leq 0\}, \quad n\in\mathbb{N},$$

where $\lambda_k(n,\mathfrak{a}_0,\mathfrak{q}_0)$, $k\in\mathbb{N}_0$, are the solutions of

$$(\lambda^2 + 2\lambda\mathfrak{a}_0 + \mathfrak{q}_0)^{n+1} = 2\lambda(-\mu_k(n))^{n+1}, \quad \operatorname{Re}\lambda \leq 0, \quad \operatorname{Im}\lambda > 0.$$

Moreover, all non-real eigenvalues satisfy

$$\sigma(G_n)\setminus\mathbb{R} \subset \{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \leq -\mathfrak{a}_0,\ |\lambda| \geq \mathfrak{q}_0\},$$

and, for any $n\in\mathbb{N}$, as $k\to\infty$,

$$\lambda_k(n,\mathfrak{a}_0,\mathfrak{q}_0) = 2^{\frac{1}{2n+1}}\mathrm{e}^{\mathrm{i}\frac{n+1}{2n+1}\pi}\big[\mu_k(n)\big]^{\frac{n+1}{2n+1}} - \frac{2(n+1)}{2n+1}\mathfrak{a}_0(1+\mathrm{o}_k(1)); \tag{6.2}$$

in particular, for $n=1$ and $\mathfrak{a}_0 = \mathfrak{q}_0 = 0$,

$$\lambda_k(n,0,0) = 2^{\frac{1}{3}}\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}(2k+1)^{\frac{2}{3}}, \quad k\in\mathbb{N}_0.$$

Proof. Both the damping a_n and the potential q clearly satisfy Assumption I.

That $(-\infty,0)\subset\sigma_{\mathrm{e}2}(G_n)$ follows from Theorem 4.2 since $a_n'(x)/a_n(x) = \mathcal{O}(1/x)$ and $q(x) = \mathrm{o}(a_n(x))$ as $x\to+\infty$; because the spectrum is closed, we obtain $(-\infty,0]\subset\sigma(G_n)$.

Since a_n is unbounded at infinity and hence satisfies Assumption II, Theorem 3.2 and Proposition 3.1 i) imply that $\sigma(G_n)\setminus(-\infty,0]$ consists only of complex conjugate pairs of eigenvalues $\lambda_k(n,\mathfrak{a}_0,\mathfrak{q}_0)$, $k\in\mathbb{N}_0$, in the closed left half plane of finite multiplicity which satisfy $0\in\sigma_{\mathrm{disc}}(T_n(\lambda))$. Thus, it suffices to consider $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\leq 0$ and $\operatorname{Im}\lambda > 0$ and we search for solutions $y\in\operatorname{Dom}(T_n(\lambda))$, $y\neq 0$, of

$$-y''(x) + 2\lambda x^{2n}y(x) = -(\lambda^2 + 2\lambda\mathfrak{a}_0 + \mathfrak{q}_0)y(x), \quad x\in\mathbb{R} \tag{6.3}$$

The (complex) change of variable

$$x = 2^{-\frac{1}{2n+2}}\lambda^{-\frac{1}{2n+2}}z, \quad x\in\mathbb{R},$$

leads to the equation

$$-w''(z) + z^{2n}w(z) = \mu w(z), \quad z\in\mathrm{e}^{\mathrm{i}\frac{\arg(\lambda)}{2n+2}}\mathbb{R}, \tag{6.4}$$

where

$$2(-\mu)^{n+1}\lambda = \left(\lambda^2 + 2\lambda\mathfrak{a}_0 + \mathfrak{q}_0\right)^{n+1}. \tag{6.5}$$

Equation (6.4) with complex z was studied extensively in [21]. It is known that every solution of (6.4) either decays or blows up exponentially in each Stokes sector

$$S_k := \left\{z\in\mathbb{C} : \left|\arg z - \frac{k\pi}{n+1}\right| < \frac{\pi}{2n+2}\right\}, \ k\in\mathbb{Z}.$$

Therefore, for (6.3) to have a solution $y\in\operatorname{Dom}(T_n(\lambda))$, it is necessary that (6.4) has a solution decaying both in S_0 and S_{n+1}. Thus, in fact it suffices to search for decaying solution of (6.4) for real z, *i.e.* to investigate the eigenvalues of (6.1), and, after several manipulations, (6.5) yields the asymptotic formula (6.2). □

As an illustration of Proposition 6.1, the spectrum of G_1 with $\mathfrak{q}_0 = 0$, and $\mathfrak{a}_0 = 0$ and $\mathfrak{a}_0 = 3$ is plotted in Figure 1.

