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BSHM Bulletin, 2018
Vol. 0, No. 0, 1–23, http://dx.doi.org/10.1080/17498430.2017.1419704

The main sources for the *Arte Mayor* in sixteenth century Spain

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One of the main changes in European Renaissance mathematics was the progressive development of algebra from practical arithmetic, in which equations and operations began to be written with abbreviations and symbols, rather than in the rhetorical way found in earlier arithmetical texts. In Spain, the introduction of algebraic procedures was mainly achieved through certain commercial or arithmetical texts, in which a section was devoted to algebra or the 'Arte Mayor'. This paper deals with the contents of the first arithmetical texts containing sections on algebra. These allow us to determine how algebraic ideas were introduced into Spain and what their main sources were. The first printed arithmetical Spanish text containing algebra was the *Libro primero de Arithmetica Algebratica* (1552) by Marco Aurel. Therefore, the aim of this paper is to analyse the possible sources of this book and show the major influence of the German text *Coss* (1525) by Christoff Rudolff, on Aurel's work.

Introduction

Mathematics in the Renaissance has specific features: the return to classical texts, new fields of application, and a new relationship within the study of nature. Nevertheless, one of the main changes was the progressive development of algebra from practical arithmetic, in which equations and operations began to be written with abbreviations and symbols rather than in the rhetorical way found in earlier texts. In the sixteenth century, algebraic procedures were developing by defining their own objects, and discerning their place between geometry and arithmetic. Therefore, some algebraic procedures began to be introduced into European arithmetical texts at this time.

In Spain, this introduction was mainly achieved through commercial or practical arithmetical texts, in which a section was devoted to algebra. In the sixteenth century, Spanish arithmetical texts can be roughly classified into two main groups: speculative or academic arithmetics, and practical arithmetics. The first group is composed of texts written in Latin, in which the contents refer to the study of numbers and proportions without any reference to algebra. The second group, practical arithmetics, is made up of texts written in the vernacular, in which the contents refer to the tools used to solve mercantile problems. The mathematical style of these texts is direct and simple, which was characteristic of books in the mercantile genre more generally.

So, the first printed texts on Spanish algebra—the Arte Mayor—appeared as chapters in books of this second kind. These works explicitly contrast the Arte Mayor with the 'Arte Menor', as arithmetic is called. There are several studies dealing with Spanish mathematics in the sixteenth century, and some dealing specifically with commercial arithmetic (Rey Pastor 1934; López Piñero 1979; Salavert Fabiani 1990, 1994; Navarro *et al.* 1999). However, these studies mainly concern the types of texts and the techniques that are used, with particular attention given to the arithmetical sections. More in-depth studies regarding the contents of the first arithmetical texts containing sections on algebra are required (cf. Romero Vallhonesta 2007, 2011, 2012; Massa-Esteve 2010, 2012; Stedall 2011, 2012; Rommevaux *et al.* 2012; Katz and ~~Hunger~~ Parshall 2014; Molina 2015, 2017; Silva 2016; Parshall 2017). These will allow us to determine how algebraic ideas were introduced into Spain and what their main sources were as regards European influence. (Note that in this period 'Spain' was a name that referred to the whole Iberian Peninsula, an area composed of several distinct kingdoms.)

Thus, this paper is focused on the first printed arithmetical Spanish text containing algebra, the *Libro primero de Arithmetica Algebratica* (1552) by Marco Aurel (*fl.* 1552) and its possible sources. Marco Aurel was German but lived in Valencia, where he worked as a teacher of mathematics (Meavilla Seguí 1991; Docampo Rey 2004). Firstly, it is important to note the existence of an earlier Catalan manuscript containing algebra that Docampo Rey (2006) has analysed. It is also interesting to remark that Gonzalo de Busto, when Juan de Ortega's *Arithmetica* was reprinted in 1552, the same year that Aurel published his work, added thirteen examples on the 'Arte Mayor' at the end of the book. However, no further explanation was given there.

The sources for the 'Arte Mayor' in Spain must be analysed in a European context. A considerable number of published treatises dealing with algebra before 1552 could be possible sources for Aurel's work. These texts sometimes present similarities in notation or in the treatment of equations. However, in most cases the notation, and the names of the unknowns and the procedures, change from one text to another. Previous research has shown some connections among algebraic texts, especially for Italian, German, and French algebra (Van Egmond 1986; Franci and Toti 1988; Cifoletti 1996; Høyrup 2010, 2015; Stedall 2011, 2012; Heeffer 2012; Katz and Parshall 2014; Parshall 2017). Looking at European algebraic influences in Spain, the present authors have examined the sections on algebra in the treatises published before Aurel's work, and a significant connection between *Coss* (1525) by Christoff Rudolff (1494–1543) and Aurel's *Libro primero* has been found. [Rudolff studied at the University of Vienna; there are a few studies on his work (Terquem 1857; Reich 1994; Heeffer 2012). In what follows we also use the complete facsimile of the original edition included in Kaunzner and Röttel (2006).] For that reason, in this paper, we analyse and compare these two works, with descriptions of their similarities and differences. Some features of both are presented in order to provide solid evidence establishing Rudolff's work as a main source for Aurel's algebra.

The aim of this paper is not only to discuss the European influence on sixteenth-century Spanish mathematics, but also to present Aurel's work as a modern text consistent with the mathematical knowledge of the time. We also attempt to reflect on the role of this work in the development of Spanish algebra.

The following section of this paper reviews the sources Aurel quotes in his algebra, and some other previous printed treatises as possible sources for his work. The later sections explore the relationship between Rudolff's *Coss* and Aurel's *Libro Primero*, looking at the structure of these works, their conception of algebra, and other specific points of contact.

Aurel's sources

In this section we discuss the quotations in Aurel's work and provide a list of previous printed works containing algebra, in order to determine the main source for his *Libro Primero*.

Quotations in the **Libro Primero**

Although the citations in Aurel's work do not point to his main source, it is important to mention them.

(1) One author quoted by Aurel is Luca Pacioli (1445–1517): specifically his *Summa de arithmetica, geometria, proportioni et proportionalità* (Venice, 1494). The *Summa* was the first work containing algebra to be printed; it is a compendium of the mathematical knowledge that was taught in the abacus schools, and is divided into *distinctioni*, *tractati*, and *articuli*. The fourth, fifth, and sixth *tractati* of the eighth *distinction* are devoted to algebra.

Aurel quotes Pacioli (as 'Fray Lucas del Burgo') twice; first when he justifies his proposal of considering eight types of equations, which he calls equalities, and second in a note after solving a problem that leads to the resolution of an equation of the fifth type (Aurel 1552, f. 77^{v}, 125^{v}). The problem is about a merchant who buys three kinds of spices, and we have to find out the price of a pound of every kind of spice. It is very similar to a problem by Pacioli, which can be found in the sixth treatise, sixth distinction, fourteenth article, twenty-first question: as Aurel indicates exactly. Aurel changes two quantities in order to obtain a result that is expressed with rational numbers. The solution of Aurel's problem is $2\frac{6}{7}$, and the solution to Pacioli's $\sqrt{\left(7\frac{135503}{361201} - \frac{100}{601}\right)}$. Aurel explains that Fray Lucas del Burgo solved this problem and he specifies that he had to change some quantities in order to obtain a rational result.

