WEAK-TYPE BOUNDEDNESS OF THE FOURIER TRANSFORM ON REARRANGEMENT INVARIANT FUNCTION SPACES

SANTIAGO BOZA AND JAVIER SORIA

ABSTRACT. We study several questions about the weak-type boundedness of the Fourier transform $\mathcal{F}$ on rearrangement invariant spaces. In particular, we characterize the action of $\mathcal{F}$ as a bounded operator from the minimal Lorentz space $\Lambda(X)$ into the Marcinkiewicz maximal space $M(X)$, both associated to a rearrangement invariant space X. Finally, we also prove some results establishing that the weak-type boundedness of $\mathcal{F}$, in certain weighted Lorentz spaces, is equivalent to the corresponding strong-type estimates.

1. INTRODUCTION

The Fourier transform of a function f in $\mathbb{R}^n$, defined as

$$\mathcal{F}(f)(x) = \widehat{f}(x) = \int_{\mathbb{R}^n} f(t) \, e^{-ix \cdot t} \, dt,$$

is a bounded operator from L^1 to L^∞ and it can also be extended continuously on L^2 where, in fact, it defines an isometry. In particular, the Fourier transform maps $L^1 + L^2$ into $L^2 + L^\infty$.

One of the main results included in the work [4] concerns the boundedness of the Fourier transform in rearrangement invariant Banach functions spaces (r.i. spaces, see [3] for standard definitions and notations) and asserts that the largest r.i. space which is mapped by the Fourier transform into a space of locally integrable functions is, in fact, $L^1 + L^2$. It is also shown that L^2 is the only r.i. space X on $\mathbb{R}^n$, on which the Fourier transform is bounded:

(1) $$\mathcal{F} : X \to X \text{ if and only if } X = L^2.$$

In this work we want to study this same kind of question, but relaxing the hypothesis, assuming now that $\mathcal{F}$ is of restricted weak-type on X. Let us first briefly recall the necessary definitions to describe this result: For a given r.i. space X, the fundamental function φ_X is defined as

$$\varphi_X(t) = \|\chi_A\|_X, \text{ where } |A| = t.$$

Note that on a r.i. space the expression above is independent of the set A, so φ_X is a well defined quasi-concave function; that is, an increasing function such that $\varphi_X(t)/t$ is non-increasing and $\varphi_X(t) \neq 0$, for $t > 0$. Let us consider the minimal

Lorentz space $\Lambda(X) = \Lambda_{\varphi_X}$ and the maximal Lorentz space (or Marcinkiewicz space) $M(X) = M_{\varphi_X}$ defined, respectively, as

$$\Lambda(X) = \left\{ f : \ \|f\|_{\Lambda(X)} = \int_0^\infty f^*(t) \ d\varphi_X(t) < \infty \right\},$$

and

$$M(X) = \left\{ f : \ \|f\|_{M(X)} = \sup_{t>0} f^{**}(t) \ \varphi_X(t) < \infty \right\},$$

where f^* is the decreasing rearrangement of f and $f^{**}(t) = \dfrac{1}{t} \int_0^t f^*(s) \ ds$.

It is well known [3, Theorem II.5.13] that for every r.i. space X we have

$$\Lambda(X) \subset X \subset M(X),$$

and all r.i. spaces Y satisfying $\Lambda(X) \subset Y \subset M(X)$ have an equivalent fundamental function: $\varphi_Y \approx \varphi_X$.

Regarding (1), and in view of what is known for the strong-type estimates [4], it is now natural to ask about weaker conditions on the r.i. space X, in terms of its fundamental function φ_X, to ensure that the Fourier transform $\mathcal{F}$ maps the minimal Lorentz space $\Lambda(X)$ into the Marcinkiewicz space $M(X)$ (i.e., it is of restricted weak-type on X). This is completely answered in Section 2 (see Theorem 2.1).