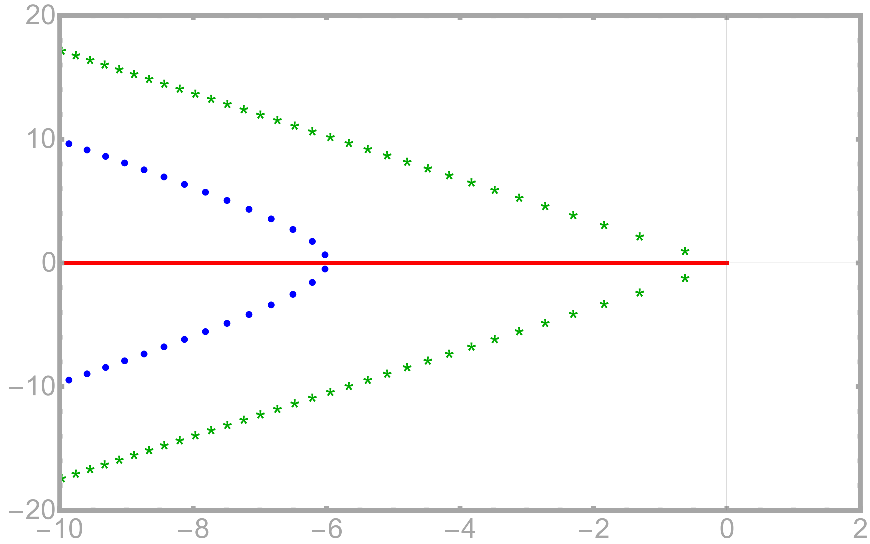

FIGURE 1. Spectrum of G_1 with $a_1(x)=x^2$ and $q\equiv 0$ in $\Omega=\mathbb{R}$, *cf.* Proposition 6.1 with $\mathfrak{a}_0=0$ (green stars), $\mathfrak{a}_0=3$ (blue dots); the essential spectrum is the semiaxis $(-\infty,0]$ (bold red) in both cases.

Remark 6.2. Proposition 6.1 illustrates the convergence of eigenvalues

$$\lambda_k(n,\mathfrak{a}_0,\mathfrak{q}_0)\longrightarrow\lambda_k(\infty,\mathfrak{a}_0,\mathfrak{q}_0),\qquad n\to\infty,$$

proved in Theorem 5.1, where $\{\lambda_k(\infty,\mathfrak{a}_0,\mathfrak{q}_0)\}$ are the solutions of $0\in\sigma_{\rm disc}(T_\infty(\lambda))$ with $\Omega_\infty=(-1,1)$ and $a_\infty=a_0$, *i.e.*

$$T_\infty(\lambda)=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\mathfrak{q}_0+2\lambda\mathfrak{a}_0+\lambda^2,\qquad\lambda\in\mathbb{C},$$

with Dirichlet boundary conditions at ±1. It is not difficult to see that the solutions of $0\in\sigma_{\rm disc}(T_\infty(\lambda))$ are given by

$$\lambda_k(\infty,\mathfrak{a}_0,\mathfrak{q}_0)=-\mathfrak{a}_0\pm\mathrm{i}\sqrt{\mu_k+\mathfrak{q}_0}\sqrt{1-\frac{\mathfrak{a}_0^2}{\mu_k+\mathfrak{q}_0}},\qquad\mu_k=\left(\frac{\pi k}{2}\right)^2,\qquad k\in\mathbb{N}.$$

Note also that we cannot expect uniform convergence in k since, for $\mathfrak{a}_0=0$, the eigenvalues of T_n lie on the two rays $\mathrm{e}^{\pm\mathrm{i}\frac{n+1}{2n+1}\pi}\mathbb{R}_+$, while the eigenvalues of T_∞, with the possible exception of finitely many, lie on the vertical line with $\operatorname{Re}\lambda=-\mathfrak{a}_0$.