Despite these quotations, and although it is evident that Aurel is very familiar with Pacioli's *Summa*, it is also clear that this was not his main source of reference, since the structure, content, and symbols for the unknowns, and the treatment of equations in Aurel's work are very different.

(2) Aurel also cited 'Albertucio de Saxonia' when considering the number of rules for solving equations. Albert of Saxony (1316–90), German philosopher, enjoyed a distinguished career in Paris; he is probably best known as the founder of the University of Vienna in 1364. Aurel (1552, f. 77^{v}) says:

> I wish to show 8 rules for the 8 equalities on which the answers of our 'rule of the thing' or 'arte mayor' are based. Since some give 6 [rules], such as Fray Lucas del Burgo, and others 10, like Albertucio de Saxonia, I regard it as appropriate to take the arithmetical mean between 10 and 6, which is 8: so you will understand the 6 by Fray Lucas and likewise the 10 by Albertucio.[1]

(3) The most-quoted work in Aurel's algebra, though, is Euclid's *Elements*. He quotes Euclid fifty-six times, eleven of them in the section on algebra. In his introduction he states that he uses the same terminology as Euclid in most of his work, especially in the eleventh and twelfth chapters. Aurel defines 'number' as Euclid and Boethius did, as he states himself.

[1]Agora te quiero mostrar 8 reglas para las 8 igualaciones en las quales estan fundadas las respuestas de nuestra regla, de la cosa, o arte mayor. Dado que algunos ponen 6, como Fray Lucas del Burgo, y otros 10, como Albertucio de Saxonia. A mi empero me ha parecido tomar el medio arithmetico entre 10, y 6, que es 8, pues por ellas entenderas las 6 de fray Lucas y por las mesmas alcançaras las 10 de Albertucio.

(4) In the third chapter, which is about proportion, Aurel cites Pythagoras as the inventor of harmonic proportion.
(5) In the sixth chapter, dealing with progressions, Aurel solves the well-known problem of finding how many wheat grains are needed to fill a chessboard, if in each square you have to put twice as many as in the previous one. Aurel explains that the contemporary Spanish mathematician Fray Ioan de Ortega made a big mistake when he gave the result of this problem.
(6) In the thirteenth chapter, Aurel defines the symbols he is going to use for the unknowns and refers to Guillelmo de Lunis as the first to translate the 'rule of the thing' from Arabic to Italian. De Lunis (variously named: see Massa-Esteve 2008), was active in the late thirteen century, and was indeed quoted in the sixteenth century as the translator of an algebra from Arabic to Italian (see Lejbowicz 2012).
(7) Finally, in the fifteenth chapter about the rules for the first equality, Aurel cites Vitruvius and his *Architecture*, and refers to how Archimedes solved the problem about the crown of king Hiero II of Syracuse. This well-known problem is about a votive crown that Hiero ordered from a goldsmith. According to Vitruvius, Hiero suspected that the goldsmith was cheating him and asked Archimedes to investigate. He discovered the fraud by dipping the crown in water, revealing that the amount of liquid displaced was not the amount expected if the crown had been made entirely of gold.

Apart from the *Summa* of Pacioli, then, the other works that Aurel quotes do not contain algebra, and the authors to whom he refers wrote no work containing algebra. Therefore, none of these works can be a main source for the algebra in Aurel's *Libro primero*. Evidently, Aurel was not inclined to quote the main source of his work.

Previous printed sources

Having set aside the works and the authors that Aurel quoted in his *Libro Primero* as possible sources for the algebra in his own work, his other possible sources are the earlier printed texts containing algebra:

- the *Summa de arithmetica, geometria, proportioni et proportionalità* (1494) by Pacioli (on which see the previous section);
- the *Ayn new Kunstlich Buech* (1518) by Heinrich Schreiber (*c.* 1492–*c.* 1526; also known as Henricus Grammateus, Heinrich Screyber or Henricus Scriptor: see Inoue 1978. He is quoted in Rudolff: Kaunzner and Röttel 2006, 263.)
- *Larismetique nouvellement compose* (1520) by Etienne de la Roche (*c.* 1470–*c.* 1530);
- the *Summa de arithmetica* (1521) by Francesco Ghaligai (d. 1536) (also published in Italian as *Pratica d'arithmetica*: 1548 and 1552);
- the *Coss* (1525) by Rudolff;
- the *Libro di arithmetica e geometria speculative e practicale* (1536) by Francesco Feliciano (born in Lazisa, near Verona, and still living in 1563). This book is a revision of his earlier *Libro de Abaco* (1517): see further Swetz and Katz 2011.
- the *Practica arithmetice & mensurandi singularis* (1539) by Girolamo Cardano (1501–76);
- the *Arithmetica integra* (1544) by Michael Stifel (1487–1567);

- the *Ars Magna* (1545) also by Cardano;
- the *Quesiti et invention diverse* (1546) by Niccolò Tartaglia (1499–1557);
- and the *Algebrae compendiosa faciliusque description* (1551) by Joannis Scheubel (1494–1570).

We have considered each of these works, taking into account their structure, the notation used, the types of equations, the practice of the 'rule of the thing' and their method of problem-solving. We have rejected the works of Cardano and Tartaglia as main sources for Aurel, because both contain geometrical demonstrations to prove some rules, while the algebra of Aurel uses no geometrical representation. We have rejected Stifel's *Arithmetica Integra* mainly because of the structure of the work and his idea of algebra (Stedall 2012, 229–231). We have rejected Scheubel's work because of the diagrams in the notation, and the operations with the characters. In the case of the work of Ghaligai, we have rejected it mainly because of the structure of the work and the notation.

We have rejected the others for combinations of these reasons: all except for Rudolff's *Coss*, which has many features in common with Aurel's work. By a closer analysis of the two works we will now show that Rudolff's *Coss* was the main source of Aurel's *Libro Primero*.

The structure of Rudolff's *Coss* and Aurel's *Libro Primero*

In this section, we compare Rudolff's *Coss* and Aurel's *Libro Primero* in terms of their structure.

Rudolff's 1525 *Coss* is one of the earliest treatises on algebra published in Germany; the full title is 'Agile and pretty calculation by the artful rules of algebra which is commonly called the Coss'.[2] It was dedicated to the Bishop of Brixen (now known as Bressanone). Michael Stiffel republished it in a revised and extended form as *Coss Christoffs Rudolffs* in 1553 (second edition 1615) adding geometrical justifications and many problems. It includes explicitly in the title the word *Coss*, the name Rudolff uses for the unknown: derived from the Italian *cosa*, 'a thing' and from which we have the terms 'cossist' for algebraists and 'cossic art' for algebra (see in general Cajori 1928).

The work is divided into two main parts. The first four chapters of the first part deal with basic operations with whole numbers and fractions, progressions, the rule of three direct and inverse, and the extraction of square and cube roots. At the beginning of the fifth chapter the author sets out the names and symbols of the powers of the unknown, referring to them as *characters* (Pacioli speaks about *caratteres*, Chuquet about *Karactes* and de la Roche *Karactes*: Pacioli 1494, f. 67^{r}; Chuquet 1484, f. 84^{r}; de la Roche 1520, f. 42^{r}), while in the rest of this chapter as well as in the sixth chapter he carries out the four operations and the rule of three on polynomials. Chapters 7 to 11 are devoted to roots, and binomial and residual expressions. The first part concludes with a short explanation of the types of proportions—multiple, superparticular, superpartiens, multiple superparticular, and multiple superpartiens—constituting the twelfth chapter.