Also, motivated by the results in [2, 13], we consider the boundedness of $\mathcal{F}$ between different weighted Lorentz spaces. If $p \in (0, \infty)$ and v is a non-negative weight, the Lorentz space $\Lambda^p(v)$ is the set of functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ for which

$$\|f\|_{\Lambda^p(v)} = \left(\int_0^\infty (f^*(t))^p \ v(t) \ dt \right)^{1/p}$$

is finite. Denoting by $V(t) = \int_0^t v(s) \ ds$ the primitive of the weight v, the weak-type Lorentz space $\Lambda^{p,\infty}(v)$ is the set of all functions f such that

$$\|f\|_{\Lambda^{p,\infty}(v)} = \sup_{t>0} f^*(t) \ V(t)^{1/p} < \infty.$$

In the particular case of $v(t) = t^{p/q-1}$, we obtain the classical Lorentz spaces $L^{q,p}$, $0 < p, q < \infty$. Similarly, the space $\Gamma^p(v)$ is the collection of those f for which

$$\|f\|_{\Gamma^p(v)} = \left(\int_0^\infty (f^{**}(t))^p \ v(t) \ dt \right)^{1/p}$$

is finite. Analogously, we define the corresponding weak-type space $\Gamma^{p,\infty}(v)$, which is the set of all measurable functions f such that

$$\|f\|_{\Gamma^{p,\infty}(v)} = \sup_{t>0} f^{**}(t) V(t)^{1/p} < \infty.$$

It is easy to see that we always have the following collection of embeddings:

$$\begin{array}{ccc} \Gamma^p(v) & \subset & \Lambda^p(v) \\ \cap & & \cap \\ \Gamma^{p,\infty}(v) & \subset & \Lambda^{p,\infty}(v) \end{array}$$

For a general review on normability properties, for these weighted Lorentz spaces, see [5].

In [2], Fourier mapping theorems in weighted Lorentz spaces were obtained in the Banach space setting. In particular the authors proved that for $1 < p \leq q$ and $q \geq 2$,

$$(2) \qquad \|\widehat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(v)},$$

provided that u is a non-increasing weight, v is in the B_p class of Ariño and Muckenhoupt [1], which is given by the following inequality

$$(3) \qquad \int_t^\infty \frac{v(s)}{s^p}\, ds \leq \frac{C}{t^p} \int_0^t v(s)\, ds,$$

and also satisfies

$$(4) \qquad \sup_{s>0} s \left(\int_0^{1/s} u(x)\, dx \right)^{1/q} \left(\int_0^s v(x)\, dx \right)^{-1/p} < \infty.$$

Conversely, if the inequality (2) holds for some weight functions u and v on $(0,\infty)$ and $1 < p,\ q < \infty$, then u and v satisfy (4). In fact, (4) is the condition obtained when (2) is tested on radial characteristic functions in $\mathbb{R}^n$.

An important technique to study the action of the Fourier transform on r.i. spaces was proved by Jodeit and Torchinsky [7], who showed that an operator T is of type $(1,\infty)$ and $(2,2)$ if and only if there is a constant C such that

$$(5) \qquad \int_0^z (Tf)^*(x)^2\, dx \leq C \int_0^z \left(\int_0^{1/t} f^*(s)\, ds \right)^2 dt, \quad f \in L^1 \cap L^2,\ z > 0.$$

In [2], and also in [13], weighted extensions of this result were obtained as part of the study of the boundedness of the operator T between weighted Lorentz spaces; namely,

$$(6) \qquad \left(\int_0^\infty (Tf)^*(x)^q\, u(x)\, dx \right)^{1/q} \leq C \left(\int_0^\infty \left(\int_0^{1/t} f^*(s)\, ds \right)^p v(t)\, dt \right)^{1/p}.$$

To interpret (6) as a Lorentz space inequality we take $w(t) = t^{p-2}v(1/t)$, we change the variable t by $1/t$, and then (6) is equivalent to the inequality

$$\|Tf\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}.$$

In the case $0 < p \leq 2 = q$, one of the main results included in [13] yields a simple condition on weights u and v which is necessary and sufficient for (6) when T is the Fourier transform (see Theorem 3.3). We observe that (5) is a particular case of (6), just by taking $p = q = 2$ and $u = v = \chi_{(0,z)}$.

As in our previous consideration for general r.i. spaces, in Section 3 we are interested in the weak-type boundedness of the Fourier transform $\mathcal{F}$ on Lorentz spaces (see Theorem 3.6) and we show that the necessary and sufficient conditions exhibited in Theorem 3.3, proving the boundedness of $\mathcal{F}$ from $\Gamma^p(w)$ into $\Lambda^2(u)$, $0 < p \leq 2$, also work for characterizing the corresponding weak-type boundedness of $\mathcal{F}$ from $\Gamma^p(w)$ into $\Lambda^{2,\infty}(u)$ or $\Gamma^{2,\infty}(u)$, in the same range of p.

