Some Turán-type Problems in Extremal Graph Theory

by

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ABSTRACT

Since the seminal work of Turán, the *forbidden subgraph* problem has been among the central questions in extremal graph theory. Let $\mathrm{ex}(n; F)$ be the smallest number m such that any graph on n vertices with m edges contains F as a subgraph. Then the *forbidden subgraph* problem asks to find $\mathrm{ex}(n; F)$ for various graphs F. The question can be further generalized by asking for the extreme values of other graph parameters like minimum degree, maximum degree, or connectivity. We call this type of question a Turán-type problem. In this thesis, we will study Turán-type problems and their variants for graphs and hypergraphs.

Chapter 2 contains a Turán-type problem for cycles in dense graphs. The main result in this chapter gives a tight bound for the minimum degree of a graph which guarantees existence of disjoint cycles in the case of dense graphs. This, in particular, answers in the affirmative a question of Faudree, Gould, Jacobson and Magnant in the case of dense graphs.

In Chapter 3, similar problems for trees are investigated. Recently, Faudree, Gould, Jacobson and West studied the minimum degree conditions for the existence of certain spanning caterpillars. They proved certain bounds that guarantee existence of spanning caterpillars. The main result in Chapter 3 significantly improves their result and answers one of their questions by proving a tight minimum degree bound for the existence of such structures.

Chapter 4 includes another Turán-type problem for loose paths of length three in a 3-graph. As a corollary, an upper bound for the multi-color Ramsey number for the loose path of length three in a 3-graph is achieved.

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INTRODUCTION

Extremal problems are at the very heart of graph theory. The basic question in extremal graph theory asks for density conditions of a host graph which guarantee existence of a certain subgraph. Turán's work (Turán, 1941), which is now called the Turán Theorem, gave birth to a large body of work in extremal graph theory. The *forbidden subgraph problem*, which is also known as the Turán problem, is as follows: Let $\mathrm{ex}(n; F)$ be the smallest number m such that any graph on n vertices with m edges contains F as a subgraph. Then the *forbidden subgraph* problem asks to find $\mathrm{ex}(n; F)$ for various graphs F.

The Turán problem can be extended by imposing additional conditions on the host graph. For example, we can consider host graphs which satisfy a certain minimum degree condition or a maximum degree condition or are, for example, highly connected. (See (Füredi, 1991), (Keevash, 2011) for general surveys of this area.)

This thesis contains some results on certain Turán-type problems for simple undirected graphs and 3-uniform hypergraph.

In Chapter 1, we introduce necessary concepts and definitions. Since we rely on Szemerédi's Regularity Lemma and some probabilistic method tools, we dedicate two sections of Chapter 1 to the Szemerédi's Regularity Lemma and Chernoff bound. The regularity lemma will be the key tool used in Chapter 2 to prove our main results and Chernoff bound will be used in Chapter 3.

In Chapter 2, we investigate families of vertex disjoint even cycles which are such that the sum of the sizes of those cycles is at least 2 times the minimum degree of the host graph. One of the motivations for this research is the following conjecture of Faudree, Gould, Jacobson and Magnant (Faudree *et al.*, 2016) on cycle spectra.

Conjecture 1.0.1. *Let* $S_e = \{|C| : C$ *is an even cycle contained in* $G\}$ *and* $S_o = \{|C| : C$ *is an odd cycle contained in* $G\}$*. If* G *is 2-connected graph, then* $\delta(G) = d \geq 3$ *implies that* $|S_e| \geq d-1$*, and, if, in addition* G *is not bipartite, then it implies that* $|S_o| \geq d$*.*

In (Faudree *et al.*, 2016), Conjecture 1.0.1 was confirmed for $d = 3$. The result proven in Chapter 2 gives an additional evidence for the conjecture as we prove it in the case of dense graphs which are sufficiently large. The main result of Chapter 2 is the following theorem.

Theorem 1.0.2. *For every* $0 < \alpha < \frac{1}{2}$*, there is a natural number* $N = N(\alpha)$ *such that the following holds. For any* $n_1, ..., n_l \in Z^+$ *such that* $\sum_{i=1}^{l} n_i = \delta(G)$ *and* $n_i \geq 2$ *for all* $i \in [l]$*, every* 2*-connected graph* G *of order* $n \geq N$ *and* $\alpha n \leq \delta(G) < n/2 - 1$ *contains* C *where* C *is a disjoint union of* $C_{2n_1}, \ldots, C_{2n_l}$ *or* G *is one of the graphs from Example 2.1.3 and* $n_1 = n_2 = \delta$ *or* G *is a subgraph of the graph from Example 2.1.4 and* $n_i = 2$ *for every* i*.*

In addition to answering Conjecture 1.0.1 for dense graphs, Theorem 1.0.2 gives the following corollary which can be viewed as a generalization of the Erdős-Faudree conjecture to the case when the minimum degree of the host graph is smaller than $n/2$.

Corollary 1.0.3. *For every* $0 < \alpha < \frac{1}{2}$*, there is a natural number* $N = N(\alpha)$ *such that the following holds. Every* 2*-connected graph* G *of order* $n \in Z$ *and minimum degree* $\delta \in Z$ *such that* $n \geq N$*,* $\alpha n \leq \delta < n/2 - 1$*, and* $\delta + n$ *is even contains* $\delta/2$ *disjoint cycles on four vertices.*

In Chapter 3, we consider a problem on spanning p-caterpillars. A p-caterpillar is a tree such that the graph induced by its internal vertices is a path and every internal vertex has exactly p leaves. Our research in Chapter 3 is motivated by the recent work of Faudree, Gould, Jacobson and West (Faudree *et al.*, 2017). In (Faudree *et al.*, 2017), the authors proved a couple of results about dominating paths. Another way of thinking about a spanning p-caterpillar is that it gives a very special dominating path in the host graph. The following theorem was proved in (Faudree *et al.*, 2017).

Theorem 1.0.4. *(Faudree* et al.*, 2017) For $p \in Z^+$ there exists n_0 such that for every $n \in (p+1)Z$ such that $n \geq n_0$ the following holds. If G is a graph on n vertices such that $\delta(G) \geq \left(1 - \frac{p}{(p+1)^2}\right) n$, then G contains a spanning p-caterpillar.*

One of the open problems from (Faudree *et al.*, 2017) asks about the sharpness of the minimum degree condition in Theorem 1.0.4, even in the case $p = 1$. In Chapter 3, we give a sharp bound for the minimum degree condition not only for the case $p = 1$ but for any $p \in Z^+$. Specifically, we prove the following theorem and show that the minimum degree can not be, in general, improved.

Theorem 1.0.5. *For $p \in Z^+$, there exists n_0 such that for every $n \in (p+1)Z$ with $n \geq n_0$ the following holds. If G is a graph on n vertices such that*

$$\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } n/(p+1) \text{ is even} \\ \frac{n+1}{2} & \text{if } n/(p+1) \text{ is odd and } p > 2 \\ \frac{n-1}{2} & \text{if } n/(p+1) \text{ is odd and } p \leq 2 \end{cases}$$

then G contains a spanning p-caterpillar.

In Chapter 4, we again study paths but, this time, in 3-uniform hypergraphs. The structure we investigate in Chapter 4 is a loose path of length three in a 3-uniform graph, denoted by P. We first obtain a Turán-type result for P, which is formally stated as follows:

Theorem 1.0.6. *Let $H = (V, E)$ be a connected 3-graph with $|H| = n \geq 7$ and $\Delta(H) \geq n - 2$. If $||H|| > 3n - 8$ then either H contains P or a critical vertex.*

By applying this result, we obtain our second main result of Chapter 4. Given a (hyper)graph F, the multicolor Ramsey number $R(F;r)$ is the least integer n such that every r-coloring of the edges of complete graph of order n yields a monochromatic copy of F. Our second contribution in Chapter 4 is the following bound for $R(P;r)$.

Theorem 1.0.7. $r+6 \leq R(P;r) \leq 2r$ *for* $r \geq 6$.

This result improves the result on the following theorem of Łuczak and Polcyn.

Theorem 1.0.8. *(Łuczak and Polcyn, 2017)*

$$R(P;r) \leq 2r + \sqrt{18r+1} + 2 \text{ for } r \in \mathbb{N}.$$

After our work was submitted for a publication, Polcyn and Łuczak (Łuczak and Polcyn, 2018) obtained another result which minimally improves the bound from Theorem 1.0.8 and shows that the upper bound is at most $\lambda_0 r + 7\sqrt{r}$ where $\lambda_0 = 1.97466..$.

In Chapter 5, we briefly review a list of our own results provided in the thesis and also suggest some research topics which we want to pursue in the future.

The results of this thesis has been presented in the papers (Yie *et al.*, 2018; Yie, 2017; Yie and Czygrinow, 2017).

1.1 Basic Definitions

The number of elements of a set X is denoted by $|X|$. If $|Y| = r$ then we say that Y is an r-set and if furthermore $Y \subset X$ then Y is an $r-$subset of X. Given a set X, we denote by $\mathcal{P}(X)$ the power-set of X. A graph G is an ordered pair (V, E) where V is a finite set, called the *vertex set* and denoted by $V(G)$, and E is a set of $2-$subsets of V, called the *edge set* and denoted by $E(G)$. The *order* of G is the number of vertices in G, which is denoted by $|G|$, so $|G| = |V(G)|$. The *size* of G is the number of edges in G, which is denoted by $||G||$ or $e(G)$, so $||G|| = e(G) = |E(G)|$. For two, not necessarily disjoint, sets $U, W \subseteq V(G)$, we will use $e(U, W) = ||U, W||$ to denote the number of edges in G with one endpoint in U, another in W.

An edge $\{x, y\}$ is said to join the vertices x and y and is denoted by xy. Thus xy and yx denote the same edge. We also say that x any y are *adjacent* vertices and the vertex x is *incident* with the edge $\{x, y\}$.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case, we write $G' \subset G$. If G' contains *all edges* of G that join two vertices in V' then G' is said to be the subgraph *induced* or *spanned* by V' and is denoted by $G[V']$. Thus, a subgraph G' of G is an induced subgraph if $G' = G[V(G')]$. If $V' = V$, then G' is said to be a *spanning* subgraph of G.

There are a few simple methods to construct new graphs from old ones. If $W \subset V(G)$, then $G - W$ means $G[V \setminus W]$. Similarly, if $E' \subset E(G)$ then $G' = G - E'$ means $G' = (V(G), E(G) \setminus E')$. If $W = \{w\}, E' = \{xy\}$, then the notation can be simplified to $G - w$, $G - xy$.

The term *independent* will be used along with vertices and edges. A set of vertices is *independent* if no two vertices of it are adjacent. A set of edges is *independent* if no two edges share a common vertex. A set of independent edges in a graph G is called a *matching* of G.

The set of vertices adjacent to a vertex $x \in V$, the *neighborhood* of x, is denoted by $N(v)$. Also $x \sim y$ means that the vertex x is adjacent to y, i.e xy, $x \sim y$ have the same meaning. For $U \subset V(G)$, we say $N(U) = \cup_{v \in U} N_G(v)$ the neighborhood of set U. The *degree* of x is $d(x) = |N(x)|$. If we want to emphasize that the underlying graph is G, then we write $N_G(v)$ and $d_G(v)$. The *minimum degree* of the vertices of a graph G is denoted by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. A similar convention will be applied for other functions depending on an underlying graph. A vertex of degree 0 is said to be an *isolated* vertex.

If $V(G) = \{v_1, v_2, \ldots, v_n\}$ then $(d(v_i))_{i=1}^n$ is a *degree sequence* of G. Usually, we order the vertices in such a way that the degree sequence obtained in this way is monotone increasing or monotone decreasing, for example, $\delta(G) = d(v_1) \leq \cdots \leq d(v_n) = \Delta(G)$.

A *path* is a graph P of the form

$$V(P) = \{v_0, v_1, \ldots, v_l\}, E(P) = \{v_0v_1, v_1v_2, \ldots, v_{l-1}v_l\}.$$

This path P is usually denoted by $v_0v_1 \ldots v_l$. The vertices v_0 and v_l are the *endvertices* of P and $l = e(P)$ is the *length* of P. There are several notions closely related to a path. A walk W in a graph is an alternating sequence of vertices and edges, say $v_0, e_1, v_1, e_2, \ldots, e_l, v_l$ where $e_i = v_{i-1}v_i, 0 < i \leq l$. With respect to the terminology above, W is also denoted by $v_0v_1 \ldots v_l$. If a walk $W = v_0v_1 \ldots v_l$ is such that $l \geq 3$, $v_0 = v_l$, and the vertices $v_i, 0 < i < l$, are distinct from each other and v_0, then W is said to be a *cycle*. For simplicity this cycle is denoted by $v_1v_2 \ldots v_l$. To emphasize the length of a path and cycle, we use notations P_l, C_l for a path of length l and a cycle of length l, respectively.

A graph is *connected* if, for every partition of its vertex into two non-empty subsets X and Y, there is an edge e such that both of $e \cap X$ and $e \cap Y$ are non-empty. A *maximal* connected subgraph is said to be a *component* of a graph. The *connectivity*(or *vertex-connectivity*) of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal results in a disconnected graph or in the trivial graph. For $k \geq 2$, we say that a graph G is k-connected if either G is a complete graph K_{k+1} or else it contains at least $k+2$ vertices and for any subset $W \in V(G)$ such that $|W| \leq k-1$, $G - W$ is still connected.

Along with the notions of connectedness and cycle, we will need the concept of a tree. A tree is a connected graph that does not contain a cycle.

A graph G is called an *r-partite* graph with vertex classes $V_1, V_2, \ldots, V_r$ if $V = V(G)$ is the disjoint union of $V_1, V_2, \ldots, V_r$ and for any edge $e \in E(G)$, there exists $i, j \in [r]$ such that $i \neq j$ and both of $e \cap V_i$ and $e \cap V_j$ are non-empty. The most interesting case is $r = 2$, and in this case, G is also called a *bipartite* graph. It is easily seen that every bipartite graph has no odd cycle and the converse is also true. To emphasize the two disjoint vertex subsets of a bipartite graph, we say that a graph G is a *bipartite* graph with bipartition V_1, V_2 or with *vertex classes* V_1, V_2.

By definition, a graph does not contain a *loop*, an edge joining a vertex to itself; neither does it contain *multiple* edges, that is, several edges joining the same two vertices.

Finally, we will introduce hypergraphs. A hypergraph is a pair (V, E) such that $V \cap E = \emptyset$ and E is a subset of $\mathcal{P}(V)$. If for every $e \in E(H)$, $|e| = k$, then we call H a k-uniform hypergraph or a k-graph. Almost all concepts introduced for graphs directly extend to k-uniform hypergraphs.

Hypergraphs will only be used in Chapter 4 where we will prove some results on 3-uniform hypergraphs. Hence some necessary definitions for hypergraphs will be arranged in Chapter 4.

1.2 The Regularity Lemma

Szemerédi's Regularity Lemma is one of the most powerful tools in extremal graph theory. In this section we will give a brief overview of the lemma.

The origin of the Regularity Lemma can be found in Szemerédi's paper (Szemerédi, 1975a) which contains a proof of Erdős-Turán conjecture about arithmetic progressions in dense subsets of integers. In (Szemerédi, 1975b), the lemma appeared in its current form.

Roughly speaking, the lemma claims that the vertex set of *every* graph can be partitioned into a constant number of almost equal classes such that most pairs of classes are *regular*, in the sense that the number of edges between two subsets of the classes is proportional to the total number of edges between two subsets, provided that the subsets are not too small. In order to formulate the lemma precisely, we need some definitions and notation. (Bollobás, 2013)

Given a graph $G = (V, E)$ and a pair (X, Y) of disjoint non-empty subsets of V, we write $d(X, Y) = d_G(X, Y) = \frac{e(X,Y)}{|X||Y|}$ for the *density* of the $X - Y$ edges of G. Call (X, Y) an $\epsilon-$regular pair if

$$|d(U, V) - d(X, Y)| < \epsilon,$$

whenever $U \subset X, V \subset Y$ are such that $|U| \geq \epsilon|X| > 0, |V| \geq \epsilon|Y| > 0$.

A partition $\mathcal{P} = (V_i)_{i=0}^t$ of the vertex set V is said to be an *equitable* partition with *exceptional* class V_0 if $|V_1| = |V_2| = \cdots = |V_t|$. Finally, an *$\epsilon$-regular-partition* is an equitable partition $(V_i)_{i=0}^t$ such that the exceptional class V_0 has at most $\epsilon|V|$ vertices and, with the exception of at most ϵt^2 pairs, the pairs (V_i, V_j), $1 \leq i \leq j \leq t$, are ϵ-regular.

Lemma 1.2.1 (Regularity Lemma,(Szemerédi, 1975b))**.** *For every $\epsilon > 0, m > 0$ there exist $N := N(\epsilon, m)$ and $M := M(\epsilon, m)$ such that every graph on at least N vertices has an $\epsilon-$regular partition $\{V_0, V_1, ..., V_t\}$ such that $m \leq t \leq M$.*

There are numerous reformulations of the regularity lemma. Here, we give two representative variations.

Lemma 1.2.2 (Regularity Lemma - alternative form 1)**.** *For every $\epsilon > 0, m > 0$ there exist $M := M(\epsilon, m)$ such that for every graph $G = (V, E)$ there is a partition $V = \cup_{i=1}^t V_i$ such that $m \leq t \leq M$, $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$ and, with the exception of at most ϵt^2 pairs, the pairs (V_i, V_j), $1 \leq i \leq j \leq t$, are ϵ-regular.*

Lemma 1.2.3 (Regularity Lemma - alternative form 2(degree form)). *For every $\epsilon > 0$ there exist $M := M(\epsilon, m)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a equitable partition $V = \cup_{i=1}^{t} V_i$ with the exception set V_0 such that $t \leq M, |V_0| \leq \epsilon |V|, |V_1| = \cdots = |V_t| \leq \lceil \epsilon |V| \rceil$ and there is a subgraph $G' \subset G$ with the following properties:*

- $d_{G'}(v) > d_G(v) - (d + \epsilon)|V|$ *for all* $v \in V$,
- $e(G'(V_i)) = 0$ *for all* $i \geq 1$.
- *all pairs* $G'(V_i, V_j)(1 \leq i \leq j \leq t)$ *are* ϵ*-regular, each with a density either 0 or a greater than* d.

We end up this section with introducing a useful fact, called the Slicing Lemma, which will be used in Chapter 2.

Lemma 1.2.4 (Slicing Lemma). *(Komlós and Simonovits, 1996). Let (U, V) be an ϵ-regular pair with density δ, and for some $\lambda > \epsilon$, let $U' \subset U, V' \subset V$ with $|U'| \geq \lambda |U|, |V'| \geq \lambda |V|$. Then (U', V') is an ϵ'-regular pair of density δ' where $\epsilon' = \max\{\frac{\epsilon}{\lambda}, 2\epsilon\}$ and $\delta' \geq \delta - \epsilon$.*

1.3 Probabilistic Methods

A *probability space* is a triple $(\Omega, \Sigma, \mathbf{P})$ where Ω is a set, Σ is a σ-field of subsets of Ω, $\mathbf{P}$ is a non-negative measure on Σ and $P(\Omega) = 1$. A real valued *random variable*(r.v.) X is a measurable real-valued function on Ω. If A is an event in some sample space, then $\mathbf{P}(A)$ denotes the probability of A. If X is a random variable such that $\Omega(X)$ is a discrete set, then the *expectation* of X is defined as follows:

$$E(X) = x\mathbf{P}(X = x).$$

Theorem 1.3.1 (Markov's Inequality)**.** *(Alon and Spencer, 2004) Let X be a non-negative random variable. Then for any positive real $\lambda > 0$,*

$$\mathbf{P}(X \geq \lambda) \leq \frac{E(X)}{\lambda}.$$

Now, we recall the linearity of expectations. Let $X_1, \ldots, X_n$ be random variables, $c_1, \ldots, c_n \in \mathbb{R}$, and $X = \sum_{i=1}^{n} c_i X_i$. Linearity of expectation states that

$$E(X) = \sum_{i=1}^{n} c_i E(X_i).$$

After the expectation the most vital statistic for a random variable X is the *variance*, denoted by $Var[X]$. It is defined by

$$Var(X) = E((X - E(X))^2).$$

Our next lemma, Chebyshev's Inequality can be easily derived from Markov's inequality.