[2] Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genennt werden.

The second part of Rudolff's book consists of three sections: theory, reduction of rules, and examples (Kaunzner and Röttel 2006, 56). In the first section of this second part, Rudolff explains the 'rule of the thing' and then discusses the eight types of equations. He explicitly uses the word *equation* (Rudolff 1525, Gvj^{v}; Kaunzner and Röttel 2006, 188). In the second section he gives four pieces of advice (*cautelae*) for reducing the twenty-four types of equations established for the earlier cossists to the eight types he considers; and in the third section, entitled 'Examples', he presents a collection of 433 problems, which exemplify the rules for solving every type of equation. According to Heeffer (2012, 139) Rudolff possibly reduces the twenty-four equation types and takes the eight cases from the Vienna 5277 codex.

Aurel's work, by comparison, was the first treatise containing algebra to be published on the Iberian Peninsula, the full title being as follows:

> First Book of Algebraic Arithmetic, containing the 'arte Mercantivol', with many other Rules of the minor art, and the Rule of Algebra, commonly called Greater Art, or Rule of the Thing: without which the tenth book of Euclid cannot be understood, nor many other exquisite skills, both in Arithmetic and in Geometry: composed, ordered and submitted for Printing by Marco Aurel, native of Germany: Entitled 'Awakener of Minds'[3]

It was dedicated to 'the magnificent Sir Father Bernardo Cimon, Citizen of the highly distinguished and crowned City of Valencia'. In the title Aurel explicitly refers to his German origins and states that the content of his book ranges from strictly mercantile arithmetic to the 'rule of the thing' (*Cosa*) or the rule of algebra, commonly called the 'Arte Mayor'.

The work consists of 140 folios divided into twenty-four chapters. In the first twelve chapters Aurel deals with numbers and their operations, rational numbers, proportion, the rule of three, the rule of one false position, progressions, square numbers and roots, irrational numbers, the binomial and residual expressions and their operations.

He concludes the twelfth chapter with the words 'end of Arte Menor'; the algebraic part begins in the thirteenth chapter and continues to the end of the work. Aurel deals with the definition of unknowns and their powers—which he calls *characters*—and their operations; and with the 'rule of the thing' with its equations—which he calls *equalities*—divided into eight types. To finish his work, Aurel solves 249 problems, which exemplify the rules for solving every type of equation.

Rudolff's and Aurel's works contain similar arithmetical and algebraic procedures with many identical numerical examples, but arranged in a different order for didactic purposes. Both authors use the same symbols, which they call *characters*, in order to refer to the unknown and its powers, and both consider eight types of equations and exemplify their rules with the resolution of many similar problems.

Despite the many similarities between the two texts, it is necessary to point out an important difference. In Rudolff's work, the arithmetical part is included in order to

[3]Libro primero de Arithmetica Algebratica, en el qual se contiene el arte Mercantívol, con otras muchas Reglas del arte menor, y la Regla del Algebra, vulgarmente llamada Arte Mayor o Regla de la cosa: sin la qual no se podra entender el décimo de Euclides, ni otros muchos primores, asi en Arithmetica como en Geometria: compuesto, ordenado, y hecho Imprimir por Marco Aurel, natural Aleman, titulado Despertador de ingenios.

understand algebraic procedures; that is to say, the book is a treatise on algebra that includes the arithmetic needed to understand it. Indeed, Rudolff (1525, f. Aiij^{v}; Kaunzner and Röttel 2006, 162) clearly states this point at the outset:

> The book is divided into two parts. The first contains eight algorithms with other preliminaries that are needed for learning the *Coss*: the second shows the rules of the *Coss*, each one separately explained by means of many examples, several of which are beautiful.[4]

In Aurel's work, however, the arithmetic and the algebra are two clearly differentiated parts: the arithmetic or 'Arte Menor' is the first part of the work, while algebra or 'Arte Mayor' constitutes the second, as stated by Aurel.

The contents of Rudolff's *Coss* and Aurel's *Libro Primero*

The idea of algebra

Both authors make mention of the word *algebra*. Rudolff refers to the different names by which *algebra* is known in different countries. Aurel only discusses its Arabic origin. Rudolff (1525, f. Aiij^{r}; Kaunzner and Röttel 2006, 162) goes on to say:

> Finally, already Plato writes that without arithmetic, music and geometry, built on number, nobody can be called wise. After the ancients had exerted themselves to understand number, they wrote a subtle art in the Arabic tongue: *gebra et almuchabala*, by the Indians called *Alboreth*, and by the Italians *de la cosa*, namely an art about things or numbers in general .... About this art I have composed this book with many beautiful examples for all who love arithmetic.[5]

Therefore, for Rudolff, algebra is an extension of arithmetic: it is first necessary to understand numbers, and then it is possible to deal with this 'Art' about things or numbers in general.

Twenty-seven years later, however, Aurel considered the 'Rule of the Thing', Algebra or the 'Arte Mayor' to be one and the same. There are no references to numbers in Aurel's explanation, but rather to more specific algebraic procedures. At the beginning of Chapter 13, quoting the first translator, Guillermo de Lunis, from Arabic to Italian, Aurel (1552, f. 68^{v}–69^{r}) says:

> The rule commonly known as the thing or greater art, which by its own name (according to Guillelmo de Lunis, who is the first translator of this rule from

[4]Das büch wirt geteilt in zwen teil: Der erst bechleüst acht algortihmos mitt etlichen andern vorleüfflen so zu erlernung der Coss nottürfftig sein. Der Ander zeigt an die reglen der Coss je eine in sunderheit erklert mit vil und mancherley schoenen exempln.

[5]Schließlich schreibt schon Plato, dass ohne Arithmetik, Musik und Geometrie, welche in der zahl gegründ, niemand weise genan sein. Demnach sich die Alten hochlich beflissen die zahl zu ergründen haben schriebè ein subtile kunst so in Arabischer zungen: Gebra et almuchabola von den Indianern Alboreth von welschen de la cose genan wurde, nämlich ein kunst von dinge oder zahlen in der gemein ... Von solcher kunst hab ich zusame gelesen diß buch mit vil schoenen exempln allen liebhabern der Arithmetic verfertigt.

Figure 1. Characters in Rudolff's *Coss* (Rudolff 1525, Dij^v; Kaunzner and Röttel 2006, 174)

> Arabic to the Italian language) is called Algebra & Almucabala, which is restoration and opposition (as you will see in the notes to the equalities) ....[6]

The first to quote Guillermo de Lunis as a translator of Algebra from Arabic into Italian was Benedetto da Firenze in his 'Practica d'arismetricha' (1463), which was never printed. We find the same quotation in Raffaello Canacci's 'Ragionamenti d'algebra' (*c.* 1485), also never printed. We find a third quotation in Francesco Ghaligai's *Summa de arithmetica*, printed in Florence in 1521 and reprinted in 1548. Aurel may have read Ghaligai or some other sources to create his idea of algebra. It is clear that in this case Aurel did not take his ideas from Rudolff.