6.2. **Examples for $d>1$.** In higher dimensions analogous examples with

$$a(x)=\sum_{j=1}^{d}x_j^{2n}+\mathfrak{a}_0,\qquad x\in\Omega=\mathbb{R}^d,\qquad\mathfrak{a}_0\geq0,$$

and a constant potential $q(x)=\mathfrak{q}_0\geq0$, $x\in\mathbb{R}^d$, can be analyzed. In particular, the case with $\mathfrak{q}_0=0$ and $d\geq3$ fits into the assumptions considered in [11] where a polynomial estimate for the energy decay of the solution was established. In fact, exponential energy decay cannot occur as Theorem 4.2 shows that the essential spectrum of the corresponding operators G_n covers $(-\infty,0]$ for each $n\in\mathbb{N}$; note that it is easy to see that the conditions of Theorem 4.2 are satisfied on U_1, which is a tube if $d=3$, *cf.* (4.1). The whole non-real part of the spectrum of G_n consists of eigenvalues satisfying

$$\sigma_{\rm p}(G_n)\subset\{\lambda\in\mathbb{C}\setminus(-\infty,0]\,:\,\operatorname{Re}\lambda\leq-\mathfrak{a}_0,\ |\lambda|^2\geq\mathfrak{q}_0\},$$

cf. Proposition 3.1. In fact, separation of variables yields that there are non-real eigenvalues asymptotically approaching rays as for $d=1$, *cf.* Proposition 6.1, except that now the corresponding multiplicities depend on the dimension d.

For $d=2$ an example with interesting spectrum is obtained for the damped wave equation on a strip of the form

$$\Omega = \mathbb{R} \times (-\ell, \ell), \quad \ell > 0,$$

and with damping a unbounded along the longitudinal direction corresponding to the first variable x. As a particular example, we consider $a(x,y) = x^2 + \mathfrak{a}_0$ and $q(x,y) = \mathfrak{q}_0 \geq 0$, $(x,y) \in \Omega$. Notice that the associated quadratic operator function can be viewed as the limit of $T_n(\lambda) = -\Delta + \mathfrak{q}_0 + 2\lambda(x^2 + \mathfrak{a}_0 + y^{2n}) + \lambda^2$ acting in $L^2(\mathbb{R}^2)$ as $n \to \infty$ in the sense of Theorem 5.1.

Proposition 6.3. *Let $\Omega = \mathbb{R} \times (-\ell, \ell)$ with $\ell > 0$, let G be as in* (2.5) *with $q(x,y) = \mathfrak{q}_0 \geq 0$ and $a(x,y) = x^2 + \mathfrak{a}_0$, $(x,y) \in \Omega$, where $\mathfrak{a}_0 \geq 0$. Then*

$$\sigma(G) = (-\infty, 0] \,\dot{\cup} \bigcup_{j \in \mathbb{N}, k \in \mathbb{N}_0} \{\lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0), \overline{\lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0)}\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 0\},$$

where $\lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0)$, $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, are the solutions of

$$2\lambda(2k+1)^2 = \left[\lambda^2 + \left(\frac{j\pi}{2\ell}\right)^2 + 2\lambda\mathfrak{a}_0 + \mathfrak{q}_0\right]^2, \quad \operatorname{Re} \lambda \leq 0, \quad \operatorname{Im} \lambda > 0. \tag{6.6}$$

Moreover, all non-real eigenvalues satisfy

$$\sigma(G) \setminus \mathbb{R} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\mathfrak{a}_0, \, |\lambda| \geq \mathfrak{q}_0\},$$

and, for fixed $k \in \mathbb{N}_0$, each sequence of eigenvalues $\{\lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0)\}_j$ satisfies

$$\operatorname{Re} \lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0) \nearrow -\mathfrak{a}_0, \quad j \to +\infty.$$

Proof. It is easy to see that a and q satisfy Assumption I and II, and hence Theorem 3.2 implies that $\sigma(G) \setminus (-\infty, 0]$ consists only of eigenvalues of finite multiplicity which may only accumulate at $(-\infty, 0]$.

It also not difficult to check that the assumptions of Theorem 4.2 are satisfied on a strip U_ω with $\omega(x) = \ell^{-1}$, *cf.* (4.1), and $B \equiv 0$, which yields $(-\infty, 0) \subset \sigma_{e2}(G)$ and hence also $0 \in \sigma(G)$ since the spectrum is closed.

Since $\operatorname{ess\,inf} a = \mathfrak{a}_0$, the first statement on the localization of the real parts of eigenvalues follows from Proposition 3.1.ii).