2. Fourier transform of restricted weak-type in rearrangement invariant spaces

For a given rearrangement invariant space X, with rearrangement invariant norm $\|\cdot\|_X$, let us denote by X' its associate space (see [3] for the definition and properties)

which is also a rearrangement invariant space whose norm, for a function g, is defined by the expression

$$\|g\|_{X'} = \sup\left\{\int_0^\infty |f^*(x)g^*(x)|\ dx:\ f \in X,\ \|f\|_X \le 1\right\}.$$

Our following result extends (1) to the a priori weaker case of restricted-weak type estimates for $\mathcal{F}$. However, we can still prove that even under these conditions, there are no other spaces where the boundedness holds, except those sharing the same fundamental function as L^2.

Theorem 2.1. *Let X be an r.i. space. Then $\mathcal{F} : \Lambda(X) \to M(X)$ is bounded if and only if $\varphi_X(t) \simeq t^{1/2}$, $t > 0$.*

Proof. The assumption that $\varphi_X(t) \simeq t^{1/2}$ implies that $\Lambda(X) = L^{2,1}$ and $M(X) = L^{2,\infty}$, since $\mathcal{F}$ maps L^2 into L^2 continuously, the assertion follows from the fact that $L^{2,1} \hookrightarrow L^2 \hookrightarrow L^{2,\infty}$.

For the converse, we observe that $\mathcal{F} : \Lambda(X) \to M(X)$ is equivalent, by duality, to $\mathcal{F} : \Lambda(X') \to M(X')$. Since $L^1 + L^2$ is the largest rearrangement invariant space which is mapped by the Fourier transform into a space of locally integrable functions [4, Theorem 1], it follows that

$$\Lambda(X) \subset L^1 + L^2 \text{ and } \Lambda(X') \subset L^1 + L^2,$$

or, equivalently,

$$\Lambda(X) \subset L^1 + L^2 \text{ and } L^2 \cap L^\infty = (L^1 + L^2)' \subset (\Lambda(X'))' = M(X).$$

Taking into account the minimal and maximal properties, respectively, of $\Lambda(X)$ and $M(X)$, we obtain that

$$\Lambda(X) \subset L^1 + L^{2,1} \text{ and } L^{2,\infty} \cap L^\infty \subset M(X).$$

These inclusions can be written in terms of the corresponding fundamental functions of the spaces involved as follows

$$\min(t, t^{1/2}) \lesssim \varphi_X(t) \lesssim \max(1, t^{1/2}).$$

Therefore, we obtain that

(7) $$\varphi_X(t) \simeq t^{1/2}, \text{ for } t \ge 1.$$

Hence, we just have to prove that for $0 < t \le 1$, $\varphi_X(t) \simeq t^{1/2}$. Now, for any $a > 0$, let us consider $f(t) = \chi_{(-a,a)}(t)$, so that

$$f^*(t) = \chi_{(0,2a)}(t) \quad \text{and} \quad \widehat{f}(\xi) = 2\frac{\sin(a\xi)}{\xi} = 2a\,\mathrm{sinc}(a\xi).$$

Since $(\widehat{f})^{**}(s) = 2a\,\mathrm{sinc}^{**}(as)$, we may take $s = (2a)^{-1}$ to get

$$\varphi_X\left(\frac{1}{2a}\right)2a\,\mathrm{sinc}^{**}(1/2) = \varphi_X\left(\frac{1}{2a}\right)(\widehat{f})^{**}\left(\frac{1}{2a}\right)$$

$$\le \|\widehat{f}\|_{M(X)} \le C\|f\|_{\Lambda(X)} = C\varphi_X(2a).$$

For all $t > 0$, replacing $2a$ by t and also by $1/t$ gives two inequalities that combine to show

$$\varphi_X(t) \simeq t\varphi_X(1/t).$$

Using (7) we see that $\varphi_X(t) \simeq t^{1/2}$ for $t < 1$ as well. $\square$

Corollary 2.2. *If w is a weight, $W(t) = \int_0^t w(s)\,ds$, $1 \leq p < \infty$, and*

$$w_p(t) = \frac{d}{dt}\left(W(t) + t^p \int_t^\infty \frac{w(s)}{s^p}\,ds\right)^{1/p},$$

then $\mathcal{F} : \Lambda^1(w_p) \to \Gamma^{1,\infty}(w_p)$ if and only if $W(t) + t^p \int_t^\infty \frac{w(s)}{s^p}\,ds \approx t^{p/2}$.