Characters in a continued proportion

One of the common features of both texts is the significance of the symbols for unknowns arranged in a continued proportion. In this case, the similarity between Rudolff's and Aurel's texts is clear. In his fifth chapter, Rudolff refers to the symbols as the *characters* he is going to use, and in his thirteenth chapter Aurel does the same (Figures 1 and 2).

As Aurel was to do later, Rudolff begins by talking about the proportion between the characters, mentioning Book 9 of Euclid's *Elements*. He writes (Rudolff 1525, f. Dij^v; Kaunzner and Röttel 2006, 174):

> Similarly, the fourth number is a cubic, after which always after two intervening it is again a cubic; (as Euclid shows in prop. 8 and 9 of the 9th book). After considerable exertion, our ancient [ancestors] invented the *Coss*; that is, the computa-

[6] La regla vulgarmente llamada de la cosa, o arte mayor, que por su propio nombre (como dice Guillelmo de Lunis, que es el que primero traslado la dicha regla de Arabigo en lengua Italiana) se llama Algebra & Almucabala, que es restauratio, & oppositio (como en los avisos de las igualaciones veras) ....

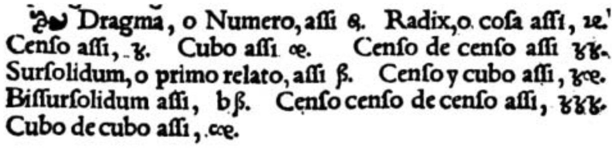
Dragma, o Numero, aſſi. Radix, o coſa aſſi,
Cenſo aſſi, Cubo aſſi. Cenſo de cenſo aſſi
Surſolidum, o primo relato, aſſi ß. Cenſo y cubo aſſi,
Biſſurſolidum aſſi, bß. Cenſo cenſo de cenſo aſſi,
Cubo de cubo aſſi,

Figure 2. Similar characters in Aurel's work (1552, f. 69[r])

> tion with a thing and numbers mentioned in natural (*natürlicher*) order, as follows: dragma, radix, zensus, cubus, zensdezens, sursolidum, zensicubus, bissursolidum, zenszensdezens, cubus de cubo.[7]

Likewise, from the beginning of his 'Arte Mayor', Aurel also reminds us of the significance of continued proportions in the 'rule of the thing', quoting Euclid Book 9, but without specific reference to propositions 8 and 9 (Aurel 1552, f. 69r):

> (This rule) is based on a continued proportion, in which there are many numbers of several genres, like squares, cubes and so on, as one may see in (Book) n° 9 by Euclid.[8]

This idea, from Euclid IX.8. (Heath 1956, 390–391), can be expressed as follows (Heath 1956, 390): If as many numbers as we please, beginning from a unit, are in continued proportion, the third from the unit will be square, as will also those which successively leave out one. The fourth will be cube, as will also those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.

$$\text{Or, in modern terms, } 1 : x = x : x^2 = x^2 : x^3 = x^3 : x^4 = ...$$

Rudolff possibly used Campanus's 1260 translation of the *Elements* (Busard 2005, 120).

References to characters in a continued proportion can be found in previous authors dating from the late fourteenth century. Continued proportion is also found in the *Triparty en la science des nombres*, a French manuscript written by Chuquet in 1484. Chuquet arranged the 'nombres' (he also calls them *karacte*) into first, second, third, and so on, according to the exponents (Chuquet 1484, f. 86[v]; Heeffer 2012). Nevertheless, no references to Euclid's *Elements* are found in this context in the earlier works we have seen. This feature thus shows a clear relationship between the works of Rudolff and Aurel.

Rudolff and Aurel employ German notation for representing unknown quantities and their powers. They use a different character for every power of the unknown, and the symbols +, − and √, for addition, subtraction and extraction of a square root, respectively. Rudolff defines the characters by means of a table (see Figure 3),

[7]Item die vierd zal ein cubic darnach alweg nach zweien dar zwischen widerumb ein cubic (wie dan Euclidin der 8 und 9 pro: des neünden büchs anzeigt) haben nach ersilichem useis erfunden die coss das ist die rechnung von einem ding un die zalen nach natürlicher ordnung genent wie hernachvolgt. Dragma radix zensus cubus zensdezens, sursolidum, zensicubus bissursolidum zenszensdezens cubus de cubo.

[8]... es fundada sobre una proporción contínua, en la qual ocurren muchos numeros de diversos generos, como quadrados, cubicos, &cc. Como en el 9° de Euclides podras ver.

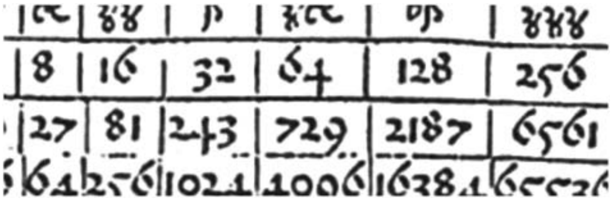

Figure 3. Rudolff's table of characters in continued proportion (Rudolff 1525, f. Diij^{v}; Kaunzner and Röttel 2006, 174)

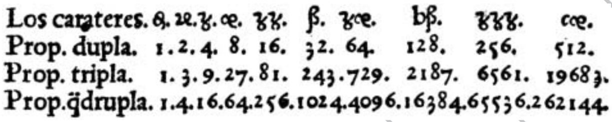

Figure 4. Aurel's table of characters in continued proportion (Aurel 1552, f. 70^{v})

and later Aurel also draws up a similar table of characters and the corresponding numbers in continued proportion (see Figure 4), likewise depending on the given base. Both authors take first 2 as the base, then 3 and finally 4. Aurel also specifies the name of each proportion: proportion duple, proportion triple, and proportion quadruple. Each author adds a brief table with fractional values and each gives the same example, the *sesquialtera* (3:2) proportion.

For calculations—addition and subtraction—with the characters, Rudolff and Aurel follow the same procedure and give some identical examples. In Figures 5 and 6 we show two examples with identical numbers, the first one also having identical symbols. In order to multiply the characters, Rudolff and Aurel construct identical tables with the symbols for the powers of the unknown, with a number above each symbol that corresponds to what nowadays is called the degree of the unknown (see Figures 7 and 8). Both authors explain that one should look for the numbers above the characters one wishes to multiply, add them together and then look for the character corresponding to this addition. This character will be the product of the two

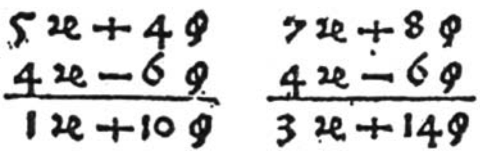

Figure 5. Subtractions in Rudolff's *Coss* (1525, f. Dv^{r}; Kaunzner and Röttel 2006, 175)

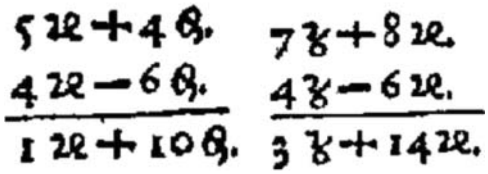

Figure 6. Subtractions in Aurel's *Libro Primero* (1552, f. 71^{v})

