# CONTINUITY OF CONDITION SPECTRUM AND ITS LEVEL SET IN BANACH ALGEBRA

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## Dedication

*Dedicated to my beloved parents*

# Abstract

**Key words :** condition spectrum, vector valued analytic maps, maximum modulus principle, upper and lower hemicontinuous correspondence, $(p, q)-\epsilon-$ pseudo spectrum.

For $0 < \epsilon < 1$ and a Banach algebra element $a$, this thesis aims to establish the results related to continuity of condition spectrum and its level set correspondence at $(\epsilon, a)$. Here we propose a method of study to achieve the continuity. We first identify the Banach algebras at which the interior of the level set of condition spectrum is empty and then we obtain the continuity results.

This thesis consists of four chapters. **Chapter 1** contains all the prerequisites which are crucial for the development of the thesis. In particular, this chapter has a quick review of the basic properties of spectrum, condition spectrum, upper and lower hemicontiuous correspondences. We also concentrate on analytic vector valued maps and generalized maximum modulus theorem for them.

For an element $a$ in $\mathcal{A}$, **Chapter 2** has the results related to interior of the level of set of the condition spectrum of $a$. At first, we focus on $1-$ level set of the condition spectrum and we prove that it has empty interior for any Banach algebra element $a$. In later portion, for $0 < \epsilon < 1$, we show that the interior of $\epsilon-$ level set of condition spectrum of $a$ is empty in the unbounded component of resolvent of $a$. For any operator $T \in B(X)$ where $X$ is complex uniformly convex or $X^*$ is complex uniformly convex, we prove $\epsilon-$condition spectrum of $T$ has an empty interior.

Contents in **Chapter 3** has the study of upper, lower hemicontinuity and joint continuity of the condition spectrum correspondence and its level set maps, in the appropriate settings. For $0 < \epsilon < 1$, we demonstrate the significant role of the empty interior of the $\epsilon-$ level set of condition spectrum at a given point $(\epsilon, a) \in (0, 1) \times \mathcal{A}$ where the continuity of the required maps are sought after. We establish the uniform continuity of the condition spectrum in the domain of normal matrices. Using the fact that $1-$ level set of condition spectrum is empty, we study the nature of $\epsilon-$ condition spectrum of an element when $\epsilon$ approaches to 1.

For the elements $a, p$ in $\mathcal{A}$ such that $p^2 = p$ and $\epsilon > 0$, **Chapter 4** is devoted to discuss some analytical and geometrical properties of $(p, e-p)-\epsilon$ pseudo spectrum of $a$ and its level sets. The efforts in the first subsection make the essential tools to show the non emptiness of $(p, e-p) - \epsilon$ pseudo spectrum of $a$. Second subsection of this chapter deals with the level sets. We show the interior of the level set of $(p, e-p)-\epsilon$ pseudo spectrum of $a$ is empty in the unbounded component of $(p, e-p)$

resolvent set of $a$. An illustration is framed to show that the condition unbounded component can not be ignored in general.
**Mathematics subject classifications :** 46H05, 47A10, 15A09.

# Contents

# List of Notation

| | |
|---|---|
| $\mathbb{C}$ | Set of complex numbers ( page 2 ). |
| $\mathcal{A}$ | Complex unital Banach algebra ( page 2 ). |
| $\|.\|$ | norm of an element in $\mathcal{A}$ ( page 2 ). |
| $\mathcal{A} \setminus \mathbb{C}e$ | Set of all non scalar elements in $\mathcal{A}$ ( page 3 ). |
| $X$ | Complex Banach space ( page 3 ). |
| $B(X)$ | Set of all bounded linear operators defined on $X$ ( page 3 ). |
| $M_n(\mathbb{C})$ | Set of all $n \times n$ matrices with complex entries ( page 4 ). |
| $s_{max}(A)$ | Maximum singular value of a matrix $A$ ( page 4 ). |
| $s_{min}(A)$ | Minimum singular value of a matrix $A$ ( page 19 ). |
| $\sigma(a)$ | Spectrum of an element $a$ in $\mathcal{A}$ ( page 4 ). |
| $\rho(a)$ | Resolvent set of $a$ in $\mathcal{A}$ (page 4). |
| $r(a)$ | spectral radius of $a$ in $\mathcal{A}$ ( page 5 ). |
| $x^*$ | Convolution of the element $x$ in a $C^*$ algebra ( page 5 ). |
| $\Lambda_\epsilon(a)$ | The pseudo spectrum of $a \in \mathcal{A}$ for $\epsilon > 0$ ( page 7 ). |
| $\sigma_\epsilon(a)$ | Condition spectrum of $a \in \mathcal{A}$ for $0 < \epsilon < 1$ ( page 8 ). |
| $B\,(a, r)$ | The open ball in $\mathbb{C}$ with center $a$ and radius $r$ ( page 9) . |
| $\phi : X \twoheadrightarrow Y$ | Correspondence between set $X$ and $Y$ ( page 12 ). |
| $Gr(\phi)$ | Graph of the correspondence $\phi$ ( page 13 ). |
| $K\,(\mathbb{C})$ | Set of all compact subsets of $\mathbb{C}$ ( page 14 ). |
| $H(A, B)$ | Hausdroff distance between two compact sets $A$ and $B$ ( page 14 ). |
| $d\,(\lambda, E)$ | distance from $\lambda$ to the set $E$, that is $\inf \{\|\lambda - \mu\| : \mu \in E\}$ ( page 14 ) . |
| $\limsup\limits_{n\to\infty} K_n$ | Limit superior of the sequence of subsets $K_n$ of a metric space $X$ ( page 15 ). |

| | |
|---|---|
| $\liminf\limits_{n\to\infty} K_n$ | Limit inferior of the sequence of subsets $K_n$ of a metric space $X$ ( page 15 ). |
| $\lim\limits_{n\to\infty} K_n$ | Limit of the sequence of subsets $K_n$ of a metric space $X$ ( page 15 ). |
| $L_\epsilon(a)$ | Level set of $\sigma_\epsilon(a)$ ( page 18 ). |
| $\overline{B}\,(a,r)$ | The closure of $B\,(a,r)$ ( page 52 ). |
| $\mathcal{C}, \mathcal{LC}$ | Condition spectrum and level set correspondence from $(0,1) \times \mathcal{A}$ to $\mathbb{C}$ ( page 41, 53 ). |
| $\mathcal{C}_a, \mathcal{LC}_a$ | Condition spectrum and level set correspondence from $(0,1)$ to $\mathbb{C}$ by fixing an $a \in \mathcal{A}$ ( page 41, 53 ). |
| $\mathcal{C}_\epsilon, \mathcal{LC}_\epsilon$ | Condition spectrum and level set correspondence from $\mathcal{A}$ to $\mathbb{C}$ by fixing an $\epsilon \in (0,1)$ ( page 41, 53 ). |
| $\mathfrak{C}, \mathfrak{C}_a, \mathfrak{C}_\epsilon$ | Condition spectrum map from $(0,1) \times \mathcal{A}$, $(0,1)$ and $\mathcal{A}$ to $K(\mathbb{C})$ ( page 47, 48 ). |
| $p, q$ | Idempotents in $\mathcal{A}$ (page 57 ). |
| $a^{(2)}_{p,q}$ | $(p,q)$ outer generalized inverse of $a$ in $\mathcal{A}$ ( page 58 ). |
| $\Lambda^{(2)}_{p,q}(a)$ | $(p,q) - \epsilon -$pseudo spectrum of $a$ in $\mathcal{A}$ for $\epsilon > 0$ ( page 58 ). |
| $L^{(2)}_{p,q}(a)$ | level set of $\Lambda^{(2)}_{p,q}(a)$. ( page 58 ) |
| $\sigma^{(2)}_{p,q}(a)$ | $(p,q)$ spectrum of $a$ in $\mathcal{A}$ ( page 59 ) . |
| $\rho^{(2)}_{p,q}(a)$ | $(p,q)$ resolvent set of $a$ in $\mathcal{A}$ ( page 59 ). |

# Chapter 1

# Prerequisites

The study of eigenvalues rooted in the literature of mathematics in the early 18th century while solving the differential equations. A brief survey about application of eigenvalues in different areas of science is given in the first chapter of the book [47]. In the beginning of 20th century, Hilbert concentrated on eigenvalues of integral operators by viewing the operators as infinite matrices which are included in more general notion called spectral values. Eigenvalues are also called characteristic values. In this report we use the notion spectral value. Set of all spectral values of an operator is called as the spectrum of the given operator.

The study of spectrum can be done more generally in a Banach algebra setting. In our context we give priority to discuss some important topological properties of a variant of the spectrum. Some of the generalized notions of spectrum are Exponential spectrum, Pseudo Spectrum and Condition spectrum. Detailed information for Exponential spectrum can be found in [38]. Pseudo spectra of a Banach algebra element is the union of all limited perturbed spectral values of the given element. The book [47] written by Trefethen and Embree consists exhaustive material about the pesudo spectra of matrices and operators defined on a Banach space. Article [28] discusses the pseudo spectra of an element in a Banach algebra. The concept of condition spectrum is introduced in [29]. The goal of this thesis is to find necessary and sufficient condition for the condition spectrum map to be continuous, under appropriate domain space and range space. Final part of this thesis devoted to $(p, q)$ outer generalized pseudo spectrum of a Banach algebra element defined with respect to the idempotents $p$ and $q$. The $(p, q)$ outer generalized inverse of a Banach algebra element is the generalization of the weighted outer generalized inverses of an operator defined on a Banach space (see [17] and [12]). Definition of $(p, q)$ outer generalized pseudo spectrum first appeared in [25]. This portion has the study of interior of level sets of $(p, q)$ outer generalized pseudo spectrum and its applications.

The main aim of this chapter is to give a brief and rapid review of some basic definitions, notions and related results which are necessary for the rest of development of the thesis.

## 1.1 Basic definitions and notions

We begin our study with the definition of an Algebra

**Definition 1.1.1** ([7], 14 in Lecture 3)**.** *Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. We say $\mathcal{A}$ is an algebra over $\mathbb{C}$ if there is a rule for multiplication on $\mathcal{A}$ that satisfies the four conditions*

1. $a(bc) = (ab)c$,
2. $a(b + c) = ab + ac$,
3. $(a + b)c = ac + bc$,
4. $\lambda(ab) = (\lambda a)b = a(\lambda b)$,

*for all $a, b, c$ in $\mathcal{A}$ and for all scalars $\lambda$.*

Our, next definition is about Banach algebras. Most of our results are derived for an element in a complex unital Banach Algebra. The concept of Banach Algebra originated in the early twentieth century. Gelfand noticed the central role of maximal ideals and using them he constructed the modern theory of Banach algebras (see [41]). The corresponding results were appeared in [20]. The book by Naimark with title "Normed Rings" [36] is the one which carried the Banach Algebra theory much widen further.

**Definition 1.1.2** ([7], 14 in Lecture 3)**.** *Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A norm $\|.\|$ on $\mathcal{A}$ satisfying*

$$\|ab\| \leq \|a\|\|b\|, \text{ for all } a, b \in \mathcal{A},$$

*is called an algebra norm and $(\mathcal{A}, \|.\|)$ is called a normed algebra. A complete normed algebra is called a complex Banach algebra. If there is an element $e \in \mathcal{A}$ such that*

$$ea = ae = a \text{ for all } a \in \mathcal{A}$$

*and* $\|e\| = 1$ *then* $\mathcal{A}$ *is called a complex unital Banach algebra.* $\mathcal{A}$ *is called commutative Banach algebra if*

$$ab = ba \text{ for all } a, b \in \mathcal{A}.$$

Since most of our results in this thesis are trivial for the elements which are scalar multiple of unity, we denote all non scalar multiple element by $\mathcal{A} \setminus \mathbb{C}e$.

The examples furnished below helps us to have a clear picture about the various kinds of Banach algebras. It is to be noted that these particular examples have a crucial role in the theory of Banach and $C^*-$ algebras.

**Example 1.1.3.**

1. *The complex field* $\mathbb{C}$ *with* $\|z\| = |z|$ *is a Complex unital Banach Algebra.*

2. *Let* $K$ *be a compact Hausdroff space and* $C(K)$ *be the set of all complex valued continuous functions on* $K$*. If* $f, g \in C(K)$ *then addition and multiplication are defined as*

$$(f+g)(x) = f(x) + g(x),\ (fg)(x) = f(x)g(x)$$

*for all* $x \in K$*. If* $f \in C(K)$ *then*

$$\|f\|_\infty = \sup_{x \in K} |f(x)|\,.$$

$C(K)$ *is a commutative complex unital Banach algebra with respect to the above norm. The map*

$$e : K \to \mathbb{C} \text{ defined by } e(x) = 1\ (x \in K)$$

*is the unit element in* $C(K)$*.*

3. *Let* $X$ *be a complex Banach space and* $B(X)$ *be the set of all bounded linear operators defined on* $X$*. If* $T_1, T_2 \in B(X)$ *then we define the addition as,*

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)\ (x \in X),$$

*multiplication is defined as,*

$$(T_1 \bullet T_2)(x) = T_1(T_2(x))\ \ (x \in X).$$

*and the operator norm of* $T \in B(X)$ *is defined as*

$$\|T\| = \sup_{x \in X, \|x\| = 1} \|T(x)\|.$$

*With the above defined norm $B(X)$ is a non commutative complex Banach algebra with unit $e \in B(X)$ where $e(x) = x$ for all $x \in X$.*

4. *Let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices. $M_n(\mathbb{C})$ forms a Banach algebra with usual matrix addition, matrix multiplication and with respect the one of following norms*

$$\|A\|_\infty = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}, \|A\|_2 = s_{max}(A) \tag{1.1}$$

*where $A \in M_n(\mathbb{C}), a_{ij} \in \mathbb{C}$ be the $(i, j)$th entry element in $A$ and $s_{max}(A)$ is the maximum singular values of $A$. Singular value of $A$ is nothing but the positive square root of the spectral values (see Definition 1.1.4) of the matrix $A^*A$ where $A^*$ denotes the conjugate transpose of $A$.*

5. *$G$ be a group and let $l^1(G)$ denote the set of mapping $f$ of $G$ into $\mathbb{C}$ such that $\sum_{s \in G} |f(s)| < \infty$. With pointwise addition and scalar multiplication, with convolution*

$$(f * g)(s) = \sum_{t \in G} f(t) g\left(t^{-1}s\right) \ \ (s \in G)$$

*as product and with the norm $\|f\| = \sum_{s \in G} |f(s)|$. $l^1(G)$ is a Banach algebra called the discrete group algebra of $G$.*

In this thesis, we explore some topological properties of some generalized spectrum. Since, we will be using the notion of spectrum and relevant elementary facts regularly, we turn our attention to the theory of spectrum.

The nature of an element in a Banach algebra can be elegantly studied by spectrum. The word spectrum was first coined by David Hilbert (see historical notes on page number 191 in [38]). The spectrum of an element is a proper subset of complex numbers which is a kind of shadow of the element. In particular, the study of spectrum facilitates a way to understand the phenomenon of invertiblity.

**Definition 1.1.4** ([14], Definition 3.1)**.** *Let $\mathcal{A}$ be a complex unital Banach Algebra with identity and $a \in \mathcal{A}$. The spectrum of $a$ is defined as*

$$\sigma(a) := \{\lambda \in \mathbb{C} : (a - \lambda) \text{ is not invertible in } \mathcal{A}\}.$$

*The complement of $\sigma(a)$ is called the resolvent set of $a$. We denote it by $\rho(a)$*

In the above definition $\lambda$ denotes $\lambda \cdot e$ where $e$ is the unit element in $\mathcal{A}$.

Definition 1.1.5 deals with spectral radius. Spectral radius is nothing but the maximum of absolute value of spectrum elements and hence this number is the radius of the smallest closed ball centered at zero which contains all spectral elements. We will be using some fundamental facts of spectral radius in certain parts of our later work.

**Definition 1.1.5** ([14], Definition 3.7)**.** *Let $\mathcal{A}$ be a complex unital Banach Algebra with identity and $a \in \mathcal{A}$, the spectral radius of $a$, $r(a)$ is defined by*

$$r(a) := \sup \{|\lambda| : \lambda \in \sigma(a)\} .$$

Theorem 1.1.6 explains some significant relationship between the spectral radius and the norm of an element.

**Theorem 1.1.6** ([14], Theorem 3.6, Proposition 3.8)**.** *Let $\mathcal{A}$ be a complex Banach Algebra with an identity, then for each $a \in \mathcal{A}$*

1. *$\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.*

2. *(Spectral mapping theorem) $\sigma(p(a)) = p(\sigma(a)) = \{p(\lambda) : \lambda \in \sigma(a)\}$ where $p(x)$ is a complex polynomial.*

3. *$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$.*

4. *$r(a) \leq \|a\|$.*

In future, we observe some of the interesting results for an element in $B(H)$. It has additional $C^*$ algebra structure. At first we study for $B(H)$ ( where $H$ is a Hilbert space) and later using GNS construction we extend these to an element in $C^*-$ algebras.

**Definition 1.1.7** ([49], Definition 8.1)**.** *A Banach algebra $\mathcal{A}$ together with a mapping (involution) $x \mapsto x^*$ on $\mathcal{A}$ satisfying the following conditions is called a $C^*$ algebra.*

- *$(x^*)^* = x$ for all $x \in \mathcal{A}$.*

- *$(ax + by)^* = \overline{a}x^* + \overline{b}y^*$ for all $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$.*

- *$(xy)^* = y^*x^*$ for all $x, y \in \mathcal{A}$.*

- *$\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$ ($C^*$ algebra condition).*

*$C^*$ algebra with unity $e$ is called unital $C^*$ algebra with $e^* = e$.*

It is already pointed out that the need of GNS construction in our work. GNS construction is completely relying on $C^*-$ homomorphisms. In this regard, we now have a look into $C^*-$ homomorphisms.

**Definition 1.1.8** ([49], Definition 9.1). *Let $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*-$ algebras. A mapping $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a $C^*-$homomorphism if $x, y \in \mathcal{A}$ and $a, b \in \mathbb{C}$ then*

1. *$\phi(ax + by) = a\phi(x) + b\phi(y)$ ( linear ).*
2. *$\phi(xy) = \phi(x)\phi(y)$ ( multiplicative ).*
3. *$\phi(x^*) = \phi(x)^*$.*
4. *$\phi$ maps the unit of $\mathcal{A}$ into the unit of $\mathcal{B}$.*

*If $\phi$ is one to one, we say that $\phi$ is a $C^*-$isomorphism.*

**Theorem 1.1.9** ([49], Theorem 14.4 (GNS)). *For every $C^*-$ algebra $\mathcal{A}$ there exists a Hilbert space $H$ such that $\mathcal{A}$ is $C^*-$ isomorphic to a $C^*-$subalgebra of $B(H)$.*

We end this section by stating of the spectrum of an element $a$ in $C^*-$ algebra $\mathcal{A}$ when the $\mathcal{A}$ is enlarged in its size. The result says the invertiblty of $a$ in $C^*-$ algebra $\mathcal{A}$ is same in the invertiblty of $a$ in the enlarged $C^*-$ algebra.

**Theorem 1.1.10** ([49], Corollary 9.11). *Let $\mathcal{B}$ be a unital $C^*-$ algebra with unit element $e$. If $\mathcal{A}$ is a $C^*-$ subalgebra of $\mathcal{B}$ which contians $e$ and $a \in \mathcal{B}$, then $\sigma(a, \mathcal{A}) = \sigma(a, \mathcal{B})$, where*

$$\sigma(a, \mathcal{A}) = \{\lambda \in \mathbb{C} : (a - \lambda) \text{ is not invertible in } \mathcal{A}\}$$

*and*

$$\sigma(a, \mathcal{B}) = \{\lambda \in \mathbb{C} : (a - \lambda) \text{ is not invertible in } \mathcal{B}\}.$$

## 1.2 Two genralization of spectrum

This section is devoted to some generalized spectrum concepts. It is known that spectrum of an element assists us to know the nature of the corresponding element. To understand spectrum of an element we need some better tool. One way is the concept of generalization of spectrum.

There are many generalizations available for the spectrum in the literature. These generalizations enhances the study of spectrum in many ways. Among the many generalized spectrum, we focus about pseudo spectrum and condition spectrum because they give some better bounds to the spectrum.

### 1.2.1 Pseudo spectrum

Let $H$ be a finite dimensional Hilbert space. If $T$ is normal (Definition 2.11 in [14]) in $B(H)$ then $T$ is unitarily diagonalizable. This is not the case with non normal operators. The study of pseudo spectrum is originated to understand the non normal operators and its spectral values. Pseudo spectra has rich application in many field of science and engineering. Our intention about the discussion for pseudo spectrum and related results is just to give the analogy between existed results in pseudo spectra and the results we prove in this thesis.

It is advisable to go through the book [47] to have a deep understanding about pseudo spectrum. At first the research article [28] intensely analyzed the pseudo spectra for a Banach algebra element.

**Definition 1.2.1** ([28], Definition 2.1 ). *For $a \in \mathcal{A}$ and $\epsilon > 0$, the $\epsilon-$ pseudo spectrum is defined as*

$$\Lambda_\epsilon(a) = \left\{ \lambda \in \mathbb{C} : \left\| (a-\lambda)^{-1} \right\| \geq \frac{1}{\epsilon} \right\}.$$

*with the convention that $\|(a-\lambda)^{-1}\| = \infty$ when $(a-\lambda)$ is not invertible.*

**Theorem 1.2.2** ([28], Theorem 2.3). *Let $a, b \in \mathcal{A}$ and $\epsilon > 0$. Then*

1. $\sigma(a) = \cap_{\epsilon > 0} \Lambda_\epsilon(a)$.
2. *If* $0 < \epsilon_1 < \epsilon_2$ *then* $\Lambda_{\epsilon_1}(a) \subseteq \Lambda_{\epsilon_2}(a)$.
3. $\Lambda_\epsilon(a+\lambda) = \lambda + \Lambda_\epsilon(a)\ (\lambda \in \mathbb{C})$.
4. $\Lambda_\epsilon(\lambda a) = \lambda \Lambda_{\frac{\epsilon}{|\lambda|}}(a)\ (\lambda \in \mathbb{C} \setminus \{0\})$.
5. $\Lambda_\epsilon(a)$ *is a nonempty compact subset of* $\mathbb{C}$.
6. $\Lambda_\epsilon(a+b) \subseteq \Lambda_{\epsilon + \|b\|}(a)$.

### 1.2.2 Condition spectrum

Consider the system of linear equation $Ax = b$ where $A \in B(\mathbb{C}^n)$, $x$ and $b \in \mathbb{C}^n$. Here $\mathbb{C}^n$ denotes the $n-$dimensional Euclidean space. The condition number of $A$ measures the amount of change in the output with respect to the change in the input. Similar to this, there is an associated and more general approach available to know the sensitivity of solution of the equations $(A-\lambda)x=b$ where $\lambda\in\mathbb{C}$. The qualitative and quantitative behaviour of $\|a-\lambda\|\|(a-\lambda)^{-1}\|$ for a Banach algebra element $a\in\mathcal{A}$ and $\lambda\in\mathbb{C}$ is answered by the study of condition spectrum. That is the scheme of this thesis.

Condition spectrum was rooted from Ransford spectrum (See [40]). By appropriately setting a Ransford open set the condition spectrum is formed in [29]. Condition spectrum is very efficiently applicable in numerical analysis of matrix equations both finite and infinite.

There are many interesting characteristics about condition spectrum, that were realized by various researchers. In [31] the authors tried to find out the condition spectra of some block Toeplitz operators with continuous symbols. The articles [30], [6] and [39] deals with preserver problem in terms of the condition spectrum. The spectral mapping theorem for condition spectrum appeared in [26]. One can see the component wise spectral mapping theorem and approximation of condition spectrum in [27]. Article [15], compared the properties of pseudo spectrum and condition spectrum for matrices.

**Definition 1.2.3.** *([29], Definition 2.5) Let $0<\epsilon<1$. The $\epsilon$-condition spectrum of $a\in\mathcal{A}$ is defined as*

$$\sigma_\epsilon(a)=\left\{\lambda\in\mathbb{C}:\|(a-\lambda)\|\left\|(a-\lambda)^{-1}\right\|\geq\frac{1}{\epsilon}\right\},$$

*with the convention that $\|(a-\lambda)\|\,\|(a-\lambda)^{-1}\|=\infty$ if $(a-\lambda)$ is not invertible.*

In chapter 2, we check the possible Banach algebras in which the interior of level sets of the condition spectrum of an element is empty. Consequently, we attempt to see the continuity of the condition spectrum. For establishing the results in these chapters, here we list out some fundamental facts of condition spectrum.

**Theorem 1.2.4** ([29], Theorem 2.7, 2.9, 3.1, Corollary 3.2, 3.4 and 3.5)**.** *Let $\epsilon\in(0,1)$. Then*

1. *$\sigma_\epsilon(0)=\{0\}$ and $\sigma_\epsilon(e)=\{1\}$.*
2. *If $0<\epsilon_1<\epsilon_2<1$ then $\sigma_{\epsilon_1}(a)\subseteq\sigma_{\epsilon_2}(a)$.*
3. *$\sigma(a)\subseteq\sigma_\epsilon(a)$ for every $a\in\mathcal{A}$. In fact $\sigma(a)=\bigcap\limits_{0<\epsilon<1}\sigma_\epsilon(a)$.*
4. *$\sigma_\epsilon(a)$ is a nonempty compact subset of $\mathbb{C}$.*

5. $\sigma_\epsilon(\alpha + \beta a) = \alpha + \beta\sigma_\epsilon(a)$ *for all* $\alpha, \beta \in \mathbb{C}$.

6. *For any* $a \in \mathcal{A} \setminus \mathbb{C}e$ *the condition spectrum* $\sigma_\epsilon(a)$ *has no isolated points.*

7. *Let* $\lambda \in \mathbb{C}$. $\sigma_\epsilon(a) = \{\lambda\}$ *if and only if* $a = \lambda$.