Proof. If we denote by $X = \Gamma^p(w)$, then $\varphi_X^p(t) = W(t) + t^p \int_t^\infty \frac{w(s)}{s^p}\,ds$. Therefore $\Lambda(X) = \Lambda^1(w_p)$, $M(X) = \Gamma^{1,\infty}(w_p)$, and the result follows from Theorem 2.1. $\square$

Remark 2.3. From the proof of Theorem 2.1, we observe that, given a rearrangement invariant space X, a necessary condition to obtain the boundedness of $\mathcal{F}$ from $\Lambda(X)$ into $M(X)$ is that both minimal Lorentz spaces $\Lambda(X)$ and $\Lambda(X')$ are contained in $L^1 + L^2$, but this is not sufficient.

As a counterexample, let us consider $X = L^{2,1} \cap L^\infty$ which coincides with its minimal Lorentz space $\Lambda(X)$, has associate space $X' = L^{2,\infty} + L^1$ and its Marcinkiewicz space is $M(X) = L^{2,\infty} \cap L^\infty$. Hence, $\Lambda(X) = L^{2,1} \cap L^\infty \subset L^1 + L^2$, $\Lambda(X') = L^{2,1} + L^1 \subset L^1 + L^2$.

Since $L^{2,1} \cap L^\infty \subset L^2 \subset L^{2,\infty}$, it holds that $\mathcal{F} : \Lambda(X) \longrightarrow L^{2,\infty}$. Therefore, in order to see that $\mathcal{F} : \Lambda(X) \not\longrightarrow M(X)$ we have to show that $\mathcal{F} : \Lambda(X) \not\longrightarrow L^\infty$.

For this purpose, let us consider $E = (-a, a)$, for $a > 1$ and the family of characteristic functions χ_E. It holds that $\widehat{\chi_E}(\xi) = 2a\,\text{sinc}(a\xi)$ and then $\|\widehat{\chi_E}\|_\infty = 2a$. Since $\|\chi_E\|_{\Lambda(X)} = \max(1, \sqrt{2a}) = \sqrt{2a}$, we conclude that there is no positive constant C such that $\|\widehat{\chi_E}\|_\infty \leq C\|\chi_E\|_{\Lambda(X)}$. Observe that $\varphi_{L^{2,1} \cap L^\infty}(t) \not\approx t^{1/2}$, for $t > 0$.

3. Weak-type boundedness on weighted Lorentz spaces

The use of level function techniques introduced in [6] and [8] was an important tool to study the boundedness of the Fourier transform in the context of Lorentz spaces, as established in [13]. The following proposition gives the basic properties needed to prove our main result, which is an extension to weighted inequalities on weak-type spaces (see [12, Proposition 2.1 and Proposition 5.1] and also [10] and [11]). Let us denote by L^+ the set of Lebesgue measurable functions $h : (0, \infty) \longrightarrow [0, \infty]$.

Proposition 3.1. [12, Proposition 2.1] *To each $h \in L^+$ there corresponds a non-increasing function $h^o \in L^+$, called the level function of h, having the following properties:*

(a) *For all non-increasing function $\varphi \in L^+$, $\int_0^\infty \varphi(x)h(x)\,dx \leq \int_0^\infty \varphi(x)h^o(x)\,dx$.*

(b) *If $0 \leq h_n \uparrow h$ pointwise then $h_n^o \uparrow h^o$ pointwise.*

(c) *If h is bounded and compactly supported then there exists a (necessarily finite or countable) collection of disjoint intervals (a_j, b_j), each of finite length, such that*

$$h^o(x) = \frac{1}{b_j - a_j} \int_{a_j}^{b_j} h(y)\,dy, \quad \text{for } a_j \leq x \leq b_j,$$

and $h^o(x) = h(x)$, for $x \notin \cup_j(a_j, b_j)$.

To use this proposition, and following [13], we introduce the class $\mathcal{A}$ of averaging operators A defined by

$$Ah(x) = \begin{cases} \dfrac{1}{b_j - a_j} \displaystyle\int_{a_j}^{b_j} h(y)\,dy, & \text{for } a_j \leq x \leq b_j \\ h(x), & \text{otherwise,} \end{cases}$$

where $\{(a_j, b_j)\}_j$ is a finite or countable collection of disjoint subintervals of $(0, \infty)$ each of finite length. Proposition 3.1 implies that, for h bounded and compactly supported there exists $A_h \in \mathcal{A}$ such that $h^o = A_h h$. The next lemma relates the level function with the averaging operators $\mathcal{A}$.