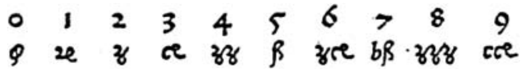

Figure 7. Rudolff's table for multiplying characters (1525, f. Dvj^{v}; Kaunzner and Röttel 2006, 176)

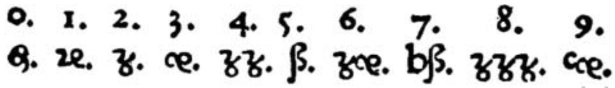

Figure 8. Aurel's table for multiplying characters (1552, f. 72^{r})

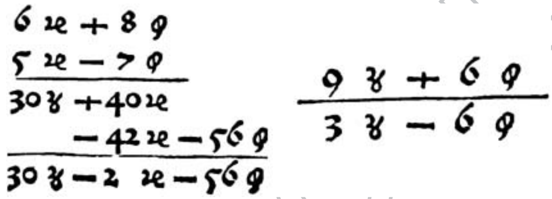

Figure 9. Multiplications and Divisions in Rudolff (1525, f. Dv^{r}; Kaunzner and Röttel 2006, 176–177)

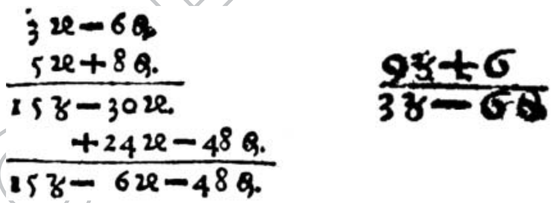

Figure 10. Multiplications and divisions in Aurel (1552, f. 73^{v}–74^{v})

previous ones (Rudolff 1525, f. Dvj^{v}; Aurel 1552, f. 72^{r}; Kaunzner and Röttel 2006, 176).[9]

For calculations of multiplication and division with the characters, Rudolff and Aurel also follow the same procedure; however, in the case of multiplication there are no identical examples. Figures 9 and 10[10] show two similar examples. Like Rudolff before him, Aurel proposes a numerical substitution to verify the result obtained in the multiplication. In the case of polynomial divisions, neither author gives a method to perform them; each says that in order to determine the value of the unknown it is necessary to assign a value to the expression. (In this case, both authors assign the value 7 to the polynomial division, obtaining the value 2 for the unknown.)

[9] Y quando tu querras multiplicar una dignidad, grado, o caracter con otro, mira lo que esta encima de cada uno, y junta lo simplemente, y aquello que verna, mira encima de qual caracter estara: tal diras que procede de tal multiplicacion.

Zu wissen den name eins products, addier die zalen so gefunde werde über den zweien quantitetn welhdu miteinander multiplicirst, das collect würt dir anzeigen den nam des products.

[10] When Aurel writes the division as a fraction, he does not put a symbol with the independent term, but puts it when he states the two expressions to be divided. It is therefore likely that Aurel forgot to put it in the final expression.

For the definitions of the characters, both authors refer to the continued proportion to which they belong, although Aurel places greater emphasis on this. Indeed, Rudolff defines the *thing* as follows (Rudolff 1525, f. Dij^{v}; Kaunzner and Röttel 2006, 174): ‘Radix is the side or root of a square’.[11]

Although Rudolff does not refer explicitly to continued proportion in this definition, he refers to propositions 8 and 9 in Book 9 of Euclid’s *Elements* before declaring the characters he is going to use, as we have seen before. These Euclidean propositions are clearly related to continued proportion.

Like Rudolff, Aurel defines the thing as a side of a square, but he explicitly quotes the continued proportion (Aurel 1552, f. 69^{v}):

> *x* is the root or side of an equilateral square. And it is the first of the numbers of one continued proportion: because d is like one, which is not a number.[12]

In fact, the relation with geometry is also found in the preface, addressed *Al Lector* (To the reader), which states that all explanations for numbers can be taken as lines. However, throughout the rest of the work he does not mention any relation with geometry again. And the essential point for using the ‘rule of the thing’ is the significance of the characters; namely, their relation to numbers in continued proportion. Aurel makes this point in the following note (Aurel 1552, f. 70^{v}):

> The character should not be understood as or taken for a number or a simple quantity, but rather for the rank, degree or place in a continued proportion. Like *z*: it is the second quantity of a continued proportion, and *R* is the fifth.[13]

It is necessary to point out a difference in the treatment of symbols. Unlike Rudolff, Aurel states that every author creates his own characters. He adds that it is not essential to use the same symbols for these characters because their operations are not based on the figure (appearance) of the characters, but rather *on their significance*. Aurel also notes that the number of characters described is not essential for the use of the ‘rule of the thing’, and that this number can be extended to infinity (Aurel 1552, f. 70^{r}–70^{v}). In Ghaligai (1521, f. 2$^{r,\,v}$), the author also describes the characters and says ‘cosí in infinito’. However, we must also recall the difference between Aurel’s notation and that of Ghaligai. Pacioli (1494, f. 143^{r}) also claims ‘one may go as long as one wants to’, and has yet another symbolism.

Thus, for Aurel, the specific symbol for the unknown, the character, is not important; the key idea is the significance of characters arranged in a continued proportion; that is to say, the rank of the character in a continued proportion that he had taken from Rudolff’s work.

[11]Radix ist die seiten oder wurzel eins quadrats.

[12]El *x*, es rayz, o lado de un cuadrado equilatero. Y es el primero de los numeros de una continua proporcion: porque *d* es como uno, el qual no es numero.

[13]Nota. El caracter no lo has de tomar, ni entender por numero o quantidad simple, sino por dignidad, grado, o casa de una continua proporcion. Como el *z*, es la segunda quantidad de una continua proporcion, y el *R* es la quinta.

Types of equations

The classification into eight types of equations—called equalities by Aurel—and the examples given for the definitions of the different types, are exactly the same in both works. Aurel may have read Rudolff's text and literally copied many of the enunciations, examples and problems, and translated them into Spanish.

When classifying equations, both authors list eight types: four of them with two terms and called simple equations (for example, in modern notation, $ax^{n+1} = bx^n$), and the four remaining, with three terms, called compound equations (for example, in modern notation, $ax^{n+1} + bx^n = c$).