8. *Let* $a \in \mathcal{A} \setminus \mathbb{C}e$. *If* $\lambda \in \mathbb{C}$ *then there exists* $r > 0$ *such that*

$$B(\lambda, r) := \{\mu \in \mathbb{C} : |\mu - \lambda| < r\} \subsetneq \sigma_\epsilon(a).$$

9. $\sup\{|\lambda| : \lambda \in \sigma_\epsilon(a)\} \leq \frac{1+\epsilon}{1-\epsilon}\|a\|$

## 1.3 Maximum modulus theorem for analytic vector valued maps

For given $\epsilon \in (0,1)$ and $a \in \mathcal{A}$, the inequalities involved in Definition 1.2.3 indicate us that the set of the form

$$\left\{\lambda \in \mathbb{C} : \|(a-\lambda)\|\|(a-\lambda)^{-1}\| = \frac{1}{\epsilon}\right\} \tag{1.2}$$

will aid us to figure out the condition spectrum. Specifically, the empty interior of the set (1.2) assures about detecting boundary of the condition spectrum easily. It is surprising that, the question "Does the set (1.2) has interior empty?" can be settled down by proving a form of maximum modulus theorem for product of analytic vector valued functions. Because of this, we allocate this section for classical maximum modulus theorem (MMT) and its considerable generalization to the vector valued mappings.

**Definition 1.3.1** ([42], Theorem 10.24 (MMT))**.** *Let* $\Omega$ *be an open connected subsets of* $\mathbb{C}$. *If* $f : \Omega \to \mathbb{C}$ *is analytic, then* $|f|$ *has no local maximum at any point in* $\Omega$ *unless* $f$ *is constant in* $\Omega$.

**Definition 1.3.2.** *(see [33], Definition 3.3) Let* $\Omega$ *be an open connected subsets of* $\mathbb{C}$. *A function* $f : \Omega \to \mathcal{A}$ *is said to be differentiable at the point* $\mu \in \Omega$ *if there exists an element* $f'(\mu) \in \mathcal{A}$ *such that*

$$\lim_{\lambda\to\mu}\left\|\frac{f(\lambda) - f(\mu)}{\lambda - \mu} - f'(\mu)\right\| = 0.$$

*If* $f$ *is differentiable at every point in* $\Omega$ *then* $f$ *is said to be analytic in* $\Omega$.

Roughly speaking, MMT does not hold for general Banach algebra set up. We construct a simple example to demonstrate the failure of MMT.

**Example 1.3.3.** *Consider the Banach algebra $\mathbb{M}_4(\mathbb{C})$ with norm $\|.\|_\infty$ and fix $\alpha \in \mathbb{C}$ Define*

$$\psi : \mathbb{C} \to \mathbb{M}_4(\mathbb{C}) \text{ by } \psi(\lambda) = \alpha \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*For any $\mu \in \mathbb{C}$, we prove that the derivative of $\psi$ at $\mu$ is $\alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.*

*Now,* $\lim\limits_{\lambda \to \mu} \left\| \dfrac{\psi(\lambda) - \psi(\mu)}{\lambda - \mu} - \alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\|_\infty$

$$\begin{aligned}
&= \lim_{\lambda \to \mu} \left\| \frac{\alpha \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \alpha \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}{\lambda - \mu} - \alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\|_\infty \\
&= \lim_{\lambda \to \mu} \left\| \frac{\begin{pmatrix} \alpha(\lambda - \mu) & 0 & 0 & 0 \\ 0 & \alpha(\lambda - \mu) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}{\lambda - \mu} - \alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\|_\infty \\
&= 0.
\end{aligned}$$

*Thus $\psi$ is analytic in $\mathbb{C}$. Moreover*

$$\|\psi(\lambda)\|_\infty = \begin{cases} |\alpha| & \text{if } |\lambda| \leq 1 \\ |\alpha\lambda| & \text{if } |\lambda| > 1. \end{cases}$$

*$\psi$ attains local maximum at $0$ but still $\psi$ is not constant.*

Thorp and Whitely in [46] realized the strong form of MMT for Banach space valued analytic functions by keeping some additional assumptions on the underlying Banach space. Though many complex Banach spaces admits the strong version of MMT, it does not fit for some special Banach spaces like $C^*-$algebras (see [24]). Theory of norm constant operator valued analytic function defined on Hilbert spaces and more general Banach spaces lie in [24] and [22].

The map $R : \rho(a) \to \mathcal{A}$ defined by $R(\lambda) = (a-\lambda)^{-1}$ is called resolvent map. The resolvent map is an analytic Banach algebra valued map (see Theorem 3.6 in [14]).

Globevnik in [23] investigated the MMT for the resolvent map. His question was "At which open set the the resolvent map does not attain local maximum?". He sensed the norm of the resolvent map is not constant in any open subset of the unbounded component of the resolvent. The same follows for any open subset of $\rho(T)$ of any element $T$ in Banach algebra $B(X)$ where the underlying space is complex uniformly convex (see definition(2.3.5)). The question about the state of norm of resolvent map for any element $a \in \mathcal{A}$ in a bounded component of $\rho(a)$ was open for long time. One can find, the history of this particular problem and some more answers related to this problem in [9], [10] and [11]. In [43] (Theorem 3.1), Shargorodsky proved, there exists an invertible bounded operator $T$ acting on the Banach space

$$\ell_\infty(\mathbb{Z}) := \left\{ x = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) \middle| \sup_{-\infty \leq i \leq \infty} |x_i| < \infty \text{ and } x_i \in \mathbb{C} \right\}$$

with norm $\|x\|_* = \sup\limits_{k\neq 0} |x_k| + |x_0|$ such that $\|(T-\lambda)^{-1}\|$ is constant in a neighborhood of $\lambda = 0$. This says that MMT fails for a particular resolvent map. Refer [16], for the results related to the level sets of resolvent norm of the linear operators.

We wind up this subsection by stating a version of maximum modulus theorem for analytic vector valued maps. This theorem is very useful in chapter 4

**Theorem 1.3.4.** *Let $\Omega_0$ be a connected open subset of $\mathbb{C}$ and $X$ be a complex Banach space. Suppose $F : \Omega_0 \to X$ is an analytic vector valued function, $\|F(\lambda)\| \leq M$ for all $\lambda$ in an* *open subset of $\Omega \subseteq \Omega_0$ and $\|F(\mu)\| < M$ for some $\mu \in \Omega_0$. Then $\|F(\lambda)\| < M$ for all $\lambda \in \Omega$.*

## 1.4 Set valued maps and continuity

### 1.4.1 Introduction

Chapter 3 of this thesis provides the study of the continuity of condition spectrum map and its level set. The corresponding maps that we study, fits into the frame work of correspondence between two topological spaces.

**Definition 1.4.1** (Definition 17.1 in [1]). *A correspondence $\phi$ from a set $X$ to a set $Y$ assigns to each $x$ in $X$ a subset $\phi(x)$ of $Y$. We denote a correspondence $\phi$ from $X$ to $Y$ by $\phi : X \twoheadrightarrow Y$.*

### 1.4.2 Upper and lower hemicontinuous maps

The continuity concept of a correspondence is not directly extendable from the continuity concept of a map between two topological spaces. There are specific approaches like hemicontinuity and demicontinuity of a correspondence are available to get the continuity. In our case, the upper and lower hemicontinuity paves a way to get the continuity of the appropriate correspondences of condition spectrum. Thus the prime objective of this subsection is to provide definitions and results which are related to upper and lower hemicontinuity of a correspondence. For the comprehensive details about various topological properties of the correspondence, we prefer the book [1].

We start with the definition of a neighborhood of a subset of a topological space.

**Definition 1.4.2** (page 558 in [1]). *Let $X$ be a topological space. A neighborhood of a subset $A$ of $X$ is any subset $B$ for which there is an open subset $V$ satisfying $A \subseteq V \subseteq B$.*

**Definition 1.4.3** (Definition 17.2 in [1]). *A correspondence $\phi : X \twoheadrightarrow Y$ between topological spaces is :*

1. *upper hemicontinuous at the point $x \in X$ if for every neighborhood $U$ of $\phi(x)$, there is a neighborhood $V$ of $x$ such that $z \in V$ implies $\phi(z) \subseteq U$*

2. *lower hemicontinuous at $x \in X$ if for every open set $U$ with $U \cap \phi(x) \neq \emptyset$, there is a neighborhood $V$ of $x$ such that $z \in V$ implies $\phi(z) \cap U \neq \emptyset$.*

3. *continuous at $x \in X$ if it is both upper and lower hemicontinuous at $x$.*

The following correspondences illustrate the definitions.

**Example 1.4.4.** 1. *Define $\phi : [0,2] \twoheadrightarrow [0,3]$ by $\phi(x) = \begin{cases} \{0\} \text{ if } x < 2 \\ [0,2] \text{ if } x = 2. \end{cases}$ The map $\phi$ is upper hemicontinuous for all $x \in [0,2]$. Consider the open set $V = \left(\frac{3}{2}, \frac{5}{2}\right)$ in $[0,3]$ such that $\phi(2) \cap V \neq \emptyset$ but for any open set $U$ around 2 in $[0,2]$, $\phi(x) \cap V = \emptyset$ for all $x \in U \setminus \{2\}$. Thus $\phi$ is not lower hemicontinuous at $x = 2$*

2. *Define $\psi : [0,2] \twoheadrightarrow [0,3]$ by $\psi(x) = \begin{cases} [0,2] \text{ if } x < 2 \\ \{0\} \text{ if } x = 2. \end{cases}$ The map $\phi$ is lower hemicontinuous for all $x \in [0,2]$. Consider the open set $V = \left[0, \frac{3}{2}\right)$ in $[0,3]$ such that $\psi(2) \subset V \neq \emptyset$ but for any open set $U$ around 2 in $[0,2]$, $\psi(x) \not\subset V$ for all $x \in U \backslash \{2\}$. Thus $\phi$ is not upper hemicontinuous at $x = 2$*

3. *Define $\chi : [0,2] \twoheadrightarrow [0,3]$ by $\chi(x) = [0,x]$. Then $\chi$ is continuous.*

The notions defined above can also be characterized using the graph of a correspondence. These equivalent characterization will have high impact on many of our results. The following is the definition of the closed graph of a correspondence.

**Definition 1.4.5** (Definition 17.9 in [1])**.** *A correspondence $\phi : X \twoheadrightarrow Y$ between two topological space is closed or has closed graph, if its graph*

$$Gr\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$$

*is a closed subset of $X \times Y$.*

**Theorem 1.4.6** (Theorem 17.20 in [1])**.** *Assume that the topological space $X$ is first countable and that $Y$ is metrizable. Then for a correspondence $\phi : X \twoheadrightarrow Y$ and a point $x \in X$ the following statements are equivalent.*

1. *The correspondence $\phi$ is upper hemicontinuous at $x$ and $\phi(x)$ is compact.*

2. *If a sequence $\{(x_n, y_n)\}$ in the graph of $\phi$ satisfies $x_n \to x$, then the sequence $\{y_n\}$ has a limit point in $\phi(x)$.*

**Theorem 1.4.7** (Theorem 17.21 in [1])**.** *Assume that the topological space $X$ is first countable and that $Y$ is metrizable. Then for a correspondence $\phi : X \twoheadrightarrow Y$ and a point $x \in X$ the following statements are equivalent.*

1. *The correspondence $\phi$ is lower hemicontinuous at $x$.*

2. *If a sequence $x_n \to x$ then for each $y \in \phi(x)$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and elements $y_k \in \phi(x_{n_k})$ for each $k$ such that $y_k \to y$.*

In the later portion an individual attention will be given to the continuity of the level set of condition spectrum map. This will be achieved by subcorrespondence notion and its properties. Theorem 1.4.9, shows that with some further assumptions, the upper hemicontinuity of the correspondence impose the same to the subcorrespondence

**Definition 1.4.8** (page 564 in [1])**.** *Let $\phi, \psi : X \twoheadrightarrow Y$ correspondences between the topological spaces $X$ and $Y$. If $\psi(x) \subseteq \phi(x)$ for each $x \in X$, then we say that $\psi$ is a subcorrespondence of $\phi$.*

**Theorem 1.4.9** (Corollary 17.18 in [1])**.** *Let $\phi, \psi : X \twoheadrightarrow Y$ correspondences between the topological spaces $X$ and $Y$ such that $\phi$ is compact valued and $\psi$ is a closed subcorrespondence of $\phi$. If $\phi$ is upper hemicontinuous at $x \in X$ then $\psi$ is also upper hemicontinuous at $x$.*

**Definition 1.4.10.** *Let $K(\mathbb{C})$ denotes the set of all compact subsets of $\mathbb{C}$. If $E, G \in K(\mathbb{C})$ then Hausdorff distance between $E$ and $G$ is defined as*

$$H(E,G) = \max\left\{\sup_{s\in E} d(s,G), \sup_{t\in G} d(t,E)\right\}. \tag{1.3}$$

We can view the condition spectrum maps $\epsilon \mapsto \sigma_\epsilon(a)$ and $a \mapsto \sigma_\epsilon(a)$ as the map between $(0,1)$ to $K(\mathbb{C})$ and between $\mathcal{A}$ to $K(\mathbb{C})$. So, the continuity of these maps can be discussed with this metric. The next theorem asserts that the continuity of a compact valued correspondence between topological spaces $X$ and $Y$ is equivalent to the continuity of a set valued map from $X$ to $K(Y)$ where $K(Y)$ denotes the space of nonempty compact subsets of $Y$ endowed with its Hausdorff metric topology (Hausdroff metric in $K(Y)$ be defined in the same way as in Equation (1.3)).

**Theorem 1.4.11** (Theorem 17.15 in [1])**.** *Let $\phi : X \twoheadrightarrow Y$ between topological space be a nonempty compact-valued correspondence from a topological space into a metrizable space. Then the function $f : X \to K(Y)$ defined by $f(x) = \phi(x)$ is continuous at $a \in X$ if and only if the correspondence $\phi$ is continuous $a \in X$.*

In some portion of our work we concentrate on the limiting nature of the $\epsilon$−condition spectrum sets as $\epsilon \to 1$. The following is the definition of limit superior, limit inferior and limit concepts involved in sequence of sets.

**Definition 1.4.12** (Definition 1.1.1 in [3]). *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of a metric space $X$. We say that the subset*

$$\limsup_{n\to\infty} K_n := \left\{x \in X : \liminf_{n\to\infty} d(x, K_n) = 0\right\}$$

*is the upper limit of the sequence $K_n$ and that the subset*

$$\liminf_{n\to\infty} K_n := \left\{x \in X : \lim_{n\to\infty} d(x, K_n) = 0\right\}$$

*is its lower limit. A subset $K$ is said to be the limit or the set limit of the sequence $K_n$ if*

$$K = \limsup_{n\to\infty} K_n = \liminf_{n\to\infty} K_n =: \lim_{n\to\infty} K_n.$$

**Note 1.4.13.** *[page 18 in [3]] It is clear from the definition that*

$$\liminf_{n\to\infty} K_n \subseteq \limsup_{n\to\infty} K_n. \tag{1.4}$$

**Note 1.4.14.** *[page 18 in [3]] If the sequence $\{K_n\}$ is decreasing, then $\lim_{n\to\infty} K_n$ exists and*

$$\lim_{n\to\infty} K_n = \bigcap_{n\geq 0} \overline{K_n}. \tag{1.5}$$

*where $\overline{K_n}$ denotes the closure of $K_n$.*

The following example illustrates the above definitions.

**Example 1.4.15.** *Consider the topological space $\mathbb{R} \times \mathbb{R}$ (take usual metric on $\mathbb{R}$) and the sequence of sets $K_n = \begin{cases} \left\{\frac{1}{n}\right\} \times [0,1] \text{ if } n \text{ is even} \\ \left\{\frac{1}{n}\right\} \times [-1,0] \text{ if } n \text{ is odd.} \end{cases}$.*
*It follows that $\liminf_{n\to\infty} K_n = \{0\} \times \{0\}$ and $\limsup_{n\to\infty} K_n = \{0\} \times [-1,+1]$.*

# Chapter 2

# Level sets of condition spectrum

Our objective in this chapter is to discuss the interior of the level of set of the condition spectrum. This chapter contains three sections. The material in the first section centers around few topological properties of level sets of the condition spectrum. The subject of the second section is 1-level set of the condition spectrum. Examples are given to illustrate the topological property of 1-level set and its variance from $\epsilon-$ level set. Mainly, we observe that 1-level set has empty interior for any Banach algebra element. In the last section we study the interior property of $\epsilon-$ level set of condition spectrum.

## 2.1 Preliminaries

In this section, for $\epsilon \in (0,1]$, we introduce the terminology $\epsilon-$ level set of condition spectrum and consequently we derive some fundamental facts of the same by applying the ideas of condition spectrum. We start by recollecting the definition of $\epsilon-$ condition spectrum.

**Definition 2.1.1.** *([29], Definition 2.5) Let $0 < \epsilon < 1$. The $\epsilon$-condition spectrum of $a \in \mathcal{A}$ is defined as*

$$\sigma_\epsilon(a) := \left\{ \lambda \in \mathbb{C} : \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| \geq \frac{1}{\epsilon} \right\},$$

*with the convention that $\|(a-\lambda)\| \, \|(a-\lambda)^{-1}\| = \infty$ if $(a-\lambda)$ is not invertible.*

The definition of the level sets of condition spectrum is the following:

**Definition 2.1.2.** *Let $0 < \epsilon \leq 1$. The $\epsilon-$ level set of condition spectrum of $a \in \mathcal{A}$ is* *defined as*

$$L_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(a-\lambda)\|\|(a-\lambda)^{-1}\| = \frac{1}{\epsilon}\right\}$$

**Note 2.1.3.** *For $0 < \epsilon < 1, L_\epsilon(a) \subset \sigma_\epsilon(a)$ for any $a \in \mathcal{A}$. If $a = \mu$ for some $\mu \in \mathbb{C}$ then $L_\epsilon(a) = \emptyset$ and so the interior of $L_\epsilon(a) = \emptyset$.*

**Proposition 2.1.4.** *Let $a \in \mathcal{A}$ and $\mu \in \mathbb{C}$. $L_1(a) = \mathbb{C} \setminus \{\mu\}$ if and only if $a = \mu$. In particular if $a = \mu$ then the interior of $L_1(a)$ is non empty.*

*Proof.* If $a = \mu$ then by direct calculations of the $1-$level set, we see that $L_1(a) = \mathbb{C} \setminus \{\mu\}$. Conversely, assume that $L_1(a) = \mathbb{C} \setminus \{\mu\}$ then $\sigma_\epsilon(a) = \{\mu\}$ for every $0 < \epsilon < 1$. Since, $\sigma(a) \neq \emptyset$ and $\sigma(a) \subseteq \sigma_\epsilon(a)$ for every $\epsilon \in (0, 1)$, we have $\sigma(a) = \sigma_\epsilon(a) = \{\mu\}$. By Theorem 1.2.4 (7), $a = \mu$. It is now obvious that if $a = \mu$ then interior of $L_1(a)$ is non empty. $\square$

Consider the Banach algebra $\mathbb{M}_2(\mathbb{C})$ with norm $\|.\|_2$. Explicit form for $\epsilon-$ level set of condition spectrum of an upper triangular $2 \times 2$ matrix is calculated in Proposition 2.1.5.

**Proposition 2.1.5.** *Let $0 < \epsilon < 1$ and $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$. Then*

$$L_\epsilon(A) = \left\{ \mu \in \mathbb{C} : \frac{\left(\sqrt{(|\mu-a| + |\mu-c|)^2 + |b|^2} + \sqrt{(|\mu-a| - |\mu-c|)^2 + |b|^2}\right)^2}{4|\mu-a||\mu-c|} = \frac{1}{\epsilon} \right\} \tag{2.1}$$

*Proof.* Let $\mu \in \mathbb{C}$. The matrix

$$\begin{aligned}(A-\mu)^*(A-\mu) &= \begin{pmatrix} \overline{a-\mu} & 0 \\ \overline{b} & \overline{c-\mu} \end{pmatrix} \begin{pmatrix} a-\mu & b \\ 0 & c-\mu \end{pmatrix} \\ &= \begin{pmatrix} |a-\mu|^2 & b(\overline{a-\mu}) \\ \overline{b}(a-\mu) & |b|^2 + |c-\mu|^2 \end{pmatrix}\end{aligned}$$

For $A \in \mathbb{M}_2(\mathbb{C})$ with 2-norm, we have the following

$$\|A-\mu\| = s_{\max}(A-\mu) \text{ and } \left\|(A-\mu)^{-1}\right\| = \frac{1}{s_{\min}(A-\mu)}.$$

where $s_{\min}(A-\mu)$ denotes the minimum singular value of $A-\mu$. Hence

$$L_\epsilon(A)=\left\{\mu\in\mathbb{C}:\frac{s_{\max}(A-\mu)}{s_{\min}(A-\mu)}=\frac{1}{\epsilon}\right\}.$$

Since, $(A-\mu)^*(A-\mu)$ has only two spectral values and their product is determinant of $(A-\mu)^*(A-\mu)$, we have that

$$\begin{aligned}
s_{\max}(A-\mu)s_{\min}(A-\mu) &= \sqrt{\det(A-\mu)^*(A-\mu)}\\
&= \sqrt{\det\begin{pmatrix}|a-\mu|^2 & b(\overline{a-\mu})\\ \overline{b}(a-\mu) & |b|^2+|c-\mu|^2\end{pmatrix}}\\
&= \sqrt{|a-\mu|^2|c-\mu|^2}\\
&= |a-\mu||c-\mu|. \qquad (2.2)
\end{aligned}$$

Sum of the spectral values of $(A-\mu)^*(A-\mu)$ is the trace of $(A-\mu)^*(A-\mu)$. Thus

$$\begin{aligned}
[s_{\max}(A-\mu)]^2+[s_{\min}(A-\mu)]^2 &= \text{trace}[(A-\mu)^*(A-\mu)]\\
&= |\mu-a|^2+|\mu-c|^2+|b|^2 \qquad (2.3)
\end{aligned}$$

From Equation (2.2) and Equation (2.3), we get
$[s_{\max}(A-\mu)\pm s_{\min}(A-\mu)]^2$

$$\begin{aligned}
&= [s_{\max}(A-\mu)]^2+[s_{\min}(A-\mu)]^2\pm 2s_{\max}(A-\mu)s_{\min}(A-\mu)\\
&= |\mu-a|^2+|\mu-c|^2+|b|^2\pm 2|a-\mu||c-\mu|.\\
&= (|\mu-a|\pm|\mu-c|)^2+|b|^2.
\end{aligned}$$

After simplification, we see $L_\epsilon(a)$ is as given in Equation (2.1) □

**Note 2.1.6.** *Since any matrix $A\in\mathbb{M}_2(\mathbb{C})$ is unitarily similar to an upper triangular matrix, the corresponding level set is also of the form given in Proposition 2.1.5.*

In order to develop the theory further, one needs to know the non emptiness of $L_\epsilon(a)$ where $0<\epsilon<1$ and $a\in\mathcal{A}\setminus\mathbb{C}e$. This will be achieved in two steps, the first one is the non emptiness of boundary of $\sigma_\epsilon(a)$ and the other one is boundary is a subset of the level set.

**Theorem 2.1.7.** *Boundary of $\sigma_\epsilon(a)$ is nonempty for every $a\in\mathcal{A}\setminus\mathbb{C}e$.*

*Proof.* Suppose, boundary of $\sigma_\epsilon(a)$ is empty, then

$$\overline{\sigma_\epsilon(a)} \cap \overline{\sigma_\epsilon(a)^c} = \emptyset.$$

Since $\sigma_\epsilon(a)$ is compact,

$$\sigma_\epsilon(a) \cap \overline{\sigma_\epsilon(a)^c} = \emptyset.$$

We get

$$\sigma_\epsilon(a)^c = \overline{\sigma_\epsilon(a)^c}.$$

Hence $\sigma_\epsilon(a)^c$ is a nonempty closed set by the above equation and by Theorem 1.2.4 (4) it is also an open set. This is a contradiction to the fact that $\mathbb{C}$ is connected. □

**Theorem 2.1.8.** *Let $0 < \epsilon < 1$. If $a \in \mathcal{A} \setminus \mathbb{C}e$ then boundary of $\sigma_\epsilon(a)$ is a subset of $L_\epsilon(a)$ and hence $L_\epsilon(a)$ is nonempty.*

*Proof.* Let $\lambda_0$ be a point in the boundary of $\sigma_\epsilon(a)$ (existence of such a point is possible by Proposition (2.1.7)). then

$$\lambda_0 \in \overline{\sigma_\epsilon(a)} \cap \overline{\sigma_\epsilon(a)^c}.$$

By Theorem 1.2.4 (4), $\lambda_0 \in \sigma_\epsilon(a)$ and hence $\|(a-\lambda_0)\| \, \|(a-\lambda_0)^{-1}\| \geq \frac{1}{\epsilon}$. We prove that $\|(a-\lambda_0)\| \, \|(a-\lambda_0)^{-1}\| = \frac{1}{\epsilon}$. Suppose, $\|(a-\lambda_0)\| \, \|(a-\lambda_0)^{-1}\| > \frac{1}{\epsilon}$. Since $\lambda_0 \in \overline{\sigma_\epsilon(a)^c}$, there exists a sequence $\{\lambda_n\} \in \sigma_\epsilon(a)^c$ such that $\lambda_n \to \lambda_0$ as $n \to \infty$. Define

$$\phi : \overline{\sigma_\epsilon(a)^c} \to \mathbb{R} \text{ by } \phi(\lambda) = \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\|.$$

Clearly $\phi(\lambda_n) \nrightarrow \phi(\lambda_0)$ as $n \to \infty$. By Theorem 1.2.4 (8), no point in $\sigma(a)$ is a limit point for $\sigma_\epsilon(a)^c$ and so $(a-\lambda)^{-1}$ exists for each $\lambda \in \overline{\sigma_\epsilon(a)^c}$, thus $\phi$ is well defined. The function $\phi$ is also continuous because the functions $a \mapsto \|a\|, \lambda \mapsto (a-\lambda)$ and $\lambda \mapsto (a-\lambda)^{-1}$ are all continuous. This is a contradiction to $\phi(\lambda_n) \nrightarrow \phi(\lambda_0)$ as $n \to \infty$. □

Following example shows that every element of $L_\epsilon(a)$ need not come from boundary of $\sigma_\epsilon(a)$.

