ON SELMER GROUPS OF ABELIAN VARIETIES OVER ℓ-ADIC LIE EXTENSIONS OF GLOBAL FUNCTION FIELDS

A. BANDINI AND M. VALENTINO

Abstract. Let F be a global function field of characteristic $p>0$ and A/F an abelian variety. Let K/F be an ℓ-adic Lie extension ($\ell\neq p$) unramified outside a finite set of primes S and such that $\mathrm{Gal}(K/F)$ has no elements of order ℓ. We shall prove that, under certain conditions, $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.

Keywords: Selmer groups, abelian varieties, function fields, pseudo-null modules.
MSC (2010): 11R23 (primary), 11G35 (secondary).

1. Introduction

Let G be a compact ℓ-adic Lie group and $\Lambda(G)$ its associated Iwasawa algebra. A crucial theme in Iwasawa theory is the study of finitely generated $\Lambda(G)$-modules and their structure, up to "pseudo-isomorphism". When $G\simeq\mathbb{Z}_\ell^d$ for some integer $d\geqslant 1$, the structure theory for finitely generated $\Lambda(G)$-modules is well known (see, e.g., [B]). For a nonabelian G, which is the case we are interested in, studying this topic is possible thanks to an appropriate definition of the concept of "pseudo-null" for modules over $\Lambda(G)$ due to Venjakob (see [V]).

Let F be a global function field of trascendence degree one over its constant field $\mathbb{F}_q$, where q is a power of a fixed prime $p\in\mathbb{Z}$, and K a Galois extension of F unramified outside a finite set of primes S and such that $G=\mathrm{Gal}(K/F)$ is an infinite ℓ-adic Lie group with $\ell\neq p$. Let A/F be an abelian variety: the structure of $\mathcal{S}:=Sel_A(K)_\ell^\vee$ (the Pontrjagin dual of the Selmer group of A over K) as a $\Lambda(G)$-module has been extensively studied, for example, in [BL], [BL2] and [T] (see also the short survey in [BBL, Section 2] and the references there) for the abelian case, and in [OT], [W] and [BV] for the noncommutative one (these results cover also the case $\ell=p$). In most cases $\mathcal{S}$ has been proved to be a finitely generated (sometimes torsion) $\Lambda(G)$-module and here we shall deal with the presence of nontrivial pseudo-null submodules in $\mathcal{S}$. For the number field setting and $K=F(A[\ell^\infty])$, this issue was studied by Ochi and Venjakob ([OV, Theorem 5.1]) when A is an elliptic curve, and by Ochi for a general abelian variety in [O] (see also [HV] and [HO] for analogous results and/or alternative proofs).

In Sections 2 and 3 we give a brief description of the objects we will work with and of the main tools we shall need, adapting some of the techniques of [OV] to our function field setting and to a general ℓ-adic Lie extension (one of the main difference being the triviality of the image of the local Kummer maps).

In Section 4 we will prove the following

Theorem 1.1 (Theorem 4.1)**.** *Let $G=\mathrm{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order ℓ and of positive dimension $d\geqslant 3$. If $H^2(F_S/K,A[\ell^\infty])=0$ and the map ψ (induced by restriction)*

$$Sel_A(K)_\ell\hookrightarrow H^1(F_S/K,A[\ell^\infty])\xrightarrow{\ \psi\ }\bigoplus_S \mathrm{Coind}_G^{G_v}H^1(K_w,A)[\ell^\infty]$$

is surjective, then $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.

For the case $d=2$ we need more restrictive hypotheses, in particular we have the following

Proposition 1.2 (Proposition 4.3). *Let $G = \mathrm{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order ℓ and of dimension $d \geqslant 2$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and $\mathrm{cd}_\ell(G_v) = 2$ for any $v \in S$, then $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.*

A few considerations and particular cases for the vanishing of $H^2(F_S/K, A[\ell^\infty])$ are included at the end of Section 4.

Acknowledgements The authors thank F. Bars for valuable discussions and comments on earlier drafts of this paper.

2. Setting and notations

Here we fix notations and conventions that will be used through the paper.

2.1. **Fields and extensions.** Let F be a global function field of trascendence degree one over its constant field $\mathbb{F}_F = \mathbb{F}_q$, where q is a power of a fixed prime $p \in \mathbb{Z}$. We put $\overline{F}$ for an algebraic closure of F.
For any algebraic extension L/F, let $\mathfrak{M}_L$ be the set of places of L: for any $v \in \mathfrak{M}_L$ we let L_v be the completion of L at v. Let S be a finite nonempty subset of $\mathfrak{M}_F$ and let F_S be the maximal Galois extension of F unramified outside S with $G_S(F) := \mathrm{Gal}(F_S/F)$. Put $\mathcal{O}_{L,S}$ as the ring of S-integers of L and $\mathcal{O}_S^\times$ as the units of $\mathcal{O}_S = \bigcup_{L \subset F_S} \mathcal{O}_{L,S}$. Finally, $C\ell_S(L)$ denotes the *S-ideal class group of* $\mathcal{O}_{L,S}$: since S is nonempty, $C\ell_S(L)$ is finite.
For any place $v \in \mathfrak{M}_F$ we choose (and fix) an embedding $\overline{F} \hookrightarrow \overline{F_v}$, in order to get a restriction map $G_{F_v} := \mathrm{Gal}(\overline{F_v}/F_v) \hookrightarrow G_F := \mathrm{Gal}(\overline{F}/F)$.

We will deal with *ℓ-adic Lie extensions* K/F, i.e., Galois extensions with Galois group an ℓ-adic Lie group with $\ell \neq p$. We always assume that our extensions are unramified outside a finite set S of primes of $\mathfrak{M}_F$.
In what follows $\mathrm{Gal}(K/F)$ is an ℓ-adic Lie group *without points of order* ℓ, then it has finite ℓ-cohomological dimension, which is equal to its dimension as an ℓ-adic Lie group ([Se, Corollaire (1) p. 413]).

2.2. **Ext and duals.** For any ℓ-adic Lie group G we denote by

$$\Lambda(G) = \mathbb{Z}_\ell[[G]] := \varprojlim_U \mathbb{Z}_\ell[G/U]$$

the associated *Iwasawa algebra* (the limit is on the open normal subgroups of G). From Lazard's work (see [L]), we know that $\Lambda(G)$ is Noetherian and, if G is pro-ℓ and has no elements of order ℓ, then $\Lambda(G)$ is an integral domain.

For a $\Lambda(G)$-module M we consider the extension groups

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_{\Lambda(G)}(M, \Lambda(G))$$

for any integer i and put $\mathrm{E}^i(M) = 0$ for $i < 0$ by convention.
Since in our applications G comes from a Galois extension, we denote with G_v the decomposition group of $v \in \mathfrak{M}_F$ for some prime $w|v$, $w \in \mathfrak{M}_L$, and we use the notation

$$\mathrm{E}^i_v(M) := \mathrm{Ext}^i_{\Lambda(G_v)}(M, \Lambda(G_v)) .$$

Let H be a closed subgroup of G. For every $\Lambda(H)$-module N we consider the $\Lambda(G)$-modules

$$\mathrm{Coind}^H_G(N) := \mathrm{Map}_{\Lambda(H)}(\Lambda(G), N) \quad \text{and} \quad \mathrm{Ind}^G_H(N) := N \otimes_{\Lambda(H)} \Lambda(G) \text{ [1]}.$$

[1] We use the notations of [OV], some texts, e.g. [NSW], switch the definitions of $\mathrm{Ind}^H_G(N)$ and $\mathrm{Coind}^H_G(N)$.