The four types of simple equations are accurately described, as well as the algorithms for solving them, with examples for each type. In modern notation we would write these types as follows:

$$1.\quad ax^{n+1} = bx^n; \quad x = \frac{b}{a}$$

$$2.\quad ax^{n+2} = bx^n; \quad x = \sqrt{\frac{b}{a}}$$

$$3.\quad ax^{n+3} = bx^n; \quad x = \sqrt[3]{\frac{b}{a}}$$

$$4.\quad ax^{n+4} = bx^n; \quad x = \sqrt[4]{\frac{b}{a}}$$

But we can find differences in the rules for solving the different types of equations in the two works. One substantial difference is that Aurel emphasizes the idea of continued proportion in contrast with Rudolff's explanation. For the first type, for example, Rudolff gives (Rudolff 1525, f. Gvij^{v}; Kaunzner and Röttel 2006, 188):

> When two natural ordered quantities are equal to each other, divide the lesser by the greater quantity, the quotient showing the value 1*x*, as [you can see] in the following examples.[14]

And Aurel (1552, f. 77^{v}–78^{r}):

> When two quantities, characters or differences of numbers are equal, and there is no missing quantity between these two terms, I say that one follows the other according to the rule of continued proportion: as *d* to *x*, [so is] *x* to *z*, [and so is] *z* to *&c*. Divide the lesser [quantity] by the greater; the quotient of this division will tell you the value of *x*.[15]

In the definition for the second type of equations, as Aurel would explain later, Rudolff added that there was one intermediate missing character between two equal characters. For the third type, he also added that there were two intermediate missing characters, and so on (Rudolff 1525, f. Gvij^{v}; Aurel 1552, f. 78^{r}; Kaunzner and Röttel

[14]Wan zwo quantitetn natürlicher ordnung einan der gleich werden/dividir die steiner in die grösser quantitet der quocient zeigt an den werdt 1x us in disen exempln.

[15]Quando se ygualaren dos quantidades, caracteres o diferencias de nombres, y no faltare alguna entremedias de las dos: digo que la una siga a la otra, en regla de contínua proporcion: como *d* a *x*; *x*a *z*, *z*a &c. Partiras la menor por la mayor, el quociente de tal particion te dira quanto vale la *x*.

2006, 188). Thus—they both explain—for the second type of simple equations (with one intermediate missing character), the square root is extracted from the quotient; for the third type (with two intermediate missing characters) the cube root; and for the fourth type (with three intermediate missing characters) the fourth root.

In fact, as Rudolff before him, Aurel tries to state a generalization of this algorithm (Aurel 1552, f. 78^{v}):

> As in the first equality you have seen that no character is missing in between, the quotient assumes the value of an *x*; in the second equality, a character is missing between the two; the quotient assumes the value of a *z*; in the third, two are missing, and it will assume the value of a ❧ ... From each quotient you will extract the root according to what comes next, providing that those that are missing between the two are equidistant.[16]

As for the compound equations, both authors give statements for the four types that can be written in modern notation as follows:

$$5.\quad ax^{n+2} + bx^{n+1} = cx^n; \quad x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} - \frac{b}{2a}$$

$$6.\quad ax^{n+2} + cx^n = bx^{n+1}; \quad x = \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \pm \frac{b}{2a}$$

$$7.\quad bx^{n+1} + cx^n = ax^{n+2}; \quad x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} + \frac{b}{2a}$$

$$8.\quad ax^{n+2k} + bx^{n+k} = cx^n; \quad ax^{n+2k} + cx^n = bx^{n+k}; \quad bx^{n+k} + cx^n = ax^{n+2k}$$

As an example, for solving the first of these compound types, this is Rudolff's rule (Rudolff 1525, Gviij^{v}; Kaunzner and Röttel 2006, 189):

> If three quantities in natural order are compared to each other, in such a way that the two greater are equal to the lesser one. Divide the lesser and middle ones, separately, by the greater quantity. Multiply half of the coefficient of the middle one by itself as square; to the square add the quotient of the lesser quantity; diminish the square root of this sum by half of the quotient of the middle one; this indicates the value of 1*x*.[17]

And this is Aurel's (1552, f. 78^{v}):

> When three quantities or differences of equally distant numbers are equal, and there is none missing in between, then the two greater [quantities] are equal to the

[16]Nota. Así como en la primera igualación has visto que no falta ningun caracter entre medios, el quociente dize la valor de una x, en la segunda igualacion falta un caracter entre medio de los dos; el quociente dize la valor de un z, en la tercera faltan 2, verna la valor de un ∞ ... De cada quociente sacaras la rayz conforme a lo que viene, con tal que los que faltaren entre medio de los 2, sean equidistantes.

[17]Werden einander vergleicht dren quantiteten natürlicher ordnung also das die grössern zwo werdè gleich gesprochen der steiner. Us dann dividir die steiner unnd mitter je eine in sunderheit durch die grosser quantitet. Multiplicir des mittern quociets halbenteil in sich quadrate zum quadrat addir den quocient der steinern quantitet radir quadrata dis ser summa minder $\frac{1}{2}$ des mittern quocients zeigt an den Wert 1x.

> lesser. You must divide the lesser and the middle [quantity] each by the greater, and you must multiply half the quotient of the middle one by itself. And to this product you must add the quotient of the division of the lesser; $\surd$ of all this sum less $\frac{1}{2}$ of the quotient of the median will be the value of one x.[18]

As regards the last type of equation, Aurel has already explained that it is a reduced type. He uses the preceding three types and also mentions that one must extract the corresponding root from the result, depending on the number of intermediate missing characters. Nevertheless, Aurel does not specify that the last type can be considered as a reduction of all the types of compound equations.

The idea of a continued proportion in the arrangement of characters is mentioned by both authors and they are aware of its significance. In other words, the identification of each character with a number, which we would now call the exponent of the power, is essential for operating and reducing the number of types of equations, and consequently the number of rules. For each type of compound equation, the authors make no distinction between successive powers like x^n, x^{n+1}, x^{n+2} or any 'equally equidistant terms', like x^n, x^{n+k}, x^{n+2k} in the treatment of the first, second or third rule; one must simply extract the *k*th root from the result.

A precedent for this kind of reduction is found in an anonymous manuscript from the Biblioteca Estense in Modena: MS 578, entitled *Della Radice dei Numeri e Metodo di trovarla* (*c.* 1475–1500; see Van Egmond 1986; Høyrup 2010). In this manuscript we do not find the idea of a continued proportion, but rather the idea of associating each character with its degree ('gradi'); 0 with N, 1 with C, 2 with Z and so on. The author divides the equations into eighteen basic types and gives one general rule that could be used to solve any equation of each type. In fact, as Van Egmond (1986, XVI) states: 'Although the author recognizes the relationship between equations that can be reduced to a single type, he does not actually perform any reductions or even seem to recognize the possibility of doing so'. This point establishes an essential difference with Aurel's work, because he clearly recognizes and explains the possibility of performing this reduction.

The rule of quantity

The 'rule of quantity' or the 'rule of the second quantity' are the expressions used in the first treatises on algebra to refer to a procedure for solving problems in which more than one unknown is involved. The first appearance of the second unknown in Western culture was probably around 1225 in the *Flos* of Fibonacci, and later in 1373 in the *Trattato di Fioretti* by Antonio de Mazzinghi (Pisano 1225, 236; also see Franci 1988, 29). The use of more than one unknown would lead to the solution of simultaneous linear equations, whose discussion represented a big step forward in the process of the algebraicization of mathematics (Romero Vallhonesta 2011; Heeffer 2012).