**Example 2.1.9.** *Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm*

$$\|x\|_* = |x_0| + \sup_{n \neq 0} |x_n|$$

*where* $x = \left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right)$*, the box represents the zero*$^{th}$ *coordinate of an element in* $\ell_\infty(\mathbb{Z})$*.*
*Take an operator* $A \in B\left(\ell_\infty(\mathbb{Z})\right)$ *such that*

$$A\left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right) = \left(\cdots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{5}}, x_2, x_3, \cdots\right).$$

*For* $\epsilon = \dfrac{1}{6}$*, we prove that the scalar 0 belongs to* $L_\epsilon(A)$ *but not in the boundary of* $\sigma_\epsilon(A)$*. By Theorem 3.1 in [43],*

$$\left\|(A-\lambda)^{-1}\right\| = 5 \text{ for } \lambda \in B\left(0, \frac{1}{5}\right). \tag{2.4}$$

*For any* $x \in \ell_\infty(\mathbb{Z})$*,*

$$\begin{aligned} \|Ax\|_* &= \left|\frac{x_1}{5}\right| + \sup_{n\neq 1}|x_n| \\ &\leq \frac{1}{5}\sup_{n\neq 0}|x_n| + \sup_{n\neq 1}|x_n| \\ &\leq \frac{1}{5}\|x\|_* + \|x\|_* \\ &= \frac{6}{5}\|x\|_*. \end{aligned} \tag{2.5}$$

*Take the unit norm element* $y = (y_k)_{k=-\infty}^{\infty}$ *such that* $y_k = \begin{cases} 1, \text{ for } k = 1,2 \\ 0, \text{ otherwise} \end{cases}$.
*It is easy to see* $\|Ay\|_* = \dfrac{6}{5}$*, thus* $\|A\| = \dfrac{6}{5}$*. Equation(2.4) and the fact* $\|A\| = \dfrac{6}{5}$ *together implies* $\|A\|\|A^{-1}\| = 6$*. Hence* $0 \in L_\epsilon(A)$*. Consider the unit norm element* $y = (y_k)_{k=-\infty}^{\infty}$ *such that*

$$y_k = \begin{cases} 1, \text{ for } k = 1,4 \\ -\overline{\lambda} \text{ for } k = 3 \\ 0, \text{ otherwise.} \end{cases}$$

*where* $\lambda \in B\left(0, \dfrac{1}{5}\right) \setminus \{0\}$*. Then*

$$\begin{aligned} \|(A-\lambda)y\|_* &= \left\|\left(\cdots, y_{-1}-\lambda y_{-2}, y_0 - \lambda y_{-1}, \boxed{\frac{y_1}{5} - \lambda y_0}, y_2 - \lambda y_1, y_3 - \lambda y_2, \cdots\right)\right\|_* \\ &= \left|\frac{y_1}{5} - \lambda y_0\right| + \sup_{n\neq 0}|y_{n+1} - \lambda y_n| > \frac{6}{5}. \end{aligned}$$

*Hence*

$$\|A-\lambda\| > \frac{6}{5}, \textit{ for } \lambda \in B\left(0, \frac{1}{5}\right) \setminus \{0\}. \tag{2.6}$$

*From Equation (2.4) and Equation (2.6), we get*

$$\|A-\lambda\|\|(A-\lambda)^{-1}\| > 6, \textit{ for } \lambda \in B\left(0, \frac{1}{5}\right) \setminus \{0\}.$$

*Thus* $B\left(0, \frac{1}{5}\right) \subset \sigma_\epsilon(A)$*, and this clearly tells us* $0$ *is not a boundary point of* $\sigma_\epsilon(A)$*.*

The same example can be done with any $\epsilon = \frac{1}{k}, k \in \mathbb{N}, k \geq 2$.

**Note 2.1.10.** *From Theorem 3.1 in [29], we know that* $\sigma_\epsilon(a)$ *is a perfect set, for any* $a \in \mathcal{A} \setminus \mathbb{C}e$*. But from the last example, we observe* $L_\epsilon(a)$ *need not to be a perfect set. Whereas the following proposition shows that* $L_\epsilon(a)$ *is an uncountable compact set.*

**Proposition 2.1.11.** *Let* $0 < \epsilon < 1$*. If* $a \in \mathcal{A} \setminus \mathbb{C}e$ *then* $L_\epsilon(a)$ *is a compact subset of* $\mathbb{C}$ *with uncountable number of elements.*

*Proof.* For $a \in \mathcal{A} \setminus \mathbb{C}e$, We know $L_\epsilon(a)$ is a nonempty subset of $\sigma_\epsilon(a)$ for each $a \in \mathcal{A} \setminus \mathbb{C}e$. Define the map

$$\phi : \mathbb{C} \setminus \sigma(a) \to \mathbb{R} \text{ defined by } \phi(\lambda) = \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\|.$$

It is clear that $\phi$ is continuous, because the function $\lambda \mapsto (a-\lambda)$ and $\lambda \mapsto (a-\lambda)^{-1}$ are continuous. Consequently, $\phi^{-1}\left(\left\{\frac{1}{\epsilon}\right\}\right)$ is closed. Hence $L_\epsilon(a)$ is a closed subset of $\sigma_\epsilon(a)$.

Suppose $L_\epsilon(a)$ has countable number of elements then $\mathbb{C} \setminus L_\epsilon(a)$ is connected. We note the following,

$$\mathbb{C}\setminus L_\epsilon(a) = \left\{\lambda \in \mathbb{C} : \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| < \frac{1}{\epsilon}\right\} \cup \left\{\lambda \in \mathbb{C} : \|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| > \frac{1}{\epsilon}\right\}. \tag{2.7}$$

The right hand side sets of the above Equation (2.7) are non empty open sets in $\mathbb{C}$. This leads to a contradiction to the fact $\mathbb{C} \setminus L_\epsilon(a)$ is connected. □

## 2.2 Interior of 1-level set of condition spectrum

Recall that, the notion condition spectrum is not defined for $\epsilon = 1$. But in the study of level sets the case $\epsilon = 1$ is also taken into account. This indicate us that, $1-$ level set is not the subset of condition spectrum and so to deal its interior property, a different treatment is essential.

By calculating the number of elements and the geometric picture of $1-$level set of condition spectrum, we conclude that the interior of $1-$level set is always empty (see Theorem 2.2.5 and Theorem 2.2.8). Examples in the beginning of this subsection aid us to find the contrast behavior of $1-$ level set. These examples mainly target on the size of the $1-$ level sets.

**Example 2.2.1.** *Let $n \in \mathbb{N}$ and $n \geq 1$. The set*

$$C^n[a,b] = \{f \in C([a,b]) \mid f \text{ is } n \text{ times continuously differentiable } \}$$

*forms a complex unital Banach algebra with respect to pointwise addition, pointwise multiplication and with the norm*

$$\|f\| = \sum_{k=0}^{n} \sup_{t\in[a,b]} \left|f^{(k)}(t)\right|$$

*where $f^{(k)}(t)$ denotes the $k^{th}$ derivative of $f$ at the point $t \in [a,b]$. If $f$ is non-constant invertible element in $C^n[a,b]$ then $\sup\limits_{t\in[a,b]} \left|f^{(1)}(t)\right| \neq 0$.*

$$\begin{aligned}
\|f\|\|f^{-1}\| &\geq \sup_{t\in[a,b]} \left|f^{(0)}(t)\right| \sup_{t\in[a,b]} \left|\left(f^{-1}\right)^{(0)}(t)\right| + \sup_{t\in[a,b]} \left|\left(f^{-1}\right)^{(0)}(t)\right| \sup_{t\in[a,b]} \left|f^{(1)}(t)\right| \\
&> 1.
\end{aligned}$$

*Thus $L_1(f) = \emptyset$ for every non scalar invertible element $f \in \mathcal{D}$.*

**Example 2.2.2.** *Consider the complex Hilbert space*

$$\ell^2(\mathbb{N}) := \left\{ x = (x_1, x_2, x_3, x_4, \dots) \middle| \sum_{i=1}^{\infty} |x_i|^2 < \infty \text{ and } x_i \in \mathbb{C} \right\}$$

*with norm*

$$\|x\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}.$$

*For some fixed $n \in \mathbb{N}$ with $n \geq 2$, consider an operator $T$ in $B(\ell^2(\mathbb{N}))$ defined as*

$$T(e_i) = \begin{cases} 2e_{(n+1)-i}, \text{ for } 1 \leq i \leq n \\ 2e_i, \text{ for all } i \geq n+1. \end{cases}$$

*where the $e_i's$ form the standard orthonormal basis for $\ell^2(\mathbb{N})$. It is easy to see that $T = T^*$ and $\sigma(T) = \{-2, 2\}$. For any $\lambda \in \rho(T)$ the operators $T - \lambda$ and $(T - \lambda)^{-1}$ are normal and so their norms are equal to its spectral radius. We have*

$$\|T - \lambda\| = \max\{|\lambda - 2|, |\lambda + 2|\} \text{ and } \left\|(T - \lambda)^{-1}\right\| = \max\left\{\frac{1}{|\lambda - 2|}, \frac{1}{|\lambda + 2|}\right\}.$$

*Hence*

$$\begin{aligned} L_1(T) &= \left\{\lambda : \frac{|\lambda - 2|}{|\lambda + 2|} = 1\right\} \bigcup \left\{\lambda : \frac{|\lambda + 2|}{|\lambda - 2|} = 1\right\} \\ &= \{\lambda : |\lambda - 2| = |\lambda + 2|\}. \end{aligned}$$

*This shows that $L_1(T)$ is unbounded.*

**Example 2.2.3.** *For $n \geq 2$, consider Banach space $\mathbb{C}^n$ with infinity norm. Take an operator $S \in B(\mathbb{C}^n)$ such that $S(e_i) = e_{(n+1)-i} + 3e_i$ where $e_i$ is the standard basis of $\mathbb{C}^n$. It is clear that $\|S\| = 4$ and $\sigma(S) = \{2, 4\}$. For any $\lambda \in \rho(S)$, we observe the following*

$$(S - \lambda)(e_i) = (3 - \lambda)\, e_i + e_{(n+1)-i} \text{ with } \|(S - \lambda)\| = 1 + |3 - \lambda|.$$

*and*

$$(S-\lambda)^{-1}(e_i) = \frac{1}{(3 - \lambda)^2 - 1}\left((3 - \lambda)\, e_i + e_{(n+1)-i}\right) \text{ with } \|(S-\lambda)^{-1}\| = \frac{1 + |3 - \lambda|}{|\,(3 - \lambda)^2 - 1|}.$$

*It is easy to verify that $L_1(S) = \{3\}$.*

**Lemma 2.2.4.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. Fix $n \in \mathbb{N}$, consider the following set*

$$E = \left\{\mu \in \mathbb{C} : \|\,(a - \mu)^n\,\| \|(a - \mu)^{-n}\| = 1\right\}.$$

*If $E$ is nonempty then for each $\mu \in E$*

$$\|\,(a - \mu)^n\,\| = |(\lambda - \mu)^n| \text{ and } \|(a - \mu)^{-n}\| = \frac{1}{|(\lambda - \mu)^n|} \text{ for all } \lambda \in \sigma(a).$$

*Proof.* Let $\mu \in E$ and $\lambda \in \sigma(a)$. By Theorem 1.1.6 (2), it is clear that

$$(\lambda - \mu) \in \sigma(a - \mu), (\lambda - \mu)^n \in \sigma\left((a - \mu)^n\right) \text{ and } \frac{1}{(\lambda - \mu)^n} \in \sigma\left((a - \mu)^{-n}\right).$$

We observe the following,

$$\begin{aligned}
\| (a-\mu)^n \| &\geq |(\lambda-\mu)^n| \\
&= \frac{1}{\frac{1}{|(\lambda-\mu)^n|}} \\
&\geq \frac{1}{\|(a-\mu)^{-n}\|} \\
&= \| (a-\mu)^n \|.
\end{aligned}$$

Hence, $|\lambda-\mu|^n = \| (a-\mu)^n \|$ and $\dfrac{1}{|\lambda-\mu|^n} = \|(a-\mu)^{-n}\|$ for all $\lambda \in \sigma(a)$. $\square$

**Theorem 2.2.5.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a)$ has more than two element then $L_1(a)$ has at most one element.*

*Proof.* Let $\lambda_1, \lambda_2, \lambda_3 \in \sigma(a)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3$. Suppose $L_1(a)$ has two distinct elements $z_1$ and $z_2$ then by Lemma 2.2.4,

$$|z_1-\lambda_1| = |z_1-\lambda_2| = |z_1-\lambda_3| = \|a-z_1\|,$$

and

$$|z_2-\lambda_1| = |z_2-\lambda_2| = |z_2-\lambda_3| = \|a-z_2\|.$$

The above two equations imply that two circles with distinct centers intersect in three distinct points. This is a contradiction. $\square$

**Theorem 2.2.6.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a)$ has more than one element. Then the interior of $L_1(a)$ is empty.*

*Proof.* Let $b \in \mathcal{A}$. Suppose interior of $L_1(b)$ is nonempty, then there exists $\eta_0 \in L_1(b)$ and $r > 0$ such that $B(\eta_0, r) \subseteq L_1(b)$. Define

$$a \coloneqq \frac{b-\eta_0}{r}.$$

It is clear that $a \in \mathcal{A}$. For any $\mu \in \triangle \coloneqq \{\lambda \in \mathbb{C} : |\lambda| < 1\}$,

$$\begin{aligned}
\|(a-\mu)\| \left\|(a-\mu)^{-1}\right\| &= \left\|\left(\frac{b-\eta_0}{r}-\mu\right)\right\| \left\|\left(\frac{b-\eta_0}{r}-\mu\right)^{-1}\right\| \\
&= \left\|\left(\frac{b-(\eta_0-r\mu)}{r}\right)\right\| \left\|\left(\frac{b-(\eta_0-r\mu)}{r}\right)^{-1}\right\| \\
&= \|(b-(\eta_0-r\mu))\| \left\|(b-(\eta_0-r\mu))^{-1}\right\| \qquad (2.8)
\end{aligned}$$

Since $|\eta_0 - (\eta_0 - r\mu)\,| = |r\mu| < r, \eta_0 - r\mu \in B\,(\eta_0, r)$, we get

$$\|(b - (\eta_0 - r\mu))\| \left\|(b - (\eta_0 - r\mu))^{-1}\right\| = 1$$

By Equation (2.8), we must have

$$\|(a - \mu)\| \left\|(a - \mu)^{-1}\right\| = 1.$$

Hence $\triangle \subseteq L_1(a)$ and $L_1(a)$ has nonempty interior. Apply Lemma 2.2.4 to $a \in \mathcal{A}$, to the scalar $0 \in L_1(a)$ and for $n = 1$, we get

$$|\lambda| = \|a\| \text{ for all } \lambda \in \sigma(a). \tag{2.9}$$

Consider two different points $\mu_1, \mu_2$ in $\sigma(b)$ (existence of such two points is possible by our assumption). By Theorem 1.1.6 (2), the scalars

$$\lambda_1 = \frac{\mu_1 - \eta_0}{r}, \lambda_2 = \frac{\mu_2 - \eta_0}{r} \text{ are in } \sigma(a).$$

It is clear that $\lambda_1 \neq \lambda_2$. Let

$$\mu \in \{(1-t)\lambda_1 : t \in (0,1)\} \cap \triangle.$$

Take $|\mu| = \delta$. Since the scalars $0, \lambda_1$ and $\mu$ are collinear and by Equation (2.9), we have

$$|\lambda_1 - \mu| = \|a\| - \delta.$$

By Lemma 2.2.4, $|\lambda - \mu|$ is constant for all $\lambda \in \sigma(a)$, we must have

$$|\lambda_1 - \mu| = |\lambda_2 - \mu| = \|a\| - \delta. \tag{2.10}$$

We achieve contradiction in the following two cases
**case : 1** Suppose $\lambda_2 \notin \{(1-t)\lambda_1 : t \in (0,1)\} \cap \triangle$.
By triangular inequality, we have

$$|\lambda_2| \quad < \quad |\mu| + |\mu - \lambda_2|$$

By Equation (2.9), the above equation becomes

$$\|a\| < \|a\|.$$

Which is a contradiction.
**case : 2** Suppose $\lambda_2 \in \{(1-t)\lambda_1 : t \in (0,1)\} \cap \triangle$.
Then there exists $t_0 \in (0,1)$ such that

$$\lambda_2 = (1-t_0)\,\lambda_1. \tag{2.11}$$

Now

$$|\lambda_2| = (1-t_0)\,|\lambda_1|$$

By Equation (2.9), we have

$$\begin{aligned} \|a\| &= (1-t_0)\,\|a\| \\ 1 &= (1-t_0) \end{aligned}$$

Hence $t_0 = 0$ and by Equation (2.11), we get $\lambda_1 = \lambda_2$. This is a contradiction. $\square$

The above result is true only if the number of elements in the spectrum is more than one. If $a \in \mathcal{A}$ and spectrum of $a$ is containing only one element then by Theorem 2.2.7 we analyze the nature of $1-$level set.

For $T \in B(X)$, the following question is the classical one in the field operator theory

"If $\sigma(T) = \{1\}$ then can we conclude $T = I$?"

We suggest the article [48] for complete survey about this problem. Theorem 2.2.7, gives a sufficient condition to answer this question for a general Banach algebra element. A different perspective of this problem in terms of the condition spectrum is discussed in Corollary 3.5 of [29].

**Theorem 2.2.7** ([2], Theorem 1.1). *Let $a \in \mathcal{A}$. If $\sigma(a) = \{1\}$ and $a$ is doubly power bound element of $\mathcal{A}$, which means $\sup\{\|a^n\| : n \in \mathbb{Z}\} < \infty$, then $a = e$.*

**Theorem 2.2.8.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ with $\sigma(a) = \{\lambda\}$. For any $k \in \mathbb{N}$, the set*

$$E = \left\{ \mu \in \mathbb{C} : \|\,(a-\mu)^k\,\|\|(a-\mu)^{-k}\| = 1 \right\}.$$

*is empty and in particular the interior of $E$ is also empty.*

*Proof.* Suppose $E \neq \emptyset$, then there exists $\mu \in E$ such that by Lemma 2.2.4, we have

$$\|(a-\mu)^k\| = \left|(\mu-\lambda)^k\right|. \tag{2.12}$$

Consider the element

$$b := \frac{(a-\mu)}{\lambda-\mu}.$$

It is clear that $b \in \mathcal{A}$ and from Equation (2.12), we get $\|b\| = 1$. We have that $\sigma(a) = \{\lambda\}$ and by Theorem 1.1.6 (2), we get $\sigma(b) = \{1\}$. For all positive integers $n$,

$$\|b^n\| \leq \|b\|^n \leq 1.$$

By Lemma 2.2.4, $\|(a-\mu)^{-1}\| = \dfrac{1}{|\lambda-\mu|}$. Now

$$\|b^{-1}\| = \left\| \left( \frac{a-\mu}{\lambda-\mu} \right)^{-1} \right\| = 1.$$

For any negative integer $n$

$$\|b^n\| \leq \|b^{-1}\|^{-n} = 1.$$

Thus $\|b^n\| \leq 1$ for all $n \in \mathbb{Z}$. Hence by Theorem 2.2.7, we conclude that $b = e$. Thus $a = \lambda$, which is a contradiction. $\square$

**Corollary 2.2.9.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $\sigma(a) = \{\lambda\}$ then $L_1(a)$ is empty and in particular interior of $L_1(a)$ is empty.*

*Proof.* Follows from Theorem 2.2.8, by taking $k = 1$. $\square$

For $a \in \mathcal{A} \setminus \mathbb{C}e$, Corollary 2.2.9 insists that $\sigma(a)$ contains more than one element if $L_1(a)$ is non empty. By example 2.2.1, it is notable that the converse of this statement is false.

The following Theorem and Example 2.2.1 proves that $L_1(a) = \emptyset$ for some elements of every Banach algebra and every element of some Banach algebra.

**Theorem 2.2.10.** *For any complex unital Banach algebra $\mathcal{A}$, there always exists an $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $L_1(a) = \emptyset$.*

*Proof.* Suppose there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a) = \{\lambda\}$ then by Corollary 2.2.9, $L_1(a) = \emptyset$. If there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\sigma(a) = \{\lambda_1, \lambda_2\}$ with $\lambda_1 \neq \lambda_2$, then by Proposition 9 in §7 of [19], there exists idempotents $e_1$ and $e_2$ such that $\sigma(ae_1) = \{\lambda_1\}, \sigma(ae_2) = \{\lambda_2\}$ and $a = ae_1 + ae_2$. We must have either $ae_1 \in \mathcal{A} \setminus \mathbb{C}e$ or $ae_2 \in \mathcal{A} \setminus \mathbb{C}e$, otherwise $a \notin \mathcal{A} \setminus \mathbb{C}e$. Hence by Corollary 2.2.9, we get $L_1(ae_1) = \emptyset$ or $L_1(ae_2) = \emptyset$. If there exists $a \in \mathcal{A} \setminus \mathbb{C}e$ such that $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \sigma(a)$ with $\lambda_1 \neq \lambda_2 \neq \lambda_3$

then consider the following polynomial

$$p(z) = \frac{(z-\lambda_2)(z-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} - \frac{(z-\lambda_1)(z-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}.$$

Clearly $\{-1,0,1\} \subseteq \sigma(p(a))$. We prove $L_1(p(a)) = \emptyset$. Suppose $\lambda \in L_1(p(a))$, then Lemma 2.2.4

$$|\lambda - 1| = |\lambda| = |\lambda + 1|.$$

We know

$$|\lambda - 1|^2 + |\lambda + 1|^2 = 2|\lambda|^2 + 2.$$

Apply $|\lambda - 1| = |\lambda|$ and $|\lambda + 1| = |\lambda|$ in the above equation, we get $0 = 2$. This is absurd. Hence $L_1(p(a)) = \emptyset$. □

Corollary 2.2.9 is fully depending the doubly power bound property of the element $a \in \mathcal{A} \setminus \mathbb{C}e$. Now, we can think of its converse statement: If $a \in \mathcal{A} \setminus \mathbb{C}e$ with $L_1(a)$ empty then $a$ is a doubly power bounded element. The following illustration is framed to show the converse is no longer true.

**Example 2.2.11.** *Consider the Banach algebra $C[0,1]$ and element $g \in C[0,1]$ such that $g(x) = 2x$. We know that $\sigma(g) = [0,2]$. We prove that $L_1(g) = \emptyset$. If $\lambda \in L_1(g)$ then by Lemma 2.2.4*

$$|\lambda - \mu| \text{ for all } \mu \in [0,2]$$

*In particular*

$$|\lambda - 1| = |\lambda| = |\lambda - 2|$$

*We have the following,*

$$|(\lambda - 1) + 1|^2 + |(\lambda - 1) - 1|^2 = 2|\lambda - 1|^2 + 2$$

*Apply $|\lambda| = |\lambda - 1|$ and $|\lambda - 2| = |\lambda - 1|$. We get $0 = 2$. This has no sense. Hence $L_1(g) = \emptyset$. We also have $\|g^n\|_\infty \to \infty$ as $n \to \infty$. Thus $g$ is not a doubly power bound element.*

## 2.3 Interior of $\epsilon-$ level set of condition spectrum

In the previous subsection we proved that the $1-$level set of condition spectrum has empty interior for all non scalar elements of a Banach algebra. For $0 < \epsilon < 1$, in this subsection we establish the same for $\epsilon-$ level set of condition spectrum in the appropriate settings.

We first develop a tool which is a version of maximum modulus Theorem for product of $n$ analytic vector valued functions (where $n \in \mathbb{N}$) to get the required results. This proof is similar to the proof of Theorem 1.3.4 which is available in [43].

**Lemma 2.3.1.** *Let $\Omega_0$ be a connected open subset of $\mathbb{C}$, let $\Omega$ be an open subset of $\Omega_0$ and let $X$ be a complex Banach space and $k \in \mathbb{N}$. For $i = 1, \cdots, n$, suppose that we have the following:*

1. $\psi_i : \Omega_0 \to X$ *are analytic vector valued functions.*
2. $\prod_{i=1}^{n} (\|\psi_i(\lambda)\|)^k \leq M$ *for all* $\lambda \in \Omega$.
3. $\prod_{i=1}^{n} (\|\psi_i(\mu)\|)^k < M$ *for some* $\mu \in \Omega_0$

*Then* $\prod_{i=1}^{n} (\|\psi_i(\lambda)\|)^k < M$ *for all* $\lambda \in \Omega$.