Theorem 1.3.2 (Chebyshev's Inequality)**.** *(Alon and Spencer, 2004) Let X be a random variable. Then for any positive real $\lambda > 0$,*

$$\mathbf{P}(|X - E(X)| \geq \lambda) \leq \frac{Var(X)}{\lambda^2}.$$

In same cases, the bound from the above theorem can be improved significantly. This is done in Chernoff bound.

There are many different forms of Chernoff bounds, each tuned to slightly different assumptions. We only provide the statement of the bound for the simple case of a sum of independent Bernoulli trials, i.e. the case in which each random variable only takes the values 0 or 1. For example, this corresponds to the case of tossing unfair coins, each with its own probability of heads, and counting the total number of heads.

Definition 1.3.3. *Consider* n *discrete random variables* $X_1, \ldots, X_n$. *We say that* $X_1, X_2, \ldots, X_n$ *are independent if*

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2)\ldots\mathbb{P}(X_n = x_n),$$

for all $x_i, \ldots, x_n$.

Theorem 1.3.4 (Chernoff Bound - Bernoulli trials). *(Chernoff, 1952) Let* $X = \sum_{i=1}^{n} X_i$ *where* $X_i = 1$ *with probability* p_i *and* $X_i = 0$ *with probability* $1 - p_i$, *and all* X_i *are independent. Let* $\mu = E(X) = \sum_{i=1}^{n} p_i$. *Then for all* $\delta > 0$,

1. $\mathbb{P}(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2}{2+\delta}\mu}$.

2. $\mathbb{P}(X \leq (1-\delta)\mu) \leq e^{-\frac{\mu\delta^2}{2}}$.

Following simple and useful bound can be obtained by combining upper and lower tails in Theorem 1.3.4 with the setting $\delta \in (0, 1)$.

Corollary 1.3.5. *With* X *and* $X_1, \ldots, X_n$ *defined as in Theorem 1.3.4 and* $\mu = E(X)$, *for all* $0 < \delta < 1$,

$$\mathbb{P}(|X - \mu| \geq \delta\mu) \leq 2e^{\frac{-\mu\delta^2}{3}}.$$

Chapter 2

EVEN CYCLES IN DENSE GRAPHS

2.1 Introduction

Throughout this chapter we discuss simple undirected graphs, and the basic graph notations we use here are already described in Section 1.1.

For a graph G we use $c(G)$ to denote the circumference of G, $oc(G)$ ($ec(G)$) to denote the length of the longest odd (even) cycle in G. If G is a graph of minimum degree d, then $c(G) \geq d$ which is best possible. However, additional assumptions on the connectivity of G usually lead to better bounds for $c(G)$ or $ec(G)$ and $oc(G)$. For example, Dirac's theorem states that if G is a 2-connected graph on n vertices, then $c(G) \geq \min\{n, 2\delta(G)\}$. Voss and Zuluaga (Voss and Zuluaga, 1977) proved the corresponding results for $ec(G)$ and $oc(G)$.

Theorem 2.1.1. *(Voss and Zuluaga, 1977) Let G be a 2-connected graph on $n \geq 2\delta(G)$ vertices. Then $ec(G) \geq 2\delta(G)$ and $oc(G) \geq 2\delta(G) - 1$.*

Dirac's Theorem gave birth to a large body of research centered around determining the length of the longest cycle in a graph satisfying certain conditions; we direct the interested reader to, e.g., (West *et al.*, 2001). Indeed, one could even search for graphs which contain cycles of all possible lengths. Such graphs are called pancyclic, and they, too, are well studied (see, e.g., (Bondy, 1971; Bondy and Simonovits, 1974; Bondy and Vince, 1998; Brandt *et al.*, 1998)). Bondy observed that in many cases a minimum degree which implies the existence of a spanning cycle also implies that the graph is pancyclic. For example, it follows from the result in (Bondy, 1971) that if G

is a graph on n vertices with minimum degree at least $n/2$ then G is either pancyclic or $G = K_{n/2,n/2}$. It's natural to ask if analogous statements are true for graphs with smaller minimum degree. In (Gould *et al.*, 2002), Gould et. al. proved the following result.

Theorem 2.1.2. *(Gould* et al.*, 2002) For every $\alpha > 0$ there is K such that if G is graph on $n > 45K\alpha^{-4}$ vertices with $\delta(G) \geq \alpha n$, then G contains cycles of every even length from $[4, ec(G) - K]$ and every odd length from $[K, oc(G) - K]$.*

Nikiforov and Shelp (Nikiforov and Schelp, 2006) proved that if G is a graph on n vertices with $\delta(G) \geq \alpha n$, then G contains cycles of every even lengths from $[4, \delta(G) + 1]$ as well as cycles of odd lengths from $[2k - 1, \delta(G) + 1]$ where $k = \lceil 1/\alpha \rceil$ unless G is one of standard counterexamples.

Faudree, Gould, Jacobson and Magnant (Faudree *et al.*, 2016) made a conjecture which motivates our work in Chapter 2. We recall it.

Conjecture 1.0.1. *Let $S_e = \{|C| : C \text{ is an even cycle contained in } G\}$ and $S_o = \{|C| : C \text{ is an odd cycle contained in } G\}$. If G is 2-connected graph, then $\delta(G) = d \geq 3$ implies that $|S_e| \geq d - 1$, and, if, in addition G is not bipartite, then it implies that $|S_o| \geq d$.*

Another line of research which motivates our work comes from problems on 2-factors. Erdős and Faudree conjectured that every graph on $4n$ vertices with minimum degree at least $2n$ contains a 2-factor consisting of $\frac{n}{4}$ copies of C_4, cycle on four vertices. This was proved by Wang in (Wang, 2010). A special case of El-Zahar's conjecture states that any graph G on $2n$ vertices with minimum degree at least n contains any 2-factor consisting of even cycles $C_{2n_1}, \ldots, C_{2n_l}$ such that $n = \sum n_i$. It's natural to ask if analogous statements can be proved in the case when the minimum degree of G is smaller. As we will show, this is true to some extent. We will prove that for almost all values of $n_1, \ldots, n_l$ such that $\sum n_i = \delta(G)$, G indeed contains C where C is the union of disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$. There are however two obstructions, of which one is well-known, when G is a graph on n vertices with minimum degree which satisfies $\alpha n < \delta(G) < (n-1)/2$.

Example 2.1.3. *Let $l \geq 2$, $q \geq 4$ be even. We first construct graph H on $l(q-2)+3$ vertices as follows. Let $V_1, \ldots, V_l$ be disjoint sets each of size $q-2$ such that $H[V_i] = K_{q-2}$ and let u_1, u_2, u_3 be three distinct vertices and let $vu_i \in E(H)$ for every $v \in V(H) \setminus \{u_1, u_2, u_3\}$ and every $i = 1, 2, 3$. Finally let G_k be obtained from H by adding exactly k out of three possible edges between vertices from $\{u_1, u_2, u_3\}$. Then $\kappa(G_k) = 3$, $\delta(G_k) = q$ but G_k does not contain two disjoint copies of C_q. Indeed, any copy of C_q in G_k contains at least two vertices from $\{u_1, u_2, u_3\}$.*

In addition to the obstruction from Example 2.1.3, another one arises when G is very close to being a complete bipartite graph.

Example 2.1.4. *Let $q = 2k$ for some $k \in Z^+$ and let U, V be disjoint and such that $|U| = q-1, |V| = n-q+1$ where $n-q+1$ is even. Now $G[U,V] = K_{q-1,n-q+1}$, $G[U] \subset K_{q-1}$ and $G[V]$ is a perfect matching. Then G is a 2-connected graph on n vertices with $\delta(G) = q$ which doesn't have $q/2$ disjoint copies of C_4. Indeed, if there are $q/2$ disjoint copies of C_4, then at least one must contain at least three vertices from V which is not possible.*

We recall the main result of this chapter.

Theorem 1.0.2. *For every $0 < \alpha < \frac{1}{2}$, there is a natural number $N = N(\alpha)$ such that the following holds. For any $n_1, ..., n_l \in Z^+$ such that $\sum_{i=1}^{l} n_i = \delta(G)$ and $n_i \geq 2$ for all $i \in [l]$, every 2-connected graph G of order $n \geq N$ and $\alpha n \leq \delta(G) < n/2 - 1$ contains C where C is a disjoint union of $C_{2n_1}, \ldots, C_{2n_l}$ or G is one of the graphs* *from Example 2.1.3 and* $n_1 = n_2 = \delta$ *or* G *is a subgraph of the graph from Example 2.1.4 and* $n_i = 2$ *for every* i.

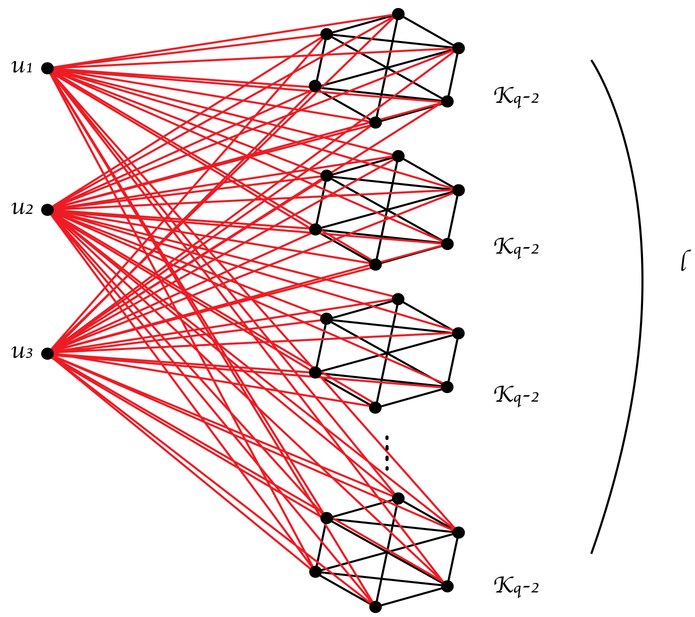

Figure 2.1: Example 2.1.3

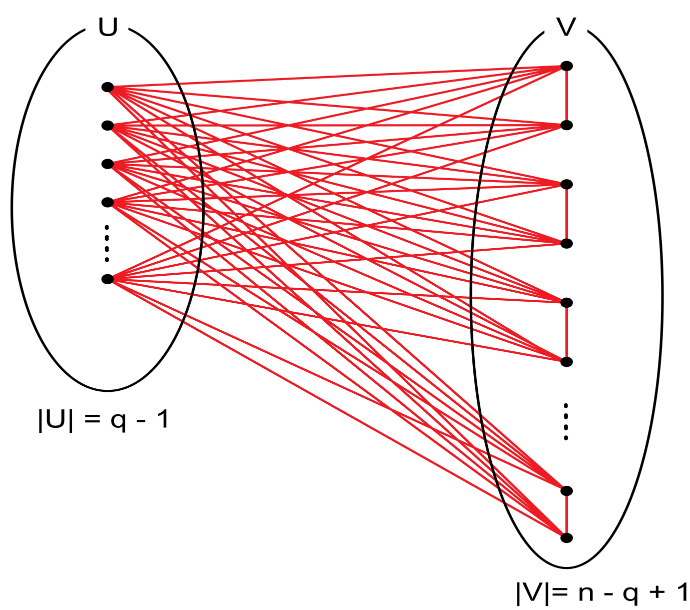

Figure 2.2: Example 2.1.4

As a corollary, we have the following fact which answers a question of Gould et. al. in the case of dense graphs.

Corollary 2.1.5. *For every* $0 < \alpha < \frac{1}{2}$, *there is a natural number* $M = M(\alpha)$ *such that the following holds. Every 2-connected graph* G *of order* $n \geq M$ *and* $\sum_{v \in V(G)} d(v) \geq \alpha n^2$ *contains a cycle of length* $2m$ *for every* $m \in \{2, \dots, \delta(G)\}$.

In addition, the following generalization of the Erdős-Faudree conjecture follows from Theorem 1.0.2.

Corollary 2.1.6. *For every* $0 < \alpha < \frac{1}{2}$, *there is a natural number* $M = M(\alpha)$ *such that the following holds. Every 2-connected graph* G *of order* $n \in Z$ *and minimum degree* $\delta \in Z$ *such that* $n \geq N$, $\alpha n \leq \delta < n/2 - 1$, *and* $\delta + n$ *is even, contains* $\delta/2$ *disjoint cycles on four vertices.*

Indeed, then $n - \delta + 1$ is odd and so it's not possible to end up in Example 2.1.4.

The proof of Theorem 1.0.2 uses the regularity method. The obstruction from Example 2.1.3 appears in the proof of the non-extremal case and the obstruction from Example 2.1.4 comes up when dealing with the extremal case. The proof is divided into several sections. In Section 2.2, we review Szemerédi's celebrated Regularity Lemma, as well as a special case of the well-known Blow-Up Lemma which is of particular use to us. In Section 2.3, we make use of regularity and results from (Czygrinow and Kierstead, 2002) to find cycles of many different lengths. Following this, we consider several cases depending on the structure of the reduced graph, and whether or not the graph is near what we call the extremal graph. The non-extremal cases are proven in Section 2.4 and Section 2.5, while the extremal cases follow in Section 2.6. The combination of these proves Theorem 1.0.2 for every sufficiently large graph.

2.2 The Blow-Up lemma

In section 1.2, we introduce some basic definitions and notations for regularity Lemma. In this section, we see more definitions, notations and some auxiliary facts for another important lemma, called Blow-up lemma.

The pair (U, V) is called $(\epsilon, \delta)-$*super-regular* if it is both $\epsilon-$regular and furthermore

$$|N(u) \cap V| \geq \delta|V| \text{ for all } u \in U, |N(v) \cap U| \geq \delta|U| \text{ for all } v \in V.$$

In addition to the regularity lemma, we will need a few well-known facts about ϵ-regular pairs (see, e.g., (Komlós and Simonovits, 1996)) and the blow-up lemma of Komlós, Sárkozy and Szemerédi from (Komlós *et al.*, 1997). Further, it is not difficult to see that an ϵ-regular pair of density δ contains a large (ϵ', δ')-super-regular pair for some δ', ϵ'.

Lemma 2.2.1. *Let $0 < \epsilon < \delta/3 < 1/3$ and let (U, V) be an $\epsilon-$regular pair with density δ. Then there exist $A' \subset A$ and $B' \subset B$ with $|A'| \geq (1-\epsilon)|A|$ and $|B'| \geq (1-\epsilon)|B|$ such that (A', B') is a $(2\epsilon, \delta - 3\epsilon)-$super-regular pair.*

Let $0 < \epsilon \ll \delta < 1$. For an ϵ-regular partition $\{V_0, V_1, \ldots, V_t\}$ of G we will consider the *reduced graph* (or *cluster graph*) of G, $R_G = R_{\epsilon,d}(V_0, V_1, \ldots, V_t)$ where $V(R_G) = \{V_1, \ldots, V_t\}$ and $V_iV_j \in E(R_G)$ if (V_i, V_j) is ϵ-regular with density at least d. When clear for the context, we will omit the subscript, writing R for the cluster graph at hand.

Finally, we conclude this section with the statement of a special case of the blow-up lemma.

Lemma 2.2.2 (Blow-Up Lemma, (Komlós *et al.*, 1997))**.** *Given $d > 0, \Delta > 0$ and $\rho > 0$ there exists $\epsilon > 0$ and $\eta > 0$ such that the following holds. Let $S = (W_1, W_2)$*

be an (ϵ, d)-super-regular pair with $|W_1| = n_1$ and $|W_2| = n_2$. If T is a bipartite graph with bipartition A_1, A_2, with maximum degree at most Δ, and T is embeddable into the complete bipartite graph K_{n_1,n_2}, then it is also embeddable into S. Moreover, for all ηn_i sized subsets $A'_i \subset A_i$ and functions $f_i : A'_i \to \binom{W_i}{\rho n_i}, i = 1, 2$, T can be embedded into S so that the image of each $a_i \in A'_i$ is in the set $f_i(a_i)$.

2.3 Preliminaries

In this section, we prove a few auxiliary facts which will be useful in the main argument. Let $V_0, V_1, \ldots, V_t$ be an ϵ-regular partition.

Lemma 2.3.1. *Let $\Delta \geq 1$ and let $0 < \epsilon \ll \delta \ll 1/\Delta$ be such that $10\epsilon\Delta \leq \delta$. Let H be graph on $\{V_1, \ldots, V_q\}$ where $|V_i| = l$ with $V_iV_j \in E(H)$ if (V_i, V_j) is ϵ-regular with density at least δ, and assume that H has maximum degree Δ. Let $\epsilon' = 5\Delta\epsilon, \delta' = \delta/2$. Then for any $i \in [t]$ there exist sets $V'_i \subset V_i$ such that $|V'_i| \geq (1 - \epsilon')l$ and (V'_i, V'_j) is (ϵ', δ')-super-regular for every $V_iV_j \in E(H)$.*

Proof. Note that $E(H)$ can be decomposed into $\Delta + 1$ matchings and so Lemma 2.3.1 follows directly from Lemma 2.2.1 and Lemma 1.2.4. □

An n-ladder, denoted by L_n is a balanced bipartite graph with vertex sets $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$ such that $\{a_i, b_j\}$ is an edge if and only if $|i - j| \leq 1$. We refer to the edges a_ib_i as rungs and the edges $\{a_1, b_1\}, \{a_n, b_n\}$ as the first and last rung respectively. Let L_{n_1}, L_{n_2} be two ladders with $n_1 \leq n_2$ and $\{a_1, b_1\}, \{a'_1, b'_1\}$ be the first rung of L_{n_1}, L_{n_2}, respectively. If there exist $a_1 - a'_1$ path P_1, $b_1 - b'_1$ path P_2 such that $\mathring{P}_1 \cap \mathring{P}_2 = \emptyset, (\mathring{P}_1 \cup \mathring{P}_2) \cap (L_1 \cup L_2) = \emptyset, |\mathring{P}_1| + |\mathring{P}_2| = 2k$ then we call $L_{n_1} \cup L_{n_2} \cup P_1 \cup P_2$ an $(n_1 + n_2, k)-$weak ladder. Obviously, an $n-$ladder is an $(n, 0)-$weak ladder.

We have following useful lemmas.

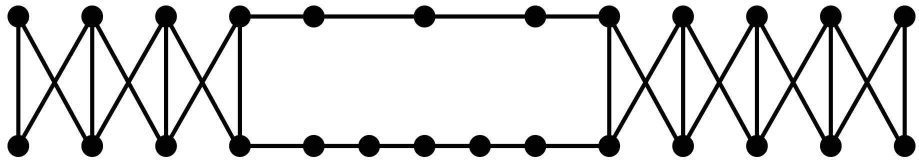

Figure 2.3: Weak ladder

Lemma 2.3.2. *Let* $2 \leq n_1 \leq \cdots \leq n_l \in Z^+$ *and let* $n = \sum_{i=1}^{l} n_i$*. If* G *contains a* $(n', k)-$*weak ladder for some* $n', k \in \mathbb{N}$ *such that* $n' \geq n + k$*, then* G *contains disjoint cycles* $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$.

Proof. If $k = 0$ then $(n', 0)$-weak ladder is actually an $L_{n'}$ where $n' \geq n$ then it is trivial that G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$. So we may assume that $k \geq 1$. Suppose G contains an $(n', k)-$weak ladder L and L consists of two ladders L_{a_1}, L_{a_2} such that $a_1 + a_2 = n'$ and disjoint paths P, Q such that $|\mathring{P}_1| + |\mathring{P}_2| = 2k$. Let $N = \{n_i : i \in [l]\}$ and choose $N' \subset N$ such that $\sum_{x \in N'} x < a_1$ and $a_1 - \sum_{x \in N'} x$ is as small as possible. Let $t := a_1 - \sum_{x \in N'} x > 0$. By the construction of N', for any $y \in N \setminus N'$, $y > t$. If $t \leq k$ then

$$a_2 = n' - a_1 = n' - (t + \sum_{x \in N'} x) \geq n' - k - \sum_{x \in N'} x \geq n - \sum_{x \in N'} x = \sum_{x \in N \setminus N'} x,$$

which implies that L_{a_2} contains remaining cycles. Hence we may assume that $t \geq k+1$ and so for any $y \in N \setminus N'$, $y \geq t + 1 \geq k + 2$. If there exists $y \in N \setminus N'$ such that $y \leq k + t + 1$, then the sub weak-ladder consisting of the last t rungs of L_{a_1}, the first rung of L_{a_2}, and P, Q contains C_{2y}. In addition,

$$n - \sum_{x \in N' \cup \{y\}} x \leq n - (a_1 - t + y) \leq (n' - k) - a_1 - 1 \leq a_2 - k - 1,$$

so $L_{a_2 - 1}$ contains remaining cycles. Otherwise, let $y = k + t + c$ where $c \geq 2$, then the sub weak-ladder consisting of the last t rungs of L_{a_1}, first c rungs of L_{a_2} and P, Q contains C_{2y}, and we have

$$n - \sum_{x \in N' \cup \{y\}} = n - (a_1 - t + y) \leq (n' - k) - a_1 - k - c \leq a_2 - k - c,$$

so L_{a_2-c} contains the remaining cycles. □

For the proof of Theorem 2.4.2, 2.5.1, our plan is to seek a (n', r)-weak ladder such that $n' \geq \delta + r$ in G, and then applying Lemma 2.3.2 to obtain disjoint cycles. In some situations, it is not possible to obtain neither a $L_{\delta(G)}$ nor a weak ladder of enough size to apply Lemma 2.3.2, the followings are useful for the cases.