Rudolff addresses the *regula quantitatis* after solving the first type of equation. The method for solving these kinds of problems consists of putting the second

[18]Quando se ygualaren tres quantidades o differencias de nombres ygualmente distantes, y que no falte ninguna entre medias, desta manera que las dos mayores se ygualen a la menor. Partiras la menor, y mediana de cada una por si por la mayor, y multiplicaras la metad del quociente, dela particion del mediano en si mesmo; y al dicho producto juntaras el quociente de la particion del menor; $\surd$ de toda esta summa, menos la $\frac{1}{2}$ del quociente del mediano, sera la valor de una x.

unknown in terms of the first one, the same for the third unknown, and so on. Thus, he puts *quant* for the second unknown, *x* being the first unknown, and when the second is expressed in terms of the first he also puts *quant* for the third: and so on.

Aurel, too, addresses *la regla de la quantidad* after solving the first type of equations. He says (Aurel 1552, f. 108):

> Chapter XVI deals with the rule of quantity, with some rules and requirements by which they are done, also known as the rule of the second thing.[19]

The method used by Aurel is the same as that of Rudolff, only with the difference that Aurel named the second unknown and those that followed *q*.

The practice of the 'Rule of the Thing'

Another feature for comparing the two works concerns the practice of the 'Rule of the Thing', which both authors regard as a complete method for solving problems rather than merely an algorithm or rule for solving the reduced equation once it has been stated. In most of the algebraic texts of the sixteenth century, the method consists of putting the unknown for the number to be found, operating to derive an equation, then reducing and solving it. Aurel, however, prior to dealing with the algorithms for types of equations, explicitly emphasizes that the 'rule of the thing' is a complete method that also includes stating an equation from the unknown and known quantities, assuming that the question is already solved and working through it in accordance with the given instructions. Rudolff (1525, Gvj^{v}–Gvij^{r}; Kaunzner and Röttel 2006, 188) and Aurel (1552, f. 76^{v}) describe the procedure similarly.

> Rudolff: This art, as stated above, is founded on eight rules of equation or comparison. Thus in working through [practising] every example, 1x must be substituted for the hidden [unknown] thing at the beginning which one wishes to know. With such a substituted root, one must proceed in every way thereafter as if it were the correct number, until the thing is brought to the point where two orders of numbers become equal to each other. At that moment the comparison will be carried out by one of the equations given below, just as it emerged. By such methods the value and meaning of the root first substituted becomes evident.[20]
>
> Aurel: And I say that to pose a question by this rule ('of the thing'), you have to imagine that such an account or question has already been posed and answered, and now you want to prove (check) it. And you will put that the answer is an *x*, with which you must proceed with the given advice and rules as if it were the known quantity itself or true answer, until finally you come to the last answer, under characters or hidden quantities, which will be the one or ones you will

[19]Capit. XVI. Trata de la regla de la quantidad, con algunas reglas, y demandas que por ella se hazen, que por otro nombre se puede llamar, regla de la segunda cosa.

[20]Dise Kunst: wie obgemelt: ist gegründt in 8 regln der equation oder vergleichung. Dann in practicirung eins jeden exempls an stat des verborgnen dings so man zu wissen begert mutz anfengklich gesetzt werden 1 *x*. Mit sölchen gesetzten radix mütz man darnach procediern in aller gestalt sam wer es die rechte zal so lang bitz die sach dahin bracht das zwo ordnung der zalen eine der andern gleich werde. Als dann würt die vergleichung practiciert durch eine autz den untergeschribhen equation so sje eingefallen ist. Durch sölche practiken kompt an tag der wendtunnd bedeütnutz des erstgesetzten radicis.

> say are those you wished to arrive at. And then you will make (practise) an equality – by one of the eight following equalities – to which it will be subjected, and thereby the value of the hidden and first proposed *x* will be declared.[21]

Later, also in Spain, Pérez de Moya (1558, 62) paraphrases Aurel, describing the method similarly:

> And so I say that to pose any question by this rule, you have to assume that such a question has already been posed and answered, and you want to check (prove) it. Putting, for example, that the answer is a 'thing', with which you will proceed by doing what the question requires, and whatever comes out as the 'thing' (1. *co*.) you will say is equal to what you wished to arrive at.[22]

Antic Roca in his work *Arithmetica* (1564, f. 262r) also gives the same explanation, in nearly the same words as Pérez de Moya:

> I say that to pose any question by this 'rule of the thing', you have to imagine that such a question has already been posed and answered, but you want to check (prove) it; and you shall first put that the answer is 'one thing', with which you have to proceed by doing what the question requires, and whatever comes out as the 'thing' (1.*cosa*) you will say is equal to what you wished to arrive at.[23]

There are hints of this method in earlier texts and, as Stedall (2011) has pointed out, Stifel explicitly described a similar method, albeit less clearly, in his *Arithmetica integra* (1544, f. 227v) as the 'Rule of Algebra'. Therefore, in this case, we may again assume that Aurel, in his *Libro Primero*, adopted the idea of this practice of the 'Rule of the Thing' from Rudolff's *Coss*, and modified it in such a manner as to pave the way to an analytical approach.

Thus, François Viète (1540–1603) on the first page of *In Artem Analyticen Isagoge* (1591, 3) clearly explains his analytical approach:

> There is a certain way of searching for the truth in mathematics that Plato is said first to have discovered. Theon called it 'analysis', which he defined as assuming

[21]Y digo que para hacer una demanda, por la dicha regla (de la cosa), has de imaginar que tal cuenta o demanda ya eshecha, y respondido, y tu agora la quieres provar. Y pornas que la respuesta fuesse una x, con la qual has de proceder con los avisos y reglas dadas, como si fuere la propia quantidad sabida, o respuesta verdadera, hasta tanto que venga a la postre la ultima respuesta, debaxo de caracteres o quantidades ocultas. La qual o las quales diras ser igual a lo que tu querrias que viniese. Y luego practicaras esta tal igualacion, por una de las 8 igualaciones siguientes, a la que sera sujeta, y por ella te sera declarada la valor de la x oculta, y primero propuesta.

[22]Y assi digo que para hazer qualquier demanda por esta regla, has de presuponer que la tal demanda es ya hecha y respondida, y que la quieres provar. Poniendo por ejemplo que la respuesta fuesse una cosa, con la qual procederas, haziendo lo que la demanda pidiere, y lo que te viniere con la 1. *co*. Diras ser ygual a lo que quisieras que viniera.

[23]'Digo que para hazer qualquier demanda por esta regla dela Cossa, has de imaginar que la tal demanda es ya hecha y respondida, empero tu la quieres provar; y pornas primeramente que la respuesta fuesse una cosa, con la qual has de proceder haziendo lo que la demanda pidiere, y lo que te viniere con la 1.cosa diras ser ygual a lo que quisieras que viniera.'

> that which is sought as if it were admitted [and working] through the consequences [of that assumption] to what is admittedly true.[24]

Forty years later again, Pierre Hérigone (1580–1643) in his *Cursus* (1634, vol 2, 1; see Massa-Esteve 2008) adopted this idea when he explained Viète's algebra, using similar words: 'Analytical doctrine or algebra, called "cosa" in Italy, is the art of finding the unknown magnitude by taking it as if it were known, and finding the equality between this and given magnitudes'. These examples show the importance of Aurel's *Libro Primero*, especially its approach to the 'Rule of the Thing' as an analytic method for solving problems in Spanish algebra.

Problem-solving

Concerning the relationship between Rudolff's and Aurel's work, we may also remark that in many problems we find the same enunciation and the same resolution: one in German and the other in Spanish. In total 143 problems are common to Rudolff and Aurel; they are broken down as follows.