*Proof.* Suppose there exists $\lambda_0 \in \Omega$ such that

$$\prod_{i=1}^{n} (\|\psi_i(\lambda_0)\|)^k = M$$

then by the Hahn-Banach Theorem for each $\psi_i(\lambda_0)$ there exists $g_i \in X^*$ such that $\|g_i\| = 1$ and

$$g_i\left(\psi_i(\lambda_0)\right) = \left\|\psi_i(\lambda_0)\right\|. \tag{2.13}$$

Consider the function

$$\phi : \Omega_0 \to \mathbb{C} \text{ defined by } \phi(\lambda) = \prod_{i=1}^{n} \left(g_i(\psi_i(\lambda))\right)^k.$$

$\phi$ is analytic because $g_i(\psi_i)$ is analytic on $\Omega_0$ for each $i$. By assumption (2)

$$\begin{aligned} |\phi(\lambda)| &= \left| \prod_{i=1}^{n} \left(g_i(\psi_i(\lambda))\right)^k \right| \\ &\leq \prod_{i=1}^{n} \|g_i\|^k \|\psi_i(\lambda)\|^k \leq M \text{ for all } \lambda \in \Omega. \end{aligned}$$

Particularly for $\lambda_0 \in \Omega$ and from Equation (2.13), we get

$$|\phi(\lambda_0)| = \left| \left( \prod_{i=1}^{n} g_i(\psi_i(\lambda_0)) \right)^k \right| = \left( \prod_{i=1}^{n} |g_i(\psi_i(\lambda_0))| \right)^k = \prod_{i=1}^{n} \|\psi_i(\lambda_0)\|^k = M,$$

Thus $|\phi|$ attains local maximum at $\lambda_0$. Since $\Omega_0$ is connected by Maximum modulus Theorem $\phi$ is constant and $\phi \equiv M$. On the other hand, by assumption (3) and by the definition of all $g_i$, we have

$$\begin{aligned} M &= |\phi(\mu)| = \left| \prod_{i=1}^{n} (g_i(\psi_i(\mu)))^k \right| \\ &\leq \prod_{i=1}^{n} \|g_i\|^k \|\psi_i(\mu)\|^k = \prod_{i=1}^{n} \|\psi_i(\mu)\|^k \\ &< M. \end{aligned}$$

This is a contradiction. □

**Theorem 2.3.2** ([35], Theorem 25.3). *A topological space $X$ is locally connected if and only if each component of an open subset is open in $X$.*

**Theorem 2.3.3.** *Let $M > 1, a \in \mathcal{A} \setminus \mathbb{C}e$ and $\Omega$ be an open subset in the unbounded component of $\rho(a)$. If*

$$\|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| \leq M \textit{ for all } \lambda \in \Omega$$

*then*

$$\|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| < M \textit{ for all } \lambda \in \Omega.$$

*Proof.* Let the unbounded component of $\rho(a)$ be $\Omega_0$. By Theorem 2.3.2, $\Omega_0$ is open. By our assumption, $\Omega \subset \Omega_0$ and

$$\|(a-\lambda)\| \left\|(a-\lambda)^{-1}\right\| \leq M, \text{ for all } \lambda \in \Omega.$$

Since $M > 1$, there exists $\delta > 0$ such that $M - \delta > 1$. Choose $\mu \in \Omega_0$ such that

$$|\mu| > \|a\| \left(1 + \frac{1}{M-\delta}\right) \left(1 - \frac{1}{M-\delta}\right)^{-1}$$

(It is possible to choose such a $\mu$ from $\Omega_0$, since $\Omega_0$ is unbounded). We observe the following

$$
\begin{aligned}
|\mu| > \|a\| \left(1 + \frac{1}{M-\delta}\right)\left(1 - \frac{1}{M-\delta}\right)^{-1} \quad &\Longleftrightarrow \quad |\mu|\left(1 - \frac{1}{M-\delta}\right) > \|a\| \left(1 + \frac{1}{M-\delta}\right) \\
&\Longleftrightarrow \quad |\mu| - \frac{|\mu|}{M-\delta} > \|a\| + \frac{\|a\|}{M-\delta} \\
&\Longleftrightarrow \quad |\mu| > \|a\| + \frac{\|a\|}{M-\delta} + \frac{|\mu|}{M-\delta} \\
&\Longleftrightarrow \quad |\mu| > \|a\| + \frac{\|a\| + |\mu|}{M-\delta} \qquad (2.14)
\end{aligned}
$$

Since $|\mu| > \|a\|$, $(a-\mu)^{-1}$ exists and

$$
\begin{aligned}
\left\|(a-\mu)^{-1}\right\| &= \left\| \frac{-1}{\mu} \sum_{n=0}^{\infty} \left(\frac{a}{\mu}\right)^n \right\| \\
&\leq \frac{1}{|\mu|} \left( \frac{|\mu|}{|\mu| - \|a\|} \right) \\
&= \frac{1}{|\mu| - \|a\|}
\end{aligned}
$$

From Equation (2.14), we have $\dfrac{1}{|\mu| - \|a\|} < \dfrac{M-\delta}{\|a\| + |\mu|}$. Substitute this value in the previous equation, we get

$$
\left\|(a-\mu)^{-1}\right\| < \frac{M-\delta}{|\mu| + \|a\|} \leq \frac{M-\delta}{\|a-\mu\|}.
$$

Thus we have

$$
\left\|(a-\mu)^{-1}\right\| < \frac{M-\delta}{\|a-\mu\|} \Longrightarrow \|(a-\mu)\| \left\|(a-\mu)^{-1}\right\| < M.
$$

Apply Theorem 2.3.1 to the analytic functions

$$
\psi_1 : \Omega_0 \to \mathcal{A} \text{ defined by } \psi_1(\lambda) = (a-\lambda)
$$

and

$$
\psi_2 : \Omega_0 \to \mathcal{A} \text{ defined by } \psi_2(\lambda) = (a-\lambda)^{-1},
$$

to get $\|(a-\lambda)\| \, \|(a-\lambda)^{-1}\| < M$ for all $\lambda \in \Omega$. This proves the theorem. $\square$

**Corollary 2.3.4.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ and $0 < \epsilon < 1$. Then $L_\epsilon(a)$ has empty interior in the unbounded component of $\rho(a)$. In particular interior of $L_\epsilon(a)$ is empty if $\rho(a)$ is connected.*

*Proof.* Immediate from Theorem 2.3.3. □

Now it is clear that, the left over part of our problem in our concern is looking the nature of $\epsilon-$ level set in the bounded component of the resolvent set. Here, we partially answer this problem by focusing into a particular Banach algebra. For $T \in B(X)$ where $X$ is a complex uniformly convex Banach space then we prove that interior of $L_\epsilon(T)$ is empty for $0 < \epsilon < 1$. We first see the notion of complex uniformly convex Banach space and some important remarks related to them.

**Definition 2.3.5** ([21], Definition 2.4 (ii))**.** *A complex Banach space $X$ is said to be complex uniformly convex (uniformly convex) if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$x, y \in X,\ \|y\| \geq \epsilon \textit{ and } \|x + \zeta y\| \leq 1,\ \forall \zeta \in \mathbb{C}\,(\zeta \in \mathbb{R}),\ \textit{with } |\zeta| \leq 1 \Rightarrow \|x\| \leq 1 - \delta.$$

There are many typical Banach spaces, those turns out to be complex uniformly convex Banach spaces. In [21], Theorem 1, Globevnik showed $L_1$ space is complex uniformly convex. From the definition it follows that every uniformly convex Banach space is complex uniformly convex space. It is proved in [13] that Hilbert spaces and $L_p$ (with $1 < p < \infty$) spaces are uniformly convex Banach spaces and hence they are all complex uniformly convex Banach spaces. An example of the Banach space which is not complex uniformly convex is $L_\infty$. It is to be noted that the dual space $L_\infty^*$ is isometrically isomorphic to a space of bounded finitely additive set functions (see [18], Chapter IV, section 8, Theorem 16 and Chapter III, section 1, Lemma 5). The space of bounded finitely additive set functions are complex uniformly convex space is proved in Proposition 1.1 in [32] and so $L_\infty^*$ is complex uniformly convex.

**Definition 2.3.6** ([21], Remark)**.** *Consider a complex Banach space $X$ and $\delta > 0$. We define $\omega_c(\delta)$ as follows*

$$\omega_c(\delta) = \sup\{\|y\| : x, y \in X \textit{ with } \|x\| = 1, \|x + \zeta y\| \leq 1 + \delta, (\zeta \in B(0,1))\}$$

**Remark 2.3.7.** *([21], Remark) Let $X$ be a complex Banach space. Then $X$ is complex uniformly convex if and only if $\lim_{\delta \to 0} \omega_c(\delta) = 0$.*

Proof of the following theorem is similar to the proof of Proposition 2 in [23].

**Theorem 2.3.8.** *Let $X$ be a complex uniformly convex Banach space and $M > 1$. If $T \in B(X)$ with*

$$\|T-\lambda\| \left\|(T-\lambda)^{-1}\right\| \leq M \textit{ for all } \lambda \in \triangle$$

*then*

$$\|T-\lambda\| \left\|(T-\lambda)^{-1}\right\| < M \textit{ for all } \lambda \in \triangle.$$

We use the following lemma to prove Theorem 2.3.8.

**Lemma 2.3.9** ([21], Theorem 2)**.** *Let $X$ be a complex Banach space and $\psi : \triangle \to X$ be an analytic function, satisfying $\|\psi(\lambda)\| \leq 1$, for all $\lambda \in \triangle$. Then*

$$\|\psi(\lambda) - \psi(0)\| \leq \left(\frac{2|\lambda|}{1-|\lambda|}\right) \omega_c(1 - \|\psi(0)\|)$$

*for all $\lambda \in \triangle$.*

*Proof of Theorem 2.3.8.* Suppose, there exists $\lambda_0 \in \triangle$ such that

$$\|T-\lambda_0\| \left\|(T-\lambda_0)^{-1}\right\| = M. \tag{2.15}$$

We arrive at a contradiction in four steps.
**Step 1:** In this step, we show that $\|\,(T-\lambda)\,\| \, \|(T-\lambda)^{-1}\| = M,$ forall $\lambda \in \triangle$
By the Hahn-Banach Theorem there exists $g_1, g_2 \in B(X)^*$ having the property that

$$g_1\left(T-\lambda_0\right) = \|T-\lambda_0\|, g_2\left((T-\lambda_0)^{-1}\right) = \left\|(T-\lambda_0)^{-1}\right\| \text{ with } \|g_1\| = \|g_2\| = 1.$$

We define the following function

$$\phi : \triangle \to \mathbb{C} \text{ by } \phi(\lambda) = g_1\left(T-\lambda\right) g_2\left((T-\lambda)^{-1}\right)$$

Since the function $\lambda \mapsto T-\lambda$ and $\lambda \mapsto (T-\lambda)^{-1}$ are analytic, it follows that $\phi$ is analytic. For all $\lambda \in \triangle \setminus \{\lambda_0\}$, we observe the following

$$\begin{aligned} |\phi(\lambda)| &= \left|g_1\left(T-\lambda\right) g_2\left((T-\lambda)^{-1}\right)\right| \\ &\leq \|g_1\| \|\,(T-\lambda)\,\| \|g_2\| \left\|(T-\lambda)^{-1}\right\| \\ &= \|\,(T-\lambda)\,\| \left\|(T-\lambda)^{-1}\right\| \\ &= M. \end{aligned}$$

In particular, for $\lambda_0 \in \triangle$, by the choice of $g_1, g_2$ and by Equation (2.15), we get

$$|\phi(\lambda_0)| = \left|g_1\left(T-\lambda_0\right) g_2\left((T-\lambda_0)^{-1}\right)\right| = \|T-\lambda_0\| \left\|(T-\lambda_0)^{-1}\right\| = M.$$

Hence

$$|\phi(\lambda)| \leq M \text{ for all } \lambda \in \triangle \tag{2.16}$$

Also $|\phi|$ attains local maximum in $\triangle$. By Maximum modulus theorem, $\phi$ is constant and $\phi \equiv M$. For any $\lambda \in \triangle$ by Equation (2.16), we have

$$M = |\phi(\lambda)| \leq \|\left(T-\lambda\right)\| \left\|(T-\lambda)^{-1}\right\| \leq M$$

Thus,

$$\|\left(T-\lambda\right)\| \left\|(T-\lambda)^{-1}\right\| = M \text{ for all } \lambda \in \triangle. \tag{2.17}$$

**Step 2:** In this step we define sequence of function $\psi_n$ from $\triangle$ to $X$ for each $n \in \mathbb{N}$ and using step 1, we prove that each $\psi_n$ is a bounded analytic vector valued function.

We know that there exists a sequence $\{x_n\}$ with $\|x_n\| = 1$ such that

$$\lim_{n\to\infty} \left\|T^{-1}(x_n)\right\| = \left\|T^{-1}\right\|.$$

By the Hahn-Banach theorem, there exists $g \in B(X)^*$ such that $g(T) = \|T\|$ with $\|g\| = 1$. For each $x_n$, we define the following function

$$\psi_n : \triangle \to X \text{ by } \psi_n(\lambda) = \frac{\left[g(T-\lambda)\right](T-\lambda)^{-1} x_n}{\|T\|\|T^{-1}\|}.$$

We prove that $\psi_n$ is an analytic bounded vector valued function on $\triangle$ in 4 claims.

**Claim 1 :** We prove $\psi_n$ is bounded.

$$\begin{aligned} \|\psi_n(\lambda)\| &= \left\|\frac{\left[g(T-\lambda)\right](T-\lambda)^{-1} x_n}{\|T\|\|T^{-1}\|}\right\| \\ &\leq \frac{\|g\|\|T-\lambda\| \, \|(T-\lambda)^{-1}\| \, \|x_n\|}{\|T\|\|T^{-1}\|} \end{aligned} \tag{2.18}$$

By the choice of $g$ and by Equations (2.17), (2.18), we get

$$\|\psi_n(\lambda)\| \leq 1. \tag{2.19}$$

**Claim 2:** We write $\psi_n$ as sum of two functions which are defined on $\triangle$.
Apply $g(T) = \|T\|$ and $\|T\|\|T^{-1}\| = M$ in the definition of $\psi_n$, we get

$$\begin{aligned}
\psi_n(\lambda) &= \frac{[g(T-\lambda)]\,(T-\lambda)^{-1}\,x_n}{\|T\|\|T^{-1}\|} \\
&= \frac{[g(T)+\lambda g(-I)]\,(T-\lambda)^{-1}\,x_n}{M} \\
&= \frac{[\|T\|+\lambda g(-I)]\,(T-\lambda)^{-1}\,x_n}{M} \\
&= \frac{\|T\|\,(T-\lambda)^{-1}\,x_n}{M} + \frac{g(-I)\lambda\,(T-\lambda)^{-1}\,x_n}{M}
\end{aligned}$$

Denote $R(\lambda) = (T-\lambda)^{-1}$ and define the following functions

$$_n\chi_1 : \triangle \to X \text{ defined by } {}_n\chi_1(\lambda) = \frac{\|T\|R(\lambda)x_n}{M}$$

and

$$_n\chi_2 : \triangle \to X \text{ defined by } {}_n\chi_2(\lambda) = \frac{g(-I)\lambda R(\lambda)x_n}{M}.$$

Since $\psi_n = {}_n\chi_1 + {}_n\chi_2$, if we prove that ${}_n\chi_1$ and ${}_n\chi_2$ are analytic, then it will imply that $\psi_n$ be analytic.
**Claim 3 :** We prove ${}_n\chi_1$ is analytic.
For any $\lambda, \mu \in \triangle$, we observe that

$$\begin{aligned}
\left\|\frac{{}_n\chi_1(\lambda) - {}_n\chi_1(\mu)}{\lambda-\mu} - \frac{\|T\|\,(-R(\lambda)^2)\,x_n}{M}\right\| &\leq \frac{\|T\|}{M}\left\|\frac{R(\lambda)-R(\mu)}{\lambda-\mu} - \left(-R(\lambda)^2\right)\right\|\|x_n\| \\
&= \frac{\|T\|}{M}\left\|\frac{R(\lambda)-R(\mu)}{\lambda-\mu} - \left(-R(\lambda)^2\right)\right\|
\end{aligned}$$

By the resolvent identity [see 7 of Lecture 16 in [7]], we have

$$\lim_{\lambda\to\mu}\left\|\frac{{}_n\chi_1(\lambda) - {}_n\chi_1(\mu)}{\lambda-\mu} - \frac{\|T\|\,(-R(\lambda)^2)\,x_n}{M}\right\| = 0 \tag{2.20}$$

Hence ${}_n\chi_1$ is analytic.
**Claim 4 :** We show that ${}_n\chi_2$ is analytic.

For every $\lambda, \mu \in \triangle$, we observe the following

$$\begin{aligned}
\frac{{}_n\chi_2(\lambda) - {}_n\chi_2(\mu)}{\lambda - \mu} &= \left(\frac{g(-I)}{M}\right)\left(\frac{\lambda R(\lambda)x_n - \mu R(\mu)x_n}{\lambda - \mu}\right) \\
&= \left(\frac{g(-I)}{M}\right)\left(\frac{R(\lambda)\left[\lambda(T-\mu) - \mu(T-\lambda)\right]R(\mu)x_n}{\lambda - \mu}\right) \\
&= \left(\frac{g(-I)}{M}\right) R(\lambda)TR(\mu)x_n.
\end{aligned}$$

Now, $\left\| \frac{{}_n\chi_2(\lambda) - {}_n\chi_2(\mu)}{\lambda - \mu} - \left(\frac{g(-I)}{M}\right) R(\mu)TR(\mu)x_n \right\|$

$$\begin{aligned}
&= \left|\frac{g(I)}{M}\right| \|R(\lambda)TR(\mu)x_n - R(\mu)TR(\mu)x_n\| \\
&\leq \left|\frac{g(I)}{M}\right| \|R(\lambda) - R(\mu)\| \, \|T\| \|R(\mu)\| \|x_n\|
\end{aligned}$$

The map $\lambda \mapsto R(\lambda)$ is analytic and so it continuous. Thus, $\lambda \to \mu$ we get $R(\lambda) \to R(\mu)$. We get

$$\lim_{\lambda \to \mu} \left\| \frac{{}_n\chi_2(\lambda) - {}_n\chi_2(\mu)}{\lambda - \mu} - \left(\frac{g(-I)}{M}\right) R(\mu)TR(\mu)x_n \right\| = 0. \tag{2.21}$$

Hence we conclude that $\psi_n$ is an analytic bounded vector valued map.

**Step 3 :** In this step we apply Lemma 2.3.9 to the functions $\psi_n$ and we will see the consequence.

Applying Lemma 2.3.9 to the function $\psi_n$, we get

$$\|\psi_n(\lambda) - \psi_n(0)\| \leq \left(\frac{2|\lambda|}{1 - |\lambda|}\right) w_c \left(1 - \|\psi_n(0)\|\right) \text{ for all } \lambda \in \triangle.$$

By applying the corresponding values for $\psi_n$, we get

$$\left\| \frac{\left[g(T-\lambda)\right](T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|} \right\| \leq \left(\frac{2|\lambda|}{1-|\lambda|}\right) w_c \left(1 - \left\|\frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|}\right\|\right)$$

Apply $g(T) = \|T\|$ to the right side of the above inequality, we get

$$\left\| \frac{\left[g(T-\lambda)\right](T-\lambda)^{-1}x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|} \right\| \leq \left(\frac{2|\lambda|}{1-|\lambda|}\right) w_c \left(1 - \frac{\|T^{-1}x_n\|}{\|T^{-1}\|}\right).$$

Using Remark $(2.3.7)$ and the fact $1 - \dfrac{\|T^{-1}x_n\|}{\|T^{-1}\|} \to 0$, we observe that

$$\lim_{n\to\infty} \left\| \frac{[g(T-\lambda)]\,(T-\lambda)^{-1}\,x_n}{\|T\|\|T^{-1}\|} - \frac{g(T)T^{-1}x_n}{\|T\|\|T^{-1}\|} \right\| = 0$$

The above equation implies

$$\lim_{n\to\infty} \left\|[g(T-\lambda)]\,(T-\lambda)^{-1}\,x_n - g(T)T^{-1}x_n\right\| = 0. \tag{2.22}$$

Now,

$$\begin{aligned} T^{-1} &= T^{-1}(T-\lambda)(T-\lambda)^{-1} \\ &= (T-\lambda)^{-1} - \lambda T^{-1}(T-\lambda)^{-1} \\ (T-\lambda)^{-1} &= T^{-1} + \lambda T^{-1}(T-\lambda)^{-1} \end{aligned} \tag{2.23}$$

We substitute this value to simplify the expression $[g(T-\lambda)]\,(T-\lambda)^{-1} - g(T)T^{-1}]$.
Now, $[g(T-\lambda)]\,(T-\lambda)^{-1} - g(T)T^{-1}$

$$\begin{aligned} &= g(T)\,(T-\lambda)^{-1} - \lambda g(I)\,(T-\lambda)^{-1} - g(T)T^{-1} \\ &= g(T)\left[T^{-1} + \lambda T^{-1}\,(T-\lambda)^{-1}\right] - \lambda g(I)\,(T-\lambda)^{-1} - g(T)T^{-1} \\ &= g(T)\left[T^{-1} + \lambda T^{-1}\,(T-\lambda)^{-1} - T^{-1}\right] - \lambda g(I)\,(T-\lambda)^{-1} \\ &= \lambda\left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1}. \end{aligned} \tag{2.24}$$

For any $\lambda \in \triangle$, from Equation (2.22) and Equation (2.24), we get,

$$\lim_{n\to\infty} \left\|\left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1}\,x_n\right\| = 0. \tag{2.25}$$

and

$$\lim_{n\to\infty} \left\|[g(T) - g(I)T]\,(T-\lambda)^{-1}\,x_n\right\| = 0. \tag{2.26}$$

**Step 4 :** In this step, we get the required contradiction by applying the appropriate value for $g(I)$ to the Equation (2.25) and the Equation (2.26).
**Case 1 :** $g(I) = 0$ Equation (2.26) becomes,

$$\lim_{n\to\infty} \left\|g(T)\,(T-\lambda)^{-1}\,x_n\right\| = 0.$$

Since the operator $T-\lambda$ is continuous for any $\lambda \in \triangle$, and so by previous equation, we get

$$\lim_{n\to\infty} \|x_n\| = 0$$

Which is a contradiction to $\|x_n\| = 1$.

**Case 2 :** $|g(I)| \leq 1$

From Equation (2.25), we get

$$\lim_{n\to\infty} \left\|(T-\lambda)\left[g(T)T^{-1} - g(I)\right](T-\lambda)^{-1} x_n\right\| = 0. \tag{2.27}$$

Since the operators $(T-\lambda)$ and $T^{-1}$ commutes, we have

$$\lim_{n\to\infty} \left\|\left(g(T)T^{-1} - g(I)\right) x_n\right\| = 0.$$

By the triangular inequality we have that

$$\lim_{n\to\infty} \left(\|g(T)T^{-1}x_n\| - \|g(I)x_n\|\right) = 0.$$

The above equation implies,

$$\lim_{n\to\infty} \|T\| \left\|T^{-1}x_n\right\| = |g(I)|.$$

Thus

$$\lim_{n\to\infty} \left\|T^{-1}x_n\right\| = \frac{|g(I)|}{\|T\|}.$$

We also know that $\lim_{n\to\infty} \left\|T^{-1}x_n\right\| = \left\|T^{-1}\right\|$. Hence $\|T\|\,\|T^{-1}\| = |g(I)| \leq 1$. But in Step 1, we observed that $\|T\|\,\|T^{-1}\| = M$. This is a contradiction to $M > 1$. Hence we have proved the theorem. □

**Note 2.3.10.** *The above result holds for any open ball in the resolvent set of $T$. Suppose, we have $\|T-\lambda\|\|\,(T-\lambda)^{-1}\,\| \leq M$ for all $\lambda \in B(\mu, r)$, and $M > 1$. We know that $B(\mu, r) = \mu + rB(0,1)$. Define the operator*

$$S := \frac{T-\mu}{r} \in B(X).$$

*Then $S \in B(X)$ and*

$$\|S-\lambda\| \left\|(S-\lambda)^{-1}\right\| \leq M \text{ for all } \lambda \in \triangle.$$

*We apply Theorem (2.3.8) to the operator $S$ to get the required result.*

**Corollary 2.3.11.** *Let $X$ be a complex Banach space such that the dual space $X^*$ is complex uniformly convex and $M > 1$. Suppose $T \in B(X)$ with*

$$\|T - \lambda\| \left\|(T - \lambda)^{-1}\right\| \leq M \text{ for all } \lambda \in B(0, 1),$$

*then*

$$\|T - \lambda\| \left\|(T - \lambda)^{-1}\right\| < M \text{ for all } \lambda \in B(0, 1).$$

*Proof.* Consider the transpose linear map $T^* \in B(X^*)$. For any $\lambda \in \mathbb{C}$, by Proposition 1.4 in [14], we get

$$\|T - \lambda\| = \|(T - \lambda)^*\| = \|T^* - \lambda\| \tag{2.28}$$

and if $T - \lambda$ is invertible, then

$$\left\|(T - \lambda)^{-1}\right\| = \left\|\left((T - \lambda)^{-1}\right)^*\right\| = \left\|((T - \lambda)^*)^{-1}\right\| = \left\|(T^* - \lambda)^{-1}\right\| \tag{2.29}$$

By Proposition 6.1 in [14], we also have

$$\sigma(T^*) = \sigma(T) \tag{2.30}$$

From Equations (2.28), (2.29) and (2.30), we get

$$\|(T - \lambda)\| \left\|(T - \lambda)^{-1}\right\| = \|(T^* - \lambda)\| \left\|(T^* - \lambda)^{-1}\right\| \text{ for all } \lambda \in \triangle$$

Now, we apply the Theorem (2.3.8) to the operator $T^*$ and the Banach space $X^*$. This completes the proof. □

**Corollary 2.3.12.** *Let $X$ be a complex Banach space, $T \in B(X)$ and $0 < \epsilon < 1$. If either $X$ (or) $X^*$ is complex uniformly convex then $L_\epsilon(T)$ has empty interior in the resolvent set of $T$.*

*Proof.* Immediate consequence of Theorem 2.3.8 and Corollary 2.3.11. □

**Corollary 2.3.13.** *Let $0 < \epsilon < 1$ and $\mathcal{A}$ be a unital $C^*$ algebra. If $a \in \mathcal{A} \setminus \mathbb{C}e$ then the interior of $L_\epsilon(a)$ is empty.*

*Proof.* We know that there exists a $C^*$ isomorphism $\psi$ form $\mathcal{A}$ to $C^*$ subalgebra of $B(H)$ for some Hilbert space $H$. For any $a \in \mathcal{A}$, we have $\sigma(a) = \sigma(\psi(a))$ and

$\|a\| = \|\psi(a)\|$. These imply that,

$$L_\epsilon(a) = L_\epsilon(\psi(a)).$$

Since, the Hilbert space $H$ is complex uniformly convex and by the Theorem 2.3.8, we get interior of $L_\epsilon(\psi(a))$ is empty. Hence the interior of $L_\epsilon(a)$ is empty. $\square$

# Chapter 3

# Continuity of condition spectrum and its level sets

This chapter is devoted to continuity of condition spectrum and its level set correspondences. In chapter 2 we classified the Banach algebras and identified the appropriate components of the spectrum of an element in which the interior of the level set of condition spectrum is empty. We noticed that the empty interior of level set of condition spectrum at given $(\epsilon, a)$ plays an essential role to discuss the continuity of the required correspondences at $(\epsilon, a)$ . Hence, we can feel the derived results in this chapter are application of results in chapter 2. Further in the domain of normal matrices, we get more than continuity namely uniform continuity of the condition spectrum correspondences . With the aid of the fact interior of $1-$ level set of condition spectrum is empty, we show that $\epsilon-$ condition spectrum of an element grows to $\mathbb{C}$ as $\epsilon$ approaches to $1$.