For a $\Lambda(G)$-module M, we denote its Pontrjagin dual by $M^\vee := \mathrm{Hom}_{cont}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In this paper, M will be a (mostly discrete) topological $\mathbb{Z}_\ell$-module, so $M^\vee$ has a natural structure of $\mathbb{Z}_\ell$-module.
If M is a discrete $G_S(F)$-module, finitely generated over $\mathbb{Z}$ and with no p-torsion, in duality theorems we shall use also the dual $G_S(F)$-module of M, i.e.,

$$M' := \mathrm{Hom}(M, \mathcal{O}_S^\times)\ (= \mathrm{Hom}(M, \boldsymbol{\mu}) \text{ if } M \text{ is finite})\ .$$

2.3. **Selmer groups.** Let A be an abelian variety of dimension g defined over F: we denote by A^t its dual abelian variety. For any positive integer n we let $A[n]$ be the scheme of n-torsion points and, for any prime ℓ, we put $A[\ell^\infty] := \varinjlim A[\ell^n]$.

The *local Kummer maps* (for any $w \in \mathfrak{M}_L$)

$$\kappa_w : A(L_w) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow \varinjlim_n H^1(L_w, A[\ell^n]) := H^1(L_w, A[\ell^\infty])$$

(arising from the cohomology of the exact sequence $A[\ell^n] \hookrightarrow A \stackrel{\ell^n}{\twoheadrightarrow} A$) enable one to define the *ℓ-part of the Selmer group* of A over L as

$$Sel_A(L)_\ell = Ker \left\{ H^1(L, A[\ell^\infty]) \to \prod_{w \in \mathfrak{M}_L} H^1(L_w, A[\ell^\infty])/Im\,\kappa_w \right\}$$

(where the map is the product of the natural restrictions between cohomology groups).
For infinite extensions $\mathcal{L}/F$ the Selmer group $Sel_A(\mathcal{L})_\ell$ is defined, as usual, via direct limits.

Since $\ell \neq p$, the $Im\,\kappa_w$ are trivial and, assuming that S contains also all primes of bad reduction for A, we have the following equivalent

Definition 2.1. The *ℓ-part of the Selmer group* of A over L is

$$Sel_A(L)_\ell = Ker \left\{ H^1(F_S/L, A[\ell^\infty](F_S)) \to \bigoplus_S \mathrm{Coind}_G^{G_v} H^1(L_w, A[\ell^\infty]) \right\}\ .$$

Letting L vary through subextensions of K/F, the groups $Sel_A(L)_\ell$ admit natural actions by $\mathbb{Z}_\ell$ (because of $A[\ell^\infty]$) and by $G = \mathrm{Gal}(K/F)$. Hence they are modules over the Iwasawa algebra $\Lambda(G)$.

3. Homotopy theory and pseudo-nullity

We briefly recall the basic definitions for pseudo-null modules over a non-commutative Iwasawa algebra: a comprehensive reference is [V].

3.1. **Pseudo-null $\Lambda(G)$-modules.** Let G be an ℓ-adic Lie group without ℓ-torsion, then $\Lambda(G)$ is an Auslander regular ring of finite global dimension $\mathfrak{d} = \mathrm{cd}_\ell(G) + 1$ ([V, Theorem 3.26], cd_ℓ denotes the ℓ-cohomological dimension).
For any finitely generated $\Lambda(G)$-module M, there is a canonical filtration

$$T_0(M) \subseteq T_1(M) \subseteq \cdots \subseteq T_{\mathfrak{d}-1}(M) \subseteq T_{\mathfrak{d}}(M) = M\ .$$

Definition 3.1. We say that a $\Lambda(G)$-module M is *pseudo-null* if

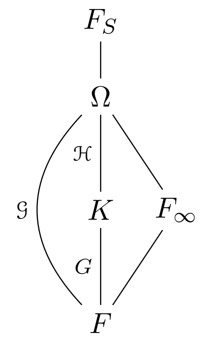
$$\delta(M) := \min\{i \,|\, T_i(M) = M\} \leqslant \mathfrak{d} - 2\ .$$

The quantity $\delta(M)$, called the *δ-dimension* of the $\Lambda(G)$-module M, is used along with the *grade* of M, that is

$$j(M) := \min\{i \mid \mathrm{E}^i(M) \neq 0\} \ .$$

As $j(M) + \delta(M) = \mathfrak{d}$ ([V, Proposition 3.5 (ii)]) we have that M is a pseudo-null module if and only if $\mathrm{E}^0(M) = \mathrm{E}^1(M) = 0$.
Since $\delta(T_i(M)) \leqslant i$ and every $T_i(M)$ is the maximal submodule of M with δ-dimension less or equal to i ([V, Proposition 3.5 (vi) (a)]), only $T_0(M), \dots, T_{\mathfrak{d}-2}(M)$ can be pseudo-null. If $T_0(M) = \cdots = T_{\mathfrak{d}-2}(M) = 0$, M does not have any nonzero pseudo-null submodule. This is the case when $\mathrm{E}^i\mathrm{E}^i(M) = 0\ \forall\, i \geqslant 2$ ([V, Proposition 3.5 (i) (c)]).

3.2. **The powerful diagram and its consequences.** In [OV, Lemma 4.5] Ochi and Venjakob generalized a result of Jannsen (see [J]) which is very powerful in applications (they call it "powerful diagram"). We provide here the statements we shall need later: for the missing details of the proofs the reader can consult [NSW, Chapter V, Section 5] and/or [OV, Section 4] (those results hold in our setting as well because we work with the $\Lambda(G)$-module $A[\ell^\infty]$, with $\ell \neq p$).
Replacing, if necessary, F by a finite extension we can (and will) assume that K is contained in the maximal pro-ℓ extension of $F_\infty := F(A[\ell^\infty])$ unramified outside S. Then we have the following



where Ω is the maximal pro-ℓ extension of F_∞ contained in F_S. We put $\mathcal{G} = \mathrm{Gal}(\Omega/F)$, $\mathcal{H} = \mathrm{Gal}(\Omega/K)$ and $G = \mathrm{Gal}(K/F)$. The extension F_∞/F will be called the *trivializing extension.*

Tensoring the natural exact sequence $I(\mathcal{G}) \hookrightarrow \Lambda(\mathcal{G}) \twoheadrightarrow \mathbb{Z}_\ell$ with $A[\ell^\infty]^\vee \simeq \mathbb{Z}_\ell^{2g}$, one gets

$$I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee \hookrightarrow \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee \twoheadrightarrow A[\ell^\infty]^\vee \ .$$

Since the mid term is projective ([OV, Lemma 4.2]), the previous sequence yields

(1) $$H_1(\mathcal{H}, A[\ell^\infty]^\vee) \hookrightarrow (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \to (\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \twoheadrightarrow (A[\ell^\infty]^\vee)_{\mathcal{H}} \ .$$

In order to shorten notations we put:

- $X = H_1(\mathcal{H}, A[\ell^\infty]^\vee)$;
- $Y = (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}}$;
- $J = Ker\{(\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \twoheadrightarrow (A[\ell^\infty]^\vee)_{\mathcal{H}}\}$.