The more detailed properties of the eigenvalue sequences $\{\lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0)\}_j$ do not follow from our abstract results. They will be obtained using the associated quadratic operator function $T(\lambda)$ and separation of variables, *i.e.* by searching for eigenfunctions of the form $\psi(x,y) := f(x)g(y)$, $(x,y) \in \Omega$. The spectral problem in the y-variable reduces to the problem

$$-g''(y) = \sigma g(y), \qquad g(\pm\ell) = 0,$$

which has the Dirichlet eigenvalues $\sigma_j = (j\pi/(2\ell))^2$, $j \in \mathbb{N}$. In the x-variable, we are left with a family of spectral problems in $L^2(\mathbb{R})$ for $T_1(\lambda) + \sigma_j$ where $T_1(\lambda) = -\mathrm{d}^2/\mathrm{d}x^2 + \mathfrak{q}_0 + 2\lambda(x^2 + \mathfrak{a}_0) + \lambda^2$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, is the quadratic operator function analyzed in Proposition 6.1. The change of variables and Stokes sectors argument as in the proof of Proposition 6.1 yield the algebraic equation (6.6) for the values of λ for which $0 \in \sigma(T_1(\lambda) + \sigma_j)$.

Since the eigenfunctions of the Dirichlet problem in $L^2((-\ell, \ell))$ form an orthonormal basis of this space, it can be shown that indeed $\sigma(T) = \cup_{j \in \mathbb{N}} \sigma(T_1 + \sigma_j)$.

Finally, a (formal) inspection of equation (6.6) shows that for fixed $k \in \mathbb{N}_0$ the eigenvalues with positive real parts are of the form

$$\lambda_{jk}(\mathfrak{a}_0, \mathfrak{q}_0) = \frac{\pi \mathrm{i}}{2\ell} - \mathfrak{a}_0 + \mathcal{O}\left(j^{-1/2}\right), \quad j \to +\infty, \tag{6.7}$$

which proves the last claim. □

The eigenvalues of G in Proposition 6.3, computed from equation (6.6), are shown in Figure 2 for the case $\mathfrak{q}_0 = 0$ and $\mathfrak{a}_0 = 0$; there the sequences given by (6.7) are clearly visible for each value of $k \in \mathbb{N}_0$.

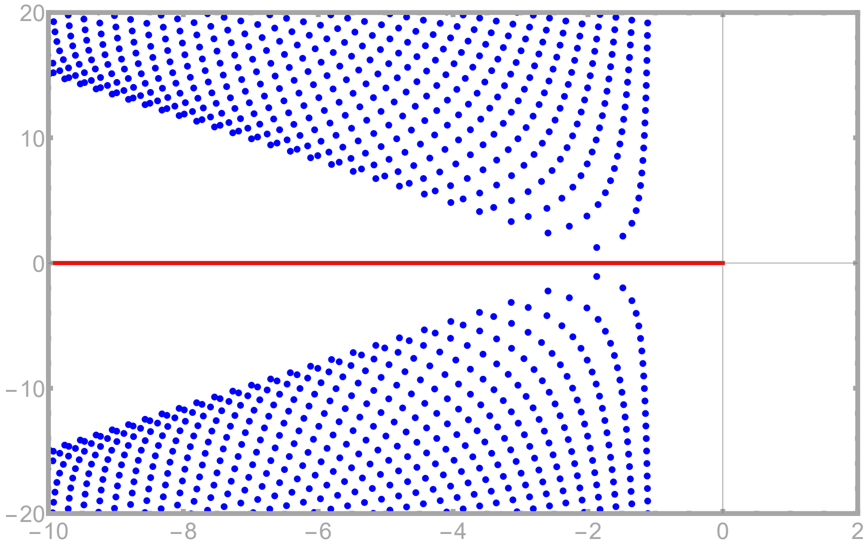

FIGURE 2. Spectrum of G with $a(x,y) = x^2$ and $q \equiv 0$ in $\Omega = \mathbb{R} \times (-\ell, \ell)$, *cf.* Proposition 6.3 with $\mathfrak{a}_0 = 1$, with eigenvalues $\{\lambda_{jk}(1,0)\}_{jk} \cup \{\overline{\lambda_{jk}(1,0)}\}_{jk}$ (blue dots) and the essential spectrum on the semiaxis $(-\infty, 0]$ (bold red).

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