## 3.1 Continuity of the condition spectrum correspondence

Let $0 < \epsilon < 1$ and $a \in \mathcal{A}$. The following correspondence evolves from the definition of $\epsilon-$condition spectrum of $a$

$$\mathcal{C} : (0,1) \times \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}(\epsilon, a) = \sigma_\epsilon(a).$$

where $((0,1) \times \mathcal{A})$ is a metric space with the following metric

$$d\left(\left(\epsilon_1, a_1\right), \left(\epsilon_2, a_2\right)\right) = |\epsilon_1 - \epsilon_2| + \|a_1 - a_2\|. \tag{3.1}$$

We obtain two more correspondences from $\mathcal{C}$. They are as follows, fix $a \in \mathcal{A}$, then

$$\mathcal{C}_a : (0,1) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}_a(\epsilon) = \sigma_\epsilon(a).$$

Fix $\epsilon \in (0,1)$, then

$$\mathcal{C}_\epsilon : \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}_\epsilon(a) = \sigma_\epsilon(a)$$

The prime of this section is to study the continuity of $\mathcal{C}$.

It is showed that the map $\mathcal{C}_a$ is upper hemicontinuous (Theorem 3.1.3). We observe emptiness of the interior of $L_\epsilon(a)$ at given $\epsilon$ and $a$ turns out to be a necessary and sufficient condition for the lower hemicontinuity of $\mathcal{C}_a$ (Theorem 3.1.4). By assuming $\mathcal{C}_a$ is continuous, we conclude that $\mathcal{C}_\epsilon$ and $\mathcal{C}$ are continuous (Theorem 3.1.11). By proving a characterization for normal matrices in terms of $\epsilon-$ condition spectrum (Theorem 3.1.17), it is found that $\mathcal{C}_\epsilon$ is uniformly continuous on the set of normal matrices (Theorem 3.1.18). Finally, we observe $\epsilon-$ condition spectrum grows to $\mathbb{C}$ when $\epsilon$ approaches $1$.

We prove some of our results for the maps $\mathfrak{C}$, $\mathfrak{C}_a$ and $\mathfrak{C}_\epsilon$ for some fixed $a \in \mathcal{A}$ and $\epsilon \in (0,1)$. These maps are defined from $(0,1) \times \mathcal{A}$ to the metric space $K(\mathbb{C})$ (see Note 1.4.10). In particular these are not correspondences. But by using theorem 1.4.11 we can get the desired results for conditions spectrum correspondences. Hence, we emphasize the reader to have a careful attention regarding the notation used in the following subsection.

### 3.1.1 Continuity of the correspondence $\mathcal{C}_a$

First, we establish a lemma which says that the graph of $\mathcal{C}$ is closed. This lemma will be applied in almost all the results in the rest of the section. We prove the results of these correspondences using the methods given in introduction chapter subsection 1.4

**Lemma 3.1.1.** *The graph of the correspondence $\mathcal{C}$ is closed. Further the correspondences $\mathcal{C}_\epsilon$ and $\mathcal{C}_a$ are closed for fixed $\epsilon \in (0,1)$ and fixed $a \in \mathcal{A}$.*

*Proof.* Consider the sequence $\{(\epsilon_n, a_n), \lambda_n\}$ in $Gr(\mathcal{C})$ and $((\epsilon_0, a), \lambda) \in ((0,1) \times \mathcal{A}) \times$

$\mathbb{C}$ where $((0,1)\times\mathcal{A})\times\mathbb{C}$ is a metric space with the following metric

$$d\left(\left(\left(\epsilon_1,a_1\right),\lambda\right),\left(\left(\epsilon_2,a_2\right),\mu\right)\right)=\left|\epsilon_1-\epsilon_2\right|+\left\|a_1-a_2\right\|+\left|\lambda-\mu\right|. \tag{3.2}$$

Suppose, $\left(\left(\epsilon_n,a_n\right),\lambda_n\right)\to\left(\left(\epsilon_0,a\right),\lambda\right)$ as $n\to\infty$ then $\epsilon_n\to\epsilon_0, a_n\to a$ and $\lambda_n\to\lambda$ as $n\to\infty$. We need to prove $\lambda\in\sigma_{\epsilon_0}(a)$. If $\lambda\in\sigma(a)$ then $\lambda\in\sigma_{\epsilon_0}(a)$. If $\lambda\notin\sigma(a)$, for $\dfrac{1}{\left\|(a-\lambda)^{-1}\right\|}$, there exists $n_0\in\mathbb{N}$ such that

$$\left\|(a_n-\lambda_n)-(a-\lambda)\right\|<\frac{1}{\left\|(a-\lambda)^{-1}\right\|}\text{ for all } n\geq n_0.$$

Hence $(a_n-\lambda_n)$ is invertible for $n\geq n_0$. Consequently, there exists a subsequence $\{a_{n_k}-\lambda_{n_k}\}$ of $\{a_n-\lambda_n\}$ such that $\left(a_{n_k}-\lambda_{n_k}\right)^{-1}\to(a-\lambda)^{-1}$ as $k\to\infty$. Since,

$$\left\|(a_{n_k}-\lambda_{n_k})\right\|\left\|\left(a_{n_k}-\lambda_{n_k}\right)^{-1}\right\|\geq\frac{1}{\epsilon_{n_k}}$$

and

$$\left\|\left(a_{n_k}-\lambda_{n_k}\right)^{-1}\right\|\to\left\|(a-\lambda)^{-1}\right\|,\left\|(a_{n_k}-\lambda_{n_k})\right\|\to\left\|(a-\lambda)\right\|,\frac{1}{\epsilon_{n_k}}\to\frac{1}{\epsilon_0}\text{ as } k\to\infty.$$

Hence, we have $\lambda\in\sigma_{\epsilon_0}(a)$. In the similar fashion, we can prove the correspondences $\mathcal{C}_\epsilon$ and $\mathcal{C}_a$ are closed. □

In the next few results we discuss the upper, lower hemicontinuity and continuity of $\mathcal{C}_a$ for fixed $a\in\mathcal{A}\setminus\mathbb{C}e$ at given $\epsilon\in(0,1)$. The following notes briefly explains about the results for elements which are scalar multiple of the identity.

**Note 3.1.2.** *The continuity of $\mathcal{C}$ for any scalar element $a\in\mathcal{A}$ and $\epsilon\in(0,1)$ follows Theorem 3.1.8. From that the upper and lower hemicontinuity of $\mathcal{C}_a$ and $\mathcal{C}_\epsilon$ at $a$ are also assured. Due to this reason, we prove the continuity of $\mathcal{C}_a$ only for non scalar $a\in\mathcal{A}$.*

**Theorem 3.1.3.** *Let $a\in\mathcal{A}\setminus\mathbb{C}e$. If $\epsilon_0\in(0,1)$ then $\mathcal{C}_a$ is upper hemicontinuous at $\epsilon_0$.*

*Proof.* Let $\epsilon_n\in(0,1)$ and $\lambda_n\in\sigma_{\epsilon_n}(a)$ such that $\epsilon_n\to\epsilon_0$. If there exists a subsequence $\{n_k\}$ such that $\epsilon_{n_k}<\epsilon_0$ then $\lambda_{n_k}\in\sigma_{\epsilon_0}(a)$. Since $\sigma_{\epsilon_0}(a)$ is compact, $\lambda_n$ has a limit point $\lambda\in\sigma_{\epsilon_0}(a)$. By Theorem 1.4.6, $\mathcal{C}_a$ is upper hemicontinuous at $\epsilon_0$.

Suppose there are only finitely many $\epsilon_n<\epsilon_0$, then there exists a decreasing subsequence $\{\epsilon_{n_k}\}$ with $\epsilon_{n_k}>\epsilon_0$. Fix $n_1$, clearly $\lambda_{n_k}\in\sigma_{\epsilon_{n_1}}(a)$ for all $n_k\geq n_1$. Since $\sigma_{\epsilon_{n_1}}(a)$ is compact, $\lambda_{n_k}$ has a limit point $\lambda$ in $\sigma_{\epsilon_{n_1}}(a)$. We prove that $\lambda\in\sigma_{\epsilon_0}(a)$. If

$\lambda \in \sigma(a)$ then $\lambda \in \sigma_{\epsilon_0}(a)$. If $\lambda_{n_k}$'s are not from $\sigma(a)$ and $\lambda \notin \sigma(a)$, then by our assumption $\lambda_{n_k} \in \sigma_{\epsilon_{n_k}}(a)$ and

$$\|a - \lambda_{n_k}\| \left\|(a - \lambda_{n_k})^{-1}\right\| \geq \frac{1}{\epsilon_{n_k}} \text{ for all } n_k \geq n_1.$$

By Lemma 3.1.1, graph of $\mathcal{C}_a$ is closed and we have, $(\epsilon_{n_k}, \lambda_{n_k}) \to (\epsilon_0, \lambda)$ as $k \to \infty$. Hence $\lambda \in \sigma_{\epsilon_0}(a)$. By Theorem 1.4.6, $\mathcal{C}_a$ is upper hemicontinuous at $\epsilon_0$. □

From the above theorem, it is evident that the upper hemicontinuity of $\mathcal{C}_a$ at $\epsilon_0$ follows without any additional assumptions. Unlike the upper hemicontinuity, the lower hemicontinuous of $\mathcal{C}_a$ at given $\epsilon_0$ needs an extra assumption. Theorem 3.1.4 reveals that the assumption is nothing but emptiness of interior of $L_{\epsilon_0}(a)$. The intuition to think about the level set of condition spectrum in the continuity of the condition spectrum arise from the article [28].

**Theorem 3.1.4.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. The map $\mathcal{C}_a$ is lower hemicontinuous at $\epsilon_0 \in (0,1)$ if and only if the interior of $L_{\epsilon_0}(a)$ is empty.*

*Proof.* Assume that the interior of $L_{\epsilon_0}(a)$ is empty. Let $V$ be a nonempty open subset in $\mathbb{C}$ such that $\sigma_{\epsilon_0}(a) \cap V \neq \emptyset$. For any $\epsilon > \epsilon_0$, by Theorem 1.2.4, $\sigma_\epsilon(a) \cap V \neq \emptyset$.

If there exists $\epsilon \in (0,1)$ with $\epsilon < \epsilon_0$ such that $\sigma_\epsilon(a) \cap V \neq \emptyset$, then choose $\delta = \dfrac{\epsilon_0 - \epsilon}{2}$. By Theorem 1.2.4, $\sigma_\epsilon(a) \cap V \neq \emptyset$ for all $\epsilon \in (\epsilon_0 - \delta, \epsilon_0 + \delta)$, this yields the lower hemicontinuity of $\mathcal{C}_a$ at $\epsilon_0$.

Suppose, for every $\epsilon < \epsilon_0$, we have

$$\sigma_\epsilon(a) \cap V = \emptyset. \tag{3.3}$$

Now for any $\mu \in \sigma_{\epsilon_0}(a) \cap V$,

$$\frac{1}{\epsilon_0} \leq \|a - \mu\| \left\|(a - \mu)^{-1}\right\| < \frac{1}{\epsilon_0 - \frac{1}{m}} \text{ for all } m > \frac{1}{\epsilon_0}.$$

This gives us, $\mu \in L_{\epsilon_0}(a)$. There exists $r > 0$ such that $B(\mu, r) \subseteq V$. By Equation (3.3)

$$\|a - \lambda\| \left\|(a - \lambda)^{-1}\right\| \leq \frac{1}{\epsilon_0} \text{ for all } \lambda \in B(\mu, r).$$

Since, interior of $L_{\epsilon_0}(a)$ is empty, there exists $\lambda_0 \in B(\mu, r)$ such that

$$\|a - \lambda_0\| \left\|(a - \lambda_0)^{-1}\right\| < \frac{1}{\epsilon_0}$$

Take $\Omega = \Omega_0 = B(\mu, r)$ and apply Lemma 2.3.1, to the analytic vector valued maps

$$\psi_1 : \Omega_0 \to \mathcal{A} \text{ defined by } \psi_1(\lambda) = (a - \lambda)$$

and

$$\psi_2 : \Omega_0 \to \mathcal{A} \text{ defined by } \psi_2(\lambda) = (a - \lambda)^{-1}.$$

We get $\mu \notin L_{\epsilon_0}(a)$ which is a contradiction.

Conversely, we assume that $\mathcal{C}_a$ is lower hemicontinuous at $\epsilon_0$, we prove that the interior of $L_{\epsilon_0}(a)$ is empty. Suppose if interior of $L_{\epsilon_0}(a)$ is nonempty, then there exists $\mu \in L_{\epsilon_0}(a)$ and $r > 0$ such that $B(\mu, r) \subsetneq L_{\epsilon_0}(a) \subseteq \sigma_{\epsilon_0}(a)$. Clearly $B(\mu, r) \cap \sigma_\epsilon(a) = \emptyset$ for all $0 < \epsilon < \epsilon_0$. This is a contradiction to $\mathcal{C}_a$ is lower hemicontinuous at $\epsilon_0$. □

**Theorem 3.1.5.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. The correspondence $\mathcal{C}_a$ is continuous at $\epsilon_0 \in (0, 1)$ if and only if the interior of $L_{\epsilon_0}(a)$ is empty.*

*Proof.* Immediate from Theorem 3.1.3 and Theorem 3.1.4. □

**Corollary 3.1.6.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. The map $\mathfrak{C}_a : (0, 1) \to K(\mathbb{C})$ defined by $\mathfrak{C}_a(\epsilon) = \sigma_\epsilon(a)$ is continuous at $\epsilon_0 \in (0, 1)$ if and only if the interior of $L_{\epsilon_0}(a)$ is empty.*

*Proof.* Immediate from Theorem 3.1.5 and Theorem 1.4.11. □

### 3.1.2 Continuity of the correspondences $\mathcal{C}_\epsilon$ and $\mathcal{C}$

We start this subsection by recollecting the upper hemicontinuity of the map $\mathcal{C}_\epsilon$. The following is the corresponding result.

**Theorem 3.1.7** (Theorem 2.7(5) in [29])**.** *The correspondence $\mathcal{C}_\epsilon$ is upper hemicontinuous at $a \in \mathcal{A}$.*

**Theorem 3.1.8.** *The function $\mathfrak{C} : (0, 1) \times \mathcal{A} \to K(\mathbb{C})$ defined by $\mathfrak{C}(\epsilon, b) = \sigma_\epsilon(b)$ is continuous at $(\epsilon_0, \lambda) \in (0, 1) \times \mathcal{A}$ where $\lambda \in \mathbb{C}$.*

*Proof.* Consider a sequence $\{(\epsilon_n, a_n)\}$ where $a_n \in \mathcal{A}$ and $\epsilon_n \in (0, 1)$ such that $(\epsilon_n, a_n) \to (\epsilon_0, \lambda)$ as $n \to \infty$. This implies $a_n \to \lambda, \epsilon_n \to \epsilon$ as $n \to \infty$. We claim that $\sigma_{\epsilon_n}(a_n) \to \sigma_\epsilon(\lambda)$ as $n \to \infty$ in the Hausdroff metric on $K(\mathbb{C})$. We first prove this theorem for $\lambda = 0$. If $\lambda = 0$ then $\sigma_\epsilon(\lambda) = \{0\}$ for all $\epsilon \in (0,1)$. We observe the following,

$$\begin{aligned}
H\left(\sigma_{\epsilon_n}(a_n), \sigma_\epsilon(\lambda)\right) &= H\left(\sigma_{\epsilon_n}(a_n), \{0\}\right). \\
&= \max\left\{ \sup_{\lambda\in\sigma_{\epsilon_n}(a_n)} d(\lambda, \{0\}), \sup_{\mu\in\{0\}} d(\mu, \sigma_{\epsilon_n}(a_n)) \right\}. \\
&= \max\left\{ \sup_{\lambda\in\sigma_{\epsilon_n}(a_n)} \inf |\lambda|, \sup_{\mu\in\{0\}} \inf_{\lambda\in\sigma_{\epsilon_n}} |\mu - \lambda| \right\} \\
&= \sup\left\{|\mu| : \mu \in \sigma_{\epsilon_n}(a_n)\right\}.
\end{aligned}$$

By Property 9 of Theorem 1.2.4

$$\sup\left\{|\mu| : \mu \in \sigma_{\epsilon_n}(a_n)\right\} \leq \frac{1+\epsilon_n}{1-\epsilon_n}\|a_n\|.$$

Since $\|a_n\| \to 0$ and the sequence $\left\{\dfrac{1+\epsilon_n}{1-\epsilon_n}\right\}$ is bounded, we have

$$H\left(\sigma_{\epsilon_n}(a_n), \sigma_\epsilon(\lambda)\right) \to 0 \text{ as } n \to \infty.$$

Suppose $\lambda \neq 0$, then we consider the sequence, $b_n = a_n - \lambda$. By our assumption $b_n \to 0$ and by the above argument the proof of this theorem follows. □

**Corollary 3.1.9.** *Let $a \in \mathcal{A}$ such that $a = \lambda$ for some $\lambda \in \mathbb{C}$ and $\epsilon_0 \in (0,1)$. The correspondence $\mathcal{C}$ is continuous at $(\epsilon_0, a) \in (0,1) \times \mathcal{A}$. Furthermore, $\mathcal{C}_a$ is continuous at $\epsilon_0$.*

*Proof.* Follows from Theorem 3.1.8 and Theorem 1.4.11. □

The following lemma is crucial in proving the joint continuity of condition spectrum map. This gives an upper bound for certain perturbed condition spectrum. Proof of this lemma is similar to the proof of Theorem 2.3 (7) in [28].

**Lemma 3.1.10.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ with $\eta := \inf\{\|\mu - a\| : \mu \in \mathbb{C}\}$ and $\epsilon \in (0,1)$. Let $n_0 \in \mathbb{N}$ such that $n_0\eta > 2$. If $b \in \mathcal{A}$ such that $\|b\| < \min\left\{\dfrac{1-\epsilon}{n_0}, \epsilon\eta\right\}$ then $\sigma_\epsilon(a+b) \subseteq \sigma_{\epsilon+n_0\|b\|}(a)$.*

*Proof.* If $b = 0$ then the result is immediate. Assume $b \neq 0$. Suppose $\lambda \notin \sigma_{\epsilon+n_0\|b\|}(a)$, then $\|a-\lambda\| \left\|(a-\lambda)^{-1}\right\| < \dfrac{1}{\epsilon + n_0\|b\|}$. It follows that

$$\left\|(a-\lambda)^{-1}\right\| < \frac{1}{(\epsilon + n_0\|b\|)\,\|a-\lambda\|}. \tag{3.4}$$

We observe the following,

$$\begin{aligned} \|(a+b-\lambda)-(a-\lambda)\| &= \|b\| \\ &< (\epsilon + n_0\|b\|)\,\eta \\ &\leq (\epsilon + n_0\|b\|)\,\|a-\lambda\|\,. \end{aligned}$$

By Equation (3.4), $\|(a+b-\lambda)-(a-\lambda)\| < \dfrac{1}{\left\|(a-\lambda)^{-1}\right\|}$. Hence, $\lambda \notin \sigma(a+b)$. Next

$$\begin{aligned} \left\|(a+b-\lambda)^{-1} - (a-\lambda)^{-1}\right\| &= \left\|(a+b-\lambda)^{-1}\left((a-\lambda)-(a+b-\lambda)\right)(a-\lambda)^{-1}\right\| \\ &\leq \left\|(a+b-\lambda)^{-1}\right\| \|b\| \left\|(a-\lambda)^{-1}\right\|. \end{aligned}$$

By Equation (3.4),

$$\left\|(a+b-\lambda)^{-1} - (a-\lambda)^{-1}\right\| < \frac{\|(a+b-\lambda)^{-1}\|\,\|b\|}{(\epsilon + n_0\|b\|)\,\|a-\lambda\|}.$$

Now, $\left\|(a+b-\lambda)^{-1}\right\| \|(a+b-\lambda)\|$

$$\begin{aligned} &= \|(a+b-\lambda)\| \left\|(a+b-\lambda)^{-1} - (a-\lambda)^{-1} + (a-\lambda)^{-1}\right\| \\ &< \|(a+b-\lambda)\| \frac{\|(a+b-\lambda)^{-1}\|\,\|b\|}{(\epsilon + n_0\|b\|)\,\|a-\lambda\|} + \frac{\|(a+b-\lambda)\|}{(\epsilon + n_0\|b\|)\,\|a-\lambda\|}. \end{aligned}$$

From the above inequality, it follows that

$$\left\|(a+b-\lambda)^{-1}\right\| \|(a+b-\lambda)\| \left(1 - \frac{\|b\|}{(\epsilon + n_0\|b\|)\,\|a-\lambda\|}\right) < \frac{\|(a+b-\lambda)\|}{(\epsilon + n_0\|b\|)\,\|a-\lambda\|}.$$

After simplification,

$$\left\|(a+b-\lambda)^{-1}\right\| \|(a+b-\lambda)\| < \frac{\|(a+b-\lambda)\|}{(\epsilon + n_0\|b\|)\,\|a-\lambda\| - \|b\|}.$$

Since,

$$\begin{aligned}
(\epsilon + n_0\|b\|)\,\|a-\lambda\| - \|b\| &= \epsilon\,\|a-\lambda\| + n_0\|b\|\,\|a-\lambda\| - \|b\|. \\
&\geq \epsilon\,\|a-\lambda\| + n_0\eta\|b\| - \|b\|. \\
&= \epsilon\,\|a-\lambda\| + (n_0\eta - 1)\,\|b\|. \\
&\geq \epsilon\,\|a-\lambda\| + \|b\|. \\
&> \epsilon\,\|a-\lambda\| + \epsilon\|b\|. \\
&\geq \epsilon\,\|a+b-\lambda\|\,.
\end{aligned}$$

Hence, $\lambda \notin \sigma_\epsilon(a+b)$. □

Next, we establish the continuity of $\mathfrak{C}$ and $\mathfrak{C}_\epsilon$ (which are defined in Theorem 3.1.11) whose range space $K(\mathbb{C})$ with Hausdroff metric by assuming $\mathfrak{C}_a$ (defined in Theorem 3.1.11) is continuous for all $a \in \mathcal{A}$. The central ideal of the proof can be found in Theorem 4.3 of [28].