So the sequence (1) becomes

(2) $$X \hookrightarrow Y \twoheadrightarrow J \ .$$

For our purpose it is useful to think of X as $H^1(F_S/K, A[\ell^\infty])^\vee$ (note that $H_1(\mathcal{H}, A[\ell^\infty]^\vee) \simeq H^1(\Omega/K, A[\ell^\infty])^\vee \simeq H^1(F_S/K, A[\ell^\infty])^\vee$).
Let $\mathcal{F}(d)$ denote a free pro-ℓ-group of rank $d = \dim \mathcal{G}$ and denote by $\mathcal{N}$ (resp. $\mathcal{R}$) the kernel of the natural map $\mathcal{F}(d) \to \mathcal{G}$ (resp. $\mathcal{F}(d) \to G$). For any profinite group H, we denote by

$H^{ab}(\ell)$ the maximal pro-ℓ-quotient of the maximal abelian quotient of H. With this notations the powerful diagram reads as

$$
\begin{array}{ccccccc}
H^2(\mathcal{H}, A[\ell^\infty])^\vee & \hookrightarrow & (H^1(\mathcal{N}^{ab}(\ell), A[\ell^\infty])^{\mathcal{H}})^\vee & \longrightarrow & H^1(\mathcal{R}, A[\ell^\infty]) & \twoheadrightarrow & X \\
\| & & \simeq\downarrow & & \downarrow & & \downarrow \\
H^2(\mathcal{H}, A[\ell^\infty])^\vee & \hookrightarrow & (\mathcal{N}^{ab}(\ell)\otimes A[\ell^\infty]^\vee)_{\mathcal{H}} & \longrightarrow & \Lambda(G)^{2gd} & \twoheadrightarrow & Y \\
 & & & & \downarrow & & \downarrow \\
 & & & & J & = & J\ .
\end{array} \tag{3}
$$

Moreover, since $cd_\ell(\mathcal{G}) \leqslant 2$ (just use [NSW, Theorem 8.3.17] and work as in [OV, Lemma 4.4, (iv)]), the module $\mathcal{N}^{ab}(\ell)\otimes A[\ell^\infty]^\vee$ is free over $\Lambda(\mathcal{G})$ ([OV, Lemma 4.2]), hence $(\mathcal{N}^{ab}(\ell)\otimes A[\ell^\infty]^\vee)_{\mathcal{H}}$ is projective as a $\Lambda(\mathcal{G}/\mathcal{H}) = \Lambda(G)$-module. Therefore, if $H^2(F_S/K, A[\ell^\infty]) = 0$, the module Y has projective dimension $\leqslant 1$. Whenever this is true the definition of J provides the isomorphisms

$$\mathrm{E}^i(X) \simeq \mathrm{E}^{i+1}(J) \qquad \text{and} \qquad E^i(J) \simeq \mathrm{E}^{i+1}((A[\ell^\infty]^\vee)_{\mathcal{H}}) \quad \forall\, i \geqslant 2\ , \tag{4}$$

which will be repeatedly used in our computations.
We shall need also a "localized" version of the sequence (2). For every $v \in S$ and a $w \in \mathfrak{M}_K$ dividing v, we define

$$X_v = H^1(K_w, A[\ell^\infty])^\vee \qquad \text{and} \qquad Y_v = (I(\mathcal{G}_v)\otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v}$$

(with $\mathcal{G}_v$ the decomposition groups of v in $\mathcal{G}$ and $\mathcal{H}_v = \mathcal{H}\cap\mathcal{G}_v$). The exact sequence

$$X_v \hookrightarrow Y_v \twoheadrightarrow J_v \tag{5}$$

fits into the localized version of diagram (3). If K_w is still a local field, then Tate local duality ([NSW, Theorem 7.2.6]) yields

$$H^2(K_w, A[\ell^\infty]) = H^2(K_w, \varinjlim_n A[\ell^n]) \simeq \varprojlim_n H^0(K_w, A^t[\ell^n])^\vee = 0\ .$$

If K_w is not local, then ℓ^∞ divides the degree of the extension K_w/F_v and $H^2(K_w, A[\ell^\infty]) = 0$ by [NSW, Theorem 7.1.8 (i)]. Therefore Y_v always has projective dimension $\leqslant 1$ and

$$\mathrm{E}^i(X_v) \simeq \mathrm{E}^{i+1}(J_v) \simeq \mathrm{E}^{i+2}((A[\ell^\infty]^\vee)_{\mathcal{H}_v}) \quad \forall\, i \geqslant 2\ . \tag{6}$$

We note that, since $\ell \neq p$, the image of the local Kummer maps is always 0, hence

$$X_v = H^1(K_w, A[\ell^\infty])^\vee = (H^1(K_w, A[\ell^\infty])/Im\,\kappa_w)^\vee \simeq H^1(K_w, A)[\ell^\infty]^\vee\ .$$

Then Definition 2.1 for $L = K$ can be written as

$$Sel_A(K)_\ell = Ker\left\{\psi\ :\ X^\vee \longrightarrow \bigoplus_S \mathrm{Coind}_G^{G_v} X_v^\vee\right\}$$

and, dualizing, we get a map

$$\psi^\vee\ :\ \bigoplus_S \mathrm{Ind}_{G_v}^G X_v \longrightarrow X$$

whose cokernel is exactly $\mathcal{S} := Sel_A(K)_\ell^\vee$.

The following result will be fundamental for our computations.

Theorem 3.2 (U. Jannsen)**.** *Let G be an ℓ-adic Lie group without elements of order ℓ and of dimension d. Let M be a $\Lambda(G)$-module which is finitely generated as $\mathbb{Z}_\ell$-module. Then $\mathrm{E}^i(M)$ is a finitely generated $\mathbb{Z}_\ell$-module and, in particular,*

1. *if M is $\mathbb{Z}_\ell$-free, then $E^i(M) = 0$ for any $i \neq d$ and $E^d(M)$ is free;*

2. *if M is finite, then $E^i(M)=0$ for any $i\neq d+1$ and $E^{d+1}(M)$ is finite.*

Proof. See [J, Corollary 2.6]. ☐

Corollary 3.3. *With notations as above:*

1. *if $H^2(F_S/K, A[\ell^\infty])=0$, then, for $i\geqslant 2$,*

$$\mathrm{E}^i(X) \ is \left\{ \begin{array}{ll} finite & if\ i=d-1 \\ free & if\ i=d-2 \\ 0 & otherwise \end{array} \right. ;$$

2. $\mathrm{E}^i_v\mathrm{E}^{i-1}_v(X_v)=0$ *for $i\geqslant 3$.*

Proof. **1.** The hypothesis yields the isomorphism $\mathrm{E}^i(X)\simeq E^{i+2}((A[\ell^\infty]^\vee)_{\mathcal{H}})$. Since

$$(A[\ell^\infty]^\vee)_{\mathcal{H}} \simeq (A[\ell^\infty]^{\mathcal{H}})^\vee = A[\ell^\infty](K)^\vee \simeq \mathbb{Z}_\ell^r \oplus \Delta$$

(with $0\leqslant r\leqslant 2g$ and Δ a finite group) and $\mathrm{E}^i(\mathbb{Z}_\ell^r\oplus\Delta)=\mathrm{E}^i(\mathbb{Z}_\ell^r)\oplus\mathrm{E}^i(\Delta)$, the claim follows from Theorem 3.2.