Lemma 2.3.3. *Let $2 \leq n_1 \leq \cdots \leq n_l \in Z^+$ and $n = \sum_{i=1}^{l} n_i$. If G contains a $(n, 1)-$weak ladder and there exists $i \in [l]$ such that $n_i > 2$ then G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$.*

Proof. We argue by induction on n. Since $n \geq 3$, $(n, 1)$-weak ladder contains C_{2n}, we may assume that $l \geq 2$ and therefore, $n \geq 5$. If $n = 5$ then $n_1 = 2, n_2 = 3$ and then it is easy to see that G contains C_4, C_6. Now, assume for an inductive case that $n \geq 6$ and let the weak ladder contain L_{a_1}, L_{a_2} and disjoint paths P, Q which connects L_{a_1}, L_{a_2} with $|\mathring{P}_1| + |\mathring{P}_2| = 2$ where $a_1 + a_2 = n, a_1 \leq a_2$. Note that $a_2 \geq n_1$. If $a_2 = n_1$ then $l = 2$, $a_1 = n_2$ then L_{a_2} contains C_{2n_1} and L_{a_1} contains C_{2n_2}. Hence we may assume that $n_1 \leq a_2 - 1$. If $n_1 = 2$ then the first n_1 rungs of L_{a_2} contains C_{2n_1} and since there exists $i \in [l] \setminus \{1\}$ such that $n_i > 2$ by the induction hypothesis the remaining $(n - n_1, 1)$-weak ladder contains $C_{2n_2}, \ldots, C_{2n_l}$. Hence we may assume that $n_1 > 2$, i.e, for any $i \in [l]$, $n_i > 2$. Since $n_1 \leq a_2 - 1$, the first n_1 rungs of L_{a_2} contains C_{2n_1} and by the induction hypothesis, $(n - n_1, 1)$-weak ladder contains $C_{2n_2}, \ldots, C_{2n_l}$. □

Corollary 2.3.4. *Let $r \in [2]$. Let $2 \leq n_1 \leq \cdots \leq n_l \in Z^+$ and let $n = \sum_{i=1}^{l} n_i$. Suppose that G contains a $(n', k)-$weak ladder such that $n' \geq n - r$, $k \geq r, n \geq 6k + 12$ and a disjoint ladder $L_{n''}$ where $n'' \geq n/3$. If G does not contain disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$, then $l = 2$ and $\lfloor \frac{n+1-r}{2} \rfloor \leq n_1 \leq \frac{n}{2} \leq n_2 \leq \lceil \frac{n+r-1}{2} \rceil$.*

Proof. Suppose G contains an $(n', k)-$weak ladder L and L consists of two ladders L_{a_1}, L_{a_2} such that $a_1 + a_2 = n'$ and disjoint paths P, Q such that $|\mathring{P}_1| + |\mathring{P}_2| = 2k$. Let $N = \{n_i : i \in [l]\}$ and let $N_0 = \{n_i \in N : n_i \leq k + r - 1\}$. Note that $n' + k \geq n$. If there exists $N' \subset N$ such that $k + r \leq \sum_{x \in N'} x \leq \frac{n}{3}$ then $L_{n''}$ contains disjoint C_{2x} for all $x \in N'$, and then by Lemma 2.3.2, (n', k)-weak ladder contains remaining cycles. Note that $\sum_{x \in N_0} x \leq \frac{n}{3}$. Indeed, if $\sum_{x \in N_0} x > n/3$, then there exists $N_0' \subset N_0$ such that $k + r \leq (n/3 + 1) - (k + r - 1) \leq \sum_{x \in N_0'} x \leq n/3$

If $|N \setminus N_0| \geq 3$ then there exists $x \in N \setminus N_0$ such that $k + r \leq x \leq \frac{n}{3}$. If $|N \setminus N_0| = 1$, say $N \setminus N_0 = \{y\}$, then we are done as well.

Finally suppose $N \setminus N_0 = \{y_1, y_2\}$ and without loss of generality $y_1 \leq y_2$. Since $\sum_{x \in N_0} \leq n/3$, $L_{n''}$ contains C_{2x} for all $x \in N_0$. If $y_1 \leq a_2 - 1$ then $C_{2y_1} \subseteq L_{y_1} \subseteq L_{a_2 - 1}$ and the $(n' - y_1, k)$-weak-ladder obtained by deleting L_{y_1} contains C_{2y_2}. Suppose that $y_1 \geq a_1 + 1 + r$, say $y_1 = a_1 + 1 + t$, $t \geq r$. Let $s := \max\{0, t - k\}$. We have two cases.

- Let $s = 0$. Since $y_1 \leq a_1 + 1 + k$, $(a_1 + 1, k)$-weak ladder consisting of L_{a_1} and the first rung of L_{a_2} contains C_{2y_1}. Moreover,

  $$y_2 \leq n - (a_1 + 1 + r) \leq n' + r - (a_1 + 1 + r) = n' - a_1 - 1 = a_2 - 1,$$

  $L_{a_2 - 1}$ contains C_{2y_2}.

- Let $s > 0$. Since $y_1 = a_1 + 1 + k + s$, $(a_1 + 1 + s, k)$-weak ladder consisting of L_{a_1} and the first $s + 1$ rung of L_{a_2} contains C_{2y_1}. Moreover,

  $$y_2 = n - (a_1 + 1 + k + s) \leq n' + r - (a_1 + 1 + r + s) \leq (n' - a_1) - (1 + s) \leq a_2 - (1 + s),$$

  $L_{a_2 - 1 - s}$ contains C_{2y_2}.

Thus $a_2 \leq y_1 \leq a_1 + r$ and the argument also works for y_2, so we obtain

$$a_2 \leq y_1, y_2 \leq a_1 + r.$$

Moreover, if there exists $y \in N_0$, then $L_{n''}$ contains C_{2y} and $y_1 + y_2 \leq n - y \leq n - 2 \leq n' = a_1 + a_2$, and then $y_1 = a_1, y_2 = a_2$, which implies that L_{a_i} contains C_{2y_i} for $i \in [2]$. Therefore, $a_1 \leq a_2 \leq n_1 \leq n_2 \leq a_1 + r \leq a_1 + 2$, which implies that

$$\lfloor \frac{n+1-r}{2} \rfloor \leq n_1 \leq \frac{n}{2} \leq n_2 \leq \lceil \frac{n+r-1}{2} \rceil.$$

□

We will next show that in special situations it is easy to find a weak ladder. To prove our next lemma we will need the following theorem of Posa (Posa, 1962).

Theorem 2.3.5. *(Posa, 1962) [L.Posa] Let G be a graph on $n \geq 3$ vertices. If for every positive integer $k < \frac{n-1}{2}, |\{v : d_G(v) \leq k\}| < k$ and if, for odd $n, |\{v : d_G(v) \leq \frac{n-1}{2}\}| \leq \frac{n-1}{2}$, then G is Hamiltonian.*

First, we will address the case of almost complete graph.

Lemma 2.3.6. *Let $\tau \in (0, 1/10)$, $n \geq \frac{100}{\tau}$. Let $G = (V, E)$ be a graph of order n such that there exists $V' \subset V$ such that $|V'| \geq (1-\tau)n$ and for any $w \in V \setminus V', |N(w) \cap V'| \geq 4\tau|V'|$ where $V' = \{v \in V : |N(v) \cap V'| \geq (1-\tau)|V'|\}$. Let $u_1, v_1, u_2, v_2 \in V$. Then following holds:*

1. *G contains a ladder L_{n_1} in $G[V \setminus \{u_1, v_1, u_2, v_2, z\}]$ having x_1y_1, x_2y_2 as its first, last rung where $x_1 \in N(u_1), y_1 \in N(v_1), z \in N(u_1) \cap N(x_1), x_2 \in N(u_2), y_2 \in N(v_2)$ and $n_1 \geq \lfloor \frac{n-5}{2} \rfloor$.*

2. *Let $x \in N(u_1), y \in N(v_1)$ be such that $xy \in E$. G contains $L_{\lfloor \frac{n-2}{2} \rfloor}$ in $G[V \setminus \{u_1, v_1\}]$ having xy as its first rung.*

3. *Let $x \in N(u_1)$ and $y \in N(v_1)$ be such that $xy \in E$. For any $z \in N(u_1) \cap N(x)$, G contains $L_{\lfloor \frac{n-3}{2} \rfloor}$ in $G[V \setminus \{u_1, v_1, z\}]$ having x, y as its first rung.*

4. Let $x \in N(u_1) \cap N(v_1)$. G contains $L_{\lfloor \frac{n-1}{2} \rfloor}$ in $G[V \setminus \{u_1\}]$ having xv_1 as its first rung.

5. G contains a Hamilton path P having u_1, v_1 as its end vertices.

We call the vertex z in case 1,3 the parity vertex.

Proof. We will only prove part (1) as the other parts are very similar. Fix u_1, v_1, u_2, v_2. Let $V'' = V \setminus V'$. Since $|N(u_1) \cap V'|, |N(v_1) \cap V'| \geq 4\tau|V'| > 3\tau n$, there exists $x_1 \in N(u_1) \cap V', y_1 \in N(v_1) \cap V'$ such that $x_1y_1 \in E$ and the same is true for vertices u_2, v_2. Let $e_1 = x_1y_1, e_2 = x_2y_2$. Moreover, since $|N(u_1) \cap N(x_1)| \geq 3\tau n > 8$, we can choose $z \in N(u_1) \cap N(x_1)$ which is different than any other vertex already chosen.

Now, let $G' = G[V \setminus \{u_1, v_1, u_2, v_2, x_1, y_1, x_2, y_2, z\}]$ and redefine $V' := V' \cap V(G'), V'' := V'' \cap V(G')$. For any $w \in V'', |N(w) \cap V'| \geq 3\tau n - 9 > \tau n \geq |V''|$, so there exists a matching $M_1 \in E(V'', V')$ saturating V''. Note that $|M_1| \leq |V''| \leq \tau n$. Let $G'' = G[V' \setminus V(M_1)]$. Since

$$\delta(G'') \geq (1-\tau)^2 n - (2\tau n + 9) > (1 - 5\tau)n > \frac{n}{2} > \frac{|G''|}{2},$$

G'' is Hamiltonian, so there exists a matching of size $\lfloor \frac{|G''|}{2} \rfloor$ in G'', say M_2. Let $M = M_1 \cup M_2$ and define the auxiliary graph $H = (M, E')$ with the vertex set M and the edge set E' as follows: Let $e' = x'y', e'' = x''y'' \in M$. If $e', e'' \in M_1$ then $e'e'' \notin E'$. Otherwise, $e'e'' \in E'$ if $G[e', e'']$ contains a matching of size 2.

If $e \in M_1$ then $d_H(e) \geq |N_H(e) \cap M_2| > \tau n$, and for any other $e \in M$, $d_H(e) \geq |N_H(e) \cap M_2| \geq (\frac{1}{2} - 3\tau)n > \frac{|H|}{2}$. Since $|M_1| \leq \tau n$, by Theorem 2.3.5, H contains a Hamiltonian cycle C, say $C := u_1 \dots u_{n'}$ where $n' = \lfloor \frac{n-9}{2} \rfloor$.

Since $d_H(e_1), d_H(e_2) > \frac{|H|}{2}$, there exists $i \in [n']$ such that $u_i \in N(e_1), u_{i+1} \in N(e_2)$ then $e_1u_iCu_{i+1}e_2$, gives a ladder L_{n_1} having e_1, e_2 as a first, last rung where $n_1 \geq n' + 2 \geq \lfloor \frac{n-5}{2} \rfloor$. □

We call the graph satisfying the condition in Lemma 2.3.6, τ-complete graph and the vertex set V' the major set and V'' the minor set.

Fact 2.3.7. *If G is τ-complete then for any subset U of the minor set, $G[V \setminus U]$ is still τ-complete.*

Moreover,

Corollary 2.3.8. *Let $\tau \in (0, 10)$. Let $G = (V, E)$ be a graph and $X_1 \subset V, X_2 \subset V$ be two disjoint vertex subsets such that $G[X], G[Y]$ are τ-complete and $|X_1|, |X_2| \geq \frac{100}{\tau}$. Suppose that there exist $u_1u_2, v_1v_2 \in E(X_1, X_2)$. Then $G[X_1 \cup X_2]$ contains $(n', 2)$-weak ladder where $n' \geq \lfloor \frac{|X_1|}{2} \rfloor + \lfloor \frac{|X_2|}{2} \rfloor - 2$. Furthermore, if $u_1v_1 \in E$ or $u_2v_2 \in E$ then $G[X_1 \cup X_2]$ contains $(n', 1)$-weak ladder where $n' \geq \lfloor \frac{|X_1|}{2} \rfloor + \lfloor \frac{|X_2|}{2} \rfloor - 1$.*

Proof. For $i \in [2]$, by Lemma 2.3.6 (2), $G[X_i]$ contains $L_{\lfloor \frac{|X_i|-2}{2} \rfloor}$ having x_iy_i as its first rung where $x_i \in N(u_i), y_i \in N(v_i)$. By attaching two ladders with u_1u_2, v_1v_2, we obtain a $(n', 2)$-weak ladder where $n' \geq \lfloor \frac{|X_1|}{2} \rfloor + \lfloor \frac{|X_2|}{2} \rfloor - 2$ and the "Furthermore" is obvious. □

Next, we will address the case of almost complete bipartite graph.

Lemma 2.3.9. *Let $\tau \in (0, \frac{1}{100})$. Let $G = (X, Y, E)$ be a bipartite graph with bipartition X, Y such that $n = |X| = |Y|$ and $\tau n \geq 100$. Suppose that $|X'|, |Y'| \geq (1 - \tau)n$ where $X' = \{x \in X : |N(x) \cap Y'| \geq (1 - \tau)n\}, Y' = \{y \in Y : |N(y) \cap X'| \geq (1 - \tau)n\}$, and for any $x \in X \setminus X', y \in Y \setminus Y', |N(x) \cap Y'| \geq 4\tau n, |N(y) \cap X'| \geq 4\tau n$. Let e_1, e_2, e_3, e_4 be such that for any e_i, $i \in [4]$, $|e_i \cap (X' \cup Y')| \geq 1$. Then G contains L_n having e_i as its $f(i)$th rung where $f(1) = 1$ and for any $i \in [3]$, $0 < f(i+1) - f(i) \leq 3$. Furthermore, if $|e_i \cap (X' \cup Y')| = 2$ then we have $|f(i + 1) - f(i)| \leq 2$.*

Proof. Let $V' = X' \cup Y', V'' = X'' \cup Y''$. Let $i \in [3]$. If $|e_i \cap (X' \cup Y')| = 2$ then we can choose $e \in E(X', Y')$ such that $G[e, e_i] \cong K_{2,2}$ and $G[e, e_{i+1}] \cong K_{2,2}$. Otherwise, we can choose $e', e'' \in E(X', Y')$ such that $G[e_i, e'], G[e', e''], G[e'', e_{i+1}] \cong K_{2,2}$. Hence we obtain a L_q where $q \leq 10$ having having e_i as its $f(i)$th rung such that $f : [4] \to [q]$ satisfies the condition in the lemma.

Now, let $X' = X' \backslash V(L_q), Y' = Y' \backslash V(L_q), X'' = X'' \backslash V(L_q), Y'' = Y'' \backslash V(L_q)$ and $X = X' \cup X'', Y = Y' \cup Y'', V = X \cup Y$. For any $x \in X''$, since $|N(x) \cap Y'| \geq 3\tau n > |X''|$, there exists a matching $M_{X''}$ saturating X''. Similarly, there exists matching $M_{Y''}$ saturating Y''.

Let $M_1 = M_{X''} \cup M_{Y''}$ and $G' = G[V \setminus V(M_1)]$. For each $e = x_i y_i \in M_1$, we can pick $x_i', x_i'' \in N(y_i) \cap X'$, $y_i', y_i'' \in N(x_i) \cap Y'$, so that all vertices are distinct and $x_i' y_i', x_i'' y_i'' \in E$. This is possible because $|N(y_i) \cap X'| > 3\tau n \geq 3|M_{X''}|$ and $|N(x_i) \cap Y'| > 3\tau n \geq 3|M_{Y''}|$. Then $G[\{x_i, y_i, x_i', x_i'', y_i', y_i''\}]$ contains a 3-ladder, which we will denote by L_i. We have $|X''| + |Y''| = m$ 3-ladders each containing exactly one vertex from $X'' \cup Y''$.

Let $X''' = X' \setminus (\cup_{i \in [m]} L_i)$, $Y''' = Y' \setminus (\cup_{i \in [m]} L_i)$. Then $|Y'''| = |X'''| \geq (1 - 3\tau)n - q > n/2$. For any $x \in X'''$,

$$\begin{aligned} |N(x) \cap Y'''| &\geq |Y'''| - \tau n - |(V(\bar{M}_1) \cap Y'''| \\ &> |Y'''| - 4\tau n \\ &> (1 - 8\tau)|Y'''| > \frac{|Y'''|}{2}, \end{aligned}$$

so there exists a matching M_2 saturating X'''. Define the auxiliary graph H as follows. For every L_i, consider vertex v_{L_i} and let

$$V(H) = \{v_{L_i} : i \in [m]\} \cup \{e : e \in M_2\}.$$

For $e = a_i b_i, e' = a_j b_j \in M_2$, $ee' \in E(H)$ if $G[\{a_i, a_j\}, \{b_i, b_j\}] = K_{2,2}$ and for $v_{L_i} \in V(H), e = a_j b_j \in M_2$, $v_{L_i} e \in E(H)$ if $a_j \in N(y_i') \cap N(y_i'')$, $b_j \in N(x_i') \cap N(x_i'')$. Then $\delta(H) \geq |H| - 10\tau n > |H|/2$ and then H is Hamiltonian, which gives a desired ladder L_n by attaching L_q as its first q rungs. □

Similarly, we also have another lemma for the case that G is almost complete bipartite, but the sizes of the sets in the bipartition differ.

Lemma 2.3.10. *Let $\tau \in (0, \frac{1}{100})$ and $C \in \mathbb{R}$ be such that $\tau C \leq \frac{1}{300}$. Let $G = (X, Y, E)$ be a bipartite graph with bipartition X, Y such that $n = |Y| \leq |X| \leq Cn$ and $\tau n \geq 100$. Suppose that $|Y'| \geq (1-\tau)n$ where $Y' = \{y \in Y : |N(y) \cap X| \geq (1-\tau)|X|\}$, for any $x \in X, |N(x) \cap Y'| \geq (1-\tau)|Y'|$. and for any $y \in Y \setminus Y', |N(y) \cap X| \geq 4\tau|X|$. Let e_1, e_2, e_3, e_4 be such that for any e_i, $i \in [4]$, $|e_i \cap (X' \cup Y')| \geq 1$. Then G contains L_n having e_i as its $f(i)$th rung where $f(1) = 1$ and for any $i \in [3]$, $0 < f(i+1) - f(i) \leq 3$. Furthermore, if $|e_i \cap (X' \cup Y')| = 2$ then we have $|f(i+1) - f(i)| \leq 2$.*

Proof. The proof is basically similar as the proof of Lemma 2.3.9. So with the same way, we obtain L_q containing e_1, e_2, e_3, e_4 in desired positions and let $X = X \setminus V(L_q), Y' = Y' \setminus V(L_q), Y'' = (Y \setminus Y') \setminus V(L_q)$ and $V = X \cup Y' \cup Y''$. For any $y \in Y''$, since $|N(y) \cap X| \geq 3\tau|X| > |Y''|$, there exists a matching M saturating Y''.