**Theorem 3.1.11.** *Suppose the map*

$$\mathfrak{C}_a : (0,1) \to K(\mathbb{C}) \textit{ defined by } \mathfrak{C}_a(\epsilon) = \sigma_\epsilon(a)$$

*is continuous at $\epsilon_0$ for every $a \in \mathcal{A}$, then the map*

$$\mathfrak{C}_{\epsilon_0} : \mathcal{A} \to K(\mathbb{C}) \textit{ defined by } \mathfrak{C}_{\epsilon_0} = \sigma_{\epsilon_0}(a)$$

*is continuous at $a$ with respect to the norm on $\mathcal{A}$ and the map*

$$\mathfrak{C} : (0,1) \times \mathcal{A} \to K(\mathbb{C}) \textit{ defined by } \mathfrak{C}(\epsilon, a) = \sigma_\epsilon(a)$$

*is continuous at $(\epsilon_0, a)$ with respect to the metric defined in 3.2.*

*Proof.* Suppose $a = \lambda$ for some $\lambda \in \mathbb{C}$, then conclusion follows from Theorem 3.1.8. Assume that $a \in \mathcal{A} \setminus \mathbb{C}e$ and $\epsilon_0 \in (0,1)$. For $r > 0$, consider the open ball

$$B\left(\sigma_{\epsilon_0}(a), r\right) \coloneqq \left\{E \subseteq \mathbb{C} : H\left(\sigma_{\epsilon_0}(a), E\right) < r\right\}.$$

Since $\mathcal{C}_a$ is continuous at $\epsilon_0$, there exists $\delta \in (0,1)$ with $\epsilon_0 - \delta > 0$ and $\epsilon_0 + \delta < 1$ such that,

$$\sigma_\epsilon(a) \in B\left(\sigma_{\epsilon_0}(a), \frac{r}{2}\right) \text{ for all } \epsilon \in (\epsilon_0 - \delta, \epsilon_0 + \delta). \tag{3.5}$$

Take

$$\eta := \inf \{\|\mu - a\| : \mu \in \mathbb{C}\}.$$

Note that $\eta > 0$ and for any $b \in B\left(a, \frac{\eta}{2}\right)$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|b-\lambda\| &= \|b-a+a-\lambda\| \geq \|a-\lambda\| - \|b-a\| \\ &\geq \frac{\eta}{2}. \end{aligned}$$

Hence

$$\inf_{b \in B\left(a, \frac{\eta}{2}\right)} \{\inf \{\|b-\lambda\| : \lambda \in \mathbb{C}\}\} \geq \frac{\eta}{2}. \tag{3.6}$$

Let $n_0 \in \mathbb{N}$ such that $n_0 \frac{\eta}{2} > 2$. Choose $\nu > 0$ such that

$$\nu < \min\left\{\frac{\epsilon_0}{4n_0}, \frac{\delta}{4n_0}, \frac{1-\epsilon_0}{4n_0}, \frac{\epsilon_0 \eta}{4}\right\}.$$

We claim that, if $(\epsilon, b) \in B((\epsilon_0, a), \nu)$ then $\sigma_\epsilon(b) \in B(\sigma_{\epsilon_0}(a), r)$. Let $(\epsilon, b) \in B((\epsilon_0, a), \nu))$ We calculate $H(\sigma_{\epsilon_0}(a), \sigma_\epsilon(b))$. Take $c = b - a$, we observe the following,

$$\begin{aligned} \epsilon_0 - \nu - n_0\|c\| &\geq \epsilon_0 - \frac{\epsilon_0}{4n_0} - n_0 \frac{\epsilon_0}{2n_0} \\ &\geq \frac{\epsilon_0}{2} - \frac{\epsilon_0}{4n_0} \\ &\geq \frac{\epsilon_0}{2}\left(1 - \frac{1}{2n_0}\right) \\ &> 0. \end{aligned} \tag{3.7}$$

$$\begin{aligned} \epsilon_0 + \nu + n_0\|c\| &\leq \epsilon_0 + \frac{1-\epsilon_0}{4n_0} + n_0 \frac{1-\epsilon_0}{4n_0} \\ &\leq \epsilon_0 + \frac{1-\epsilon_0}{4} + \frac{1-\epsilon_0}{4} \\ &< \epsilon_0 + 1 - \epsilon_0 \\ &< 1. \end{aligned} \tag{3.8}$$

$$\begin{aligned} \epsilon_0 - \nu + \|c\| &< \epsilon_0 - \nu + \nu - |\epsilon - \epsilon_0| \\ &= \epsilon_0 - |\epsilon - \epsilon_0| \\ &\leq \epsilon_0 + (\epsilon - \epsilon_0). \\ &= \epsilon < 1. \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\epsilon + \|c\| &< \epsilon + \nu - |\epsilon - \epsilon_0| \\ &< \epsilon + \nu + (\epsilon_0 - \epsilon) \\ &< 1.\end{aligned}$$

By Equation (3.7) and Equation (3.8) the sets $\sigma_{\epsilon_0 - \nu - n_0\|c\|}(a)$ and $\sigma_{\epsilon_0+\nu+n_0\|c\|}(a)$ are well defined.
Now, we prove the following inclusion,

$$\sigma_{\epsilon_0-\nu-n_0\|c\|}(b-c) \subseteq \sigma_{\epsilon_0-\nu}(b).$$

Take $\eta_0 = \inf\{\|b-\lambda\| : \lambda \in \mathbb{C}\}$. By our choice of $b$, we have $\eta_0 \geq \frac{\eta}{2}$. Since $n_0\frac{\eta}{2} > 2$, we must have $n_0\eta_0 > 2$. Now

$$\begin{aligned}\frac{1-\epsilon}{n_0} &= \frac{1-(\epsilon_0 - \nu - n_0\|c\|)}{n_0} \\ &= \frac{1-\epsilon_0}{n_0} + \frac{\nu}{n_0} + \|c\| \\ &> \|c\|.\end{aligned}$$

and

$$\begin{aligned}(\epsilon_0 - \nu - n_0\|c\|)\,\eta_0 &> \left(\epsilon_0 - \frac{\epsilon_0}{4} - n_0\frac{\epsilon_0}{4n_0}\right)\eta_0 \\ &= \frac{\epsilon_0}{2}\eta_0 \\ &\geq \frac{\epsilon_0\eta}{4} \\ &> \nu \\ &> \|c\|.\end{aligned}$$

Hence $\|c\| < \min\left\{\frac{1-\epsilon}{n_0}, \epsilon\eta_0\right\}$. Apply Lemma 3.1.10 for $a = b, b = -c, \eta_0 = \inf\{\|b-\lambda\| : \lambda \in \mathbb{C}\}$ and $\epsilon = \epsilon_0 - \nu - n_0\|c\|$ we have

$$\sigma_{\epsilon_0-\nu-n_0\|c\|}(b-c) \subseteq \sigma_{\epsilon_0-\nu}(b).$$

Next, we prove the following inclusion,

$$\sigma_\epsilon(a+c) \subseteq \sigma_{\epsilon+n_0\|c\|}(a). \tag{3.10}$$

We observe that,

$$\begin{aligned}
\frac{1-\epsilon}{n_0} &> \frac{1-(\epsilon_0+\nu-\|c\|)}{n_0} \\
&= \frac{1-\epsilon_0}{n_0} - \frac{\nu}{n_0} + \frac{\|c\|}{n_0} \\
&> 3\frac{1-\epsilon_0}{4n_0} + \frac{\|c\|}{n_0} \\
&> 3\nu + \frac{\|c\|}{n_0} \\
&> \|c\|.
\end{aligned}$$

and

$$\begin{aligned}
\epsilon\eta &> (\epsilon_0-\nu+\|c\|)\eta \\
&> (\epsilon_0-\frac{\epsilon_0}{4}+\|c\|)\eta = \left(\frac{3\epsilon_0}{4}+\|c\|\right)\eta \\
&> 3\nu+\|c\|\eta > \|c\|.
\end{aligned}$$

Hence $\|c\| < \min\left\{\frac{1-\epsilon}{n_0}, \epsilon\eta\right\}$. Apply Lemma 3.1.10 for $a = a, b = c, \eta = \inf\{\|a-\lambda\| : \lambda \in \mathbb{C}\}$ and $\epsilon \in (0,1)$ such that $|\epsilon - \epsilon_0| < \nu$, we have

$$\sigma_\epsilon(a+c) \subseteq \sigma_{\epsilon+n_0\|c\|}(a).$$

Now,

$$\begin{aligned}
\sigma_{\epsilon_0-\nu-n_0\|c\|}(a) &= \sigma_{\epsilon_0-\nu-n_0\|c\|}(a-b+b). \\
&= \sigma_{\epsilon_0-\nu-n_0\|c\|}(b-c) && [\text{ by assumption } c=b-a] \\
&\subseteq \sigma_{\epsilon_0-\nu-n_0\|c\|+n_0\|c\|}(b) && [\text{ by Lemma 3.1.10 by Equation (3.6)}]. \\
&\subseteq \sigma_{\epsilon_0-\nu+\|c\|}(b) && [\text{ by Theorem 1.2.4}] \\
&\subseteq \sigma_\epsilon(b) && [\text{ by Equation (3.9) and by Theorem 1.2.4}]. \\
&= \sigma_\epsilon(a+c) && [\text{ by assumption } c=b-a]. \\
&\subseteq \sigma_{\epsilon+n_0\|c\|}(a) && [\text{ by Lemma 3.1.10}] \\
&\subseteq \sigma_{\epsilon_0+\nu+n_0\|c\|}(a) && [\text{ by Equation (3.10) and by Theorem 1.2.4}].
\end{aligned}$$

From the above calculations, we got

$$\sigma_{\epsilon_0-\nu-n_0\|c\|}(a) \subseteq \sigma_\epsilon(b) \subseteq \sigma_{\epsilon_0+\nu+n_0\|c\|}(a).$$

Now, $H(\sigma_{\epsilon_0}(a), \sigma_\epsilon(b))$

$$\begin{aligned}
&= \max\left\{\sup_{\lambda\in\sigma_{\epsilon_0}(a)} d\left(\lambda, \sigma_\epsilon(b)\right), \sup_{\mu\in\sigma_\epsilon(b)} d\left(\mu, \sigma_{\epsilon_0}(a)\right)\right\}. \\
&\leq \max\left\{\sup_{\lambda\in\sigma_{\epsilon_0}(a)} d\left(\lambda, \sigma_{\epsilon_0-\nu-n_0\|c\|}(a)\right), \sup_{\mu\in\sigma_{\epsilon_0+\nu+n_0\|c\|}(a)} d\left(\mu, \sigma_{\epsilon_0}(a)\right)\right\}. \\
&\leq \max\left\{H\left(\sigma_{\epsilon_0}(a), \sigma_{\epsilon_0-\nu-n_0\|c\|}(a)\right), H\left(\sigma_{\epsilon_0+\nu+n_0\|c\|}(a), \sigma_{\epsilon_0}(a)\right)\right\}. \qquad (3.11)
\end{aligned}$$

We observe the following,

$$|\epsilon_0 - (\epsilon_0 - \nu - n_0\|c\|)| = \nu + n_0\|c\| < (n_0+1)\,\nu \leq (2n_0)\,\nu < (2n_0)\,\frac{\delta}{4n_0} = \frac{\delta}{2}.$$

Similarly, $|\epsilon_0 - (\epsilon_0 + \nu + n_0\|c\|)| = \nu + n_0\|c\| < \frac{\delta}{2}$. By Equation (3.5),

$$H\left(\sigma_{\epsilon_0}(a), \sigma_{\epsilon_0-\nu-n_0\|c\|}(a)\right) \leq \frac{r}{2} \text{ and } H\left(\sigma_{\epsilon_0}(a), \sigma_{\epsilon_0+\nu+n_0\|c\|}(a)\right) \leq \frac{r}{2}.$$

By Equation (3.11), $H(\sigma_{\epsilon_0}(a), \sigma_\epsilon(b)) < r$. This proves the theorem. □

**Corollary 3.1.12.** *Suppose $\mathcal{C}_a$ is continuous at $\epsilon_0 \in (0,1)$ then $\mathcal{C}_{\epsilon_0}$ is continuous at $a \in \mathcal{A}$ with respect to the norm on $\mathcal{A}$. Further, the correspondence $\mathcal{C}$ is jointly continuous at $(\epsilon_0, a)$.*

*Proof.* Theorem follows from Theorem 3.1.11 and Theorem 1.4.11. □

**Remark 3.1.13.** *The assumption $\mathfrak{C}_a$ and $\mathcal{C}_a$ are continuous for all $a \in \mathcal{A}$ can be replaced by interior of $L_\epsilon(a)$ is empty for given $\epsilon$ and $a$. After replacement, proof of Theorem 3.1.11 and Corollary 3.1.12 follows from Corollary 3.1.6 and Theorem 3.1.5.*

Following lemma gives a fine picture about the growth of the condition spectrum when the value of $\epsilon$ approaches $1$. With the aid of this lemma we look at the limiting behaviour of $\sigma_\epsilon(a)$ as $\epsilon \to 1$.

**Lemma 3.1.14.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$. If $K \subsetneq \mathbb{C}$ is compact with $K \cap L_1(a) = \emptyset$ then there exists an $\epsilon \in (0,1)$ such that $K \subsetneq \sigma_\epsilon(a)$.*

*Proof.* Take $\lambda \in K$. We first prove that there exists $\delta' \in (0,1)$ such that $\lambda \in \sigma_{\delta'}(a)$. If $\lambda \in \sigma(a)$ then $\lambda \in \sigma_{\delta'}(a)$ for all $0 < \delta' < 1$. If $\lambda \notin \sigma(a)$ then

$$\|a - \lambda\| \left\|(a-\lambda)^{-1}\right\| = M$$

for some $M > 1$ (Since $K \cap L_1(a) = \emptyset$). Take $\delta' = \frac{1}{M}$. It follows that $\lambda \in \sigma_{\frac{1}{M}}(a)$. Since for any $0 < \delta' < \delta < 1$,

$$\|a - \lambda\| \left\|(a-\lambda)^{-1}\right\| > \frac{1}{\delta}.$$

Hence, $\lambda$ is an interior point of $\sigma_\delta(a)$. There exists an $r_\lambda > 0$ such that $B(\lambda, r_\lambda) \subsetneq \sigma_\delta(a)$. Thus the collection $\{B(\lambda, r_\lambda) : \lambda \in K\}$ is an open cover for $K$. Since $K$ is compact, we have $K \subseteq \bigcup_{i=1}^{n} B(\lambda_i, r_{\lambda_i})$. Since each $B(\lambda_i, r_{\lambda_i}) \subsetneq \sigma_{\delta_i}(a)$, by taking $\epsilon = \max\{\delta_i : i = 1 \text{ to } n\}$, we get $K \subsetneq \sigma_\epsilon(a)$. □

**Note 3.1.15.** *For any fixed $a \in \mathcal{A}$, by Theorem 1.2.4 and note 1.4.14, we have $\lim_{\epsilon \to 0} \sigma_\epsilon(a) = \sigma(a)$.*

**Theorem 3.1.16.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ and consider the sequence $\{\epsilon_n\}$ where $\epsilon_n \in (0,1)$. If $\lim_{n\to\infty} \epsilon_n = 1$ then $\lim_{n\to\infty} \sigma_{\epsilon_n}(a) = \mathbb{C}$.*

*Proof.* Let $\lambda \in \mathbb{C} \setminus L_1(a)$. There exists a compact set $K$ such that $\lambda \in K$ and $K \cap L_1(a) = \emptyset$. By Lemma 3.1.14, $K \subset \sigma_\epsilon(a)$ for some $\epsilon \in (0,1)$. Consequently, there exists $n_0$ such that $\sigma_\epsilon(a) \subseteq \sigma_{\epsilon_{n_0}}(a)$. Hence

$$d(\lambda, \sigma_{\epsilon_n}(a)) = 0 \text{ for all } n \geq n_0.$$

and so, $\lambda \in \liminf_{n\to\infty} \sigma_{\epsilon_n}(a)$. If $\lambda \in L_1(a)$ then by Lemma 3.1.14, there exists a sequence $\{\lambda_k\}$ and a subsequence $\{\epsilon_{n_k}\}$ of $\{\epsilon_n\}$ such that $\lambda_k \in \sigma_{\epsilon_{n_k}}(a)$ and $|\lambda - \lambda_k| < \dfrac{1}{n_k}$. From this, it follows that $\lim_{k\to\infty} d\left(\lambda, \sigma_{\epsilon_{n_k}}(a)\right) = 0$. For given $\delta > 0$ there exists $n_k > 0$ such that $\dfrac{1}{n_k} < \delta$. By Theorem 1.2.4, for any $n$ which satisfies $n_k \leq n$ such that $d(\lambda, \sigma_{\epsilon_n}(a)) \leq d\left(\lambda, \sigma_{\epsilon_{n_k}}(a)\right) < \dfrac{1}{n_k} < \delta$. This gives $\lambda \in \liminf_{n\to\infty} \sigma_{\epsilon_n}(a)$. By note 1.4.13, proof of the Theorem follows. □

Next, we prove the uniform continuity of condition spectrum map defined on set of all normal matrices for a fixed $\epsilon \in (0,1)$. Due to this, Theorem 3.1.17 identifies the condition spectrum set for the given normal matrix. The uniform continuity is discussed in Theorem 3.1.18. In the following two results $M_n(\mathbb{C})$ denote the the set of all $n \times n$ complex matrices with $2-$norm (see example 1.1.3 (4) in chapter 1).

**Theorem 3.1.17.** *Suppose $A \in M_n(\mathbb{C})$ is normal and $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ then*

$$\sigma_\epsilon(A) = \bigcup_{i,j\in\underline{k}} \overline{B}\left(\frac{\lambda_i - \epsilon^2\lambda_j}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}|\lambda_i - \lambda_j|\right). \tag{3.12}$$

*where $\underline{k} = \{1, 2, \cdots, k\}$.*

*Proof.* Assume that $A$ is normal and $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$. Since $A$ is normal, it is unitarily diagonalizable, and note that the 2-norm is unitarily invariant, we observe the following,

$$\begin{aligned} \lambda \in \sigma_\epsilon(A) \setminus \sigma(A) \quad &\Leftrightarrow \quad \|(A-\lambda)\| \left\|(A-\lambda)^{-1}\right\| \geq \frac{1}{\epsilon} \\ &\Leftrightarrow \quad \frac{\max_{i\in\underline{k}} |\lambda - \lambda_i|}{\min_{j\in\underline{k}} |\lambda - \lambda_j|} \geq \frac{1}{\epsilon}. \\ &\Leftrightarrow \quad \frac{|\lambda - \lambda_m|}{|\lambda - \lambda_l|} \geq \frac{1}{\epsilon}(\text{ for some } l, m \in \underline{k}). \end{aligned}$$

Consider the following inequality

$$\epsilon\,|\lambda - \lambda_m| \geq |\lambda - \lambda_l|$$

Apply $\lambda = x + iy$, $\lambda_m = a_r + ia_i$ and $\lambda_l = b_r + ib_i$ and square both sides. We get,

$$\epsilon^2\left[(x-a_r)^2 + (y-a_i)^2\right] \geq (x-b_r)^2 + (y-b_i)^2$$

Rearrange and expand to get

$$(1-\epsilon^2)x^2 + (1-\epsilon^2)y^2 - 2(b_r - \epsilon^2 a_r)x - 2(b_i - \epsilon^2 a_i)y + |\lambda_l|^2 - \epsilon^2|\lambda_m|^2 \leq 0.$$

Now complete the square.

$$\begin{aligned} &(1-\epsilon^2)\left(x - \frac{b_r - \epsilon^2 a_r}{1-\epsilon^2}\right)^2 + (1-\epsilon^2)\left(y - \frac{b_i - \epsilon^2 a_i}{1-\epsilon^2}\right)^2 \\ &\quad \leq \quad \frac{(b_r - \epsilon^2 a_r)^2 + (b_i - \epsilon^2 a_i)^2}{(1-\epsilon^2)} - (|\lambda_l|^2 - \epsilon^2|\lambda_m|^2) \\ &\quad = \quad \frac{\epsilon^2}{(1-\epsilon^2)}|\lambda_m - \lambda_l|^2. \end{aligned}$$

From here,

$$\left|\lambda - \frac{\lambda_l - \epsilon^2\lambda_m}{1-\epsilon^2}\right| \le \frac{\epsilon}{1-\epsilon^2}|\lambda_m - \lambda_l|$$

Hence, $\lambda \in \sigma_\epsilon(A) \setminus \sigma(A)$ if and only if $\lambda \in \bigcup_{i,j\in\underline{k}, i\neq j} \overline{B}\left(\frac{\lambda_i - \epsilon^2\lambda_j}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}|\lambda_i - \lambda_j|\right)$.
Since $\sigma_\epsilon(A)$ has no isolated points, we have

$$\sigma_\epsilon(A) = \bigcup_{i,j\in\underline{k}} \overline{B}\left(\frac{\lambda_i - \epsilon^2\lambda_j}{1-\epsilon^2}, \frac{\epsilon}{1-\epsilon^2}|\lambda_i - \lambda_j|\right).$$

□

**Theorem 3.1.18.** *Let $A$ and $B$ are two normal matrices. If $\epsilon \in (0,1)$ then*

$$H\left(\sigma_\epsilon(A), \sigma_\epsilon(B)\right) \le \frac{1+\epsilon}{1-\epsilon}\|A-B\|.$$

*Proof.* By Theorem 17, Lecture 18 of [7],

$$H(\sigma(A), \sigma(B)) \le \|A-B\|.$$

Let $\mu \in \sigma_\epsilon(A)$. Since, $A$ and $B$ are normal, by Equation (3.12) $\mu = \frac{\mu_i - \epsilon^2\mu_j}{1-\epsilon^2} + \delta_1$ for some $\delta_1 \le \frac{\epsilon}{1-\epsilon^2}\left|\mu_i - \mu_j\right|$ for some $\mu_i, \mu_j \in \sigma(A)$. For any $\lambda_k, \lambda_l \in \sigma(B)$, consider the following scalar

$$\lambda = \frac{\lambda_k - \epsilon^2\lambda_l}{1-\epsilon^2} + \frac{\epsilon}{1-\epsilon^2}\left|\lambda_k - \lambda_l\right|$$

Clearly, $\lambda \in \sigma_\epsilon(B)$ and

$$\begin{aligned}
|\lambda - \mu| &= \left|\left(\frac{\mu_i - \epsilon^2\mu_j}{1-\epsilon^2} - \frac{\lambda_k - \epsilon^2\lambda_l}{1-\epsilon^2}\right) + \delta_1 - \frac{\epsilon}{1-\epsilon^2}\left|\lambda_k - \lambda_l\right|\right| \\
&\le \left|\frac{\mu_i - \epsilon^2\mu_j}{1-\epsilon^2} - \frac{\lambda_k - \epsilon^2\lambda_l}{1-\epsilon^2}\right| + \frac{\epsilon}{1-\epsilon^2}\left(|\mu_i - \mu_j| - |\lambda_k - \lambda_l|\right) \\
&\le \left|\frac{(\mu_i - \lambda_k) + \epsilon^2(\lambda_l - \mu_j)}{1-\epsilon^2}\right| + \frac{\epsilon}{1-\epsilon^2}\left(|\mu_i - \lambda_k| + |\mu_j - \lambda_l|\right) \\
&\le \frac{(1+\epsilon)^2\|A-B\|}{1-\epsilon^2} \qquad \text{(By Theorem Theorem 17 in Lecture 18 of [7])} \\
&= \frac{1+\epsilon}{1-\epsilon}\|A-B\|.
\end{aligned}$$

Hence the proof. □

## 3.2 Continuity of the correspondences $\mathcal{LC}_\epsilon, \mathcal{LC}_a$ and $\mathcal{LC}$

This section deals with the results for continuity of level sets of $\epsilon-$ condition spectrum and role of empty interior of level sets in continuity. Since, the level set is compact proper subset of the condition spectrum, using the sub correspondence concept ( see 1.4.8) we derive many of our results.

For $0 < \epsilon < 1$ and $a \in \mathcal{A}$ the following level set correspondence arise naturally

$$\mathcal{LC} : (0,1) \times \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{LC}(\epsilon, a) = L_\epsilon(a).$$

where $((0,1) \times \mathcal{A})$ is a metric space with the metric defined in 3.2. We get two more correspondences from $\mathcal{C}$. They are as follows, fix $a \in \mathcal{A}$, then

$$\mathcal{LC}_a : (0,1) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{LC}_a(\epsilon) = L_\epsilon(a).$$

Fix $\epsilon \in (0,1)$, then

$$\mathcal{LC}_\epsilon : \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{LC}_\epsilon(a) = L_\epsilon(a).$$

Note that, if $a = \lambda$ for some $\lambda \in \mathbb{C}$ then $L_\epsilon(a) = \emptyset$ for any $\epsilon \in (0,1)$. Thus it is trivial that the correspondence $\mathcal{LC}_a, \mathcal{LC}_\epsilon$ and $\mathcal{LC}$ are continuous at $a$ and at given $\epsilon \in (0,1)$. Because of this reason, we concentrate only on the non scalar elements in $\mathcal{A}$.

**Lemma 3.2.1.** *The graph of the correspondence $\mathcal{LC}$ is closed.*

*Proof.* Consider the sequence $\{(\epsilon_n, a_n), \lambda_n\}$ in $Gr\,(\mathcal{LC})$ and $((\epsilon_0, a), \lambda) \in ((0,1) \times \mathcal{A}) \times \mathbb{C}$. Suppose, $((\epsilon_n, a_n), \lambda_n) \to ((\epsilon_0, a), \lambda)$ as $n \to \infty$ then $\epsilon_n \to \epsilon_0, a_n \to a$ and $\lambda_n \to \lambda$ as $n \to \infty$. Clearly

$$\left\|(a_n - \lambda_n)^{-1}\right\| = \frac{1}{\epsilon_n \left\|(a_n - \lambda_n)\right\|}$$

and $\epsilon_n \left\|(a_n - \lambda_n)\right\| \to \epsilon_0 \left\|a - \lambda\right\|$. If we take $b_n = (a_{n_k} - \lambda_{n_k})$ and $b = a - \lambda$ then by Lemma 5 in [37], $a - \lambda$ is invertible. Since $\left\|(a_n - \lambda_n)^{-1}\right\| \to \|(a-\lambda)^{-1}\|$, $\lambda \in L_{\epsilon_0}(a)$. Hence, the Graph of $\mathcal{LC}$ is closed. □

**Remark 3.2.2.** *For $\epsilon \in (0,1)$ and $a \in \mathcal{A} \setminus \mathbb{C}e$, it is evident that $L_\epsilon(a) \subsetneq \sigma_\epsilon(a)$. By Lemma 3.2.1, we understand that the correspondence $\mathcal{LC}_a, \mathcal{LC}_\epsilon$ and $\mathcal{LC}$ are closed subcorrespondences of $\mathcal{C}_a, \mathcal{C}_\epsilon$ and $\mathcal{C}$ respectively.*

**Theorem 3.2.3.** *Let $a \in \mathcal{A} \setminus \mathbb{C}e$ and $\epsilon_0 \in (0,1)$. The map $\mathcal{LC}_a$ is upper hemicontinuous at $\epsilon_0$ and $\mathcal{LC}_{\epsilon_0}$ is upper hemicontinuous at $a$.*

*Proof.* Follows from Remark 3.2.2, Theorem 1.4.9, Theorem 3.1.3 and Theorem 3.1.7. □

In general the correspondence $\mathcal{LC}_a$ and $\mathcal{LC}_\epsilon$ need not to be lower hemicontinuous even though the interior of $L_\epsilon(a)$ is empty for given $a$ and $\epsilon$. By looking at a particular Banach algebra $\mathcal{A}$ and an appropriate element $a \in \mathcal{A}$, we furnish an example to show the correspondence $\epsilon \mapsto L_\epsilon(a)$ is not lower hemicontinuous at some $\epsilon_0$ and for a fixed $\epsilon_0 \in (0,1)$ the correspondence $b \mapsto L_\epsilon(b)$ is not lower hemicontinuous at $a$.