2. Use Theorem 3.2 and the isomorphism in (6). ☐

Lemma 3.4. *If $H^2(F_S/K, A[\ell^\infty])=0$, then there is the following commutative diagram*

$$\begin{array}{ccccc}
\mathrm{E}^1(Y) & \xrightarrow{g_1} & \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^1_v(Y_v) & \twoheadrightarrow & Coker(g_1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{E}^1(X) & \xrightarrow{h_1} & \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^1_v(X_v) & \twoheadrightarrow & Coker(h_1) \\
\downarrow & & \downarrow & & \downarrow f \\
\mathrm{E}^2(J) & \xrightarrow{\bar g_1} & \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2_v(J_v) & \twoheadrightarrow & Coker(\bar g_1)\,.
\end{array}$$

Proof. The inclusions $\mathcal{G}_v\subseteq\mathcal{G}$ and $\mathcal{H}_v\subseteq\mathcal{H}$ induce the maps

$$(I(\mathcal{G}_v)\otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v} \to (I(\mathcal{G})\otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v} \to (I(\mathcal{G})\otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}}\ .$$

We have a homomorphism of $\Lambda(G)$-modules $g:\bigoplus_S \mathrm{Ind}_{G_v}^G Y_v\to Y$ which, restricted to the X_v's, provides the map $h:\bigoplus_S \mathrm{Ind}_{G_v}^G X_v\to X$. So we have the following situation

(7)
$$\begin{array}{ccc}
X & \xleftarrow{h} & \bigoplus_S \mathrm{Ind}_{G_v}^G X_v \\
\downarrow & & \downarrow \\
Y & \xleftarrow{g} & \bigoplus_S \mathrm{Ind}_{G_v}^G Y_v \\
\downarrow & & \downarrow \\
J & \xleftarrow{\bar g} & \bigoplus_S \mathrm{Ind}_{G_v}^G J_v
\end{array}$$

where $\bar g$ is induced by g and the diagram is obviously commutative.

Since Y and the Y_v's have projective dimension $\leqslant 1$ (i.e., $\mathrm{E}^2(Y)=\mathrm{E}^2(Y_v)=0$), the lemma follows by taking Ext in diagram (7) and recalling that, for any $i\geqslant 0$, $\mathrm{E}^i_v(\mathrm{Ind}_{G_v}^G(X_v)) = \mathrm{Ind}_{G_v}^G\mathrm{E}^i_v(X_v)$ (see [OV, Lemma 5.5]). ☐

In the next subsection we are going to describe the structure of $Coker(g_1)$.

3.3. **Homotopy theory and** $Coker(g_1)$**.** For every finitely generated $\Lambda(G)$-module M choose a presentation $P_1 \to P_0 \to M \to 0$ of M by projectives and define the *transpose* functor DM by the exactness of the sequence

$$0 \to \mathrm{E}^0(M) \to \mathrm{E}^0(P_0) \to \mathrm{E}^0(P_1) \to DM \to 0.$$

Then it can be shown that the functor D is well-defined and one has $D^2 = Id$ (see [J]).

Definition 3.5. Let L be an extension of F contained in F_S. Then we define

$$Z(L) := H^0(F_S/L, \varinjlim_m D_2(A[\ell^m]))^\vee$$

where

$$D_2(A[\ell^m]) = \varinjlim_{F \subset E \subset F_S} (H^2(F_S/E, A[\ell^m]))^\vee$$

and the limit in $\varinjlim_m D_2(A[\ell^m])$ is taken with respect to the ℓ-power map $A[\ell^{m+1}] \xrightarrow{\ell} A[\ell^m]$. In the same way we define $Z(L)$ for any Galois extension L of F_v.

An alternative description of the module Z is provided by the following

Lemma 3.6. *Let K be a fixed extension of F contained in F_S and K_w its completion for some $w|v \in S$. Then*

$$Z(K) \simeq \varprojlim_{F \subseteq L \subseteq K} H^2(F_S/L, T_\ell(A)) \quad \text{and} \quad Z(K_w) \simeq \varprojlim_{F_v \subseteq L \subseteq K_w} H^2(L, T_\ell(A)) \ .$$

Proof. **Global case.** For any global field L, let

$$Ш^i(F_S/L, A[\ell^\infty]) := Ker \left\{ H^i(F_S/L, A[\ell^\infty]) \to \bigoplus_S H^i(L_w, A[\ell^\infty]) \right\} \ .$$

We have already seen that $H^2(L_w, A[\ell^\infty]) = 0$, hence $H^2(F_S/L, A[\ell^\infty]) = Ш^2(F_S/L, A[\ell^\infty])$. Using the pairing of [M, Ch. I, Proposition 6.9], we get

$$\begin{aligned} Z(K) &= H^0(F_S/K, \varinjlim_m \varinjlim_{F \subseteq L \subseteq F_S} Ш^2(F_S/L, A[\ell^m])^\vee)^\vee \\ &= H^0(F_S/K, \varinjlim_m \varinjlim_{F \subseteq L \subseteq F_S} Ш^0(F_S/L, A^t[\ell^m]))^\vee \\ &= (\varinjlim_m \varinjlim_{F \subseteq L \subseteq F_S} Ш^0(F_S/L, A^t[\ell^m])^{\mathrm{Gal}(F_S/K)})^\vee \\ &= (\varinjlim_m \varinjlim_{F \subseteq L \subseteq K} Ш^0(F_S/L, A^t[\ell^m]))^\vee \\ &= \varprojlim_m \varprojlim_{F \subseteq L \subseteq K} (H^2(F_S/L, A[\ell^m])^\vee)^\vee \\ &= \varprojlim_{F \subseteq L \subseteq K} H^2(F_S/L, T_\ell(A)) \ . \end{aligned}$$

Local case. The proof is similar (using Tate local duality). □

We recall that our group G has no elements of order ℓ, hence $\Lambda(G)$ is a domain. Moreover for any open subgroup U of G we have that (see [J, Lemma 2.3])

$$\mathrm{E}^i(U) \simeq \mathrm{E}^i(G) \quad \forall\, i \in \mathbb{Z}$$

is an isomorphism of $\Lambda(U)$-modules. An ℓ-adic Lie group G always contains an open pro-ℓ subgroup ([DdSMS, Corollary 8.34]), so, in order to use properly the usual definitions of "torsion submodule" and "rank" for a finitely generated $\Lambda(G)$-module, with no loss of generality, we will assume that G is pro-ℓ.

Proposition 3.7. *Let M be a finitely generated $\Lambda(G)$-module. Then $\mathrm{E}^i(M)$ is a finitely generated torsion $\Lambda(G)$-module for any $i \geqslant 1$.*

Proof. Take a finite presentation $P_1 \to P_0 \to M \to 0$ with finitely generated and projective $\Lambda(G)$-modules P_1 and P_0, and the consequent exact sequence

$$0 \to R_1 \to P_0 \to M \to 0 \tag{8}$$

for a suitable submodule R_1 of P_1. Since M and $\mathrm{Hom}_{\Lambda(G)}(M, \Lambda(G))$ have the same $\Lambda(G)$-rank, computing ranks in the sequence coming from (8)

$$\begin{aligned} &\mathrm{Hom}_{\Lambda(G)}(M,\Lambda(G)) \hookrightarrow \mathrm{Hom}_{\Lambda(G)}(P_0,\Lambda(G)) \to \mathrm{Hom}_{\Lambda(G)}(R_1,\Lambda(G)) \to \mathrm{E}^1(M) \to \\ &\qquad \to 0 \to \mathrm{E}^1(R_1) \to \mathrm{E}^2(M) \to 0 \to \cdots \to 0 \to \mathrm{E}^{i-1}(R_1) \to \mathrm{E}^i(M) \to 0 \to \cdots \end{aligned}$$

one finds $\mathrm{rank}_{\Lambda(G)}(\mathrm{E}^1(M)) = 0$ for any finitely generated $\Lambda(G)$-module M. Therefore $\mathrm{E}^1(R_1)$ is torsion, which yields $\mathrm{E}^2(M) \simeq \mathrm{E}^1(R_1)$ is torsion. Iterating $\mathrm{E}^i(M) \simeq \mathrm{E}^{i-1}(R_1)$ is $\Lambda(G)$-torsion $\forall\, i \geqslant 2$. □

Lemma 3.8. *Let F_n be subfields of K such that $\mathrm{Gal}(K/F) = \varprojlim_n \mathrm{Gal}(F_n/F)$. Then*

$$H^2_{Iw}(K_w, T_\ell(A)) := \varprojlim_{n,m} H^2(F_{v_n}, A[\ell^m])$$

is a torsion $\Lambda(G_v)$-module. If $H^2(F_S/K, A[\ell^\infty]) = 0$, then

$$H^2_{Iw}(K, T_\ell(A)) := \varprojlim_{n,m} H^2(F_S/F_n, A[\ell^m])$$

is a $\Lambda(G)$-torsion as well.