Let $G' = G[V \setminus V(M)]$. For each $e = x_i y_i \in M$, we can pick $x'_i, x''_i \in N(y_i)$, $y'_i, y''_i \in N(x_i) \cap Y'$, so that all vertices are distinct and $x'_i y'_i, x''_i y''_i \in E$. This is possible because $|N(y_i) \cap X'| > 3\tau|X| \geq 3|M|$ and for any x_i, x'_i, x''_i, $|N_{G'}(x_i) \cap Y'|, |N_{G'}(x'_i) \cap Y'|, |N_{G'}(x''_i) \cap Y'| > (1-\tau)n > (\frac{1}{2} + 3\tau)n$. Then $G[\{x_i, y_i, x'_i, x''_i, y'_i, y''_i\}]$ contains a 3-ladder, which we will denote by L_i. We have $|Y''| = m$ 3-ladders each containing exactly one vertex from Y''.

Let $Y''' = Y' \setminus (\cup_{i \in [m]} L_i)$ and choose $X''' \subset X \setminus (\cup_{i \in [m]} L_i)$ such that $|X'''| = |Y'''|$. Then $|Y'''| = |X'''| \geq (1-3\tau)n - q > n/2$. For any $x \in X'''$,

$$\begin{aligned}
|N(x) \cap Y'''| &\geq |Y'''| - \tau n - |(V(\bar{M}_1) \cap Y'''| \\
&> |Y'''| - 4\tau n \\
&> (1-8\tau)|Y'''| > \frac{|Y'''|}{2},
\end{aligned}$$

so there exists a matching M_2 saturating X'''. Define the auxiliary graph H as follows. For every L_i, consider vertex v_{L_i} and let

$$V(H) = \{v_{L_i} : i \in [m]\} \cup \{e : e \in M_2\}.$$

For $e = a_i b_i, e' = a_j b_j \in M_2$, $ee' \in E(H)$ if $G[\{a_i, a_j\}, \{b_i, b_j\}] = K_{2,2}$ and for $v_{L_i} \in V(H), e = a_j b_j \in M_2$, $v_{L_i} e \in E(H)$ if $a_j \in N(y_i') \cap N(y_i'')$, $b_j \in N(x_i') \cap N(x_i'')$. Then $\delta(H) \geq |H| - 20C\tau n > |H|/2$ and then H is Hamiltonian, which gives a desired ladder L_n by attaching L_q as its first q rungs. □

A T-graph is graph obtained from two disjoint paths $P_1 = v_1, \ldots, v_m$ and $P_2 = w_1, \ldots, w_l$ by adding an edge $w_1 v_i$ for some $i = 1, \ldots, m$. In (Czygrinow and Kierstead, 2002), it is shown that if $P = V_1, \ldots, V_{2s}$ is a path consisting of pairwise-disjoint sets V_i such that $|V_1| = l - 1$, $|V_{2s-1}| = l + 1$, $|V_i| = l$ for every other i, and in which (V_i, V_{i+1}) is (ϵ, δ)-super regular for suitably chosen ϵ and δ, then $G\left[\bigcup V_i\right]$ contains a spanning ladder. We will use this result in one part of our argument but in many other places the following, much weaker statement will suffice.

Lemma 2.3.11. *There exist $0 < \epsilon$, $10\sqrt{\epsilon} < d < 1$, and l_0 such that the following holds. Let $P = V_1, \ldots, V_r$ be a path consisting of pairwise-disjoint sets V_i such that $|V_i| = l \geq l_0$ and in which (V_i, V_{i+1}) is (ϵ, d)-super regular. In addition, let $x_1 \in V_1, x_2 \in V_2$. Then $G\left[\bigcup V_i \setminus \{x_1, x_2\}\right]$ contains a ladder L such that the first rung of L is in $N(x_1) \cap V_2$, $N(x_2) \cap V_1$ and $|L| \geq (1 - 5\sqrt{\epsilon}/d)rl$.*

Proof. We will construct L in a step by step fashion. Initially, let $L := \emptyset$ and let $k \in [2]$. We have $|N(x_k) \cap V_{3-k}| \geq dl > \epsilon l$ and so there exist x_1', x_2' such that $x_k' \in N(x_k)$, $x_1' x_2' \in E$ and $|N(x_k') \cap V_k \setminus L| \geq dl - 1 \geq 2\sqrt{\epsilon} l$. For the general step, suppose $x_1 \in V_1, x_2 \in V_2$ are the endpoints of L and $|N(x_k) \cap V_{3-k} \setminus L| \geq 2\sqrt{\epsilon} l$. Let $U_k := V_k \setminus L$ and suppose $|U_k| \geq 5\sqrt{\epsilon} l/d$. Then, by Lemma 1.2.4, $(U_k, N(x_k) \cap V_{3-k} \setminus L)$ is $\sqrt{\epsilon}$-regular with density at least $d/2$. Thus all but at most $\sqrt{\epsilon} l$ vertices $v \in N(x_k) \cap V_{3-k} \setminus L$

have $|N(v) \cap U_k| \geq (\frac{d}{2} - \sqrt{\epsilon})|U_k| \geq 2\sqrt{\epsilon}l + 1$. Since $|N(x_k) \cap V_{3-k} \setminus L| \geq 2\sqrt{\epsilon}l$, there are $A_k \subset N(x_k) \cap V_{3-k} \setminus L$ such that $|A_k| \geq \sqrt{\epsilon}l$ and every vertex $v \in A_k$ has $|N(v) \cap U_k| \geq 2\sqrt{\epsilon}l + 1$. Hence there exist $x'_1 \in A_1, x'_2 \in A_2$ such that $x'_1x'_2 \in E$ and $|N(x'_k) \cap V_k \setminus (L \cup \{x_k\})| \geq 2\sqrt{\epsilon}l$ and we can add one more rung to L from $V_1 \times V_2$. To move from (V_1, V_2) to (V_3, V_4) suppose L ends in $x_1 \in V_1, x_2 \in V_2$. Pick $x'_1 \in N(x_1) \cap V_2 \setminus L$ so that $|N(x_2) \cap N(x'_1) \cap V_3| \geq 2\sqrt{\epsilon}l$. Note that $|N(x_2) \cap V_3| \geq dl, |N(x_1) \cap V_2 \setminus L| \geq 2\sqrt{\epsilon}l$ and so x'_1 can be found in the same way as above. Next find $x'_2, x_3 \in N(x_2) \cap N(x'_1) \cap V_3$ such that $|N(x'_2) \cap N(x_3) \cap V_4| > 0$, and finally let $x_4 \in N(x'_2) \cap N(x_3) \cap V_4$. Then $x_3x_4 \in E, x_3 \in N(x_2) \cap N(x'_1) \cap V_3, x_4 \in N(x'_2) \cap V_4$ and $|N(x_3) \cap (V_4 \setminus \{x_4\})|, |N(x_4) \cap (V_3 \setminus \{x'_2, x_3\})| \geq dl - 2 \geq 2\sqrt{\epsilon}l$. □

We will need the following observation.

Fact 2.3.12. *Let G be a 2-connected graph on n vertices such that $\delta(G) \geq \alpha n, n > \frac{10}{\alpha^2}$ and let U_1, U_2 be two disjoin sets such that $|U_i| \geq 2$. Then there exist two disjoint $U_1 - U_2$ paths P_1, P_2 such that $|P_1| + |P_2| \leq \frac{10}{\alpha}$.*

Proof. Let P_1, P_2 be two $U_1 - U_2$ paths such that $|P_1| + |P_2|$ is the smallest. Without loss of generality, $|P_1| \leq |P_2|$. Note that both paths are induced subgraphs and suppose $P_2 := v_1 \ldots v_l$, $l > 5/\alpha$. Let $A = \{v_{3i} : i \in [\frac{l}{3}]\}$. If for any $x, y \in A$, $|N(x) \cap N(y)| \leq 1$ then

$$|\cup_{v \in A} N(v) \cap (V \setminus V(P_2))| \geq \sum_{i=0}^{|A|} \max\{(\alpha n - 2 - i), 0\} > n,$$

a contradiction. Hence there exist two vertices x, y in P_2 such that $dist_{P_2}(x, y) > 2$ and $|N_G(x) \cap N_G(y)| \geq 2$. Then $N_G(x) \cap N_G(y) \cap (V \setminus V(P_1)) = \emptyset$ or we get a shorter $U_1 - U_2$. Thus $|N_G(x) \cap N_G(y) \cap V(P_1)| \geq 2$ and we again get shorter disjoint $U_1 - U_2$ paths. □

In our last fact in the introductory section we will show that a component in a graph either contain two disjoint paths of total length much bigger than its minimum degree or the component has a very specific structure.

Theorem 2.3.13. *Let C be a component in a graph G which satisfies $|C| \geq 2\delta(G)$. If $G[C]$ does not contain a Hamiltonian path then either there exist a path P_1 such that for any $v \in V(C)\backslash V(P_1)$, $N(v) \subset V(P_1)$ or there exists two disjoint paths P_1, P_2 such that $|V(P_1)| + |V(P_2)| > 3\delta(G)$.*

Proof. Let P_1 be a maximum path in C, say $P_1 = v_1, \ldots, v_r$. If P_1 is a Hamiltonian path or $G[V(C)\backslash V(P_1)]$ is independent then we are done. Thus we may assume that there exists a path in $G[V(C)\backslash V(P_1)]$, say $P_2 = u_1, \ldots, u_s$ such that $s \geq 2$. Let

$$A = \{i : v_i \in N(v_1) \cap V(P_1)\}, A^- = \{i - 1 : i \in A\},$$
$$B = \{i : v_i \in N(v_r) \cap V(P_1)\}, B^+ = \{i + 1 : i \in B\}.$$

If $G[V(P_1)]$ contains a cycle of length at least $|V(P_1)| - 1$ then it gives a longer path by attaching P_2 to the cycle. Therefore,

$$A^- \cap B^+ = \emptyset,$$

which implies that

$$|A^- \cup B^+| \geq 2\delta(G).$$

By the maximality of P_2,

$$N(u_1) \subset V(P_2) \cup V(P_1).$$

By the maximality of P_1,

$$N(u_1) \cap (A^- \cup B^+) = \emptyset.$$

Therefore,

$$\delta(G) \leq d(u_1) \leq r - 2\delta(G) + s - 1,$$

which implies that

$$|V(P_1)| + |V(P_2)| = r + s \geq 3\delta(G) + 1.$$

□

2.4 The first non-extremal case

In this section we will address the case when G is non-extremal and $\alpha n \leq \delta(G) \leq (1/2 - \gamma)n$ for some $\alpha, \gamma > 0$. For the clarity, we define $\beta-$extremal as follows.

Definition 2.4.1. *Let G be a graph with $\delta(G) = \delta$. We call that G is β-extremal if there exists a set $B \subset V(G)$ such that $|B| \geq (1 - \delta/n - \beta)n$ and all but at most $4\beta n$ vertices $v \in B$ have $|N(v) \cap B| \leq \beta n$.*

Then the main theorem in this section follows.

Theorem 2.4.2. *Let $\alpha, \gamma \in (0, \frac{1}{2})$ and let $\beta > 0$ be such that $\beta < (\frac{\alpha}{400})^2 \leq \frac{1}{640000}$. Then there exists $N(\alpha, \gamma) \in \mathbb{N}$ such that for all $n \geq N$ the following holds. For every 2-connected graph G on n vertices with $\alpha n \leq \delta(G) \leq (1/2 - \gamma)n$ which is not β-extremal and every $n_1, \ldots, n_l \geq 2$ such that $\sum n_i = \delta$*

(i) G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$ or

(ii) δ is even, $n_1 = n_2 = \frac{\delta}{2}$ and G is one a graph from Example 2.1.3.

Proof. Fix constants $d_1 := \min\{\frac{\alpha^6}{10^{10}}, \frac{\gamma}{10}, \beta^2\}, d_2 := \frac{d_1}{2}$ and let $\epsilon_1, \epsilon_2, \epsilon_3$ be such that $\epsilon_1 < 300\epsilon_1 < \epsilon_2 < \epsilon_2^{1/4} < \frac{\epsilon_3}{10} < 10\epsilon_3 < d_2$. Applying Lemma 1.2.1 with parameters ϵ_1 and m, we obtain our necessary $N = N(\epsilon_1, m), M = M(\epsilon_1, m)$. Let $N(\alpha) = \max\{N, \lceil\frac{100M}{\alpha\epsilon_3}\rceil\}$ and let G be an arbitrary graph with $|G| = n \geq N(\alpha)$ and $\delta = \delta(G) \geq \alpha n$. By Lemma 1.2.1 and some standard computations, we can obtain an

ϵ_1-regular partition $\{V_0, V_1, ..., V_t\}$ of G with $t \in [m, M]$, $|V_0| \leq \epsilon_1 n$ and such that there are at most $\epsilon_1 t$ pairs of indexes $\{i, j\} \in \binom{[t]}{2}$ such that (V_i, V_j) is not ϵ_1-regular.

Let $l := |V_i|$ for $i \geq 1$ and note that

$$(1-\epsilon_1)\frac{n}{t} \leq l \leq \frac{n}{t}.$$

Now, let R be the cluster graph with threshold d_1, that is, given $\{V_0, V_1, ..., V_t\}$ as above, $V(R) = \{V_1, \ldots, V_t\}$ and $E(R) = \{V_iV_j : (V_i, V_j)$ is ϵ_1-regular with $d(V_i, V_j) \geq d_1\}$. In view of the definition of ϵ_1 and d_1 we have the following,

$$\delta(R) \geq (\delta/n - 2d_1)t.$$

Lemma 2.4.3. *Let C be a component in R which contains a T-graph H with $|H| \geq (2\delta/n + \epsilon_3)t$. Then G contains a $(n', r)-$weak ladder where $n' \geq \delta + r$.*

Proof. Since $\Delta(H) \leq 3$, by Lemma 2.3.1 applied to H there exist subsets $V_i' \subseteq V_i$ for every $V_i \in V(H)$ such that (V_i', V_j') is (ϵ_2, d_2)-super-regular for every $V_iV_j \in H$ and

$$|V_i'| \geq (1-\epsilon_2)l.$$

Let $P = U_1', \ldots, U_s'$, $Q = U_i', W_1', \ldots, W_r'$ denote the two paths forming H. Note that if $i + r \geq (2\delta/n + \epsilon_3)$, then $G[\bigcup_{j=1}^{i} U_j' \cup \bigcup_{j=1}^{r} W_j']$ contains a ladder on m vertices where

$$m \geq (2\delta/n + \epsilon_3)(1-\epsilon_2)(1-\epsilon_1)n \geq 2\delta.$$

Otherwise, let $x \in U_{i+1}', y \in U_{i+2}'$. There is an x, z-path P on $r+1$ vertices for some $z \in W_{r-1}'$ and a y, w-path Q on $r+1$ vertices for some $w \in W_{r-2}'$ which is disjoint from P. By Lemma 2.3.11, there is a ladder L' on $(i+r)(1-\epsilon_2)(1-5\sqrt{\epsilon_2}/d_2)l$ vertices in $G[U_1' \cup \cdots U_i' \cup W_1' \cup \ldots W_r']$ which ends at $z' \in N(z) \cap W_r'$ and $w' \in N(w) \cap W_{r-1}'$ and a ladder L'' on $(s-i)(1-\epsilon_2)(1-5\sqrt{\epsilon_2}/d_2)l$ vertices in $G[U_{i+2}' \cup U_s']$ which ends at $x' \in N(x) \cap U_{i+2}'$ and $y' \in N(y) \cap U_{i+3}'$ such that $L \cap (P \cup Q), L' \cap (P \cup Q) = \emptyset$.

Then $|L'| + |L''| \geq (2\delta/n + \epsilon_3)t(1-\epsilon_2)(1-5\sqrt{\epsilon_2}/d_2)l \geq 2\delta + \frac{\epsilon_3 n}{2}, |P| = |Q| = r+1$ and $\frac{\epsilon_3 n}{4} - (r+1) \geq 0$. Thus $L_1 \cup P' \cup Q' \cup L_2$ contains a $(n', r+1)-$weak ladder where $P' = x'Pz', Q' = y'Qw'$ and $n' - (r+1) \geq \delta$. □

Lemma 2.4.4. *Let C be a component in R and suppose $|C| \geq (2\delta/n + \epsilon_3)t$. Then either there is a T-graph H such that $|H| \geq (2\delta/n + \epsilon_3)t$, or there is a set $\mathcal{I} \subset V(C)$ such that $|\mathcal{I}| \geq |C| - (\delta/n + 8d_1)t$ and $||R[\mathcal{I}]|| = 0$.*

Proof. Let $P_1 = V_1, \ldots, V_s$ be a path of maximum length in C and subject to this is such that $||R[V(C) \setminus V(P_1)]||$ is maximum. By Theorem 2.3.13 we may assume that $s < (2\delta/n + \epsilon_3)t$ and that for any $W \in V(C)\backslash V(P_1)$, $N(W) \subset V(P_1)$ (i.e. $||R[V(C) \setminus V(P_1)]|| = 0$). Let $W \in V(C)\backslash V(P_1)$ be arbitrary and let

$$\mathcal{W} = \{i \in [s] : V_i \in N(W)\},$$
$$\mathcal{W}^+ = \{i \in [s] : i-1 \in \mathcal{W}, i+1 \in \mathcal{W}\},$$
$$\mathcal{W}^{++} = \{i \in [s] : i \in \mathcal{W}, i-1, i-2 \notin \mathcal{W}\}.$$

Since P_1 is a longest path, $\mathcal{W} \cap \mathcal{W}^+ = \emptyset$.

In addition, note that $|\mathcal{W}^+| + |\mathcal{W}^{++}| + 1 = |N_R(W)|$. As a result, if $|\mathcal{W}^+| = (\delta/n - Cd_1)t, C \geq 7$ then $|\mathcal{W}^{++}| \geq (C-2)d_1 t$. But then,

$$\begin{aligned} |V(P_1)| &\geq 2|\mathcal{W}^+| + 3|\mathcal{W}^{++}| \\ &\geq 2(\delta/n - Cd_1)t + 3 \cdot (C-2)d_1 t \\ &\geq (2\delta/n + (C-6)d_1)t > 2(\delta/n + \epsilon_3)t > |V(P_1)|. \end{aligned}$$

Thus we may assume that $|\mathcal{W}^+| > (\delta/n - 7d_1)t$. Let $\mathcal{I} := \{V_i | i \in \mathcal{W}^+\} \cup (V(R) \backslash V(P_1))$. Then $|\mathcal{I}| \geq |C| - (|V(P_1)| - |\mathcal{W}^+|) \geq |C| - (\delta/n + 8d_1)t$. We will show that $\mathcal{I}$ is an independent set in R. Clearly $V(C) \setminus V(P_1)$ is independent. Suppose there is $W' \in V(C)\backslash V(P_1)$ such that for some $i \in \mathcal{W}^+$, $V_i \in N_R(W')$. Let P_1' be obtained from P_1 by exchanging V_i with W and note that the length of P_1' is equal to the length of P_1 but $||R[V(C) \setminus V(P_1')]|| \neq 0$ contradicting the choice of P_1. Now suppose $V_iV_j \in R$ for some $i, j \in \mathcal{W}^+$, with $i < j$. Then $P_1' := V_s P_1 V_{j+1} W V_{i+1} P_1 V_j V_i P_1 V_1$ is a longer path. □

In the following lemma, we show that for graphs whose reduced graphs are connected, either the graph contains a δ-weak ladder, hence it includes the claimed number of cycle lengths, or again it is very nearly our extremal structure.