Lemma 3.2. [13, Lemma 2.5] *If $u \in L^+$, for all $x > 0$*

$$\frac{1}{x} \int_0^x u^o(t) \, dt = \sup_{A \in \mathcal{A}} \frac{1}{x} \int_0^x Au(t) \, dt \le 2 \sup_{y \ge x} \frac{1}{y} \int_0^y u(t) \, dt \le \frac{2}{x} \int_0^x u^o(t) \, dt.$$

For u and v positive measurable functions defined on $(0, \infty)$ and u^o the level function of the weight u, let us consider the constant:

$$K_0 = \sup_{x>0} \left(\int_0^\infty \min(x^{-2}, t^{-2}) \, u^o(t) \, dt \right)^{1/2} \left(\int_0^\infty \min(x^{-p}, t^{-p}) \, v(t) \, dt \right)^{-1/p}$$

$$\simeq \sup_{x>0} \left(x^{-2} \int_0^x u^o(s) \, ds \right)^{1/2} \left(\int_x^\infty \left(\frac{1}{t} \int_0^t v(s) \, ds \right) \frac{dt}{t^p} \right)^{-1/p}.$$

Using Lemma 3.2, one can prove that K_0 is equivalent to

$$K_1 = \sup_{x \le y} \left(\frac{x}{y} \int_0^y u(s) \, ds \right)^{1/2} \left(x^p \int_x^\infty \left(\frac{1}{t} \int_0^t v(s) \, ds \right) \frac{dt}{t^p} \right)^{-1/p},$$

which is independent of u^o. We will also need to recall the following result, whose extension to weak-type boundedness on Lorentz spaces is the main goal of our Theorem 3.6:

Theorem 3.3. [13, Corollary 5.2] *Suppose $0 < p \le 2$ and $u, w \in L^+$. The following conditions are equivalent:*

(i) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^2(u)$.
(ii) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^2(u^o)$.
(iii) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^2(u^o)$.
(iv) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^2(u)$.
(v)

$$(8) \qquad \sup_{x \le y} \left(\frac{x}{y} \int_0^y u(t) \, dt \right)^{1/2} \left(x^p \int_0^{1/x} \left(t^p \int_t^\infty w(s) \frac{ds}{s^p} \right) \frac{dt}{t} \right)^{-1/p} < \infty.$$

Remark 3.4. As we have already mentioned in the Introduction, condition (4) gives another characterization of the boundedness of $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^2(u)$, whenever $1 < p \le 2$, $w \in B_p$ (since $\Lambda^p(w) = \Gamma^p(w)$ [9, Theorem 4]) and u is decreasing (and hence, $u \in B_2$ and again $\Lambda^2(u) = \Gamma^2(u)$). Therefore, under these hypotheses (4) and

(8) are equivalent. In fact, we can give a direct proof:

$$\sup_{0<x\leq y}\left(\frac{x}{y}\int_0^y u(t)\,dt\right)^{1/2}\left(x^p\int_0^{1/x}\left(t^p\int_t^\infty w(s)\frac{ds}{s^p}\right)\frac{dt}{t}\right)^{-1/p}$$

$$\approx\sup_{0<x}\left(U(x)\right)^{1/2}\left(x^p\int_0^{1/x}\left(t^p\int_t^\infty w(s)\frac{ds}{s^p}\right)\frac{dt}{t}\right)^{-1/p}$$

$$\approx\sup_{0<x}\left(U(x)\right)^{1/2}\left(x^p W(1/x)+\int_{1/x}^\infty\frac{w(s)}{s^p}\,ds\right)^{-1/p}$$

$$\approx\sup_{0<x}\left(U(x)\right)^{1/2}\left(x^p W(1/x)\right)^{-1/p}$$

$$\approx\sup_{0<s}sU^{1/2}(1/s)W^{-1/p}(s),$$

which is (4).