Proof. The proofs are identical so we only show the second statement. From the spectral sequence

$$\mathrm{E}_2^{p,q} = E^p(H^q(F_S/K, A[\ell^\infty])^\vee) \implies H^{p+q}_{Iw}(K, T_\ell(A))$$

due to Jannsen (see [J1]), we have a filtration for $H^2_{Iw}(K, T_\ell(A))$

$$0 = H^2_3 \subseteq H^2_2 \subseteq H^2_1 \subseteq H^2_0 = H^2_{Iw}(K, T_\ell(A)) \ , \tag{9}$$

which provides the following sequences:

$$\begin{aligned} &\mathrm{E}^0(H^1(F_S/K, A[\ell^\infty])^\vee) \to \mathrm{E}^2(H^0(F_S/K, A[\ell^\infty])^\vee) \to H^2_1 \\ &\qquad\qquad \to \mathrm{E}^1(H^1(F_S/K, A[\ell^\infty])^\vee) \to \mathrm{E}^3(H^0(F_S/K, A[\ell^\infty])^\vee) \end{aligned}$$

and

$$H^2_1 \hookrightarrow H^2_{Iw}(K, T_\ell(A)) \twoheadrightarrow \mathrm{E}_\infty^{0,2} \ .$$

By hypothesis $\mathrm{E}_\infty^{0,2} \simeq \mathrm{E}_2^{0,2} = 0$, so $H^2_1 \simeq H^2_{Iw}(K, T_\ell(A))$.
Since $H^i(F_S/K, A[\ell^\infty])^\vee$ is a finitely generated $\Lambda(G)$-module for $i \in \{0,1\}$ (for $i = 1$ just look at X in diagram (3)), Proposition 3.7 yields that the groups $\mathrm{E}^2(H^0(F_S/K, A[\ell^\infty])^\vee)$ and $\mathrm{E}^1(H^1(F_S/K, A[\ell^\infty])^\vee)$ are $\Lambda(G)$-torsion. Hence H^2_1 is torsion as well. □

Lemma 3.9. *With notations and hypotheses as in Lemma 3.4, $Coker(g_1)$ is finitely generated as $\mathbb{Z}_\ell$-module.*

Proof. Lemma 3.6 yields $Z(K) = H^2_{Iw}(K, T_\ell(A))$ so, using [OV, Proposition 4.10], one has $DH^2_{Iw}(K, T_\ell(A)) \simeq Y$. Therefore $\mathrm{E}^1(DH^2_{Iw}(K, T_\ell(A))) \simeq \mathrm{E}^1(Y)$. Since $H^2_{Iw}(K, T_\ell(A))$ is a $\Lambda(G)$-torsion module, [OV, Lemma 3.1] implies $\mathrm{E}^1(DH^2_{Iw}(K, T_\ell(A)) \simeq H^2_{Iw}(K, T_\ell(A))$, i.e.,

$$H^2_{Iw}(K, T_\ell(A)) \simeq \mathrm{E}^1(Y)$$

(the same holds for the "local" modules). The map g_1 of Lemma 3.4 then reads as

$$g_1 : \varprojlim_n H^2(F_S/F_n, T_\ell(A)) \to \bigoplus_S \mathrm{Ind}^G_{G_v} \varprojlim_n H^2(F_{v_n}, T_\ell(A)) \ .$$

The claim follows from the Poitou-Tate sequence (see [NSW, 8.6.10 p. 488]), since

$$Coker(g_1) \simeq \varprojlim_{n,m} H^0(F_S/F_n, (A[\ell^m])') \ .$$

□

4. Main Theorem

We are now ready to prove the following

Theorem 4.1. *Let $G = \mathrm{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order ℓ and of positive dimension $d \geqslant 3$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and the map ψ in the sequence*

$$Sel_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \mathrm{Coind}^{G_v}_G H^1(K_w, A)[\ell^\infty] \tag{10}$$

is surjective, then $\mathcal{S} := Sel_A(K)^\vee_\ell$ has no nontrivial pseudo-null submodule.

Proof. We need to prove that

$$\mathrm{E}^i\mathrm{E}^i(\mathcal{S}) = 0 \quad \forall\, i \geqslant 2 \ ,$$

and we consider two cases.
<u>Case $i = 2$.</u> Let $\mathcal{D} := \bar{g}_1(\mathrm{E}^2(J))$. Then

$$Coker(\bar{g}_1) = \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v)/\mathcal{D} \ .$$

Observe that $\mathcal{D} \simeq \bar{g}_1(\mathrm{E}^3(A[\ell^\infty]^\vee_{\mathcal{H}}))$ is a finitely generated $\mathbb{Z}_\ell$-module (it is zero if $d \neq 3$ and free as $\mathbb{Z}_\ell$-module if $d = 3$), so $\mathrm{E}^1(\mathcal{D}) = 0$. Even if the theorem is limited to $d \geqslant 3$ we remark here that, for $d = 2$, $\mathcal{D}$ is finite and, for $d = 1$, $\mathcal{D} = 0$: hence $\mathrm{E}^1(\mathcal{D}) = 0$ in any case.
Moreover

$$\begin{aligned}\mathrm{E}^2(\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v)) &= \mathrm{E}^2(\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^3(A[\ell^\infty]^\vee_{\mathcal{H}_v})) \\ &= \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2\mathrm{E}^3(A[\ell^\infty]^\vee_{\mathcal{H}_v}) = 0 \ ,\end{aligned}$$

so, taking Ext in the sequence,

$$\mathcal{D} \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v) \twoheadrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v)/\mathcal{D} \ , \tag{11}$$

one finds

$$\mathrm{E}^1(\mathcal{D}) \to \mathrm{E}^2(\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v)/\mathcal{D}) \to \mathrm{E}^2(\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v)) \ .$$

Therefore

$$\mathrm{E}^2(\bigoplus_S \mathrm{Ind}^G_{G_v} \mathrm{E}^2(J_v)/\mathcal{D}) = 0 \ . \tag{12}$$

Recall the sequences

$$\bigoplus_S \mathrm{Ind}_{G_v}^G X_v \hookrightarrow X \twoheadrightarrow \mathcal{S} \tag{13}$$

$$Ker(f) \hookrightarrow Coker(h_1) \twoheadrightarrow Coker(\bar{g}_1) \tag{14}$$

provided (respectively) by the hypothesis on ψ and by Lemma 3.4. Take Ext on (13) to get

$$\mathrm{E}^1(X) \xrightarrow{h_1} \mathrm{E}^1(\bigoplus_S \mathrm{Ind}_{G_v}^G X_v) \to \mathrm{E}^2(\mathcal{S}) \to \mathrm{E}^2(X)\ .$$