**Example 3.2.4.** *Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm*

$$\|x\|_* = |x_0| + \sup_{n \neq 0} |x_n| \textit{ where } x = \left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right),$$

*where the box represents the zero$^{th}$ coordinate of an element in $\ell_\infty(\mathbb{Z})$. For $M > 2$, consider an operator $A \in B(\ell_\infty(\mathbb{Z}))$ such that*

$$A\left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right) = \left(\cdots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{M}}, x_2, x_3, \cdots\right).$$

*We first show that interior of $L_{\epsilon_0}(A)$ is empty for $\epsilon_0 = \dfrac{1}{M+1}$.*
*It is proved in Theorem 3.1 of [44] that*

$$\left\|(A-\lambda)^{-1}\right\| = M \textit{ for } |\lambda| < \min\left\{\frac{1}{M}, \frac{1}{2} - \frac{1}{M}\right\}. \tag{3.13}$$

*Take $r = \min\left\{\dfrac{1}{M}, \dfrac{1}{2} - \dfrac{1}{M}\right\}$. From Example 4.9 in [28], we have*

$$\sigma(A) = \{z \in \mathbb{C} : |z| = 1\}$$

*and*

$$\left\|(A-\lambda)^{-1}\right\| \geq M \textit{ for } |\lambda| < 1. \tag{3.14}$$

*It is easy to see, with unit vectors $y = (y_k)_{k=-\infty}^{\infty}$ such that*

$$y_k = \begin{cases} 1, \textit{ for } k = 1, 2 \\ 0, \textit{ otherwise} \end{cases}$$

*and* $z = (z_k)_{k=-\infty}^{\infty}$ *such that* $z_k = \begin{cases} 1, \textit{ for } k = 1, 4 \\ -\overline{\lambda} \textit{ for } k = 3 \\ 0, \textit{ otherwise.} \end{cases}$ *where* $0 < |\lambda| < 1$ *that*

$$\|A\| = \frac{1}{M} + 1 \text{ and } \|A - \lambda\| \geq \frac{1}{M} + 1 + |\lambda|^2 \text{ for } 0 < |\lambda| < 1. \tag{3.15}$$

*By Equation(3.13) and Equation(3.15), we get* $\|A\|\|A^{-1}\| = M + 1$ *and by Equation(3.14) and Equation(3.15) we have*

$$\|A - \lambda\| \left\|(A - \lambda)^{-1}\right\| \geq M + 1 + M|\lambda|^2 > M + 1 \text{ for } 0 < |\lambda| < 1. \tag{3.16}$$

*Thus* $B(0,1) \cap L_{\epsilon_0}(A) = \{0\}$. *Hence interior of* $L_{\epsilon_0}(A)$ *is empty in the set* $B(0,1)$. *By Corollary 4.3 in [45], interior of* $L_{\epsilon_0}(A)$ *is empty in the set* $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$.
*Now, we show that the correspondence* $\epsilon \mapsto L_\epsilon(A)$ *is not lower hemicontinuous at* $\epsilon_0$.
*For any* $\epsilon > \dfrac{1}{M+1}$, *it is clear that* $L_\epsilon(A) \cap B(0,r) = \emptyset$ *but* $L_{\epsilon_0}(A) \cap B(0,r) \neq \emptyset$. *Hence the correspondence* $\mathcal{LC}_A$ *is not lower hemicontinuous at* $\epsilon_0$.
*Next, we prove the correspondence* $\mathcal{LC}_{\epsilon_0}$ *is not lower hemicontinuous at* $A$.
*Let* $\delta > 0$ *and consider the set* $\{S \in B(\ell_\infty(\mathbb{Z})) : \|A - S\| < \delta\}$. *Choose* $N \in \mathbb{N}$ *such that* $N > M > 2$ *and* $\dfrac{1}{M} - \dfrac{1}{N} < \delta$. *Take* $B \in B(\ell_\infty(\mathbb{Z}))$ *such that*

$$B\left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right) = \left(\cdots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{N}}, x_2, x_3, \cdots\right).$$

*By Example 4.9 in [28],* $\sigma(B) = \{z \in \mathbb{C} : |z| = 1\}$. *Apply* $A = B$ *and* $M = N$ *in Equations (3.13), (3.14),(3.15) and (3.16), we get*

$$\left\|(B - \lambda)^{-1}\right\| = N \text{ for } |\lambda| < \min\left\{\frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right\}$$

*and*

$$\|B - \lambda\| \left\|(B - \lambda)^{-1}\right\| \geq N + 1 > M + 1 \text{ for } 0 \leq |\lambda| < 1.$$

*Thus* $L_{\epsilon_0}(B) \cap B(0,r) = \emptyset$ *but* $\|A - B\| < \delta$ *and* $L_{\epsilon_0}(A) \cap B(0,r) \neq \emptyset$. *Hence the correspondence* $\mathcal{LC}_{\epsilon_0}$ *is not lower hemicontinuous at* $A$.

Next, we pay our attention to continuity of map $\mathcal{LC}$. We show that empty interior of $L_\epsilon(a)$ is sufficient condition for the continuity.

**Theorem 3.2.5.** *Let* $(\epsilon_0, a)$ *in* $(0,1) \times \mathcal{A}$. *If interior of* $L_{\epsilon_0}(a)$ *is empty then* $\mathcal{LC}$ *is jointly* *upper hemicontinuous at* $(\epsilon_0, a)$.

*Proof.* Assume $a \in \mathcal{A} \setminus \mathbb{C}e$. Consider the sequence $(\epsilon_n, a_n)$ in $(0,1) \times \mathcal{A}$ such that $(\epsilon_n, a_n) \to (\epsilon_0, a)$ and the sequence $\{\lambda_n\}$ in $L_{\epsilon_n}(a_n)$. Since $\lambda_n \in \sigma_{\epsilon_n}(a_n)$, by Theorem 3.1.8, the sequence $\{\lambda_n\}$ has a limit point $\lambda \in \sigma_{\epsilon_0}(a)$. Consequently there exists a subsequence $\{\lambda_{n_k}\}$ such that $\lambda_{n_k} \to \lambda$

Clearly

$$\left\|(a_{n_k} - \lambda_{n_k})^{-1}\right\| = \frac{1}{\epsilon_{n_k} \left\|(a_{n_k} - \lambda_{n_k})\right\|}$$

and $\epsilon_{n_k} \left\|(a_{n_k} - \lambda_{n_k})\right\| \to \epsilon_0 \left\|a - \lambda\right\|$. If we take $b_n = (a_{n_k} - \lambda_{n_k})$ and $b = a - \lambda$ then by Lemma 5 in [37], $a - \lambda$ is invertible. By Lemma 3.2.1, $\lambda \in L_{\epsilon_0}(a)$. $\square$

# Chapter 4

# $(p, q)$ outer generalized pseudo spectrum

The theme of this chapter is to understand the $(p, q)$ outer generalized pseudo spectrum by studying the interior of its level sets. For $\epsilon > 0$ and $a \in \mathcal{A}$, the $\epsilon-$ pseudo spectrum is defined as

$$\Lambda_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \left\|(a-\lambda)^{-1}\right\| \geq \frac{1}{\epsilon}\right\}.$$

It is clear from the definition that, while computing the pseudo spectrum, if we assure that the set

$$\left\{\lambda \in \mathbb{C} : \left\|(a-\lambda)^{-1}\right\| = \frac{1}{\epsilon}\right\}. \tag{4.1}$$

does not have any interior point then it is easy to trace it out. Problems and answers related to interior of level set of pseudo spectrum can be seen in [23], [28],[43] and [44].

In this chapter we study the elementary properties of a more generalized pseudo spectra and the interior of its level sets. Hence, the established results in this chapter are more general as compared with the results in [23], [28] and [43]. The objects in this chapter are independent of the first two chapters but with a common base idea .

The concept of generalized inverses arise in the history of mathematics while solving the system of linear equation $Ax = b$ when $A$ is rectangular or non invertible. It is well known that there are more than one generalized inverse of $A$. Among them the outer generalized inverse is also one. Let $A$ be an $m \times n$ matrix.

If there exists a $n \times m$ matrix $B$ such that

$$BAB = B$$

then we say $B$ is the outer generalized inverse of $A$. We denote $B$ as $A^{(2)}$. See [5] for more details of all kinds of generalized inverses and its applications.

For a given $m \times n$ matrix $A$, the outer generalized inverse $B$ with prescribed range space $P$ and null space $Q$ is discussed in [5]. In this case, we denote the $B$ as $A^{(2)}_{P,Q}$. This expected weighted outer generalized inverse assures the uniqueness but not the existence. outer generalized inverse with prescribed range space and null space for a given operator $T$ defined on a Banach space $X$ is given in [12]. This can be extended to Banach algebras.

Consider two idempotent elements $p, q \in \mathcal{A}$ i.e. $p^2 = p$ and $q^2 = q$.

**Definition 4.0.6.** *([25], Definition 1.1) Let $a \in \mathcal{A}$. An element $b \in \mathcal{A}$ satisfying,*

$$bab = b, ba = p \text{ and } e - ab = q$$

*will be called a $(p, q)$ outer generalized inverse of $a$ and it is denoted by $a^{(2)}_{p,q}$.*

In [25], Kolundžija introduced the concept of the $(p, q) - \epsilon$-pseudo spectrum of $a$ in $\mathcal{A}$. Let $\epsilon > 0$ and $a \in \mathcal{A}$. The $(p, q) - \epsilon$-pseudo spectrum is defined as

$$\Lambda^{(2)}_{(p,q)-\epsilon}(a) := \left\{ \lambda \in \mathbb{C} : (a - \lambda)^{(2)}_{p,q} \text{ does not exist or } \left\|(a - \lambda)^{(2)}_{p,q}\right\| \geq \frac{1}{\epsilon} \right\}.$$

In the same paper, Kolundžija studied about $(p, q) - \epsilon$-pseudo spectrum of elements of the Banach algebra which are in the block matrix form. For the geometric understanding of $(p, q)-\epsilon$-pseudo spectrum, because of the inequalities in $(p, q)-\epsilon$-pseudo spectrum and in order to understand it, one has to know more about its boundary set. It is clear that the boundary sets are subsets of the set (see Theorem 4.1.15),

$$L^{(2)}_{(p,q)-\epsilon}(a) = \left\{ \lambda \in \mathbb{C} : \left\|(a - \lambda)^{(2)}_{p,q}\right\| = \frac{1}{\epsilon} \right\}.$$

The above set is called level set of $(p, q) - \epsilon$ pseudo spectrum. In computational point of view, if we are sure that the level sets do not contain any interior point then it can help us to trace out the boundary sets of $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$. Because of the reasons so far discussed, here we study about the interior property of $L^{(2)}_{(p,e-p)-\epsilon}(a)$ for given $a \in \mathcal{A}$.

Preliminary subsection concentrates on the non emptiness of $\Lambda^{(2)}_{(p,e-p)-\epsilon}(a)$ and the analyticity of $(p, e-p)$ resolvent map. Second subsection focuses on the interior property of the level set of $(p, e-p)-\epsilon$ pseudo spectrum set. Theorems which are in this section ( Theorem 4.2.3, Theorem 4.2.6 ) are extended version of the results of Globevnik. Using these results we prove $(p,q)-\epsilon$ pseudo spectrum has finite number of components and each component has nonempty intersection with $(p,q)$ spectrum (Theorem 4.2.9). Example is constructed to show that $L^{(2)}_{(p,e-p)-\epsilon}(a)$ may have nonempty interior ( Example 4.2.10 ) for some Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$.

## 4.1 Elementary topological properties

The main aim of this subsection is to prove the non-emptiness of $(p, e-p)-\epsilon$ pseudo spectrum. In order to achieve this, we first introduce some terminologies and establish basic results regarding the $(p, e-p)$ resolvent set.

**Definition 4.1.1.** *For an element $a \in \mathcal{A}$, the $(p,q)$-resolvent set is defined as*

$$\rho^{(2)}_{p,q}(a) := \left\{\lambda \in \mathbb{C} : (a-\lambda)^{(2)}_{p,q} \text{ exists } \right\}.$$

*The complement of the set $\rho^{(2)}_{p,q}(a)$ is called $(p,q)$-spectrum and it is denoted by $\sigma^{(2)}_{p,q}(a)$. The map $\lambda \mapsto (a-\lambda)^{(2)}_{p,q}$ defined from $\rho^{(2)}_{p,q}(a)$ to $\mathcal{A}$ is called the $(p,q)$- resolvent map.*

From now onwards, we consider the idempotent $p \neq 0$ and $p \neq e$ and we fix the idempotent element $q := e-p$. If $\lambda \in \rho^{(2)}_{p,q}(a)$ then we denote the element $(a-\lambda)^{(2)}_{p,q}$ by $R_a(\lambda)$.

**Note 4.1.2.** *For a given $a \in \mathcal{A}$, if $R_a(\lambda)$ exists for some $\lambda \in \mathbb{C}$ then from Definition 4.0.6,*

$$[R_a(\lambda)]\,(a-\lambda) = p \text{ and } (a-\lambda)\,[R_a(\lambda)] = p. \tag{4.2}$$

*By equation (4.2), $ap = pa$. Consequently, if $ap \neq pa$ then $\sigma^{(2)}_{p,q}(a) = \mathbb{C}$. Because of this reason, in the rest of this chapter, we assume $ap = pa$ for given $a \in \mathcal{A}$.*

**Note 4.1.3.** *If $R_a(\lambda)$ exists for some $\lambda \in \mathbb{C}$ then by equation (4.2), $R_a(\lambda)$ and $a$ commutes and $[(a-\lambda)^n]^{(2)}_{p,q}$ exists for any $n \in \mathbb{N}$. Moreover, $[(a-\lambda)^n]^{(2)}_{p,q} = [R_a(\lambda)]^n$.*

**Note 4.1.4.** *If $\lambda \in \sigma^{(2)}_{p,q}(a)$ then we assume that $\|R_a(\lambda)\| = \infty$.*

The following lemma and theorem is an analog of the well known result that $\rho(a)$ is nonempty open subset of $\mathbb{C}$ for any $a \in \mathcal{A}$.

**Lemma 4.1.5.** *Let $a \in \mathcal{A}$. If $\lambda \in \rho(a)$ then $\lambda \in \rho_{p,q}^{(2)}(a)$*

*Proof.* It is easy to see that $R_a(\lambda) = p(a-\lambda)^{-1}$ for any $\lambda \in \rho(a)$ □

**Theorem 4.1.6.** *The set $\rho_{p,q}^{(2)}(a)$ is a nonempty open subset of $\mathbb{C}$, for any $a \in \mathcal{A}$.*

*Proof.* By Lemma 4.1.5, $\rho_{p,q}^{(2)}(a)$ is nonempty. Take $\mu \in \rho_{p,q}^{(2)}(a)$, for any $\lambda \in \mathbb{C}$ satisfies

$$|\mu - \lambda| < \frac{1}{\|R_a(\mu)\|}$$

we have $e + [R_a(\mu)]\left((a-\lambda) - (a-\mu)\right)$ is invertible. From equation (4.2),

$$(a-\lambda)\left[R_a(\mu)\right](a-\mu) = (a-\mu)\left[R_a(\mu)\right](a-\lambda).$$

Hence by Theorem 4.1 in [17], $\lambda \in \rho_{p,q}^{(2)}(a)$. □

If $\lambda \in \mathbb{C}$ is from the $(p,q)$ spectrum set then Corollary 4.1.7 explains the nature of $(p,q)$ resolvent norm in a neighborhood of $\lambda$.

**Corollary 4.1.7.** *Let $\{\lambda_n\}$ be a sequence from $\rho_{p,q}^{(2)}(a)$. If $\lambda_n \to \lambda$ for some $\lambda \in \sigma_{p,q}^{(2)}(a)$ then $\|R_a(\lambda_n)\| \to \infty$.*

*Proof.* Suppose $\|R_a(\lambda_n)\| \leq M$ for some $M \in \mathbb{R}$ then $\dfrac{1}{\|R_a(\lambda_n)\|} \geq \dfrac{1}{M}$. Since $\lambda_n \to \lambda$, for the real number $\dfrac{1}{M+1}$, there exists $n_0 \in \mathbb{N}$ such that

$$|\lambda - \lambda_n| < \frac{1}{M+1} < \frac{1}{M} \leq \frac{1}{\|R_a(\lambda_n)\|} \text{ for all } n \geq n_0.$$

By Theorem 4.1.6, $\lambda \in \rho_{p,q}^{(2)}(a)$. This is a contradiction. □

**Theorem 4.1.8.** *The map $f : \rho_{p,q}^{(2)}(a) \to \mathcal{A}$ defined by $f(\lambda) = [R_a(\lambda)]^n$ is analytic for each $n \in \mathbb{N}$.*

*Proof.* We first prove this theorem for $n = 1$. For any $\lambda, \mu \in \rho_{p,q}^{(2)}(a)$, by Theorem 4.2 (a) in [17],

$$[R_a(\lambda)] - [R_a(\mu)] = (\lambda - \mu)\left[R_a(\lambda)\right]\left[R_a(\mu)\right]. \tag{4.3}$$

Fix $\mu \in \rho_{p,q}^{(2)}(a)$ and consider the open set $B\left(\mu, \dfrac{1}{\|R_a(\mu)\|}\right)$. By Theorem 4.1.6, $B\left(\mu, \dfrac{1}{\|R_a(\mu)\|}\right)$ is a subset of $\rho_{p,q}^{(2)}(a)$. Since $e - [R_a(\mu)](\lambda - \mu)$ is invertible for any $\lambda \in B\left(\mu, \dfrac{1}{\|R_a(\mu)\|}\right)$ and from Equation (4.3),

$$R_a(\lambda) = \sum_{n=0}^{\infty} (\lambda - \mu)^n \left[R_a(\mu)\right]^{n+1}.$$

Hence the map $\lambda \mapsto R_a(\lambda)$ is analytic. The map $\lambda \mapsto [R_a(\lambda)]^n$ is also analytic because it is the product of $n$ analytic functions of the form $\lambda \mapsto [R_a(\lambda)]$. $\square$

The following are some examples of $(p, q)$-resolvent set and $(p, q)$-spectrum for given $a \in \mathcal{A}$ and $p \in \mathcal{A}$.

**Example 4.1.9.** *Let $a = \lambda$ for some $\lambda \in \mathbb{C}$. It is easy to see, $\rho^{(2)}_{p,q}(a) = \mathbb{C} \setminus \{\lambda\}$ and $\sigma^{(2)}_{p,q}(a) = \{\lambda\}$.*

Our next example shows that $\rho^{(2)}_{p,q}(a)$ may have multiple components.

**Example 4.1.10.** *Consider the set $E = \{z \in \mathbb{C} : 4 \leq |z| \leq 5\} \cup \{z \in \mathbb{C} : 6 \leq |z| \leq 7\}$. Take the operator $T \in B\left(\ell^2(\mathbb{N})\right)$ with,*

$$T(e_{2i-1}) = r_i e_{2i-1} \textit{ and } T(e_{2i}) = q_i e_{2i} \textit{ for all } i \in \mathbb{N}$$

*where $\{e_i : i \in \mathbb{N}\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$, $\{r_i \in \mathbb{C} : i \in \mathbb{N}\}$ is countable dense subsets of $\{z \in \mathbb{C} : 4 \leq |z| \leq 5\}$ and $\{q_i \in \mathbb{C} : i \in \mathbb{N}\}$ is a countable dense subset of $\{z \in \mathbb{C} : 6 \leq |z| \leq 7\}$. Take the projection operator $P \in B\left(\ell^2(\mathbb{N})\right)$*

$$P(e_{2i-1}) = e_{2i-1} \textit{ and } P(e_{2i}) = 0 \textit{ for all } i \in \mathbb{N}$$

*Take $Q = I - P$. It is evident that $PT = TP$ and $\sigma(T) = E$. By Lemma 4.1.5,*

$$\{z \in \mathbb{C} : |z| > 7\} \cup \{z \in \mathbb{C} : 5 < |z| < 6\} \cup \{z \in \mathbb{C} : |z| < 4\} \subset \rho^{(2)}_{(P,Q)}(T).$$

*We prove, $\{z \in \mathbb{C} : 4 \leq |z| \leq 5\} \subseteq \sigma^{(2)}_{(P,Q)}(T)$. Suppose $R_T(r_i)$ exists for some $r_i$, then from the equation $[R_T(r_i)]\,(T - r_i) = P$,*

$$Ker(T - r_i) \cap Ran(P) = \{0\}.$$

*where $Ker(T - r_i)$ denotes the null space of $T - r_i$ and $Ran(P)$ denotes the range space of $P$. But for every $i \in \mathbb{N}$,*

$$e_{2i-1} \in Ker(T - r_i) \cap Ran(P)$$

*which is a contradiction. Hence* $\{r_i \in \mathbb{C} : i \in \mathbb{N}\} \subset \sigma^{(2)}_{(P,Q)}(T)$*. Since* $\{r_i \in \mathbb{C} : i \in \mathbb{N}\}$ *is dense in* $\{z \in \mathbb{C} : 4 \leq |z| \leq 5\}$ *and* $\rho^{(2)}_{(P,Q)}(T)$ *is open,*

$$\{z \in \mathbb{C} : 4 \leq |z| \leq 5\} \subseteq \sigma^{(2)}_{(P,Q)}(T)$$

*From this we also conclude,* $\rho^{(2)}_{(P,Q)}(T)$ *has more than one component.*

Next, we prove our major goal of this subsection $(p,q)-\epsilon$ pseudo spectrum is non empty. This can be settled by the help of the results we proved so far.

**Definition 4.1.11** ([25], Definition 3.3)**.** *Let* $\epsilon > 0$*. The* $(p,q)-\epsilon-$ *pseudospectrum of an element* $a \in \mathcal{A}$ *is defined as*

$$\Lambda^{(2)}_{(p,q)-\epsilon}(a) = \left\{\lambda \in \mathbb{C} : (a-\lambda)^{(2)}_{p,q} \text{ does not exist (or) } \left\|(a-\lambda)^{(2)}_{p,q}\right\| \geq \frac{1}{\epsilon}\right\}.$$

In the following example, we find the $(p,q)-\epsilon$ pseudo spectrum explicitly.

**Example 4.1.12.** *Consider the Banach algebra* $B(\mathbb{C}^n)$ *where* $\mathbb{C}^n$ *is the Euclidean space. Let* $T \in B(\mathbb{C}^n)$ *such that* $T(e_i) = \alpha_i e_i$ *for some* $\alpha_i \in \mathbb{C}$ *and the projection operator* $P \in B(\mathbb{C}^n)$ *defined as* $P(e_1) = e_1$ *and* $P(e_i) = 0$ *for all* $i = 2$ *to* $n$*. For any* $\lambda \in \mathbb{C} \setminus \{\alpha_1\}$*, we define the operator* $S(\lambda) \in B(\mathbb{C}^n)$ *by*

$$[S(\lambda)](e_i) = \begin{cases} \frac{1}{\alpha_1-\lambda}e_1 & \text{for } i = 1 \\ 0 & \text{otherwise }. \end{cases}$$

*It is easy to see,* $R^{(2)}_{(P,I-P)}(\lambda) = S(\lambda)$ *for any* $\lambda \in \mathbb{C} \setminus \{\alpha_1\}$*. Hence*

$$\Lambda^{(2)}_{(P,I-P)-\epsilon}(T) = \left\{\lambda \in \mathbb{C} : |\lambda - \alpha_1| \leq \frac{1}{\epsilon}\right\}.$$

**Theorem 4.1.13.** *The set* $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ *is a compact subset of* $\mathbb{C}$*.*

*Proof.* We know, $\Lambda^{(2)}_{(p,q)-\epsilon}(a) = \sigma^{(2)}_{p,q}(a) \cup \left\{\lambda \in \mathbb{C} \mid \|R_a(\lambda)\| \geq \frac{1}{\epsilon}\right\}$. By Theorem 4.1.6, $\sigma^{(2)}_{p,q}(a)$ is closed. The set $\left\{\lambda \in \mathbb{C} : \|R_a(\lambda)\| \geq \frac{1}{\epsilon}\right\}$ is closed, because the map $\lambda \mapsto \|R_a(\lambda)\|$ is continuous. By Lemma 4.1.5, for any $\lambda \in \rho(a) \cap \Lambda^{(2)}_{(p,q)-\epsilon}(a)$ with $|\lambda| > \|a\|$, we have

$$\epsilon \leq \|R_a(\lambda)\| = \|p(a-\lambda)^{-1}\| \leq \|p\| \frac{1}{|\lambda| - \|a\|}.$$

The above equation implies, $|\lambda| \leq \dfrac{\|p\|}{\epsilon} + \|a\|$. Hence $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ is compact. $\square$

**Theorem 4.1.14.** *The set $\sigma^{(2)}_{p,q}(a)$ is a nonempty subset of $\mathbb{C}$. In particular, $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ is a nonempty subset of $\mathbb{C}$.*

*Proof.* Suppose $\sigma^{(2)}_{p,q}(a) = \emptyset$ then $\Lambda^{(2)}_{(p,q)-\epsilon}(a) = \left\{\lambda \in \rho^{(2)}_{p,q}(a) : \|R_a(\lambda)\| \geq \frac{1}{\epsilon}\right\}$. Since $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ is compact, there exists $M > 0$ such that $\|R_a(\lambda)\| \leq M$ for all $\lambda \in \Lambda^{(2)}_{(p,q)-\epsilon}(a)$. Consequently,

$$\|R_a(\lambda)\| \leq M \text{ for every } \lambda \in \mathbb{C}. \tag{4.4}$$

Since $\rho^{(2)}_{p,q}(a) = \mathbb{C}$ and the map $\lambda \mapsto R_a(\lambda)$ is analytic and bounded on $\mathbb{C}$, by Theorem 19.1 in [4], there exists a constant $K$ such that

$$\|R_a(\lambda)\| \equiv K \text{ for all } \lambda \in \mathbb{C}.$$

If $K = 0$ then $R_a(\lambda) = 0$, this implies $p = 0$, which is a contradiction to our assumption $p \neq 0$. If $K > 0$ then $\Lambda^{(2)}_{(p,q)-K}(a)$ is unbounded, which is a contradiction to Theorem 4.1.13. Hence $\sigma^{(2)}_{p,q}(a) \neq \emptyset$. By Definition 4.1.11, $\sigma^{(2)}_{p,q}(a) \subseteq \Lambda^{(2)}_{(p,q)-\epsilon}(a)$. Thus $\Lambda^{(2)}_{(p,q)-\epsilon}(a) \neq \emptyset$. □

**Theorem 4.1.15.** *Let $a \in \mathcal{A}$ and $\epsilon > 0$. Then $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ has no isolated points.*

*Proof.* Every point in $\sigma^{(2)}_{(p,q)}(a)$ is an interior point of $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$. Otherwise, there exists a sequence $\{\lambda_n\}$ with $\lambda_n \in \rho^{(2)}_{(p,q)}(a)$ and $\|R_a(\lambda_n)\| < \frac{1}{\epsilon}$ such that $\lambda_n \to \lambda$. This is a contradiction to Corollary 4.1.7. Since the map $\lambda \mapsto \|R_a(\lambda)\|$ is continuous, the set

$$\left\{\lambda \in \rho^{(2)}_{(p,q)}(a) : \|R_a(\lambda)\| > \frac{1}{\epsilon}\right\} \tag{4.5}$$

is open and hence every $\lambda$ which satisfies $\|R_a(\lambda)\| > \frac{1}{\epsilon}$ is an interior point of $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$. Next, we consider a point $\mu \in \Lambda^{(2)}_{(p,q)-\epsilon}(a)$ such that $\|R_a(\mu)\| = \frac{1}{\epsilon}$. If $\mu$ is an isolated point then there exists an $r > 0$ such that $\|R_a(\lambda)\| < \frac{1}{\epsilon}$ for every $\lambda \in B(\mu, r)$.Take $\Omega_0 = \Omega = B(\mu, r)$ define the following map

$$F : \Omega_0 \to \mathcal{A} \text{ defined by } F(\lambda) = R_a(\lambda).$$

We apply Theorem 1.3.4 and it gives us $\|R_a(\mu)\| < \frac{1}{\epsilon}$, which is a contradiction. □

## 4.2 Level sets of $(p, q)$- outer generalized pseudo spectrum

This subsection is devoted to the interior property of $L^{(2)}_{(p,q)-\epsilon}(a)$. With the aid of the maximum modulus principle (see Theorem 4.2.3) to the $(p, q)$ resolvent map, we establish the result $L^{(2)}_{(p,q)-\epsilon}(a)$ has empty interior in the unbounded component of $\rho^{(2)}_{p,q}(a)$. We prove a similar set of results to any non scalar operator $T$ acting on the complex uniformly convex Banach space $X$ irrespective of the size of component of $\rho^{(2)}_{p,q}(T)$. Finally, we prove $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ has finite number of components in the appropriate settings (see Theorem 4.2.9).