The next lemma gives a very handy expression to calculate necessary conditions for weak-type estimates by means of the action of $\mathcal{F}$ on some test functions, and is the analogue (in the weak-type setting) to [13, Corollary 4.8]:

Lemma 3.5. *If $z>0$, $w_z(t)=\min(t^{-2},z^{-2})$, and $u\in L^+$, then*

$$\sup_{A\in\mathcal{A}}\|Aw_z\|_{L^{1,\infty}(u)}\simeq\|w_z\|_{L^{1,\infty}(u^o)}.$$

Proof. For each $s>0$, define $A_{(0,s)}$ to be the averaging operator in $\mathcal{A}$ based on the single interval $(0,s)$. That is,

$$A_{(0,s)}f(t)=\begin{cases}\dfrac{1}{s}\displaystyle\int_0^s f(r)\,dr,&\text{for }0<t\leq s,\\[3mm] f(t),&\text{for }t>s.\end{cases}$$

If $t\geq z$, we may take $s=t$ to get

$$A_{(0,t)}w_z(t)=\frac{1}{t}\left(\frac{2}{z}-\frac{1}{t}\right)\simeq\frac{1}{tz}.$$

On one hand, each $A_{(0,t)}\in\mathcal{A}$, so for all $t\geq z$

$$\sup_{A\in\mathcal{A}}\|Aw_z\|_{L^{1,\infty}(u)}\geq\|A_{(0,t)}w_z\|_{L^{1,\infty}(u)}$$

$$\geq A_{(0,t)}w_z(t)\int_0^t u(r)\,dr\simeq\frac{1}{tz}\int_0^t u(r)\,dr.$$

On the other hand, since w_z is nonincreasing, $w_z(t)\leq A_{(0,t)}w_z(t)$ and, moreover, the largest average of w_z on any interval that includes t is the average over $(0,t)$. Thus, for each $A\in\mathcal{A}$,

$$Aw_z(t)\int_0^t u(r)\,dr\leq A_{(0,t)}w_z(t)\int_0^t u(r)\,dr$$

$$\simeq\begin{cases}\dfrac{1}{z^2}\displaystyle\int_0^t u(r)\,dr,&\text{for }t<z,\\[3mm]\dfrac{1}{tz}\displaystyle\int_0^t u(r)\,dr,&\text{for }t\geq z,\end{cases}\;\leq\sup_{t\geq z}\frac{1}{tz}\int_0^t u(r)\,dr.$$

Combining these two estimates, applying Lemma 3.2 and using that u^o is decreasing, we finally get

$$\sup_{A \in \mathcal{A}} \|A w_z\|_{L^{1,\infty}(u)} \simeq \sup_{t \geq z} \frac{1}{tz} \int_0^t u(r) \, dr \simeq \frac{1}{z^2} \int_0^z u^o(r) \, dr \simeq \|w_z\|_{L^{1,\infty}(u^o)}.$$

$\square$

Theorem 3.6. *Let $0 < p \leq 2$, and u, w be two functions in L^+. Then, the following conditions are equivalent:*

(i) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^{2,\infty}(u)$.
(ii) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^{2,\infty}(u^o)$.
(iii) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^{2,\infty}(u^o)$.
(iv) $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^{2,\infty}(u)$.
(v) *Condition* (8) *holds.*

Consequently, the strong-type conditions included in Theorem 3.3 are also equivalent to the corresponding weak-type boundedness.

Proof. First of all, we observe that the finiteness of condition K_1 is equivalent to condition (8) but expressed in terms of the weight $v(t) = t^{p-2} w(1/t)$. So, we will first restrict to prove that $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^{2,\infty}(u)$ if and only if $K_0 < \infty$ or, equivalently, $K_1 < \infty$.

The sufficiency of the condition follows from [13, Theorem 5.1], since $K_0 < \infty$ or $K_1 < \infty$ is equivalent to the fact $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^2(u)$. To prove the necessity, we suppose that the boundedness holds which, in terms of the weight $v(t) = t^{p-2} w(1/t)$, by a change of variables implies that

$$\sup_{t > 0} U^{1/2}(t) (\widehat{f})^*(t) \leq C \left(\int_0^\infty \left(\int_0^{1/t} f^*(s) \, ds \right)^p v(t) \, dt \right)^{1/p}.$$

Following the same idea as in [13, Corollary 4.8], let us consider $z > 0$ and $w_z(t) = \min(t^{-2}, z^{-2})$. Fix $R > 0$ and $A \in \mathcal{A}$. Then by [13, Corollary 4.7], there exists a function f such that $f^* \leq \chi_{(0,1/z)}$ and for all $y \in (0, R)$