If $d \geqslant 5$, then $\mathrm{E}^2(X) \simeq \mathrm{E}^3(J) \simeq \mathrm{E}^4(A[\ell^\infty]^\vee_{\mathcal{H}}) = 0$. When this is the case $Coker(h_1) \simeq \mathrm{E}^2(\mathcal{S})$ and sequence (14) becomes

$$Ker(f) \hookrightarrow \mathrm{E}^2(\mathcal{S}) \twoheadrightarrow \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D}\ .$$

By Lemma 3.9, $Ker(f)$ is a finitely generated $\mathbb{Z}_\ell$-module. Taking Ext, one has

$$\mathrm{E}^2(\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}^2(J_v)/\mathcal{D}) \to \mathrm{E}^2\mathrm{E}^2(\mathcal{S}) \to \mathrm{E}^2(Ker(f))\ ,$$

where the first and third term are trivial, so $\mathrm{E}^2\mathrm{E}^2(\mathcal{S}) = 0$ as well.
We are left with $d = 3, 4$. We know that $\mathrm{E}^4(A[\ell^\infty]^\vee_{\mathcal{H}}) = \mathrm{E}^2(X)$ is free over $\mathbb{Z}_\ell$ if $d = 4$ or finite if $d = 3$ (again we remark it is 0 if $d = 1, 2$). Anyway $\mathrm{E}^2\mathrm{E}^2(X) = 0$ in all cases. From the sequence

$$Coker(h_1) \hookrightarrow \mathrm{E}^2(\mathcal{S}) \xrightarrow{\eta} \mathrm{E}^2(X)$$

one writes

$$Coker(h_1) \hookrightarrow \mathrm{E}^2(\mathcal{S}) \twoheadrightarrow Im(\eta) \tag{15}$$

where $Im(\eta)$ is free over $\mathbb{Z}_\ell$ if $d = 4$ or finite if $d = 3$.
Taking Ext in (14) one has

$$\mathrm{E}^2(Coker(\bar{g}_1)) \to \mathrm{E}^2(Coker(h_1)) \to \mathrm{E}^2(Ker(f))$$

with the first (see equation (12)) and third term equal to zero, so $\mathrm{E}^2(Coker(h_1)) = 0$. This fact in sequence (15) implies

$$0 = \mathrm{E}^2(Im(\eta)) \to \mathrm{E}^2\mathrm{E}^2(\mathcal{S}) \to \mathrm{E}^2(Coker(h_1)) = 0\ ,$$

so $\mathrm{E}^2\mathrm{E}^2(\mathcal{S}) = 0$.
<u>**Case $i \geqslant 3$.**</u> From sequence (13) we get the following

$$\mathrm{E}^{i+1}(A[\ell^\infty]^\vee_{\mathcal{H}}) \simeq \mathrm{E}^{i-1}(X) \to \bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^{i-1}(X_v) \to \mathrm{E}^i(\mathcal{S}) \to \mathrm{E}^i(X) \simeq \mathrm{E}^{i+2}(A[\ell^\infty]^\vee_{\mathcal{H}})\ . \tag{16}$$

We have four cases, depending on whether $\mathrm{E}^{i-1}(X)$ and $\mathrm{E}^i(X)$ are trivial or not.
Case 1. Assume $\mathrm{E}^{i-1}(X) = \mathrm{E}^i(X) = 0$.
From (16) we obtain the isomorphism

$$\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^{i-1}(X_v) \simeq \mathrm{E}^i(\mathcal{S})\ ,$$

so

$$\bigoplus_S \mathrm{Ind}_{G_v}^G \mathrm{E}_v^i \mathrm{E}_v^{i-1}(X_v) \simeq \mathrm{E}^i\mathrm{E}^i(\mathcal{S}) = 0$$

thanks to Corollary 3.3 part **2**. We remark that this is the only case to consider when $d = 1, 2$.

Case 2. Assume $\mathrm{E}^{i-1}(X)=0$ and $\mathrm{E}^i(X)\neq 0$.
This happens when $i=d-2$ or $i=d-1$ and $A[\ell^\infty]^\vee_{\mathcal{H}}$ is finite. From (16) we have

$$\bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-3}_v \hookrightarrow \mathrm{E}^{d-2}(\mathcal{S}) \twoheadrightarrow N$$

$$(\text{ resp. } \quad \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-2}_v \hookrightarrow \mathrm{E}^{d-1}(\mathcal{S}) \twoheadrightarrow N \ )$$

where N is a submodule of the free module $\mathrm{E}^{d-2}(X)$ (resp. of the finite module $\mathrm{E}^{d-1}(X)$). Therefore $\mathrm{E}^{d-2}(N)=0$ (resp. $\mathrm{E}^{d-1}(N)=0$) and, moreover, $\mathrm{E}^{d-2}_v\mathrm{E}^{d-3}_v(X_v)=0$ (resp. $\mathrm{E}^{d-1}_v\mathrm{E}^{d-2}_v(X_v)=0$) by Corollary 3.3 part **2**. Hence $\mathrm{E}^{d-2}\mathrm{E}^{d-2}(\mathcal{S})=0$ (resp. $\mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S})=0$).
Case 3. Assume $\mathrm{E}^{i-1}(X)\neq 0$ and $\mathrm{E}^i(X)=0$.
This happens when $i=d$ or $i=d-1$ and $A[\ell^\infty]^\vee_{\mathcal{H}}$ is free. The sequence (16) gives

$$N \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-1}_v(X_v) \twoheadrightarrow \mathrm{E}^d(\mathcal{S})$$

$$(\text{ resp. } \quad N \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-2}_v(X_v) \twoheadrightarrow \mathrm{E}^{d-1}(\mathcal{S}) \ )$$

where now N is a quotient of the finite module $\mathrm{E}^{d-1}(X)$ (resp. of the free module $\mathrm{E}^{d-2}(X)$). Then $\mathrm{E}^d(N)=0$ (resp. $\mathrm{E}^{d-1}(N)=0$) and

$$\bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^d_v\mathrm{E}^{d-1}_v(X_v) \simeq \mathrm{E}^d\mathrm{E}^d(\mathcal{S})=0$$

$$(\text{ resp. } \quad \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-1}_v\mathrm{E}^{d-2}_v(X_v) \simeq \mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S})=0 \ ) \ .$$

Case 4. Assume $\mathrm{E}^{i-1}(X)\neq 0$ and $\mathrm{E}^i(X)\neq 0$.
This happens when $i=d-1$ and $A[\ell^\infty]^\vee_{\mathcal{H}}$ has nontrivial rank and torsion. From sequence (16) we have

$$\mathrm{E}^{d-2}(X) \to \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-2}_v(X_v) \to \mathrm{E}^{d-1}(\mathcal{S}) \to \mathrm{E}^{d-1}(X) \ .$$

Let N_1, N_2 and N_3 be modules such that:

- N_1 is a quotient of $\mathrm{E}^{d-2}(X)$ (which is torsion free so that $\mathrm{E}^{d-2}(N_1)=0$);
- N_2 is a submodule of $\mathrm{E}^{d-1}(X)$ (which is finite so that $\mathrm{E}^{d-1}(N_2)=0$);
- N_3 is a module such that the sequences

  $$N_1 \hookrightarrow \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-2}_v(X_v) \twoheadrightarrow N_3 \qquad \text{and} \qquad N_3 \hookrightarrow \mathrm{E}^{d-1}(\mathcal{S}) \twoheadrightarrow N_2$$

  are exact.