	Number of problems		
Type of equation	Rudolff	Aurel	Common to both
First	187	115	39
'Rule of quantity'	31	8	4
Second	30	21	17
Third	20	15	12
Fourth	20	13	12
Fifth	40	30	22
Sixth	30	19	16
Seventh	30	17	16
Eighth	24	11	5

In many problems we find not only the same wording but also the same way of arriving at the solution. For example, in his fourth problem of the first type, Rudolff (1525, f. Hvij^{v}; Kaunzner and Röttel 2006, 192) has:

> Find a number, $\frac{2}{3}$ of which is as much as if I had added 3 to the half of the same number.
> Let the number be $1x$; then I say that $\frac{2}{3}x$ is equal to $\frac{1}{2}x + 3Q$. Work according to the teaching of the first case: subtract $\frac{1}{2}x$ from $\frac{2}{3}x$; $\frac{1}{6}x$ remains, which equals $3Q$. Dividing makes 1x [equal to] $18Q$.
> Proof: $\frac{2}{3}$ of 18 is 12. Similarly, $\frac{1}{2}$ of 18 is 9, to which I add 3 [and] it also becomes 12.[25]

[24]Est veritatis inquirendae via quaedam in Mathematicis, quam Plato primus invenisse dicitur, à Theone nominate Analysis, & ab eodem definita, Adsumptio quaesiti tanquam concessi per consequential ad verum concessum.

[25]4. Such ein zal welcher 2/3 gleich sovil mache als hett ich zum halbenteil der selben zal 3 addiert. Sek die zal sei 1 x dennach sprich ich das 2/3 x ist gleich $\frac{1}{2}$ x + 3 Q. Thu nach unternicht der erstn cautel Subtrahit $\frac{1}{2}$ x von 2/3 x gleibt 1/6 x gleich 3 Q. Dividirn facit 1x 18 Q. Proba 2/3 auss 18 ist 12. Item $\frac{1}{2}$ von 18 ist 9 darzu addier ich 3 werden auch 12.

Later, Aurel in his first problem has similarly (1552, f. 82ᵛ):

> Let me give a number, $\frac{2}{3}$ of which is the same as if half of it were added to the number 3.
> Let this number be $1x$, $\frac{2}{3}$ of which is $\frac{2}{3}x$: this will be equal to $\frac{1}{2}x + 3Q$. It equals. Subtract $\frac{1}{2}x$ from $\frac{2}{3}x$, which yields $\frac{1}{6}x$. This will be equal to $3Q$. Divide 3 by $\frac{1}{6}$, which yields 18. So you will say that this is the number required.
> Try it: $\frac{2}{3}$ of 18 is 12: which will be $\frac{1}{2}$ of 18, plus 3.[26]

Or, again problem 11 in Aurel is identical to problem 9 in Rudolff, each involving the second type of equation. The problem refers to the purchase of a piece of cloth of 40 *varas*[27] that costs as many ducats as the number of *varas* that would be obtained with $5\frac{5}{8}$ ducats. The aim is to determine how much the piece measuring 40 *varas* costs. It is clear that the problem is not a realistic one. The objective of these kinds of problems was to illustrate the power of algebraic tools compared with arithmetical ones. The choice of this problem to illustrate a case that led to the resolution of an equation of the second type shows once more how much Aurel absorbed from Rudolff as regards this problem: or that both have a common source. Each author solves the problem by means of the rule of three, from which they obtain the equation that we would write as $\frac{8}{15}x^2 = 40$ and whose solution is $x = 15$.

It seems that Aurel believed that the problems solved in Rudolff's work were appropriate for people's understanding of algebra. Thus, he decided to translate many problems in Rudolff's *Coss* from German to Spanish in order, as he states at the beginning of his work, to disseminate this new science in Spain (Aurel 1552, *Al Lector*).

Concluding remarks

This paper has aimed to determine the influence of some European algebra texts on the first work in Spanish devoted to the 'Arte Mayor', as well as to think about the role of these influences in the development of Spanish algebra. On the first point, we have also presented some additional influences from Italy. It appears that Aurel knew Pacioli's *Summa*, and possibly that he was also familiar with Galighai's work or his source, mainly regarding the significance of the characters and the quotation by Guillermo de Lunis.

Taking into account the didactic aim of Aurel's algebra, he may have believed that Rudolff's *Coss* was appropriate for a better understanding of algebra and for introducing algebra into Spain. We have found many similarities between Rudolff's *Coss* and Aurel's *Libro Primero*, by comparing the notations they used, the types of equations, the rule of quantity, the practice of the 'rule of the thing', and the way of solving problems. Despite the similarities, though, we also note some differences, such as Aurel's emphasis on the idea of characters in a continued proportion. Thus, while it is true that Aurel absorbed ideas from the German text, he also added new

[26] 1. Mandadme dar un numero cuyos 2/3 sean tanto como si juntara a la mitad del mismo numero 3. Pongo que el numero sea 1 x, cuyos 2/3 es 2/3 x: estos seran iguales a 1/2x +3 *Q*. Yguala. Quita $\frac{1}{2}$x de 2/3 x, que daran 1/6 x. Este sera ygual a 3 *Q*. Parte 3 por 1/6, vernan 18. Tanto diras que es el numero demandado. Pruevolo, el 2/3 de 18, son 12: tanto sera la $\frac{1}{2}$ de 18, y mas 3.

[27] One *vara* is a measurement of length used in different regions of what now constitutes Spain, with different values ranging from 768 to 912 mm. Aurel used *varas* where Rudolff had used florins.

comments, original ideas and further problems, such as those described in the third section of this paper. Nevertheless, we may claim that the main source for the ideas in the Spanish 'Arte Mayor' was this German text.

With regard to our second aim, of clarifying the role of these influences in the development of Spanish algebra, it appears that Aurel's book was also a source for Perez de Moya's and Roca's books (Massa-Esteve 2012). For these authors, as for Aurel, the 'rule of the thing' is not only a particular rule or an algorithm for solving the equation, but a complete method, with a process that includes the statement or construction of an equation. Moreover, this statement of the equation must be made in a particular manner: that is, by operating with the unknown as if it were known, assuming the problem to be solved, and thereby achieving equality between unknown and given (in other words, by establishing an equation); finally, one must apply the corresponding algorithm to solve this equation, according to its type.

In the same way that Aurel took Rudolff's work as his main reference—although the *Libro primero* is not a mere translation—Pérez de Moya also took most of his *Compendio* (1558) from Aurel, as well as of his *Arithmetica* (1562). The fact that Pérez de Moya's work, with some improvements, later ran to more than twenty-five editions, is a measure of the ultimate influence of Aurel's text, drawing as it did from a German source, on the development of Spanish algebra.

Acknowledgements

We are grateful to Jens Høyrup, Karin Reich and Antoni Roca-Rosell who read an earlier version of this article and made some remarks concerning content and language. This research is included in the project HAR2013-44643-R and in the project HAR2016- 75871-R of the Spanish Ministerio de Economía y Competitividad.

Disclosure statement

No potential conflict of interest was reported by the authors.

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