Lemma 2.4.5. *If R is connected, then either G contains a $(n', r)-$weak ladder where $n' \geq \delta + r$, or there exists a set $V' \subset V$ such that $|V'| \geq (1 - \delta/n - \beta)n$, such that all but at most $4\beta n$ vertices $v \in V'$ have $|N_{G'}(v)| \leq \beta n$ where $G' = G[V']$.*

Proof. Since $2\delta/n + \epsilon_3 \leq 2(1/2 - \gamma) + \epsilon_3 \leq 1$, By Claim 2.4.4 and Claim 2.4.3, we may assume that there is $I \subset V(R)$ such that $|I| \geq |C| - (\delta/n + 8d_1)t = (1 - \delta/n - 8d_1)t$ and $||R[I]|| = 0$. Let $V' = \cup_{X \in I} X$. Then

$$|V'| = l|I| \geq l(1 - \delta/n - 8d_1)t \geq (1 - \delta/n - 9d_1)n \geq (1 - \delta/n - \beta)n.$$

Let $W = \{w \in V' : |N_{V'}(w)| \geq \sqrt{d_1}n\}$. We claim that $|W| < 4\sqrt{d_1}n \leq 4\beta n$. Suppose otherwise. Then we have

$$||G[V']|| \geq \frac{4\sqrt{d_1}n \cdot \sqrt{d_1}n}{2} = 2d_1 n^2.$$

which implies that there is at least one edge in $R[I]$. Indeed, there are at most $\epsilon_1 t^2 l^2 \leq \epsilon_1 n^2$ edges in irregular pairs, at most $d_1 t^2 l^2 \leq d_1 n^2$ edges in pairs (A, B) with $d(A, B) \leq d_1$, and at most $t\binom{l}{2} < \epsilon_1 n^2$ edges in $\bigcup_{i \geq 1} G[V_i]$. □

Thus from Lemma 2.4.5 we are either done or there is a set $V' \subset V$ such that $|V'| \geq (1 - \alpha - \beta)n$, such that all but at most $4\beta n$ vertices $v \in V'$ have $|N_{G'}(v)| \leq \beta n$. The latter case will be addressed in the section which contains the extremal case.

However, we are not done yet with the non-extremal case because R can be disconnected. Indeed, it is this part of the argument which requires careful analysis and uses the fact that G is 2-connected. We will split the proof into lemmas based on the nature of components in R and will assume in the rest of the section that R is disconnected.

Lemma 2.4.6. *If R is disconnected and contains a component C which is not bipartite and a component C' such that $|C'| > (\delta/n + 3d_1)t$ then G contains a $(n', r)-$weak ladder for some $n' \geq \delta + r$.*

Proof. Note that C and C' can be the same component. Let C, C' be two components such that $|C| + |C'| \geq (2\delta/n + d_1)t$ and suppose C in not bipartite path. Then there exist path $P = V_1, \dots, V_s$ in C and $Q = U_1, \dots, U_r$ in C' such that $|P| + |Q| \geq (2\delta/n + d_1)t$. In addition, C contains an odd cycle B.

Let $\bar{P}$ be obtained from P be applying Lemma 2.3.1 and let $\bar{Q}$ be obtained from Q by applying Lemma 2.3.1 and let $V_1', \dots, V_s', U_1', \dots, U_r'$ denote the modified clusters. Let $U_1 := \bigcup_{V \in \bar{P}} V$, $U_2 := \bigcup_{V \in \bar{Q}} V$. Since G is 2-connected, from Fact 2.3.12, there exist two disjoint $U_1 - U_2$ paths Q_1, Q_2 in G such that $|Q_1| + |Q_2| \leq \frac{10}{\alpha}$. Let $\{x_k, y_k\} = (V(Q_1) \cup V(Q_2)) \cap U_k$. We will extend Q_1, Q_2 to paths Q_1', Q_2', so that $Q_1' \cap Q_2' = \emptyset$, the endpoints of Q_1' are in U_1', V_1', the endpoints of Q_2' are in U_2', V_2' and $|Q_1'| = |Q_2'| \leq K$ for some constant K which depends on α only. For C' we simply find short paths from x_2, y_2 to U_1', U_2', that is, let $x_2' \in U_1'$, $y_2' \in U_2'$ and find paths S_1, S_2 so that $S_1 \cap S_2 = \emptyset$, S_1 is an x_2', x_2-path, S_2 is a y_2', y_2-path, $|S_i| \leq r$ and $||S_1| - |S_2|| \leq 1$. Let $S_i' := S_i \cup Q_i$. Note that $|S_i'| \leq r + \frac{10}{\alpha}$ but the paths can have different lengths. Let R_1 be a path in $G[C]$ on at most $|C|$ vertices from x_1 to a vertex $x_1' \in V_1'$ which does not intersect S_1'. Note that for every $V \in C$, $|V \cap (S_1' \cup S_2' \cup R_1)|$ is a constant and so if (V, W) is (ϵ, d)-super-regular then $(V \setminus (S_1' \cup S_2' \cup R_1), W \setminus (S_1' \cup S_2' \cup R_1))$ is $(2\epsilon, d/2)$-super-regular.

Consequently, using the fact that C contains an odd cycle, it is possible to find a path R_2 from y_2 to a vertex $y_2' \in V_2$ so that $|R_2| \leq |C|$, $R_2 \cap (S_1' \cup S_2' \cup R_1) = \emptyset$, and $|R_1|+|S_1'|, |R_2|+|S_2'|$ have the same parity. If $|R_1|+|S_1'| > |R_2|+|S_2'|$, then use (V_1', V_2') to extend R_2 so that the equality holds. Let Q_1', Q_2' be the resulting paths. Note that $|Q_1'|+|Q_2'|$ is constant and since $|P|+|Q| \geq (2\delta/n + d_1)t$ we can find two ladders L_{n_1} in $G[P]$, L_{n_2} in $G[Q]$ such that $n_1 + n_2 \geq \delta + d_1 n/4$, $(L_1 \cup L_2) \cap (Q_1' \cup Q_2') = \emptyset$ and such that L_i ends in $N(x_i'), N(y_i')$. $\square$

Next we will address the case when all components are bipartite.

Lemma 2.4.7. *If R is disconnected and every component is bipartite, then G contains either L_δ or a $(n', r)-$weak ladder for some n', r such that $n' \geq \delta + r$.*

Proof. Let $\xi := 20d_1/\alpha^2$, $\tau := 20\sqrt{d_1}/\alpha^2$ and let q be the number of components in R and let D be a component in R. Then D is bipartite and so $|D| \geq 2\delta(R) \geq 2(\delta/n - 2d_1)t$. Thus, in particular, $q \leq 1/(2(\delta/n - 2d_1)) \leq n/\delta$.

For a component D in R, if $|D| \geq (2\delta/n + \epsilon_3)t$, then by Lemma 2.4.4 and Lemma 2.4.3, we may assume that there is an independent set $I \subset V(D)$ such that $|I| \geq |D| - (\delta/n + 8d_1)t$. Suppose components are $D_1, D_2, \ldots, D_q$ and D_i has bipartition A_i, B_i such that $|A_i| \leq |B_i|$. Then, we have

$$(\delta/n - 2d_1)t \leq |A_i| \leq (\delta/n + 8d_1)t$$

and $|B_i| \geq (\delta/n - 2d_1)t$. Let $X_i := \bigcup_{W \in A_i} W, Y_i := \bigcup_{W \in B_i} W$ and $G_i := G[X_i, Y_i]$. Then

$$\delta - 3d_1 n \leq |X_i| \leq \delta + 8d_1 n$$

and

$$\delta - 3d_1 n \leq |Y_i|.$$

In addition, since B_i is independent in R, $||G_i|| = e(X_i, Y_i) \geq \delta|Y_i| - 2d_1n^2 \geq |X_i||Y_i| - 10d_1n^2 \geq (1-\xi)|X_i||Y_i|$.

Let $X_i' := \{x \in X_i | |N_G(x) \cap Y_i| \geq (1-\sqrt{\xi})|Y_i|\}$ and note that $|X_i'| \geq (1-\sqrt{\xi})|X_i| \geq \frac{2\delta}{3}$. Similarly let $Y_i' := \{y \in Y_i | |N_G(y) \cap X_i| \geq (1-\sqrt{\xi})|X_i|\}$ and note that $|Y_i'| \geq (1-\sqrt{\xi})|Y_i| \geq \frac{2\delta}{3}$. Let $G' := G[X_i', Y_i']$ and note that for every vertex $x \in X_i'$,

$$|N_G(x) \cap Y_i'| \geq (1 - 2\sqrt{\xi})|Y_i'|, \tag{2.1}$$

and the corresponding statement is true for vertices in Y_i'.

Let $V_0' := V_0 \cup \bigcup_i ((X_i \setminus X_i') \cup (Y_i \setminus Y_i'))$ and note that $|V_0'| \leq (2\sqrt{\xi} + \epsilon_1)n \leq 3\sqrt{\xi}n$. Then for every vertex $v \in V_0'$ we have $|N_G(v) \cap (V(G) \setminus V_0')| \geq \delta/2$. Thus, since the number of components is at most n/δ, for every $v \in V_0'$ there is $i \in [q]$ such that $|N_G(v) \cap X_i'| + |N_G(v) \cap Y_i'| \geq \delta^2/(2n)$ and we assign v to Y_i' (X_i') if $|N_G(v) \cap X_i'| \geq \delta^2/(4n)$ ($|N_G(v) \cap Y_i'| \geq \delta^2/(4n)$) so that every v is assigned to exactly one set. Let X_i'', (Y_i'') denote the set of vertices assigned to X_i' (Y_i') and let $V_i' := X_i' \cup X_i'' \cup Y_i' \cup Y_i''$.

First assume that there exists i such that $\min\{|X_i' \cup X_i''|, |Y_i' \cup Y_i''|\} \geq \delta$. If $|X_i'|, |Y_i'| \leq \delta$ then by removing some vertices from $X_i'' \cup Y_i''$, we get $|X_i' \cup X_i''| = |Y_i' \cup Y_i''| = \delta$ and by Lemma 2.3.9, we obtain L_δ. If $|X_i'| > \delta$ then choose $Z_i \subset X_i'$ such that $|Z_i| = \delta$ and then for every vertex $y \in Y_i'$,

$$|N_G(y) \cap Z_i| \geq |Z_i| - 2\sqrt{\xi}|X_i'| \geq (1 - 2 \cdot \frac{2}{\alpha}\sqrt{\xi})|Z_i| \geq (1-\tau)\delta,$$

If $|Y_i'| > \delta$ then the same is true for vertices $x \in X_i'$. Hence if $|X_i'|, |Y_i'| \geq \delta$ then we can choose $Z_i \subset X_i', W_i \subset Y_i'$ such that $|Z_i| = |W_i| = \delta$ and for any $x \in Z_i, y \in W_i$, $|N(x) \cap W_i|, |N(y) \cap Z_i| \geq (1-\gamma)\delta$, so by Lemma 2.3.9, G contains L_δ. Since $|Y_i'| \leq \frac{2}{\alpha}|X_i' \cup X_i''|, \tau \cdot \frac{2}{\alpha} \leq \frac{1}{300}$, if $|X_i'| < \delta$, $|Y_i'| \geq \delta$ then by Lemma 2.3.10, $G[X_i' \cup X_i'', Y_i']$ contains L_δ.

Now, we may assume that $\min\{|X_i' \cup X_i''|, |Y_i' \cup Y_i''|\} < \delta$ for all $i \in [q]$.

Claim 2.4.8. *Let $i \in [q]$. If there exists $j \in [q]$ such that there exists $Z_i \in \{X'_i, Y'_i\}, Z_j \in \{X'_j \cup X''_j, Y'_j \cup Y''_j\}$ such that $E(Z_i, Z_j)$ has a matching of size 2, then G contains a (n', r) weak ladder such that $n' - r \geq \delta$.*

Proof. Without loss of generality, let $i = 1, j = 2$ and $Z_1 = X'_1, Z_2 = X'_2 \cup X''_2$. Let $u_1u_2, v_1v_2 \in E(X'_1, X'_2 \cup X''_2)$. For $i \in [2]$, choose e_i such that $u_i \in e_i$ and $e_i \cap Y'_1 \neq \emptyset$, e'_i such that $v_i \in e'_i$ and $e'_i \cap Y'_2 \neq \emptyset$. For $i \in [2]$, by Lemma 2.3.10, $G[X'_i \cup X''_i \cup Y'_i]$ contains L_t having e_i, e'_i are in its first 4 rungs where $t \geq \frac{2\delta}{3}$. By attaching these two ladders with u_1u_2, v_1v_2, we obtain a (n', r)-weak ladder such that $r \leq 10$ and $n' - r \geq (2t - 10) - 10 \geq \frac{4\delta}{3} - 20 \geq \delta$. □

Claim 2.4.9. *If there exists $i \in [q]$ such that $||G[X'_i \cup X''_i]|| + ||G[Y'_i \cup Y''_i]|| \geq 2$ then G contains a $(n', r)-$weak ladder for some $n' \geq \delta + r$.*

Proof. Let $j \neq i$ and recall that $V'_i = X'_i \cup X''_i \cup Y'_i \cup Y''_i$, $V'_j = X'_j \cup X''_j \cup Y'_j \cup Y''_j$. Since G is 2-connected there are disjoint $V'_i - V'_j$-paths P, Q such that $|P| + |Q| \leq \frac{10}{\alpha}$ from Fact 2.3.12. Let $\{x_1\} = V(P) \cap V'_i, \{x_2\} = V(Q) \cap V'_i$ and $\{y_1\} = V(P) \cap V'_j, \{y_2\} = V(Q) \cap V'_j$.

Let $z_1z_2 \in E(G[X'_i \cup X''_i]) \cup E(G[Y'_i \cup Y''_i])$ be such that $|\{z_1, z_2\} \cap \{x_1, x_2\}| \leq 1$, which is possible since $||G[X'_i \cup X''_i]|| + ||G[Y'_i \cup Y''_i]|| \geq 2$. We remove some vertices in $X''_i \cup Y''_i \setminus \{x_1, x_2, z_1, z_2\}$ and some vertices in $X''_j \cup Y''_j \setminus \{y_1, y_2\}$ so that $|X'_i \cup X''_i| = |Y'_i \cup Y''_i|, |X'_j \cup X''_j| = |Y'_j \cup Y''_j|$.

For any $x \in \{x_1, x_2, z_1, z_2\}$, if $x \in X'_i \cup X''_i(Y'_i \cup Y''_i)$ choose $x' \in N(x) \cap Y'_i(X'_i)$, say $e(x) = \{x, x'\}$, then we have $E_0 = \cup_{x \in \{x_1, x_2, z_1, z_2\}} e(x)$ such that $|E_0| \leq 4$ and for any $e \in E_0$, $|e \cap (X'_i \cup Y'_i)| \geq 1$. Similarly, for any $y \in \{y_1, y_2\}$, if $y \in X'_j \cup X''_j(Y'_j \cup Y''_j)$ choose $y' \in N(y) \cap Y'_j(X'_j)$, say $e(y) = \{y, y'\}$, then we have $E'_0 = \cup_{y \in \{y_1, y_2\}} e(y)$ such that $|E'_0| = 2$ and for any $e \in E'_0$, $|e \cap (X'_j \cup Y'_j)| \geq 1$. Then by Lemma 2.3.9, there exist ladders $L_{|X'_i \cup X''_i|}$ in $G[V'_i]$ and $L_{|X'_j \cup X''_j|}$ in $G[V'_j]$ such that E_0, E'_0 are in those first

10 rungs. Since $|X_i' \cup X_i''| + |X_j' \cup X_j''| \geq \frac{4\delta}{3}$, and $20 + \frac{5}{\alpha} < \frac{\delta}{6}$, we obtain $(n', r)-$weak ladder such that $n' \geq \frac{4\delta}{3} - 20$, $r \leq 20 + \frac{10}{\alpha}$ and so $n' - r \geq \frac{4\delta}{3} - \frac{10}{\alpha} - 40 \geq \delta + \frac{\delta}{3} - \frac{30}{\alpha} \geq \delta.(\because n \geq \frac{90}{\alpha^2}.)$ $\square$

Now, we choose $i \in [q]$ such that $\min\{|X_i' \cup X_i''|, |Y_i' \cup Y_i''|\}$ is maximum and we redistribute vertices from V_j, for $j \in [q] \setminus \{i\}$ as follows. Without loss of generality, let $|X_i' \cup X_i''| < \delta$. If there exists $v \in V_j'$ such that $|N(v) \cap Y_i'| \geq 4\tau|Y_i|'$ then we move it to X_i'' until $|X_i' \cup X_i''| = \delta$. We apply the same process to $Y_i' \cup Y_i''$ if $|Y_i' \cup Y_i''| < \delta$. After the redistribution, if $\min\{|X_i' \cup X_i''|, |Y_i' \cup Y_i''|\} = \delta$ then again Lemma 2.3.9 and Lemma 2.3.10 imply existence of L_δ. Thus assume $|X_i'| + |X_i''|$ is less than δ after redistribution. Since $||G[Y_i' \cup Y_i'']|| \leq 1$, at least $|Y_i'| - 2$ vertices $y \in Y_i'$ have a neighbor in $V(G) \setminus V_i'$. Therefore for some $j \neq i$ and $Z_j' \in \{X_j' \cup X_j'', Y_j \cup Y_j''\}$, $|E_G(Y_i', Z_j')| \geq (|Y_i'| - 2)/(2q - 2) \geq \delta|Y_i'|/(3n)$. If there is a matching of size two in $G[Y_i', Z_j']$, then by Claim 2.4.8, we obtain a (n', r)-weak-ladder with $n' \geq \delta + r$. Otherwise, there is a vertex $z \in Z_j'$ such that $|N_G(z) \cap Y_i'| \geq 4\tau|Y_i'|$ and then we can move z to X_i''. $\square$

Finally, we will prove the case when all the components are small.

Lemma 2.4.10. *If every component D of R satisfies $|D| \leq (\delta/n + 3d_1)t$ then either G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$ for every $n_1, \ldots, n_l \geq 2$ such that $\sum n_i = \delta$ or δ is even, $n_1 = n_2 = \frac{\delta}{2}$ and G is one of the graphs from Example 2.1.3.*

Proof. Let $\xi = 6d_1/\alpha$, $\tau := \frac{100d_1}{\alpha^2}$. Note that $3\sqrt{\xi} \leq \tau \leq \frac{\alpha}{40} < \frac{1}{40}$. Since $d_1 < \gamma/2$, there are at least three components. Indeed, otherwise

$$|V| \leq 2(\delta/n + 3d_1)n + \epsilon_1 n = 2\delta + (3d_1 + \epsilon_1)n \leq n - (2\gamma - 3d_1 - \epsilon_1) < n.$$

Let q be a number of components. Let $V_D = \bigcup_{X \in D} X$ and let $G_D = G[V_D]$. Note that $\alpha n/2 \le \delta - 3d_1 n \le |V_D| \le \delta + 3d_1 n \le 2\delta$ and we have

$$|E_G(V_D, V \setminus V_D)| \le |D|\frac{n}{t} d_1 n + \epsilon_1 n^2 \le 2d_1 \delta n.$$

Thus

$$|E(G_D)| \ge \frac{\delta |V_D|}{2} - 2d_1\delta n \ge \binom{|V_D|}{2} - 4d_1 \delta n \ge (1-\xi)\binom{|V_D|}{2}.$$

Let $V'_D = \{v \in V_D || N_G(v) \cap V_D| \ge (1-\sqrt{\xi})|V_D|\}$ and note that $|V'_D| \ge (1-2\sqrt{\xi})|V_D| \ge (1-3\sqrt{\xi})\delta$ and for any $v \in V'_D$,

$$|N_G(v) \cap V'_D| \ge (1-3\sqrt{\xi})|V'_D| \ge (1-\tau)|V'_D|.$$

Move vertices from $V_D \setminus V'_D$ to V_0 to obtain V'_0. We have $|V'_0| \le (\epsilon_1 + 2\sqrt{\xi})n \le 3\sqrt{\xi}n$ and $|V'_D| \ge (1-2\sqrt{\xi})|V_D| \ge (1-3\sqrt{\xi})\delta$. Now we redistribute vertices from V'_0 as follows. Add v from V'_0 to V''_D if $|N(v) \cap V'_D| \ge 4\tau |V'_D|$. Since $|V'_0| \le 3\sqrt{\xi}n \le \delta/3$ and the number of components is at most $n/(\delta - 3d_1 n) \le 2n/\delta$, so for every $v \in V'_0$, there exists a component D such that $|N(v) \cap V'_D| \ge \frac{\alpha}{3}\delta \ge \frac{\alpha}{6}|V'_D| \ge 4\tau |V'_D|$. Let $V^*_D := V'_D \cup V''_D$. Note that $|V''_D| \le 3\sqrt{\xi}n \le \tau |V'_D|$, which says $|V'_D| \ge (1-\tau)|V^*_D|$. Hence for any component D, $G[V^*_D]$ is τ-complete.

Claim 2.4.11. *If D_1, D_2 are two components and there is a matching of size four between $V^*_{D_1}$ and $V^*_{D_2}$, then G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$ for every $n_1, \ldots, n_l \ge 2$ such that $\sum n_i = \delta$.*

Proof. Let D be a component which is different than D_1 and D_2. By Fact 2.3.12, there exist two $V^*_D - (V^*_{D_1} \cup V^*_{D_2})$-paths P, Q which can contain vertices from at most two edges in the matching. Let $u, v \in (V(P) \cup V(Q)) \cap V^*_D$, $x, y \in (V(P) \cup V(Q)) \cap (V^*_{D_1} \cup V^*_{D_2})$ and let $x'y', x''y''$ be two independent edges in $E(V^*_{D_1}, V^*_{D_2})$ such that $\{x', y', x'', y''\} \cap \{x, y\} = \emptyset$. Then we have two cases:

- x, y are in a same component, without loss of generality, let $x, y \in V^*_{D_1}$. By applying Lemma 2.3.6 to each component, we obtain a ladder in each component.