Applying the functor Ext we find

$$\mathrm{E}^{d-2}(N_1) \to \mathrm{E}^{d-1}(N_3) \to \bigoplus_S \mathrm{Ind}^G_{G_v}\mathrm{E}^{d-1}_v\mathrm{E}^{d-2}_v(X_v)$$

(which yields $\mathrm{E}^{d-1}(N_3)=0$), and

$$\mathrm{E}^{d-1}(N_2) \to \mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S}) \to \mathrm{E}^{d-1}(N_3)$$

which proves $\mathrm{E}^{d-1}\mathrm{E}^{d-1}(\mathcal{S})=0$. □

Remark 4.2. As pointed out in various steps of the previous proof, most of the statements still hold for $d=1,2$. The only missing part is $\mathrm{E}^2(Ker(f))=0$ for $i=2$, in that case only our calculations to get $\mathrm{E}^2\mathrm{E}^2(\mathcal{S})=0$ fail. In particular the same proof shows that $\mathrm{E}^2\mathrm{E}^2(\mathcal{S})=0$ when $Ker(f)$ is free and $d=1$ or when $Ker(f)$ is finite and $d=2$ or, obviously, for any d if f is injective.

We can extend the previous result to the $d \geqslant 2$ case with some extra assumptions.

Proposition 4.3. *Let $G = \mathrm{Gal}(K/F)$ be an ℓ-adic Lie group without elements of order ℓ and of dimension $d \geqslant 2$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and $\mathrm{cd}_\ell(G_v) = 2$ for any $v \in S$, then $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.*

Proof. Since $\mathrm{cd}_\ell(F_v) = 2$ (by [NSW, Theorem 7.1.8]), our hypothesis implies that $\mathrm{Gal}(\overline{F_v}/K_w)$ has no elements of order l (see also [NSW, Theorem 7.5.3]). Hence $H^1(K_w, A[\ell^\infty])^\vee = 0$ and $Sel_A(K)_\ell^\vee \simeq X$ embeds in Y. Now $H^2(F_S/K, A[\ell^\infty]) = 0$ yields Y has projective dimension $\leqslant 1$, so Y has no nontrivial pseudo-null submodule (by [OV, Proposition 2.5]). $\square$

4.1. **The hypotheses on $H^2(F_S/K, A[\ell^\infty])$ and ψ.** Let F_m be extensions of F such that $\mathrm{Gal}(K/F) \simeq \varprojlim_m \mathrm{Gal}(F_m/F)$. To provide some cases in which the main hypotheses hold we consider the Poitou-Tate sequence for the module $A[\ell^n]$, from which one can extract the sequence

$$\begin{array}{rcl} 0 \longrightarrow Ker(\psi_{m,n}) \longrightarrow H^1(F_S/F_m, A[\ell^n]) & \xrightarrow{\psi_{m,n}} & \prod_{\substack{v_m|v \\ v\in S}} H^1(F_{v_m}, A[\ell^n]) \\ & & \downarrow \phi_{m,n} \\ \prod_{\substack{v_m|v \\ v\in S}} H^2(F_{v_m}, A[\ell^n]) \longleftarrow H^2(F_S/F_m, A[\ell^n]) & \longleftarrow & Ker(\psi^t_{m,n}))^\vee \\ \downarrow & & \\ H^0(F_S/F_m, A^t[\ell^n])^\vee \longrightarrow 0 & & \end{array} \tag{17}$$

(where $\psi^t_{m,n}$ is the analogue of $\psi_{m,n}$ for the dual abelian variety A^t , i.e., their kernels represent the Selmer groups over F_m for the modules $A^t[\ell^n]$ and $A[\ell^n]$ respectively). Taking direct limits on n and recalling that $H^2(F_{v_m}, A[\ell^\infty]) = 0$, the sequence (17) becomes

$$\begin{array}{rcl} 0 \longrightarrow Sel_A(F_m)_\ell \longrightarrow H^1(F_S/F_m, A[\ell^\infty]) & \xrightarrow{\psi_m} & \prod_{\substack{v_m|v \\ v\in S}} H^1(F_{v_m}, A[\ell^\infty]) \\ & & \downarrow \phi_m \\ 0 \longleftarrow H^2(F_S/F_m, A[\ell^\infty]) & \longleftarrow & (\varprojlim_n Ker(\psi^t_{m,n}))^\vee \end{array} \tag{18}$$

(for more details one can consult [CS, Chapter 1]). One way to prove that $H^2(F_S/K, A[\ell^\infty]) = 0$ and ψ is surjective is to show that $(\varprojlim_n Ker(\psi^t_{m,n}))^\vee = 0$ for any m. We mention here two cases in which the hypothesis on the vanishing of $H^2(F_S/K, A[\ell^\infty])$ is verified. The following is basically [CS, Proposition 1.9].

Proposition 4.4. *Let F_m be as above and assume that $|Sel_{A^t}(F_m)_\ell| < \infty$ for any m, then*

$$H^2(F_S/K, A[\ell^\infty]) = 0\ .$$

Proof. From [M, Chapter I Remark 3.6] we have the isomorphism

$$A^t(F_{v_m})^* \simeq H^1(F_{v_m}, A[\ell^\infty])^\vee\ ,$$

where $A^t(F_{v_m})^* \simeq \varprojlim_n A^t(F_{v_m})/\ell^n A^t(F_{v_m})$.

Taking inverse limits on n in the exact sequence

$$A^t(F_m)/\ell^n A^t(F_m) \hookrightarrow Ker(\psi^t_{m,n}) \twoheadrightarrow Ш(A^t/F_m)[\ell^n]\ ,$$

and noting that $|Ш(A^t/F_m)[\ell^\infty]| < \infty$ yields $T_\ell(Ш(A^t/F_m)) = 0$, we find

$$A^t(F_m)^* \simeq \varprojlim_n Ker(\psi^t_{m,n})\ .$$

Therefore (18) becomes

$$0 \longrightarrow Sel_A(F_m)_\ell \longrightarrow H^1(F_S/F_m, A[\ell^\infty]) \xrightarrow{\psi} \prod_{\substack{v_m|v \\ v\in S}} (A^t(F_{v_m})^*)^\vee \xrightarrow{\widetilde{\phi}} (A^t(F_m)^*)^\vee \longrightarrow H^2(F_S/F_m, A[\ell^\infty]) \longrightarrow 0 \tag{19}$$

By hypothesis $A^t(F_m)^*$ is finite, therefore $H^2(F_S/F_m, A[\ell^\infty])$ is finite as well. From the cohomology of the sequence

$$A[\ell] \hookrightarrow A[\ell^\infty] \overset{\ell}{\twoheadrightarrow} A[\ell^\infty]$$

(and the fact that $H^3(F_S/F_m, A[\ell]) = 0$, because $cd_\ell(\mathrm{Gal}(F_S/F_m)) = 2$), one finds

$$H^2(F_S/F_m, A[\ell^\infty]) \overset{\ell}{\twoheadrightarrow} H^2(F_S/F_m, A[\ell^\infty]) \ ,$$

i.e., $H^2(F_S/F_m, A[\ell^\infty])$ is divisible. Being divisible and finite $H^2(F_S/F_m, A[\ell^\infty])$ must be 0 for any m and the claim follows. □

We can also prove the vanishing of $H^2(F_S/K, A[\ell^\infty])$ for the extension $K = F(A[\ell^\infty])$.