$$A w_z(y)^{1/2} \lesssim (\widehat{f})^*(y).$$

Then, we obtain the following

$$U^{1/2}(y) A w_z(y)^{1/2} \lesssim U^{1/2}(y) (\widehat{f})^*(y) \lesssim \|\widehat{f}\|_{\Lambda^{2,\infty}(u)}$$
$$\lesssim \left(\int_0^\infty \left(\int_0^{1/t} f^*(s) \, ds \right)^p v(t) \, dt \right)^{1/p}$$
$$\lesssim \left(\int_0^\infty \left(\int_0^{1/t} \chi_{(0,1/z)}(s) \, ds \right)^p v(t) \, dt \right)^{1/p}$$
$$= \left(\int_0^\infty w_z(y)^{p/2} v(y) \, dy \right)^{1/p},$$

since $y \in (0, R)$ and $R > 0$. Letting $R \to \infty$, this implies that for some positive constant $C > 0$

$$\|A w_z\|_{L^{1,\infty}(u)} \leq C \|w_z\|_{L^{p/2}(v)}.$$

Let us consider

$$D = \sup_{A \in \mathcal{A}, z > 0} \frac{\|Aw_z\|_{L^{1,\infty}(u)}}{\|w_z\|_{L^{p/2}(v)}} < \infty,$$

where the supremum is taken over A belonging to the class $\mathcal{A}$ of averaging operators. As a consequence of Lemma 3.5, we obtain that

$$D = \sup_{A \in \mathcal{A}, z > 0} \frac{\|Aw_z\|_{L^{1,\infty}(u)}}{\|w_z\|_{L^{p/2}(v)}} \simeq \sup_{z > 0} \frac{\|w_z\|_{L^{1,\infty}(u^o)}}{\|w_z\|_{L^{p/2}(v)}},$$

and this last quantity can be estimated by

$$D \simeq \sup_{z > 0} \frac{\|w_z\|_{L^{1,\infty}(u^o)}}{\|w_z\|_{L^{p/2}(v)}} = \sup_{z > 0} \frac{\sup_{t > 0} U^o(t) \min(z^{-2}, t^{-2})}{\left(\int_0^\infty \min(z^{-p}, t^{-p}) v(t) \, dt \right)^{2/p}}$$

$$\simeq \sup_{z > 0} \sup_{z \leq t} \frac{t^{-2} U^o(t)}{\left(\int_z^\infty \left(\frac{1}{s} \int_0^s v(x) \, dx \right) \frac{ds}{s^p} \right)^{2/p}} = \sup_{t > 0} \frac{t^{-2} U^o(t)}{\left(\int_t^\infty \left(\frac{1}{s} \int_0^s v(x) \, dx \right) \frac{ds}{s^p} \right)^{2/p}}.$$

It follows from Lemma 3.2 that

$$U^o(t) \simeq \sup_{y \geq t} \frac{t}{y} U(y),$$

and we obtain

$$D \simeq \sup_{t \leq y} \frac{U(y)}{y} \frac{1}{t \left(\int_t^\infty \left(\frac{1}{s} \int_0^s v(x) \, dx \right) \frac{ds}{s^p} \right)^{2/p}} = K_1^2,$$

which is finite.

Since u^o is non-increasing we have that $(u^o)^o = u^o$ and it follows that $K_0 < \infty$ for the pair of weights (u, v) if and only if it is finite for the pair (u^o, v). Arguing as before, this is equivalent to $\mathcal{F} : \Gamma^p(w) \longrightarrow \Lambda^{2,\infty}(u^o)$.

Again, the fact that u^o is non-increasing implies that $\Lambda^{2,\infty}(u^o) = \Gamma^{2,\infty}(u^o)$. Therefore, $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^{2,\infty}(u^o)$ is also equivalent to $K_0 < \infty$.

Finally, for any measurable function f, the inequalities

$$\sup_{t > 0} f^*(t) U(t)^{1/2} \leq \sup_{t > 0} f^{**}(t) U(t)^{1/2} \leq \sup_{t > 0} f^{**}(t) U^o(t)^{1/2}$$

imply $\Gamma^{2,\infty}(u^o) \subseteq \Gamma^{2,\infty}(u) \subseteq \Lambda^{2,\infty}(u)$. Thus $\mathcal{F} : \Gamma^p(w) \longrightarrow \Gamma^{2,\infty}(u)$ is also equivalent to $K_0 < \infty$. □