If n_1 is such that $n_1 > |V_D^*| - 7$ then C_{2n_i} can be obtained by attaching a ladder in $G[V_D*]$ and some first rungs in a ladder in $G[V_{D_1}^*]$ with P, Q and a parity vertex(if necessary) in $G[V_D^*]$. If n_2 is such that $n_2 > |V_{D_2}^*| - 7$ then C_{2n_i} can be obtained by attaching a ladder in $G[V_{D_2}*]$ and some last rungs in a ladder in $G[V_{D_1}^*]$ with $x'x'', y'y''$. Moreover, remaining small cycles can be obtained in a ladder remained in $G[V_{D_1}^*]$. Otherwise, the case is trivial.

- Let $x \in V_{D_1}^*, y \in V_{D_2}^*$. Since there is a matching of size four between $V_{D_1}^*$ and $V_{D_2}^*$, there is a matching $x'''y'''$ in $E(V_{D_1}^*, V_{D_2}^*)$ or remaining two matching e_1, e_2 are such that $x \in e_1, y \in e_2$, say $e_1 = xy''', e_2 = x'''y$. In both case, we obtain a ladder starting at $N(u), N(v)$ in $G[V(D^*)]$. In the first sub-case, we choose a ladder starting at $N(x), N(x')$ ending at $N(x''), N(x''')$ in $G[V_{D_1}^*]$ and a ladder starting at $N(y), N(y')$ ending at $N(y''), N(y''')$ in $G[V_{D_2}^*]$. By attaching those three ladders with using parity vertex in an appropriate manner, we obtain a desired structure containing disjoint cycles. In the other case, we choose a ladder starting at at $N(x), N(x''')$ ending at $N(x'), N(x'')$ in $G[V_{D_1}^*]$ and a ladder starting at $N(y), N(y''')$ ending at $N(y'), N(y'')$ in $G[V_{D_2}^*]$. Similarly, we are done by attaching those three ladders.

□

By Claim 2.4.11, we may assume that for any $i, j \in [q]$, $E(V_{D_i}^*, V_{D_j}^*)$ has a matching of at most 3. Then we have another claim which is useful for the arguments follow.

Claim 2.4.12. *Let D be a component. For any $X \subset V \setminus V_{D^*}$, if $|\{v \in V_{D^*} : N(v) \cap X \neq \emptyset\}| \geq \frac{|V_{D^*}|}{2}$ then there exists $x \in X$ such that $|N(v) \cap V_D'| \geq 4\tau|V_D'|$.*

Proof. Let X is a subset of $V \setminus V_{D^*}$ and assume that $|\{v \in V_{D^*} : N(v) \cap X \neq \emptyset\}| \geq$

$\frac{|V_{D^*}|}{2}$. Then there exists $i \in [q]$ and $Y \subset V^*_{D_i}$ such that

$$|\{v \in V_{D^*} : N(v) \cap Y \neq \emptyset\}| \geq \frac{|V_{D^*}|}{2} \cdot \frac{1}{q} \geq \frac{\alpha |V_{D^*}|}{2}.$$

So, there exists $v \in Y$ such that

$$|N(v) \cap V_{D^*}| \geq \frac{\alpha |V_{D^*}|}{6},$$

which implies that

$$|N(v) \cap V_{D'}| \geq 4\tau |V'_D|.$$

□

Claim 2.4.13. *Let D^*_1, D^*_2 be two components. If there exist two distinct vertices $x, y \in V^*_{D_1}$ such that $|N(x) \cap V'_{D_2}|, |N(y) \cap V'_{D_2}| \geq 4\tau |V'_{D_2}|$, then there exists a $(n', r)-$ weak ladder where $r \in \{1, 2\}$, $n' + r = \lfloor \frac{|V^*_{D_1}| + |V^*_{D_2}|}{2} \rfloor$.*

Proof. Since $|N(x) \cap V'_{D_2}|, |N(y) \cap V'_{D_2}| > \tau |V'_{D_2}|$, there is $x' \in N(x) \cap V'_{D_2}$, so there exists $y' \in N(y) \cap V'_{D_2}$ such that $x'y' \in E$. If $|V^*_{D_1}|$ is even, then by Lemma 2.3.6 (2), $G[V^*_{D_2}]$ contains a ladder $L_{\lfloor \frac{|V^*_{D_2}|}{2} \rfloor}$ having $x'y'$ as its first rung and $G[V^*_{D_1}]$ contains a ladder $L_{\frac{|V^*_{D_1}|}{2} - 1}$ starting at $N(x), N(y)$. By attaching those two ladder with xx', yy', we obtain a $(n', 1)$- weak ladder where $n' = \frac{|V^*_{D_1}|}{2} - 1 + \lfloor \frac{|V^*_{D_2}|}{2} \rfloor = \lfloor \frac{|V^*_{D_1}| + |V^*_{D_2}|}{2} \rfloor - 1$. Now, suppose that $|V^*_{D_1}|$ is odd. If $\{x, y\} \cap V''_{D_1} \neq \emptyset$, without loss of generality, $x \in V''_{D_1}$, then by Fact 2.3.7, $G[V^*_{D_1} \setminus \{x\}]$ is τ-complete. Since $|N(x) \cap V'_{D_2}| \geq 4\tau |V'_{D_2}|$, $G[V^*_{D_2} \cup \{x\}]$ is τ-complete. Since there exists $x'' \in V^*_{D_1} \cap N(x)$ such that $x'' \neq y$, by Lemma 2.3.6 (2), there exists a $L_{\frac{|V^*_{D_1}| - 1}{2} - 1}$ the first rung $e_1 = v_1 v_2$ of which is such that $v_1 \in N(x''), v_2 \in N(y)$ in $G[V^*_{D_1} \setminus \{x\}]$, and there exists a $L_{\lfloor \frac{|V^*_{D_2}| + 1}{2} \rfloor - 1}$ the first rung $e_2 = v'_1 v'_2$ of which is such that $v'_1 \in N(x), v'_2 \in N(y')$ in $G[V^*_{D_2} \cup \{x\}]$, so by attaching two ladders with xx'' and yy', we obtain a $(\lfloor \frac{|V^*_{D_1}| + |V^*_{D_2}|}{2} \rfloor - 2, 2)$-weak ladder. If $\{x, y\} \subset V'_{D_1}$ then there exists $z \in N(x) \cap N(y) \cap V^*_{D_1}$. Since $x', y' \in V'_{D_2}$, there exists $z' \in N(x') \cap N(y') \cap V^*_{D_2}$. By Lemma 2.3.6 (4), there exists a $L_{\frac{|V^*_{D_1}|-1}{2}}$ the first rung of which is zy in $G[V^*_{D_1} \setminus \{x\}]$, and there exists a $L_{\lfloor\frac{|V^*_{D_2}|-1}{2}\rfloor}$ the first rung of which is $z'y'$ in $G[V^*_{D_2} \setminus \{x'\}]$. By attaching two ladders with xx', yy', we obtain a $(\lfloor\frac{|V^*_{D_1}|+|V^*_{D_2}|}{2}\rfloor - 1, 1)$-weak ladder. □

Claim 2.4.14. *If D^*_1, D^*_2 are two components such that $|V^*_{D_1}| + |V^*_{D_2}| \geq 2K+1$, then there exists a $(n', c)-$ weak ladder where $n' \geq K-2, 2 \leq c \leq \frac{7}{\alpha}$.*

Proof. We can always delete a vertex from $V^*_{D_1}$ and so we may assume that $|V^*_{D_1}| + |V^*_{D_2}| = 2K+1$. Then exactly one of the terms $|V^*_{D_i}|$ is odd, and we have $\lfloor\frac{|V^*_{D_1}|}{2}\rfloor + \lfloor\frac{|V^*_{D_2}|}{2}\rfloor = K$. By Fact 2.3.12, there exist two disjoint path P, Q between $V^*_{D_1}, V^*_{D_2}$ such that $|V(P)| + |V(Q)| \leq \frac{10}{\alpha}$. Let x_i be the endpoints of P and y_i be the ends of Q where $x_i, y_i \in V^*_{D_i}, i \in [2]$. If $|V(P)| + |V(Q)|$ is even then by applying Lemma 2.3.6 (2), we obtain a $L_{\lfloor\frac{|V^*_{D_1}|-2}{2}\rfloor}$ in $G[V^*_{D_1}]$ and $L_{\lfloor\frac{|D^*_2|-2}{2}\rfloor}$ in $G[V^*_{D_2}]$ which start at $N(x_1), N(y_1)$, $N(x_2), N(y_2)$, respectively. By attaching these ladders, we obtain a $(n', c)-$weak ladder where $n' = \lfloor\frac{|V^*_{D_1}|-2}{2}\rfloor + \lfloor\frac{|V^*_{D_2}|-2}{2}\rfloor \geq K-2$ and $2 \leq c \leq \frac{4+\frac{10}{\alpha}}{2} \leq \frac{6}{\alpha}$. Otherwise, assume that $|V(P)| + |V(Q)|$ is odd. If $|V^*_{D_i}|$ is odd then $|V^*_{D_{3-i}}|$ is even, and we apply Lemma 2.3.6 (3) to $G[V^*_{D_i}]$ and Lemma 2.3.6 (2) to $G[V^*_{D_{3-i}}]$. Then by attaching those two ladders, we obtain a (n', c)-weak ladder where $n' \geq K-2$ and $3 \leq c \leq \frac{5+\frac{10}{\alpha}}{2} \leq \frac{7}{\alpha}$. □

Now, we move vertices between components to obtain, if possible, components of larger size. Let D be a component and suppose $|V^*_D| \leq \delta$. Then every vertex $v \in V'_D$ has a neighbor in $V \setminus V^*_D$. Thus $|E(V'_D, V \setminus V^*_D)| \geq |V'_D|$. If there is a matching of size $8n/\delta$, then for some component $F \neq D$ there is a matching of size four between V'_D and V^*_F, and we are done by Claim 2.4.11. Hence there is a vertex $v \in V \setminus V^*_D$ such that $|N(v) \cap V^*_D| > \delta|V'_D|/8n$, so $|N(v) \cap V'_D| \geq 4\tau|V'_D|$, then we can *move* v to V''_D. Thus we may assume that there is a component D such that $|V^*_D| \geq \delta+1$. We will now move vertices between components. To avoid introducing new notation, we will use D_i^* to refer to the ith component after moving vertices from and to D_i^*. Move vertices so that after renumbering of components we have $|V_{D_1}^*| \geq |V_{D_2}^*| \geq \cdots \geq |V_{D_q}^*|$ and for any $k \in [q]$, $\sum_{i=1}^{k} |V_{D_i}^*|$ is as big as possible and subject to this, for any $i, j \in [q] \setminus \{1\}$, $|E(V_{D_1}^*, V_{D_i}^*)| \leq |E(V_{D_1}^*, V_{D_j}^*)|$ where $i < j$. Note that if $i < j$ and $|V_{D_i}^*| = |V_{D_j}^*|$ then $|E(V_{D_1}^*, V_{D_i}^*)| \leq |E(V_{D_1}^*, V_{D_j}^*)|$. If $|V_{D_1}^*| \geq \delta + 14d_1 n$ then we stop moving any vertices. Hence the natural first case is that $|V_{D_1}^*| \geq \delta + 11d_1 n$. Since $q - 1 \geq 2$, there exists $i \in [q] \setminus \{1\}$ such that at most $7d_1 n$ vertices in the original $V_{D_i}^*$ were moved to $V_{D_1}^*$, $G[V_{D_i}^*]$ is 2τ-complete and

$$|V_{D_1}^*| + |V_{D_i}^*| \geq (\delta + 11d_1 n) + (\delta - 10d_1 n) = 2\delta + d_1 n.$$

By Claim 2.4.14, $G[V_{D_1}^* \cup V_{D_2}^*]$ contains a (n', c)-weak ladder where $n' \geq \delta + \frac{d_1 n}{2} - 2$, $2 \leq c \leq \frac{7}{\alpha}$. Since $\frac{d_1 n}{2} \geq \frac{8}{\alpha} \geq \frac{7}{\alpha} + 2$, by Lemma 2.3.2, $G[V_{D_1}^* \cup V_{D_2}^*]$ contains all disjoint cycles. So we may assume that all possible moves terminate and

$$\delta + 11d_1 n > |V_{D_1}^*| \geq |V_{D_2^*}| \geq \cdots \geq |V_{D_q^*}|.$$

Note that for any $i \in [3], j \in [q]$ such that $j > i$, there is no $v \in V_{D_j}^*$ such that $|N(v) \cap V_{D_i}'| \geq \delta |V_{D_i}'|/(16n)$, since otherwise, v can be moved to V_{D_i}''. There are at most $(q-1) \cdot 14d_1 n$ vertices in each original components moved to other components, but since $(q-1) \cdot 14d_1 n \leq \frac{50d_1 n}{\alpha}$, for $i \geq 2$, $G[V_{D_i}^*]$ is 2τ-complete, because $\tau \geq \frac{100d_1}{\alpha^2}$.

We will continue the analysis based on the size of $V_{D_1}^*$. Suppose $|V_{D_1}^*| = \delta + c_1$ where $c_1 \geq 1$.

Claim 2.4.15. *If $|V_{D_1}^*| = \delta + 1$ then for any $u, v \in V_{D_1}^*$, $N(u) \cap N(v) \cap V_{D_1}^* \neq \emptyset$.*

Proof. Suppose not and let $u, v \in V_{D_1}^*$ be such that $N(u) \cap N(v) \cap V_{D_1}^* = \emptyset$. Then for any $x \in V_{D_1}^* \setminus \{u, v\}$, $|N(x) \cap \{u, v\}| \leq 1$, so $|N(x) \cap V_{D_1}^*| \leq |V_{D_1}^*| - 2 = \delta - 1$, therefore

$$|E(V^*_{D_1}, V \setminus V^*_{D_1})| \geq |V^*_{D_1}| - 2 \geq \frac{|V^*_{D_1}|}{2},$$

by Claim 2.4.12, there exists $z \in V \setminus V^*_{D_1}$ which can be moved to $V^*_{D_1}$, a contradiction.

□

First, we consider the case that $|V^*_{D_2}| \geq \delta$. By Fact 2.3.12, there are $V^*_{D_1} - V^*_{D_2}$ paths P_1, P_2 such that $|V(P_1)| + |V(P_2)| \leq \frac{10}{\alpha}$. Denote by u_i, v_i the endpoints of P_i such that $u_i \in V^*_{D_1}, v_i \in V^*_{D_2}$. Since $|V^*_{D_3} \cap (V(P_1) \cup V(P_2)| \leq \frac{10}{\alpha}$, $G[V^*_{D_3} \setminus (V(P_1) \cup V(P_2))]$ is 2τ-complete (because $\tau|V^*_{D_3}| \geq \tau \cdot \frac{\delta}{2} \geq \frac{20}{\alpha}$), so by Lemma 2.3.6, $G[V^*_{D_3} \setminus (V(P_1) \cup V(P_2))]$ contains a $L_{n''}$ where $n' \geq \frac{\delta}{3}$.

Since $|V^*_{D_1}| + |V^*_{D_2}| \geq 2\delta + 1$, by Claim 2.4.14, $G[V^*_{D_1} \cup V^*_{D_2} \cup V(P_1) \cup V(P_2)]$ contains a $(\delta - 2, k)$-weak ladder where $k \geq 2$. By Corollary 2.3.4, we may assume that either $n_1 = \lfloor \frac{\delta}{2} \rfloor, n_2 = \lceil \frac{\delta}{2} \rceil$ or $n_1 = \lfloor \frac{\delta - 1}{2} \rfloor, n_2 = \lceil \frac{\delta + 1}{2} \rceil$. If $n_1 = \lfloor \frac{\delta}{2} \rfloor, n_2 = \lceil \frac{\delta}{2} \rceil$ then $G[V^*_{D_1}]$ contains C_{2n_2} and $G[V^*_{D_2}]$ contains C_{2n_1}. Otherwise, let $n_1 = \lfloor \frac{\delta - 1}{2} \rfloor, n_2 = \lceil \frac{\delta + 1}{2} \rceil$. If δ is odd then $G[V^*_{D_1}]$ contains C_{2n_2} and $G[V^*_{D_2}]$ contains C_{2n_1}, so let δ be even. If $u_1 u_2 \in E$ then by Lemma 2.3.6 (5), $G[V^*_{D_2} \cup \{u_1, u_2\}]$ contains C_{2n_2} and by Lemma 2.3.6 (2), $G[V^*_{D_1} \setminus \{u_1, u_2\}]$ contains C_{2n_1}. If $u_1 u_2 \notin E$ then by Claim 2.4.15, there exists $u_3 \in N(u_1) \cap N(u_2) \cap V^*_{D_1}$, and then $G[V^*_{D_2} \cup \{u_1, u_2, u_3\}]$ contains C_{2n_2} and $G[V^*_{D_1} \setminus \{u_1, u_2, u_3\}]$ contains C_{2n_1}.

Now, we assume that $|V^*_{D_2}| < \delta$, so let $|V^*_{D_2}| = \delta - c_2$ where $c_2 \geq 1$. By Claim 2.4.11, $|E(V^*_{D_1}, V^*_{D_2})| \leq 3|V^*_{D_2}|$, and so

$$|E(V^*_{D_2}, V \setminus (V^*_{D_1} \cup V^*_{D_2}))| \geq (c_2 + 1 - 3)|V^*_{D_2}| \geq (c_2 - 2)|V^*_{D_2}|.$$

By Claim 2.4.12, $|E(V^*_{D_2}, V \setminus (V^*_{D_1} \cup V^*_{D_2}))| < \frac{|V^*_{D_2}|}{2}$, so $c_2 \in [2]$ and there exist two distinct vertices $x, y \in V^*_{D_1}$ such that $|N(x) \cap V^*_{D_2}| + |N(y) \cap V^*_{D_2}| \geq \frac{3|V^*_{D_2}|}{2}$. It implies that $|N(x) \cap N(y) \cap V^*_{D_2}| \geq \frac{|V^*_{D_2}|}{2}$ and $|N(x) \cap V'_{D_2}|, |N(y) \cap V'_{D_2}| \geq 4\tau|V'_{D_2}|$. By Claim 2.4.13, there exists a $(n', k)-$ weak ladder such that $k \in [2]$ and $n' + k \geq \lfloor \frac{|V^*_{D_1}| + |V^*_{D_2}|}{2} \rfloor$.

We have two cases.

- **Case 1:** $c_1 \geq c_2$. Note that $|V^*_{D_1}| + |V^*_{D_2}| = (\delta + c_1) + (\delta - c_2) \geq 2\delta$. Hence $G[V^*_{D_1} \cup V^*_{D_2}]$ contains either a $(\delta-1, 1)$-weak ladder or $(\delta-2, 2)$-weak-ladder. Since $G[V^*_{D_3}]$ contains $L_{n''}$ where $n'' \geq \frac{\delta}{3}$, by Corollary 2.3.4, it suffices to show that G contains disjoint $C_{2\lfloor\frac{\delta}{2}\rfloor}$,$C_{2\lceil\frac{\delta}{2}\rceil}$ or disjoint $C_{2\lfloor\frac{\delta-1}{2}\rfloor}$,$C_{2\lceil\frac{\delta+1}{2}\rceil}$. We can choose $Z \subset \{x, y\}$ so that $G[V^*_{D_1} \setminus Z]$ contains $C_{2\lfloor\frac{\delta}{2}\rfloor}$ and $G[V^*_{D_2} \cup Z]$ contains $C_{2\lceil\frac{\delta}{2}\rceil}$. Since $N(x) \cap N(y) \cap V^*_{D_2} \neq \emptyset$, we can choose $Z \subset N(x) \cap N(y) \cap V^*_{D_2}$ so that $G[V^*_{D_1} \cup Z]$ contains $C_{2\lceil\frac{\delta+1}{2}\rceil}$ and $G[V^*_{D_2} \setminus Z]$ contains $C_{2\lfloor\frac{\delta-1}{2}\rfloor}$.