Proposition 4.5. *If $K = F(A[\ell^\infty])$, then $H^2(F_S/K, A[\ell^\infty]) = 0$.*

Proof. $\mathrm{Gal}(F_S/K)$ has trivial action on $A[\ell^\infty]$ and (by the Weil pairing) on $\boldsymbol{\mu}_{\ell^\infty}$, so

$$H^2(F_S/K, A[\ell^\infty]) \simeq H^2(F_S/K, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}) \simeq H^2(F_S/K, (\boldsymbol{\mu}_{\ell^\infty})^{2g}) \ .$$

Let $F_n = F(A[\ell^n])$, using the notations of Lemma 3.6, Poitou-Tate duality ([NSW, Theorem 8.6.7]) and the isomorphism $Ш^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z}) \simeq \mathrm{Hom}(C\ell_S(F_n), \mathbb{Z}/\ell^m\mathbb{Z})$ ([NSW, Lemma 8.6.3]), one has

$$\begin{aligned} H^2(F_S/K, \boldsymbol{\mu}_{\ell^\infty}) &\simeq Ш^2(F_S/K, \boldsymbol{\mu}_{\ell^\infty}) \simeq \varinjlim_{n,m} Ш^2(F_S/F_n, \boldsymbol{\mu}_{\ell^m}) \\ &\simeq \varinjlim_{n,m} Ш^1(F_S/F_n, \boldsymbol{\mu}'_{\ell^m})^\vee \simeq \varinjlim_{n,m} Ш^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z})^\vee \\ &\simeq \varinjlim_{n,m} \mathrm{Hom}(C\ell_S(F_n), \mathbb{Z}/\ell^m\mathbb{Z})^\vee \simeq \varinjlim_{n,m} C\ell_S(F_n)/\ell^m \\ &\simeq \varinjlim_{n} C\ell_S(F_n) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0 \end{aligned}$$

since $C\ell_S(F_n)$ is finite. □

Remark 4.6. The above proposition works in the same way for a general ℓ-adic Lie extensions, unramified outside S, which contains the trivializing extension.

Example 4.7. Let A be an abelian variety without complex multiplication: by Proposition 4.5, the extension $K = F(A[\ell^\infty])$ realizes the hypothesis of Proposition 4.3 when every bad reduction prime is of split multiplicative reduction (in order to have $\mathrm{cd}_\ell(G_v) = 2$) and $\ell > 2g+1$ (by [ST] and the embedding $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{GL}_{2g}(\mathbb{Z}_\ell)$). Therefore $Sel_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule. When $A = \mathcal{E}$ is an elliptic curve (using Igusa's theorem, see, e.g., [BLV]) one can prove that $\dim \mathrm{Gal}(K/F) = 4$ and also the surjectivity of the map ψ (which, in this case, is not needed to prove the absence of pseudo-null submodules): more details can be found in [S].
The same problem over number fields cannot (in general) be addressed in the same way and one needs the surjectivity of the map ψ. The topic is treated (for example) in [C, Section 4.2] and [HV, Section 7.1].

References

[BBL] A. Bandini, F. Bars and I. Longhi *Aspects of Iwasawa theory over function fields.* To appear in the EMS Congress Reports, arXiv:1005.2289v2 [math.NT] (2010).

[BL] A. Bandini and I. Longhi *Control theorems for elliptic curves over function fields.* Int. J. Number Theory **5** no. 2 (2009), 229–256.

[BL2] A. Bandini and I. Longhi *Selmer groups for elliptic curves in $\mathbb{Z}_l^d$-extensions of function fields of characteristic p.* Ann. Inst. Fourier **59** no. 6 (2009), 2301–2327.

[BLV] A. Bandini, I. Longhi and S. Vigni. *Torsion points on elliptic curves over function fields and a theorem of Igusa.* Expo. Math. **27** no. 3 (2009), 175–209.

[BV] A. Bandini and M. Valentino. *Control Theorems for ℓ-adic Lie extensions of global function fields.* To appear in Ann. Sc. Norm. Sup. Pisa Cl. Sci. DOI Number: 10.2422/2036-2145.201304_001 , arXiv:1206.2767v2 [math.NT] (2012).

[B] N. Bourbaki. *Commutative Algebra.* Paris: Hermann (1972).

[C] J. Coates. *Fragments of the GL_2 Iwasawa theory of elliptic curves without complex multiplication.* in *Arithmetic Theory of Elliptic Curves, (Cetraro, 1997)*, Ed. C. Viola, Lecture Notes in Mathematics 1716, Springer (1999), 1–50.

[CS] J. Coates and R. Sujatha. *Galois cohomology of elliptic curves. 2nd Edition.* Narosa (2010).

[DdSMS] J.D. Dixon, M.P.F. du Sautoy, A. Mann and D. Segal. *Analytic pro-p groups. 2nd Edition.* Cambridge Studies in Advanced Mathematics **61**, Cambridge Univ. Press (1999).

[HV] O. Hachimori and O. Venjakob *Completely faithful Selmer groups over Kummer extensions.* Doc. Math. Extra Vol. Kazuya Kato's fiftieth birthday (2003), 443-478.

[HO] O. Hachimori and T. Ochiai *Notes on non-commutative Iwasawa theory.* Asian J. Math. **14** no. 1 (2010), 11-17.

[J] U. Jannsen. *Iwasawa modules up to isomorphism.* Advanced Studies in Pure Mathematics 17 (1989), Algebraic Number Theory - in honor of K. Iwasawa, 171–207.

[J1] U. Jannsen. *A spectral sequence for Iwasawa adjoints*, arXiv:1307.6547v1 [math.NT] (2013).

[L] M. Lazard. *Groupes analytiques p-adiques.* Publ. Math. I.H.E.S. **26** (1965), 389–603.

[M] J.S. Milne. *Arithmetic Duality Theorems.* BookSurge, LLC, Second edition, 2006.

[NSW] J. Neuchirch, A. Schmidt and K. Wingberg. *Cohomology of number fields - Second edition.* GTM 323, Springer-Verlag, (2008).

[O] Y. Ochi. *A note on Selmer groups of abelian varieties over the trivializing extensions.* Proc. Amer. Math. Soc. **134** no. 1 (2006), 31–37 .

[OV] Y. Ochi and O. Venjakob. *On the structure of Selmer groups over p -adic Lie extensions.* J. Algebraic Geom. 11 no. **3** (2002), 547–580.

[OT] T. Ochiai and F. Trihan, *On the Selmer groups of abelian varieties over function fields of characteristic $p > 0$*, Math. Proc. Cambridge Philos. Soc. **146** no. 1 (2009), 23–43.

[S] G. Sechi. *GL_2 Iwasawa Theory of Elliptic Curves over Global Function Fields.* PhD thesis, University of Cambridge, (2006).

[Se] J.P. Serre. *Sur la dimension cohomologique des groupes profinis.* Topology **3** (1965), 413–420.

[ST] J.P. Serre and J. Tate. *Good reduction of abelian varieties.* Ann. of Math. **88** (1968), 492–517.

[T] K.-S. Tan *A generalized Mazur's theorem and its applications.* Trans. Amer. Math. Soc. **362** no. 8 (2010), 4433-4450.

[V] O. Venjakob. *On the structure theory of the Iwasawa algebra of a p-adic Lie group.* J. Eur. Math. Soc. (JEMS) **4** no. 3 (2002), 271–311.

[W] M. Witte, *On a noncommutative Iwasawa main conjecture for varieties over finite fields.* to appear in J. Eur. Math. Soc. (JEMS), arXiv:1004.2481v2 [math.NT] (2012).

Andrea Bandini
Università degli Studi di Parma - Dipartimento di Matematica e Informatica
Parco Area delle Scienze, 53/A - 43124 Parma - Italy
e-mail: andrea.bandini@unipr.it

Maria Valentino
Università della Calabria - Dipartimento di Matematica e Informatica
via P. Bucci - Cubo 31B - 87036 Arcavacata di Rende (CS) - Italy
e-mail: maria.valentino84@gmail.com