Remark 3.7. It is interesting to compare Theorem 2.1 and Theorem 3.6 to see which conditions we obtain for the same kind of spaces. For example, under the hypotheses and notations of Corollary 2.2, if we further assume that w satisfies (3) (i.e., $w \in B_p$), then $X = \Gamma^p(w) = \Lambda^p(w)$ [9, Theorem 4] and hence $\varphi_X(t) \approx W^{1/p}(t)$. Thus, $\mathcal{F} : \Lambda^1(w_p) \rightarrow \Gamma^{1,\infty}(w_p)$ if and only if $W(t) \approx t^{p/2}$, which for $p = 1$ gives

$$(9) \qquad\qquad W(t) \approx t^{1/2}, \quad t > 0.$$

On the other hand, under the same considerations, with $p = 1$, we have that $\Lambda^1(w_1) = \Lambda^1(w) = \Gamma^1(w)$ and using [5, Theorem 6.5] it turns out that $\Gamma^{2,\infty}(u) =$

$\Gamma^{1,\infty}(w)$ if and only if

$$(10) \qquad U^{1/2}(t) \approx t \sup_{t \le s} \frac{W(s)}{s} \approx W(t).$$

Therefore, if (10) holds and we use Theorem 3.6, we have that $\mathcal{F} : \Lambda^1(w_1) \to$ $\Gamma^{1,\infty}(w_1)$ if and only if $\mathcal{F} : \Gamma^1(w) \to \Gamma^{1,\infty}(w)$ which, using (8) and (3), is equivalent to

$$\sup_{0 < x \le y} \left(\frac{x}{y} W^2(y) \right)^{1/2} \left(x \int_0^{1/x} t \int_t^\infty w(s) \frac{ds}{s} \frac{dt}{t} \right)^{-1}$$

$$= \sup_{0 < x \le y} \left(\frac{x}{y} W^2(y) \right)^{1/2} \left(xW(1/x) + \int_{1/x}^\infty \frac{w(s)}{s} ds \right)^{-1}$$

$$(11) \qquad \approx \sup_{0 < x \le y} \frac{W(y)}{\sqrt{y}} \frac{1}{\sqrt{x}W(1/x)} < \infty.$$

Thus, we finally obtain that, for $w \in B_1$, the boundedness of the Fourier transform $\mathcal{F} : \Lambda^1(w) \to \Gamma^{1,\infty}(w)$ is equivalent to any of the two conditions (9) or (11).

Observe that $W(t) = \sqrt{t}$ satisfies (11). Conversely, if (11) holds then if we consider the values $1 = x \le y$, we get that $W(y) \lesssim \sqrt{y}$ and, with $0 < x \le y = 1$, $W(1/x) \gtrsim 1/\sqrt{x}$, which gives (9) whenever $t \ge 1$. A symmetric argument finally shows the remaining case $0 < t < 1$.

Acknowledgment. The authors would like to thank the anonymous referee for some clarifying remarks (in particular regarding Theorem 2.1 and Lemma 3.5), which have really improved the final exposition of our work.

References

[1] M.A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.

[2] J.J. Benedetto and H.P. Heinig, *Weighted Fourier inequalities: new proofs and generalizations*, J. Fourier Anal. Appl. **9** (2003), 1–37.

[3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[4] L. Brandolini and L. Colzani, *Fourier transform, oscillatory multipliers and evolution equations in rearrangement invariant function spaces*, Colloquium Math. **71** (1996), no. 2, 273–286.

[5] M.J. Carro, L. Pick, J. Soria, and V. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4** (2001), 397–428.

[6] I. Halperin, *Function spaces*, Canad. J. Math. **5** (1953), 273–288.

[7] M. Jodeit Jr. and A. Torchinsky, *Inequalities for Fourier transforms*, Studia Math. **37** (1971), 245–276.

[8] G.G. Lorentz, *Bernstein Polynomials*, University of Toronto Press, Toronto, 1953.

[9] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158

[10] G. Sinnamon, *Spaces defined by the level function and their duals*, Studia Math. **111** (1994), 19–52.

[11] G. Sinnamon, *Interpolation of spaces defined by the level function*, ICM-90 Satellite Conference Proceedings, Harmonic Analysis, S. Igari (ed.), Springer-Verlag, Tokyo, 1991, 190–193.

[12] G. Sinnamon, *The level function in rearrangement invariant spaces,* Publ. Mat. **45** (2001), 175–198.

[13] G. Sinnamon, *The Fourier transform in weighted Lorentz spaces,* Publ. Mat. **47** (2003), 3–29.

Department of Mathematics, EETAC, Polytechnical University of Catalonia, E-08860 Castelldefels, Spain.
    E-mail address: santiago.boza@upc.edu

Department of Mathematics and Informatics, University of Barcelona, Gran Via 585, E-08007 Barcelona, Spain.
    E-mail address: soria@ub.edu