- **Case 2:** $c_1 < c_2$. Then $c_1 = 1, c_2 = 2$. Since $|E(V^*_{D_2}, V \setminus (V^*_{D_1} \cup V^*_{D_2}))| < \frac{|V^*_{D_2}|}{2}$, $|E(V^*_{D_2}, V^*_{D_1})| \geq \frac{5|V^*_{D_2}|}{2}$, so again by Claim 2.4.11, there exist $x, y, z \in V^*_{D_1}$ such that $E(V^*_{D_2}, V^*_{D_1}) = E(V^*_{D_2}, \{x, y, z\})$ and for any $u \in \{x, y, z\}$, $|E(V^*_{D_2}, \{u\})| \geq \frac{|V^*_{D_2}|}{2}$. Note that $G[V^*_{D_1} \cup V^*_{D_2}]$ contains a $(n', 1)$-weak ladder where $n' \geq \delta - 2$ and $G[V^*_{D_3}]$ contains $L_{n''}$ where $n'' = \lfloor\frac{|V^*_{D_3}|}{2}\rfloor$. If there exists n_i such that $2 < n_i \leq n''$ then $G[V^*_{D_3}]$ contains C_{2n_i}, and by Lemma 2.3.2, $G[V^*_{D_1} \cup V^*_{D_2}]$ contains remaining disjoint cycles. If for every i, $n_i = 2$, then G obviously contains all C_{2n_i} for $i \in [l]$.

  Hence we may assume that $l = 2$ and $|V^*_{D_3}| < 2n_1 \leq 2n_2$.

  Claim 2.4.16. *For any $i, j \in [q] \setminus \{1\}$, the size of a maximum matching in $E(V^*_{D_i}, V^*_{D_j})$ is at most one.*

  Proof. Suppose to a contrary that there exist $i, j \in [q]\setminus\{1\}$ such that $E(V^*_{D_i}, V^*_{D_j})$ contains a matching of size at least two. If $2 \notin \{i, j\}$ then by Corollary 2.3.8, $G[V^*_{D_i} \cup V^*_{D_j}]$ contains a $(n', 2)-$weak ladder where $n' \geq \lfloor\frac{|D^*_{V_i}|}{2}\rfloor + \lfloor\frac{|D^*_{V_j}|}{2}\rfloor - 2 \geq \frac{3\delta}{4}$, and then $C_{2n_2} \subset G[V^*_{D_1} \cup V^*_{D_2}], C_{2n_1} \subset G[V^*_{D_i} \cup V^*_{D_j}]$. Otherwise, without loss of generality, $i = 2$, $j > 2$. Let e_1, e_2 be two indepdent edges in $E(V^*_{D_2}, V^*_{D_j})$. Let

$x' \in N(x) \cap V'_{D_2}, y' \in N(y) \cap V'_{D_2}$ such that $x'y' \in E$ and $\{x', y'\} \cap (e_1 \cup e_2) = \emptyset$. Then $G[V^*_{D_1} \cup \{x', y'\}]$ contains C_{2n_1}. Since $G[V^*_{D_2} \setminus \{x', y'\}]$ is 2τ-complete, $G[V^*_{D_2} \cup V^*_{D_j} \setminus \{x', y'\}]$ contains C_{2n_2}. $\square$

If $|V^*_{D_3}| \leq \delta - 4$, then by Claim 2.4.12, $|E(V^*_{D_3}, V^*_{D_1} \cup V^*_{D_2})| > \frac{9|V^*_{D_3}|}{2}$ and then $|E(V^*_{D_3}, V^*_{D_2})| \geq \frac{3|V^*_{D_3}|}{2}$, and we are done by Corollary 2.3.8. So we may assume that $|V^*_{D_3}| \in \{\delta - 2, \delta - 3\}$ which leads to two sub-cases.

- $|V^*_{D_3}| = \delta - 3$. By Claim 2.4.12, $|E(V^*_{D_3}, V^*_{D_1} \cup V^*_{D_2})| > \frac{7|V^*_{D_3}|}{2}$. By Claim 2.4.11, $|E(V^*_{D_3}, V^*_{D_1})| \leq 3|V^*_{D_3}|$, so by Claim 2.4.16, there exists $w \in V^*_{D_2}$ such that $\frac{|V^*_{D_3}|}{2} < |E(V^*_{D_2}, V^*_{D_3})| = |E(\{w\}, V^*_{D_3})| \leq |V^*_{D_3}|$. Hence $|E(V^*_{D_3}, V^*_{D_1})| > \frac{5|V^*_{D_3}|}{2}$, and then there exist $x_1, y_1 \in V^*_{D_1}$ such that $|N(x_1) \cap V^*_{D_3}|, |N(y_1) \cap V^*_{D_3}| > \frac{|V^*_{D_3}|}{2}$. In addition, there is a vertex $z \in V^*_{D_1} \setminus \{x_1, y_1\}$ such that $|N(z) \cap V^*_{D_3}| > \frac{|V^*_{D_3}|}{2}$. Choose $z' \in N(z) \cap N(w) \cap V^*_{D_3}$. Then $G[V^*_{D_2} \cup \{z, z'\}]$ contains C_{2n_1} and $G[V^*_{D_1} \cup V^*_{D_3} \setminus \{z, z'\}]$ contains C_{2n_2}.
- $|V^*_{D_3}| = \delta - 2$. Note that we may assume

$$n_1 = \lfloor \frac{\delta}{2} \rfloor, n_2 = \lceil \frac{\delta}{2} \rceil,$$

as otherwise $2n_1 \leq \delta - 2$, and $G[V^*_{D_3}]$ contains C_{2n_1}, $G[V^*_{D_1} \cup V^*_{D_2}]$ contains C_{2n_2}.

Claim 2.4.17. *For any $i \in [q] \setminus \{1\}$ such that $|V^*_{D_i}| = \delta - 2$, $|E(V^*_{D_1}, V^*_{D_i})| \geq \frac{5|V_{D_i}|}{2}$ and $E(V^*_{D_1}, V^*_{D_i}) = E(\{x, y, z\}, V^*_{D_i})$.*

Proof. Let $i \in [q] \setminus \{1\}$ be such that $|V^*_{D_i}| = \delta - 2$. Since $|V^*_{D_i}| = |V^*_{D_2}|$, by the redistribution process,

$$|E(V^*_{D_1}, V^*_{D_i})| \geq |E(V^*_{D_1}, V^*_{D_2})| \geq \frac{5|V^*_{D_2}|}{2} = \frac{5|V^*_{D_i}|}{2}.$$

There exists $x_1, y_1, z_1 \in V^*_{D_1}$ such that $E(V^*_{D_1}, V^*_{D_i}) = E(\{x_1, y_1, z_1\}, V^*_{D_i})$ and $|N(x_1)\cap V^*_{D_i}|, |N(y_1)\cap V^*_{D_i}|, |N(z_1)\cap V^*_{D_i}| \geq \frac{|V^*_{D_i}|}{2}$, so $|N(x_1 \cap V'_{D_i})|, |N(y_1 \cap V'_{D_i})|, |N(z_1 \cap V'_{D_i})| \geq 4\tau|V'_{D_i}|$, and then for any $U \subset \{x_1, y_1, z_1\}, G[U \cup V^*_{D_i}]$ is τ-complete. If $\{x_1, y_1, z_1\} \neq \{x, y, z\}$, w.l.o.g, $z_1 \notin \{x, y, z\}$, then $G[\{y_1, z_1\} \cup V^*_{D_i}]$ contains C_{2n_1} and $G[V^*_{D_2} \cup V^*_{D_1} \setminus \{y_1, z_1\}]$ contains C_{2n_2}. □

Claim 2.4.18. *For any $i \in [q] \setminus \{1\}$, $|V^*_{D_1}| = \delta - 2$.*

Proof. Suppose not and choose $i \in [q] \setminus \{1\}$ such that $|V^*_{D_i}| \leq \delta - 3$, and subject to this, i is the smallest, i.e, for any $i' \in [i-1] \setminus \{1\}$, $|V^*_{D'_i}| = \delta - 2$. Note that $i \geq 4$. First, assume that there exists $i_1, i_2 \in [i-1] \setminus \{1\}$ such that there exists $y_1 \in V^*_{D_{i_1}}$, $y_2 \in V^*_{D_{i_2}}$ such that $|N(y_1) \cap V^*_{D_i}|, |N(y_2) \cap V^*_{D_i}| > 0$. Let Q be a $N(y_1) - N(y_2)$ path in $G[V^*_{D_i}]$. Since $|V^*_{D_{i_1}}|, |V^*_{D_{i_2}}| = \delta - 2$, by Fact 2.4.17,

$$|E(V^*_{D_{i_1}}, \{x, y, z\})|, |E(V^*_{D_{i_2}}, \{x, y, z\})| \geq \frac{5(\delta - 2)}{2}.$$

Choose $x' \in N(x) \cap V_{D^*_{i_1}}$ such that $x'y_1 \in E$ and $x'' \in N(x) \cap V_{D^*_{i_2}}$ such that $x'' \neq y_2$. Then $G[\{x\} \cup \{x', y_1\} \cup V(Q) \cup V_{D_{i_2}}]$ contains C_{2n_1} and $G[V^*_{D_1} \cup V^*_{D_{i_1}} \setminus \{x, x', y_1\}]$ contains C_{2n_2}. Hence $|E(V^*_{D_1}, V^*_{D_i})| \geq \frac{5|V^*_{D_i}|}{2}$ and there exists $i' \in [i-1] \setminus \{1\}$ such that $|E(V^*_{D_{i'}}, V^*_{D_i})| \geq \frac{|V^*_{D_i}|}{2}$ and then there exists $V^*_{D_i} - V^*_{D_{i'}}$ path Q such that $V(\mathring{Q}) \subset V^*_{D_1}$ and $|V(\mathring{Q}) \cap \{x, y, z\}| = 1$. Similarly, we can find C_{2n_1} in $G[V^*_{D_{i'}} \cup V^*_{D_i} \cup V(Q)]$ and C_{2n_2} in $G[V^*_{D_1} \cup V^*_{D_j} \setminus V(Q)]$ where $j \in \{2, 3\} \setminus \{i'\}$. □

Finally, suppose there exists $i, j(> i) \in [q] \setminus \{1\}$ such that $E(V^*_{D_i}, V^*_{D_j}) \neq \emptyset$. Let $e^* \in E(V^*_{D_i}, V^*_{D_j})$. By Claim 2.4.17, $|N(x) \cap V^*_{D_i}|, |N(x) \cap V^*_{D_j}| \geq \frac{\delta-2}{2}$, so we can choose $x' \in (N(x) \cap V^*_{D_i}) \setminus e^*, x'' \in (N(x) \cap V^*_{D_j}) \setminus e^*$. Then $G[V^*_{D_i} \cup V^*_{D_j} \cup \{x\}]$ contains C_{2n_2} and $G[V^*_{D_1} \setminus \{x\}]$ contains C_{2n_1}. Therefore, for any $i, j \in [q] \setminus \{i\}$, $E(V^*_{D_i}, V^*_{D_j}) = \emptyset$, which implies that G is a graph from Example 2.1.3.

□

We can now finish the proof. If R is connected then by Lemma 2.4.5, 2.3.2, G contains cycles $C_{2n_1}, \ldots, C_{2n_l}$ or there exists a set $V' \subset V$ with $|V'| \geq (1 - \delta/n - \beta)n$, such that all but at most $4\beta n$ vertices $v \in V'$ have $|N_{G'}(v)| \leq \beta n$ where $G' = G[V']$. If R is disconnected and there is a component which is non-bipartite, then we are done by Lemma 2.4.6,2.4.10, and 2.3.2, and if all components are bipartite, then G has $C_{2n_1}, \ldots, C_{2n_l}$ by Lemma 2.4.7, 2.3.2. □

2.5 The second non-extremal case

In this section we will show that if G is non-extremal and $\delta(G) \geq (1/2 - \gamma)n$ for small enough γ, then G contains disjoint cycles $C_{2n_1}, \ldots, C_{2n_l}$.

Theorem 2.5.1. *There exists $\gamma > 0$ and N such that for every 2-connected graph G on $n \geq N$ vertices with $(1/2 - \gamma)n \leq \delta(G) < n/2 - 1$, G contains disjoint cycles $C_{2n_1}, \ldots, C_{2n_l}$ for every $n_1, \ldots, n_l$ where $n_i \geq 2$ and $n_1 + \cdots + n_l = \delta(G)$ or G is β-extremal for some $\beta = \beta(\gamma)$ such that $\beta \to 0$ as $\gamma \to 0$. In addition, if G is not β-extremal and $n/2 - 1 \leq \delta(G) \leq n/2$, then G contains a cycle on $2\delta(G)$ vertices.*

Proof. We will use the same strategy as in the proof of Theorem 2.4.2. The first part of the proof is very similar to an argument from (Czygrinow and Kierstead, 2002) and we only outline the main idea. Consider the reduced graph R as in the proof of Theorem 2.4.2.

First suppose R is connected. We will use the procedure from (Czygrinow and Kierstead, 2002) to show that either G has a ladder on at least $n - 1$ vertices or G

is β-extremal. Since R is connected and $\delta(R) \geq (\delta/n - 2d_1)t \geq (1/2 - 2\gamma)t$, there is a path in R, $P = U_1V_1, \ldots, U_sV_s$ where $s \geq (1/2 - 3\gamma)t$. As in (Czygrinow and Kierstead, 2002) we move one vertex from U_1 to U_s, and the clusters in R which are not on P to V_0 so that $|V_0| \leq 7\gamma n$ and redistribute vertices from V_0 using the following procedure from (Czygrinow and Kierstead, 2002). Let ξ, σ be two constants. The procedure is executed twice with different values of ξ and σ. Distribute two vertices at a time and assign them to U_i, V_j so that for every i, $|U_i| - |V_i|$ is constant, the number of vertices assigned to U_i and V_j is at most $O(\xi n/k)$, and if x is assigned to U_i (V_j), then $|N_G(x) \cap V_i| \geq \sigma n/k$ $(|N_G(x) \cap U_j| \geq \sigma n/k)$. Let Q denote the set of clusters X such that $\xi n/k$ vertices have been assigned to X. We have $|Q| \leq 7\gamma k/\xi$. For $X \in \{U_i, V_i\}$, let X^* be such that $\{X^*, X\} = \{U_i, V_i\}$. For a vertex z let $N_z = \{X \in V(P) \setminus Q | |N_G(z) \cap X^*| \geq \sigma n/k\}$ and $N_z^* = \{X^* | X \in N_z\}$.

Take x, y from V_0, and choose X, Y such that X, X^*, Y, Y^* are not in Q, and either $N_x^* \cap N_y \neq \emptyset$ or $N_x^* \cap N_y = \emptyset$ but $\exists_{X \in N_x, Y \in N_y} |E_G(X, Y)| \geq 2\sigma n^2/k^2$. The argument from (Czygrinow and Kierstead, 2002) shows that either G has a ladder on $2\lfloor n/2 \rfloor$ vertices or the algorithm fails. We will show that if it fails, then G is β-extremal for some $\beta > 0$. Since $|Q| \leq 7\gamma k/\xi$ and $|V_0| \leq 7\gamma n$, using the fact that $\delta(G) \geq (1/2 - \gamma)n$, we have $|N_x| \geq \left(\frac{1}{2} - \frac{10\gamma}{\xi}\right) k$. If $N_x^* \cap N_y \neq \emptyset$, then we assign x to $X \in N_x$ and y to X^* for some X such that $X^* \in N_x^* \cap N_y$. Otherwise $N_x^* \cap N_y = \emptyset$ (and so N_x and N_y are almost identical). If there is $X \in N_x$ and $Y \in N_y$ such that $|E(X, Y)| \geq 2\sigma n^2/k^2$, then assign x to X, y to Y and a vertex $y' \in Y$ such that $|N_G(y) \cap X| \geq \sigma n/k$ to X^*. Otherwise G is β-extremal for some $\beta > 0$.

We can now assume that R is disconnected so it has two components D_1, D_2. Although slightly different arguments will be needed, we will reuse some parts of the proof of Lemma 2.4.10. As in the proof of Theorem 2.4.2, we have $\delta(R) \geq (\delta/n - 2d_1)t \geq (1/2 - 3d_1)t$. We set $\xi := 12d_1$, $\tau = 400d_1$ and for a component D

define $V'_D = \{v||N_G(v) \cap V_D| \geq (1-\sqrt{\xi})|V_D|\}$ where V_D is the set of vertices in clusters from D. As in the case of the proof of Lemma 2.4.10, we have $|E(G_D)| \geq (1-\xi)\binom{|V_D|}{2}$ and similarly we also have $|V_D \setminus V'_D| \leq 2\sqrt{\xi}|V_D|$. We move vertices from $V_D \setminus V'_D$ to V_0 and redistribute them to obtain V''_D consisting of those vertices $v \in V_0$ for which $|N_G(v) \cap V'_D| \geq \frac{\delta}{6} \geq \tau|V'_D|$, and set $V^*_D := V'_D \cup V''_D$. We have $V^*_{D_1} \cup V^*_{D_2} = V(G)$ and $G[V^*_{D_1}]$, $G[V^*_{D_2}]$ are τ-complete. By Lemma 2.3.6 (2), $G[V^*_{D_1}]$ contains $L_{\lfloor\frac{|V^*_{D_1}|}{2}\rfloor}$ and $G[V^*_{D_2}]$ contains $L_{\lfloor\frac{|V^*_{D_2}|}{2}\rfloor}$.

Since $\delta < \frac{n}{2} - 1$, $n = 2\delta + K$ where $K \geq 3$. If δ is odd then there exists $i \in [l]$ such that $n_i > 2$. If δ is even, i.e, $4|2\delta$, and for any $i \in [l]$, $n_i = 2$ then G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$. Indeed, if $|V^*_{D_1}| = 4t + b, |V^*_{D_2}| = 4t' + b'$ such that $b + b' > K, b, b' < 4$ then $b + b' \geq K + 4 \geq 7$, so $b = 4$ or $b' = 4$, a contradiction. Hence by Lemma 2.3.2 and 2.3.3, it suffices to show that G contains either $(\delta+2, 2)$-weak ladder or $(\delta, 1)$-weak ladder. Since G is 2-connected, there is a matching of size two in $G[V^*_{D_1}, V^*_{D_2}]$.

Claim 2.5.2. *If there exists a matching consisting of $u_1u_2, v_1v_2 \in E(V^*_{D_1}, V^*_{D_2})$ such that $N(u_1) \cap N(v_1) \cap V^*_{D_1} \neq \emptyset$ and $N(u_2) \cap N(v_2) \cap V^*_{D_2} \neq \emptyset$ then $G[V^*_{D_1} \cup V^*_{D_2}]$ contains $(n', 1)$-weak ladder where $n' \geq \lfloor\frac{|V^*_{D_1}|-1}{2}\rfloor + \lfloor\frac{|V^*_{D_2}|-1}{2}\rfloor$.*

Proof. For $i \in [2]$, choose $z_i \in N(u_i) \cap N(v_i)$. By Lemma 2.3.6 (4), $G[V^*_{D_i} \setminus \{u_i\}]$ contains $L_{\lfloor\frac{|V^*_{D_i}|-1}{2}\rfloor}$ having z_iv_i as its first rung. By attaching these two ladders with u_1u_2, v_1v_2, we obtain a desired weak ladder. □

Claim 2.5.3. *If $|V^*_{D_1}| \leq \delta$ then there exists a matching consisting of $u_1u_2, v_1v_2 \in E(V^*_{D_1}, V^*_{D_2})$ such that $u_1v_1 \in E$.*

Proof. Let I be a maximum independent set in $G[V^*_{D_1}]$. If $G[V^*_{D_1}]$ is complete then it is trivial, so we may assume $|I| \geq 2$. Choose $u_1 \in I, v_1 \in V^*_{D_1} \setminus I$ such that $u_1v_1 \in E$.

Since $|I| \geq 2$, $|N(u_1) \cap V^*_{D_1}| \leq |V^*_{D_1}| - 2 \leq \delta - 2$, which implies that

$$|N(u_1) \cap V^*_{D_2}| \geq 2.$$

Since $|N(v_1) \cap V^*_{D_2}| \geq 1$, we can choose $v_2 \in N(v_1) \cap V^*_{D_2}$ and $u_2 \in N(u_1) \cap V^*_{D_2}$ such that $u_2 \neq v_2$. □

Claim 2.5.4. *If* $|V^*_{D_1}|, |V^*_{D_2}| \leq \delta + 8$ *then there exists a matching consisting of* $\{u_1, u_2\}, \{v_1, v_2\} \in E(V^*_{D_1}, V^*_{D_2})$ *such that one of the following holds.*

- $N(u_1) \cap N(v_1) \neq \emptyset$ *and* $N(u_2) \cap N(v_2) \neq \emptyset$.
- $u_1v_1 \in E$ *or* $u_2v_2 \in E$.