**Note 4.2.1.** *The set $L^{(2)}_{(p,q)-\epsilon}(a)$ is non empty. Otherwise $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ is a nonempty open as well as closed subset of $\mathbb{C}$. This is a contradiction to the fact $\mathbb{C}$ is connected.*

**Note 4.2.2.** *Let $\mu$ be a point of the boundary of $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$. By Theorem, 4.1.13 $\|R_a(\mu)\| \geq \frac{1}{\epsilon}$. Suppose $\|R_a(\mu)\| > \frac{1}{\epsilon}$, then by Theorem 4.1.15, $\mu$ is an interior point of $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$. This is a contradiction to the fact that $\mu$ is a boundary point. Hence $\mu \in L^{(2)}_{(p,q)-\epsilon}(a)$. Consequently, boundary set of $\Lambda^{(2)}_{(p,q)-\epsilon}(a)$ is a subset of $L^{(2)}_{(p,q)-\epsilon}(a)$.*

The following is a form of maximum modulus principle to the map $\lambda \mapsto [R_a(\lambda)]^n$.

**Theorem 4.2.3.** *Let $a \in \mathcal{A}$, $\Omega$ be an open subset in the unbounded component of $\rho^{(2)}_{(p,q)}(a)$ and $n \in \mathbb{N}$. For some $M > 0$, suppose $\left\|[(a-\lambda)^n]^{(2)}_{p,q}\right\| \leq M$ for all $\lambda \in \Omega$, then $\left\|[(a-\lambda)^n]^{(2)}_{p,q}\right\| < M$ for all $\lambda \in \Omega$.*

*Proof.* Let us take the unbounded component of $\rho^{(2)}_{(p,q)}(a)$ be $\Omega_0$. By note 4.1.3, for any $n \in \mathbb{N}$, $[(a-\lambda)^n]^{(2)}_{p,q} = [R_a(\lambda)]^n$ for all $\lambda \in \Omega_0$. We note the following,

$$\{\lambda \in \mathbb{C} : \|[R_a(\lambda)]^n\| \geq M\} \subseteq \left\{\lambda \in \mathbb{C} : \|R_a(\lambda)\| \geq M^{\frac{1}{n}}\right\}.$$

By Theorem 4.1.13, $\left\{\lambda \in \mathbb{C} : \|R_a(\lambda)\| \geq M^{\frac{1}{n}}\right\}$ is bounded and hence

$$\{\lambda \in \mathbb{C} : \|[R_a(\lambda)]^n\| < M\} \cap \Omega_0 \neq \emptyset.$$

Take $\mu \in \{\lambda \in \mathbb{C} : \|[R_a(\lambda)]^n\| < M\} \cap \Omega_0$. Proof follows by applying Theorem 1.3.4 to the analytic function $\lambda \mapsto [R_a(\lambda)]^n$ defined from $\Omega_0$ to $\mathcal{A}$, the open set $\Omega$ and to the point $\mu$. □

**Corollary 4.2.4.** *Let $a \in \mathcal{A}$ and $\epsilon > 0$. Then $L^{(2)}_{(p,q)-\epsilon}(a)$ has empty interior in the unbounded component of $\rho^{(2)}_{(p,q)}(a)$*

*Proof.* Follows from Theorem 4.2.3, by applying $n = 1$. $\square$

Next, we concentrate on interior of $L^{(2)}_{(p,q)-\epsilon}(T)$ where $T \in B(X)$ and $X$ is complex uniformly convex Banach space ( see Definition 2.3.5). We prove that, if $\|[R_T(\lambda)]^n\|$ is constant in an open set of $\rho^{(2)}_{p,q}(T)$ then it is the global minimum of $\|[R_T(\lambda)]^n\|$ for all $\lambda \in \rho^{(2)}_{p,q}(T)$.

**Lemma 4.2.5.** *( [24], Lemma 1.1) Let $\lambda \mapsto f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots$ be a function with values in a complex Banach space $X$, defined and analytic in a neighbourhood of the point $0$ in the complex plane. If $\|f(\lambda)\| \equiv \|a_0\|$ in a neighbourhood of the point $0$, then for each $a_i$ $(i = 1, 2, \cdots)$ an $r_i > 0$ exists such that $\|a_0 + \lambda a_i\| \leq \|a_0\|$ $(|\lambda| \leq r_i)$.*

Proof of the following theorem goes similar to the proof of the Theorem 3.4 in [8].

**Theorem 4.2.6.** *Let $T \in B(X)$ where $X$ be a complex uniformly convex Banach space and $n \in \mathbb{N}$. If $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| = 1$ in an open subset $U$ of $\rho^{(2)}_{p,q}(T)$ then $\left\|[(T-\lambda)^n]^{(2)}_{p,q}\right\| \geq 1$ for all $\lambda \in \rho^{(2)}_{p,q}(T)$.*

*Proof.* We know, $[(T-\lambda)^n]^{(2)}_{p,q} = [R_T(\lambda)]^n$. By Theorem 4.1.8, for every fixed $\lambda_0 \in U$, there exists an $r > 0$ such that the map,

$$f : B(0, r) \to B(X) \text{ defined by } f(\lambda) = [R_T(\lambda + \lambda_0)]^n$$

is analytic at $0$. Moreover, for any $\lambda \in B(0, r)$,

$$[R_T(\lambda + \lambda_0)]^n = \left[\sum_{i=0}^{\infty} [R_T(\lambda_0)]^{i+1} \lambda^i\right]^n = [R_T(\lambda_0)]^n + n\,[R_T(\lambda_0)]^{n+1}\lambda + \mathcal{O}(\lambda^2).$$

Take $a_0 = [R_T(\lambda_0)]^n$ and $a_1 = n\,[R_T(\lambda_0)]^{n+1}$. Since $\|f(\lambda)\| = \|a_0\|$, by Lemma 4.2.5, there exists $r_1 > 0$ such that

$$\left\|[R_T(\lambda_0)]^n + \lambda n\,[R_T(\lambda_0)]^{n+1}\right\| \leq 1 \text{ for all } |\lambda| \leq r_1.$$

Hence for any $\lambda \in B(0, 1)$,

$$\left\|[R_T(\lambda_0)]^n + r_1\lambda n\,[R_T(\lambda_0)]^{n+1}\right\| \leq 1. \tag{4.6}$$

There exists a sequence $\{e_k\}$ from $X$ with $\|e_k\| = 1$, such that

$$\lim_{k\to\infty} \|[R_T(\lambda_0)]^n (e_k)\| = \|[R_T(\lambda_0)]^n\| = 1. \tag{4.7}$$

Equation (4.6) implies,

$$\left\|[R_T(\lambda_0)]^n (e_k) + r_1\lambda n [R_T(\lambda_0)]^{n+1} (e_k)\right\| \leq 1. \tag{4.8}$$

Take $x_k = [R_T(\lambda_0)]^n (e_k)$ and $y_k = r_1 n [R_T(\lambda_0)]^{n+1} (e_k)$. We claim that $\lim_{k\to\infty} \|y_k\| = 0$. Suppose $\|y_k\| \geq \epsilon$ for some $\epsilon > 0$ then by Equation (4.8),

$$\|x_k + \lambda y_k\| \leq 1 \text{ for all } \lambda \in B(0,1). \tag{4.9}$$

From the definition of complex uniformly convex Banach space, there exists $\delta > 0$ such that

$$\|x_k\| \leq 1 - \delta.$$

This is a contradiction to Equation (4.7). Hence,

$$\lim_{k\to\infty} \|y_k\| = \lim_{k\to\infty} \left\|r_1 n [R_T(\lambda_0)]^{n+1} (e_k)\right\| = 0.$$

Therefore,

$$\lim_{k\to\infty} \left\|[R_T(\lambda_0)]^{n+1} (e_k)\right\| = 0. \tag{4.10}$$

For any $\lambda \in \rho^{(2)}_{p,q}(a)$, by Theorem 4.2 (a) in [17]

$$R_T(\lambda)-R_T(\lambda_0) = (\lambda-\lambda_0) [R_T(\lambda)] [R_T(\lambda_0)] = (\lambda-\lambda_0) [I + (\lambda - \lambda_0) [R_T(\lambda)]] [R_T(\lambda_0)]^2 . \tag{4.11}$$

where $I$ denotes the identity operator on $B(X)$. From Equation (4.11), it is easy to see,

$$[R_T(\lambda)]^n - [R_T(\lambda_0)]^n = B_n [R_T(\lambda_0)]^{n+1} \tag{4.12}$$

where $B_n := \sum_{j=0}^{n-1} \begin{pmatrix} n \\ j+1 \end{pmatrix} (\lambda - \lambda_0)^{j+1} (I + (\lambda - \lambda_0)R_T(\lambda))^{j+1} (R_T(\lambda_0))^j$. Since the operator $B_n$ is bounded and from the Equation (4.7), Equation (4.10),

$$\begin{aligned} \lim_{k\to\infty} \|[R_T(\lambda)]^n (e_k)\| &\geq \lim_{k\to\infty} \|[R_T(\lambda_0)]^n (e_k)\| - \lim_{k\to\infty} \left\|B_n [R_T(\lambda_0)]^{n+1} (e_k)\right\| \\ &\geq \lim_{k\to\infty} \|[R_T(\lambda_0)]^n (e_k)\| - \|B_n\| \lim_{k\to\infty} \left\|[R_T(\lambda_0)]^{n+1} (e_k)\right\| \\ &= 1. \end{aligned}$$

Hence the theorem follows. $\square$

**Corollary 4.2.7.** *Let $M > 0$, $T \in B(X)$ where $X$ be a complex uniformly convex Banach space and $n \in \mathbb{N}$. If $\left\|[(T-\lambda)^n]_{p,q}^{(2)}\right\| = M$ in an open subset $U$ of $\rho_{p,q}^{(2)}(T)$ then $\left\|[(T-\lambda)^n]_{p,q}^{(2)}\right\| \geq M$ for all $\lambda \in \rho_{p,q}^{(2)}(T)$.*

*Proof.* Suppose $\left\|[(T-\lambda)^n]_{p,q}^{(2)}\right\| = M$ in an open subset $U$ of $\rho_{p,q}^{(2)}(T)$, then

$$\left\|\left[\left(M^{\frac{1}{n}}T - M^{\frac{1}{n}}\lambda\right)^n\right]_{p,q}^{(2)}\right\| = 1 \text{ for all } \lambda \in U. \tag{4.13}$$

Consider the operator $S \coloneqq M^{\frac{1}{n}}T$. From Equation (4.13), for each $\mu \in M^{\frac{1}{n}}U$, we obtain $\left\|[(S-\mu)^n]_{p,q}^{(2)}\right\| = 1$. By Theorem 4.2.6, $\left\|[(S-\mu)^n]_{p,q}^{(2)}\right\| \geq 1$ for all $\mu \in \rho_{p,q}^{(2)}(S)$. Thus $\left\|[(T-\lambda)^n]_{p,q}^{(2)}\right\| \geq M$ for all $\lambda \in \rho_{p,q}^{(2)}(T)$. $\square$

**Corollary 4.2.8.** *Let $X$ be a complex uniformly convex Banach space. If $T \in B(X)$ then $L_{(p,q)-\epsilon}^{(2)}(T)$ has empty interior in $\rho_{(p,q)}^{(2)}(T)$.*

*Proof.* Immediate from Corollary 4.2.7 by applying $n = 1$. $\square$

**Theorem 4.2.9.** *Let $X$ be a complex uniformly convex Banach space, $T \in B(X)$. Then $\Lambda_{(p,q)-\epsilon}^{(2)}(T)$ has finite number of components and every component of $\Lambda_{(p,q)-\epsilon}^{(2)}(T)$ contains an element from $\sigma_{p,q}^{(2)}(T)$.*

*Proof.* Let $E$ be a component of $\Lambda_{(p,q)-\epsilon}^{(2)}(T)$. We first prove the following,

$$\text{if } E \cap \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\} \neq \emptyset \text{ then } E \cap \sigma_{p,q}^{(2)}(T) \neq \emptyset.$$

Assume to the contrary that $E$ is a component and $E \cap \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\} \neq \emptyset$ but $E \cap \sigma_{p,q}^{(2)}(T) = \emptyset$. Consider the set

$$G \coloneqq E \setminus \left(L_{(p,q)-\epsilon}^{(2)}(T)\right) = E \cap \left(L_{(p,q)-\epsilon}^{(2)}(T)\right)^c.$$

Note that, $G \subseteq \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\} \subseteq \left(L_{(p,q)-\epsilon}^{(2)}(T)\right)^c$. We prove that $G$ is open in $\mathbb{C}$. Let $\mu \in G$. Since $\left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\}$ is open, there exists $r_\mu > 0$ such that

$$B(\mu, r_\mu) \subseteq \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\} \subseteq \left(L_{(p,q)-\epsilon}^{(2)}(T)\right)^c$$

Since $E$ is a component, $\mu \in E$ and $B(\mu, r_\mu)$ is connected, we have $B(\mu, r_\mu) \subseteq E$. By the definition of $G$, $B(\mu, r_\mu) \subseteq G$, it follows that $G$ is open in $\mathbb{C}$. Let $\mu \in G$, hence there exists $F \in B(X)^*$ such that $F(R_T(\mu)) = \|R_T(\mu)\|$. Define

$$\psi : G \to \mathbb{C} \text{ by } \psi(\lambda) = F(R_T(\lambda)).$$

Clearly $\psi$ is well defined, analytic and also continuous on $\overline{G}$ ( closure of $G$ ). For any boundary point $\lambda$ of $G$ we have $\|R_T(\lambda)\| = \frac{1}{\epsilon}$, hence $|\psi(\lambda)| \leq \frac{1}{\epsilon}$ but at the point $\mu$, we have $|\psi(\mu)| = |F(R_T(\mu))| = \|R_T(\mu)\| > \frac{1}{\epsilon}$. This is a contradiction to Maximum Modulus Theorem.

By Corollary 4.1.7, for each $\lambda \in \sigma^{(2)}_{p,q}(T)$, there exists $r_\lambda > 0$ with $B(\lambda, r_\lambda) \subseteq \Lambda^{(2)}_{(p,q)-\epsilon}(T)$ and $\left\{B(\lambda, r_\lambda) : \lambda \in \sigma^{(2)}_{p,q}(T)\right\}$ is an open cover for $\sigma^{(2)}_{p,q}(T)$. Since $\sigma^{(2)}_{p,q}(T)$ is compact, there exists $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ such that $\sigma^{(2)}_{p,q}(T) \subseteq \bigcup\limits_{i=1}^{n} B(\lambda_i, r_{\lambda_i})$. Consequently, there exists components $C_1, C_2, \cdots, C_m$ of $\Lambda^{(2)}_{(p,q)-\epsilon}(T)$ with $m \leq n$ and each $C_i$ contains at least one $B(\lambda_i, r_{\lambda_i})$ such that

$$\sigma^{(2)}_{p,q}(T) \subseteq \bigcup_{i=1}^{n} B(\lambda_i, r_{\lambda_i}) \subseteq \bigcup_{i=1}^{m} C_i.$$

We claim that $\left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \dfrac{1}{\epsilon}\right\} \subseteq \bigcup\limits_{i=1}^{m} C_i$. For $\mu \in \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\}$, there exists $r > 0$ such that $B(\mu, r) \subseteq \left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\}$ hence $B(\mu, r) \subseteq E$ for some connected component $E$ of $\Lambda^{(2)}_{(p,q)-\epsilon}(T)$. We proved that $E \cap \sigma^{(2)}_{p,q}(T) \neq \emptyset$, it follows that $E \subseteq \bigcup\limits_{i=1}^{m} C_i$. Thus

$$\left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\} \subseteq \bigcup_{i=1}^{m} C_i.$$

Since each $C_i$ is closed in $\mathbb{C}$ and by Theorem 4.1.15, Corollary 4.2.8, we have

$$\overline{\left\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| > \frac{1}{\epsilon}\right\}} = \Lambda^{(2)}_{(p,q)-\epsilon}(T) = \bigcup_{i=1}^{m} C_i.$$

Hence the theorem follows. $\square$

The following is an example for interior of $L^{(2)}_{(p,q)-\epsilon}(a)$ can be nonempty in the bounded component of $\rho^{(2)}_{p,q}(a)$.

**Example 4.2.10.** *Consider the Banach space $\ell_\infty(\mathbb{Z})$ with norm*

$$\|x\|_* = |x_0| + \sup_{n\neq 0} |x_n| \text{ where } x = \left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right),$$

*and the box represents the zero$^{th}$ coordinate of an element in $\ell_\infty(\mathbb{Z})$. For $M > 2$, take an operator $A \in B\left(\ell_\infty(\mathbb{Z})\right)$ such that*

$$A\left(\cdots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \cdots\right) = \left(\cdots, x_{-2}, x_{-1}, x_0, \boxed{\frac{x_1}{M}}, x_2, x_3, \cdots\right). \tag{4.14}$$

*Take $R := \min\left\{\frac{1}{M}, \frac{1}{2} - \frac{1}{M}\right\}$ and from Theorem 3.1 in [43], we know that*

$$\left\|(A-\lambda)^{-1}\right\| = \left\|(A-\lambda)^{-1}(e_0)\right\|_* = M \tag{4.15}$$

*where $e_0 = \left(\cdots, 0, 0, \boxed{1}, 0, 0, \cdots\right)$ and $\lambda \in \mathbb{C}$ such that $|\lambda| < R$. Consider the Banach space $X = \ell_\infty(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})$ with norm $\|(x,y)\| = \left(\|x\|_*^2 + \|y\|_*^2\right)^{\frac{1}{2}}$. By Theorem 1.8.6 in [34], $X$ is a Banach space. We take the following operators*

$$T : X \to X \text{ defined by } T(x,y) = (A(x), A(y))$$

*where $A$ is an operator defined in Equation (4.14) and*

$$P : X \to X \text{ defined by } P(x,y) = (x,0).$$

*It is easy to see that $P^2 = P$ and $PT = TP$. By Theorem 1.8.12 in [34], $\sigma(T) = \sigma(A)$ and so we get,*

$$R_T(\lambda) = (T-\lambda)^{-1} P \text{ for all } \lambda \in \{\lambda \in \mathbb{C} : |\lambda| < R\}$$

*For any $(x,y) \in X$ with $\|(x,y)\| = 1$, we have*

$$\left\|(T-\lambda)^{-1} P(x,y)\right\| = \left\|(T-\lambda)^{-1}(x,0)\right\| = \left\|(A-\lambda)^{-1}(x)\right\|_* \leq M\|x\|_* \leq M\|(x,y)\|. \tag{4.16}$$

*and particularly for the unit vector $(e_0, 0) \in X$, we have*

$$\left\|(T-\lambda)^{-1} P(e_0,0)\right\| = \left\|(T-\lambda)^{-1}(e_0,0)\right\| = \left\|(A-\lambda)^{-1}(e_0)\right\|_* = M = M\|(e_0,0)\|. \tag{4.17}$$

*From Equation (4.16) and Equation (4.17), we get $\left\|(T-\lambda)^{-1}P\right\| = M$ for each $\lambda$ in*

$\{\lambda \in \mathbb{C} : |\lambda| < R\}$. *Thus interior of* $\{\lambda \in \mathbb{C} : \|R_T(\lambda)\| = M\}$ *is non empty.*

# Conclusion and Future work

We briefly summarize the results obtained in this thesis

## Summary

The following results are proved

1. Let $a \in \mathcal{A} \setminus \mathbb{C}e$. We obtained that the interior of $1-$ level set of condition spectrum is empty. If $a = \mu$ for some $\mu \in \mathbb{C}$ then $1-$ level set of condition spectrum is the set $\mathbb{C} \setminus \{\mu\}$ and for $0 < \epsilon < 1$, $\epsilon-$ level set of condition spectrum is the empty set.

2. Let $0 < \epsilon < 1$. If $a \in \mathcal{A} \setminus \mathbb{C}e$ then interior of $\epsilon-$ level set of condition spectrum is empty in the unbounded component of resolvent set of $a$

3. Let $X$ be a complex Banach space, $T \in B(X)$ and $0 < \epsilon < 1$. If either $X$ or $X^*$ is complex uniformly convex then $L_\epsilon(T)$ has empty interior in the resolvent set of $T$.

4. Fix $a \in \mathcal{A} \setminus \mathbb{C}e$. The correspondence

$$\mathcal{C}_a : (0,1) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}_a(\epsilon) = \sigma_\epsilon(a).$$

is continuous at $\epsilon_0 \in (0,1)$ if and only if interior of $\epsilon-$ level set of condition spectrum for $a$ is empty.

5. Suppose the correspondence

$$\mathcal{C}_a : (0,1) \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}_a(\epsilon) = \sigma_\epsilon(a)$$

is continuous at $\epsilon_0$ for every $a \in \mathcal{A}$, then the map

$$\mathcal{C}_{\epsilon_0} : \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}_{\epsilon_0} = \sigma_{\epsilon_0}(a)$$

is continuous at $a$ with respect to the norm on $\mathcal{A}$ and the map

$$\mathcal{C} : (0,1) \times \mathcal{A} \twoheadrightarrow \mathbb{C} \text{ defined by } \mathcal{C}(\epsilon, a) = \sigma_\epsilon(a)$$

is continuous at at $(\epsilon_0, a)$

6. All the level set correspondence are upper hemicontinuous but not lower hemicontinuous.

7. Let $a \in \mathcal{A}$ and $\epsilon > 0$. $(p,q) - \epsilon$ level set of outer generalized pseudo spectrum has empty interior in the unbounded component of $(p,q)$ resolvent set of $a$.

8. Let $X$ be a complex uniformly convex Banach space. If $T \in B(X)$ then $(p,q) - \epsilon$ level set of outer generalized pseudo spectrum has empty interior has empty interior in $(p,q)$ resolvent set of $T$.

# Future work

The following are some of the problems which we are interested to work in the near future.

1. Example of a Banach algebra and an element such that its $\epsilon-$ level set of condition spectrum has non empty interior (or) Example of a discontinuous condition spectrum map.

2. Explicit form of the condition spectrum for various types of operators defined on a Banach space.

3. Continuity of $(p, e-p) - \epsilon-$pseudo spectrum maps.

4. Discuss about the the interior of the level set of $(p, e-p) - \epsilon-$condition spectrum

5. Continuity of the $(p, e-p) - \epsilon-$condition spectrum maps.

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# Publications based on the thesis work

**Journal Publication**

1. D. Sukumar and S. Veeramani. *Level sets of the condition spectrum.* Ann. Funct. Anal., 8(3):314–328, 2017.
   https://projecteuclid.org/euclid.afa/1491280440

2. D. Sukumar and S. Veeramani. *Continuity of condition spectrum and its level sets.(Communicated)*

**Conference proceedings**

1. D. Sukumar and S. Veeramani. Level sets of $(p, e-p)$ outer generalized pseudo spectrum. *The Journal of Analysis*, May 2017.
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**Presentations in Conferences/ Workshops**

1) Title: Level sets of $(P, Q)$ outer generalized pseudo spectrum.
   Venue : International Conference on Analysis and its Applications-2015 (ICAA-2015) held at Aligarh Muslim University, Aligarh 202002, India during December 19-21, 2015.

2) Title : Level sets of condition spectrum.
   Venue : International Workshop on Operator Theory and its Applications-2016 (IWOTA-2016) held at Washington University, St. Louis, USA, during July 18-22, 2016.

3) Title : Level sets of $(P, Q)$ outer generalized pseudo spectrum.
   Venue : International Conference on Mathematical Analysis and its Applications (ICMAA-2016) held at IIT Roorkee, India, during November 28- December 02, 2016.

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