Proof. Suppose that for any two independent edges $u_1u_2, v_1v_2 \in E(V^*_{D_1}, V^*_{D_2})$, the first condition does not hold. Choose $u_1u_2, v_1v_2 \in E(V^*_{D_1}, V^*_{D_2})$. If $u_1v_1 \in E$ or $u_2v_2 \in E$ then the second condition holds, so we may assume that $u_1v_1 \notin E$ and $u_2v_2 \notin E$. Without loss of generality, $N(u_1) \cap N(v_1) = \emptyset$. Then $|N(u_1) \cap V^*_{D_1}| + |N(v_1) \cap V^*_{D_1}| \leq |V^*_{D_1}| - 2$, and then

$$|N(u_1) \cap V^*_{D_2}| + |N(v_1) \cap V^*_{D_2}| \geq 2\delta - (|V^*_{D_1}| - 2) \geq \delta - 6 \geq (1 - \tau)|V^*_{D_2}|.$$

Without loss of generality, $|N(v_1) \cap V^*_{D_2}| \geq |N(u_1) \cap V^*_{D_2}|$, so $|N(v_1) \cap V^*_{D_2}| \geq \frac{(1-\tau)|V^*_{D_2}|}{2}$. If $|N(u_1) \cap V^*_{D_2}| > \tau|V^*_{D_2}|$ then there exists $u'_2 \in N(u_1) \cap V'_{D_2}$, since $|N(u'_2) \cap V'_{D_2}| \geq (1 - \tau)|V'_{D_2}|$ there exists $v'_2 \in N(u'_2) \cap N(v_1) \cap V'_{D_2}$, so $\{u_1, v_1\}, \{u'_2, v'_2\}$ are such that the second condition holds. Otherwise, assume that $|N(u_1) \cap V^*_{D_2}| \leq \tau|V^*_{D_2}|$. Note that $|N(u_2) \cap V'_{D_2}| \geq 4\tau|V^*_{D_2}|$. Since $|N(v_1) \cap V^*_{D_2}| \geq (1 - 2\tau)|V^*_{D_2}|$, there exists $v'_2 \in N(v_1) \cap N(u_2) \cap V^*_{D_2}$, so $\{u_1, v_1\}, \{u_2, v'_2\}$ are such that the second condition holds. □

Without loss of generality $|V^*_{D_1}| \leq |V^*_{D_2}|$. If $|V^*_{D_1}| \leq \delta$ then by Claim 2.5.3 and Corollary 2.3.8, G contains $(\delta, 1)$-weak ladder. Hence we may assume that $|V^*_{D_1}| \geq$

$\delta+1$. If $|V^*_{D_2}| \geq \delta+9$ then $|V^*_{D_1}|+|V^*_{D_2}| \geq 2\delta+10$, and then by Corollary 2.3.8, G contains $(n',2)$-weak ladder where

$$n' \geq \lfloor \frac{|V^*_{D_1}|}{2} \rfloor + \lfloor \frac{|V^*_{D_2}|}{2} \rfloor - 2 \geq \delta+2.$$

Hence $\delta+1 \leq |V^*_{D_1}| \leq |V^*_{D_2}| \leq \delta+8$. By Claim 2.5.4, there exists a matching consisting of $u_1u_2, v_1v_2 \in E(V^*_{D_1}, V^*_{D_2})$ such that one of the conditions from Claim 2.5.4 holds. If $N(u_1) \cap N(v_1) \neq \emptyset$ and $N(u_2) \cap N(v_2) \neq \emptyset$ then by Claim 2.5.2, G contains $(n',1)$-weak ladder where $n' \geq \lfloor \frac{|V^*_{D_1}|-1}{2} \rfloor + \lfloor \frac{|V^*_{D_2}|-1}{2} \rfloor \geq \delta$. Otherwise, $u_1v_1 \in E$ or $u_2v_2 \in E$, then by Corollary 2.3.8, G contains $(n',1)$-weak ladder where $n' \geq \lfloor \frac{|V^*_{D_1}|}{2} \rfloor + \lfloor \frac{|V^*_{D_2}|}{2} \rfloor - 1 \geq \delta$. □

2.6 Extremal Case

In this section we will prove the extremal case.

Theorem 2.6.1. *Let $0 < \alpha < \frac{1}{2}$ be given and β be such that $\beta < (\frac{\alpha}{400})^2 \leq \frac{1}{640000}$. If G is a graph on n vertices with minimum degree $\delta \geq \alpha n$ which is β-extremal, then either G contains L_δ or G is a subgraph of the graph from Example 2.1.4. Moreover, in the case when G is a subgraph of the graph from Example 2.1.4, for every $n_1, \ldots, n_l \geq 2$ such that $\sum n_i = \delta$, G contains disjoint cycles $C_{2n_1}, C_{2n_2}, \ldots, C_{2n_l}$ if $n_i \geq 3$ for at least one i.*

Proof. Recall that G is β-extremal if the exists a set $B \subset V(G)$ such that $|B| \geq (1-\delta/n-\beta)n$ and all but at most $4\beta n$ vertices $v \in B$ have $|N(v) \cap B| \leq \beta n$. Let $A = V(G) \backslash B$ and note that $\delta - \beta n \leq |A| \leq \delta + \beta n$, because for some $w \in B$, $|N(w) \cap A| \geq \delta - \beta n$. Let $C := \{v \in B : |N(v) \cap B| > \beta n\}$, $A_1 := A, B_1 := B \setminus C$. Then $|B_1| \geq n - \delta - 5\beta n$ and $|C| \leq 4\beta n$. Consequently, we have

$$|E(A_1, B_1)| \geq (\delta - 5\beta n)|B_1| \geq (\delta - 5\beta n)(n - \delta - 5\beta n) \geq \delta n - \delta^2 - 5\beta n^2. \quad (2.2)$$

We have the following claim.

Claim 2.6.2. *There are at most $\sqrt{\beta}n$ vertices v in A_1 such that $|N(v) \cap B_1| < n - \delta - 6\sqrt{\beta}n$.*

Proof. Suppose not. Then

$$|E(A_1, B_1)| < (n - \delta - 6\sqrt{\beta}n) \cdot \sqrt{\beta}n + (n - \delta + \beta n)(\delta + \beta n - \sqrt{\beta}n) \leq \delta n - \delta^2 - 5\beta n^2.$$

This contradicts (2.2). □

Let $\gamma := 6\sqrt{\beta}$ and move those vertices $v \in A_1$ to C for which $|N(v) \cap B_1| < n-\delta-6\sqrt{\beta}n$. Let $A_2 := A_1 \setminus C$, $B_2 := B_1$. Then, by Claim 2.6.2, $|A_2| \geq \delta-(\beta+\sqrt{\beta})n$, $|B_2| \geq n - \delta - 5\beta n$ and,

$$|C| \leq (4\beta + \sqrt{\beta})n < 2\sqrt{\beta}n. \tag{2.3}$$

In addition, for every $v \in A_2$, $|N(v) \cap B_2| \geq n - \delta - 6\sqrt{\beta}n$ and for every vertex $v \in B_2$, $|N(v) \cap A_2| \geq \delta - (\beta + \sqrt{\beta})n$.

We now partition $C = A_2' \cup B_2'$ as follows. Add v to A_2' if $|N(v) \cap B_2| \geq \gamma n$, and add it to B_2' if $|N(v) \cap A_2| \geq \gamma n$ and $\min\{|A_2 \cup A_2'|, |B_2 \cup B_2'|\}$ is as large as possible. Without loss of generality assume $|A_2 \cup A_2'| \leq |B_2 \cup B_2'|$. We have two cases.

Case (i) $|A_2 \cup A_2'| \geq \delta$.

For any $v \in A_2'$, $|N(v) \cap B_2| \geq \gamma n > |C| \geq |A_2'|$. Therefore, there exists matching $M \in E(A_2', B_2)$ which saturates A_2'. Note that $q := |M| = |A_2'| \leq |C| < 2\sqrt{\beta}n$. For every $\{x_i, y_i\} \in M$, we can pick $x_i', x_i'' \in A_2, y_i', y_i'' \in B_2 \setminus V(M)$ all distinct, so that $\{x_i', y_i'\} \in E, \{x_i'', y_i''\} \in E$ and $x_i', x_i'' \in N(y_i), y_i', y_i'' \in N(x_i)$. Note that this is possible because $|N(x_i) \cap B_2| \geq 6\sqrt{\beta}n > 3|M|$. Then $G[\{x_i, y_i, x_i', x_i'', y_i', y_i''\}]$ contains a 3-ladder, which we will denote by L_i. Note that $|\bigcup_{i \leq q} V(L_i)| = 3|M| < 6\sqrt{\beta}n$. We repeat the same process to find p 3-ladders L_j for each vertex from B_2'. We have $p+q = |C| < 2\sqrt{\beta}n$ 3-ladders, each containing exactly one vertex from C. Note that $|A_2 \setminus \bigcup_{i=}^{p+q} V(L_i)| \leq |B_2 \setminus \bigcup_{i=1}^{p+q} V(L_i)|$.

For every $v \in A_2 \setminus \bigcup_{i=1}^{p+q} V(L_i)$, $|N(v) \cap (B_2 - (\bigcup_{i=1}^{p+q} V(L_i)))| \geq n - \delta - 18\sqrt{\beta}n$ and for every $v \in B_2 \setminus \bigcup_{i=1}^{p+q} V(L_i)$, $|N(v) \cap A_2| \geq \delta - 18\sqrt{\beta}n$. Therefore there exists a matching $M' = \{\{a_i, b_i\} : i = 1, \ldots, |A_2 - \cup_{i=1}^{p+q} V(L_i)|\}$ which saturates $A_2 - \bigcup_{i=1}^{p+q} V(L_i)$. Define the auxiliary graph H as follows. For every L_i consider vertex V_{L_i} and let

$$V(H) = \{v_{L_i} : i \in [p+q]\} \cup \{e : e \in M'\}.$$

For $e = a_i b_i, e' = a_j b_j \in M'$, $ee' \in E(H)$ if $G[\{a_i, a_j\}, \{b_i, b_j\}] = K_{2,2}$ and for $v_{L_i} \in V(H), e = a_j b_j \in M'$, $v_{L_i} e \in E(H)$ if $a_j \in N(y'_i) \cap N(y''_i), b_j \in N(x'_i) \cap N(x''_i)$. Then $\delta(H) \geq |H| - 100\sqrt{\beta}n > \frac{|H|}{2}$, H contains a Hamilton cycle, which gives, in turn, a $(|A_2| + |A'_2|)$-ladder in G.

Case (ii) $|A_2 \cup A'_2| = \delta - K$ for some $0 < K \leq \beta + \sqrt{\beta}n < 2\sqrt{\beta}n$.

Note that for every vertex $v \in B_2 \cup B'_2$, $K \leq |N(v) \cap (B_2 \cup B'_2)| < (\gamma + 2\sqrt{\beta})n$. Indeed, if $v \in B_2$, then $|N(v) \cap (B_2 \cup B'_2)| \leq \beta n + |B'_2|$ and if $v \in B'_2$, then $|N(v) \cap (B_2 \cup B'_2)| < \gamma n + |B'_2|$ as otherwise we would move v to A'_2. Thus, in particular, for every $v \in B_2 \cup B'_2$, $|N(v) \cap A_2| \geq 9|A_2|/10$. In addition, $|A_2 \cup A'_2| \leq |B_2 \cup B'_2| - 2K - 2$.

Let Q be a maximum triple matching in $G[B_2 \cup B'_2]$ and Q' be a maximum double matching in $G[B_2 \cup B'_2 \setminus V(Q)]$.

Claim 2.6.3. *If $|Q| + |Q'| \geq K$ and $|Q'| \leq 2$, then G contains L_δ.*

Proof. Without loss of generality, let $|Q| = K - 2$ and $|Q'| = 2$. For $i \in [K-2]$, let x_i denote the center of the ith star in the triple matching and let x'_i, y'_i, z'_i be its leaves in $G[B_2 \cup B'_2]$. Let x_{K-1}, x_K be the centers of the stars in the double matching and let $\{x'_{K-1}, y'_{K-1}\}, \{x'_K, y'_K\}$ denote the sets of leaves. Let $S := \{x_1, \ldots, x_K\}$ and note that $|S \cup A_2 \cup A'_2| = \delta$. For every $z, w \in B_2 \cup B'_2$, $|N(w) \cap N(z) \cap A_2| \geq 4|A_2|/5$. Therefore, for any $i \in [K-2]$, there exists $y_i \in N(y'_i) \cap N(x'_i) \cap A_2$ and $z_i \in N(z'_i) \cap N(x'_i) \cap A_2$, i.e $G[\{x_i, x'_i, y_i, y'_i, z_i, z'_i\}]$ forms 3-ladder and for $j \in \{K-1, K\}$, there exists $y_j \in$

$N(x'_j) \cap N(y'_j) \cap A_2$, so $G[x_j, x'_j, y_j, y'_j]$ forms a 2-ladder, say L_j. As similar as we did in the case (i), we define auxiliary graph H such that $V(H)$ consists of $K-2$ 3-ladders, 2 2-ladders and 3-ladders wrapping remaining vertices in $A'_2 \cup B'_2$ and matchings in $E(A_2, B_2)$ saturating remaining vertices in A_2. For the definition of $E(H)$, only difference with what did in case (i) is for $v_{L_{K-1}}, v_{L_K}$. For $e = ab \in M$,$j \in \{K-1, K\}$, $v_{L_j}e \in E(H)$ if $y_jb, ay'_j \in E$. Then $d_H(v_{L_{K-1}}), d_H(v_{L_K}) > \frac{|H|}{2}$ and $H - \backslash\{v_{L_{K-1}}, v_{L_K}\}$ has a Hamilton cycle and then we obtain a Hamilton path in H which has $v_{L_{K-1}}, v_{L_K}$ as its two ends. It implies that G contains L_δ. □

Claim 2.6.4. *If $K \geq 3$ then $|Q| \geq K$.*

Proof. Suppose not. Then every vertex $v \in (B_2 \cup B'_2)\backslash V(Q)$ has at least $K-2$ neighbors in $V(Q)$. Hence

$$\begin{aligned}|E(V(Q), (B_2 \cup B'_2)\backslash V(Q))| &\geq (K-2)|(B_2 \cup B'_2)\backslash V(Q)| \\ &> (K-2)(|(B_2 \cup B'_2)| - 4K) \\ &= (K-2)(n - \delta - 3K) \\ &> (K-2)(n - \delta - 6\sqrt{\beta}n) > Kn/5.\end{aligned}$$

Since for every $v \in B_2 \cup B'_2$, $|N(v) \cap (B_2 \cup B'_2)| < (\gamma + 2\sqrt{\beta})n$,

$$|E(V(Q), (B_2 \cup B'_2)\backslash V(Q)) < 4K(\gamma + 2\sqrt{\beta})n = 32K\sqrt{\beta}n.$$

By combining these two inequalities, we obtain

$$\beta > (\frac{1}{160})^2,$$

which is a contradiction to $\beta < \frac{1}{640000}$. □

By Claim 2.6.4 and 2.6.3, we may assume that $K \leq 2$. Assume that $K = 2$. By Claim 2.6.3, $|Q| + |Q'| \leq 1$ and then every vertex $v \in (B_2 \cup B'_2)\backslash(V(Q) \cup V(Q'))$ has at least $K-1$ neighbors in $V(Q)\cup V(Q')$. By the same calculation as we did in Claim 2.6.4, we will run into a contradiction.

Finally, suppose that $K=1$, i.e, $|A_2\cup A_2'|=\delta-1$ and $\delta(G[B_2\cup B_2'])\geq 1$. By Claim 2.6.3, $\Delta(G[B_2\cup B_2'])\leq 1$, which implies that $G[B_2\cup B_2']$ is a perfect matching, so $|B_2\cup B_2'|$ is even. Hence G is a subgraph of the graph from Example 2.1.4. To prove the "Moreover" part, we proceed as follows. Let $v_1v_1', v_2v_2'\in E(G[B_2\cup B_2'])$. We have $|N(v_1)\cap N(v_2)\cap A_2|\geq 4|A_2|/5$ and $|N(v_1')\cap N(v_2')\cap A_2|\geq 4|A_2|/5$. Thus there is a copy of C_6 containing v_1v_1', v_2v_2', say $C_6 : x_1v_1v_1'x_1'v_2v_2'x_1$ where $x_1, x_1'\in A_2$. Similarly, $G[A_2\cup A_2'\cup B_2\cup B_2'\setminus V(C_6)]$ contains $L_{\delta-3}$ such that $z_1\in N(x_1)\cap B_2, z_1'\in N(v_1)\cap A_2$ and z_1z_1' is the first rung of $L_{\delta-3}$. Let $n_l\geq 3$. Then the C_6 with first n_l-3 rung contains C_{2n_l} and remaining $L_{\delta-3-(n_l-3)}=L_{\delta-n_l}$ contains disjoint cycles $C_{2n_1},\ldots,C_{2n_{l-1}}$. □

2.7 Final comments

Proof of Corollary 2.1.5. Let G be a graph on $n\geq N(\alpha/8)$ vertices such that $||G||\geq \alpha n^2$. If $\delta(G)\geq n/2$, then G is pancyclic. If $n/2-1>\delta(G)\geq \alpha n/8$, then Theorem 1.0.2 implies that G contains all even cycles of length $4,\ldots,2\delta(G)$. If $\delta(G)\geq n/2-1$, then G contains all cycle if lengths $4,\ldots,2\delta(G)-2$ by Theorem 1.0.2 and a cycle on $2\delta(G)$ vertices by Theorem 2.4.2, Theorem 2.6.1 and Theorem 2.5.1.

Otherwise, by Mader's theorem, G contains a subgraph H which is $\alpha n/4$-connected. If $|H|\leq 2\delta(H)$ then H is pancyclic by Bondy's theorem and we are done since $|H|\geq \alpha n/4\geq 2\delta(G)$. If $\delta(H)\leq |H|/2$, then by Theorem 1.0.2, H contains all even cycles $4,\ldots,2\delta(H)$ and $\delta(H)\geq \alpha n/4\geq \delta(G)$. □

Chapter 3

BALANCED SPANNING CATERPILLAR

3.1 Introduction

Every connected graph contains a spanning tree, yet quite often it is desirable to find a spanning tree which satisfies certain additional conditions. There are many results giving sufficient minimum degree conditions for graphs to contain very special spanning trees. For example, Dirac's theorem from (Dirac, 1952) states that any graph on $n \geq 3$ vertices with minimum degree at least $(n-1)/2$ has a spanning path. In (Win, 1975), S. Win generalized this fact and proved the following theorem.

Theorem 3.1.1. *Let $k \geq 2$ and let G be a graph on n vertices such that $\sum_{x \in I} d(x) \geq n-1$ for every independent set I of size k. Then G contains a spanning tree of maximum degree at most k.*

In particular, if the minimum degree of G is at least $(n-1)/k$, then G contains a spanning tree of maximum degree at most k. In fact, as showed in (Czygrinow *et al.*, 2001), the degree condition from Theorem 3.1.1 implies that either G has a spanning caterpillar of maximum degree at most k or G belongs to a special exceptional class. We refer the reader to (Ozeki and Yamashita, 2011) for a comprehensive survey of spanning trees.

Another way of thinking about caterpillars is by looking at domination problems. A set $S \subseteq V$ is a dominating set in a graph $G = (V, E)$ if every vertex in $V \setminus S$ has a neighbor in S. A dominating set S is called a *connected dominating set* if, in addition, $G[S]$ is connected. In the special case when $G[S]$ contains a path, we say that G has a dominating path. In (Broersma, 1988), Broersma proved a result on cycles passing within a specified distance of a vertex and stated an analogous result for paths from which, as one of the corollaries, we get the following fact.

Theorem 3.1.2. *If G is a k-connected graph on n vertices such that $\delta(G) > \frac{n-k}{k+2} - 1$, then G contains a dominating path.*

In particular, if G is connected then $\delta(G) > \frac{n-1}{3} - 1$ implies that G has a spanning caterpillar. In this paper we will be concerned with a minimum degree condition that implies existence of spanning balanced caterpillar.

A p-caterpillar is a tree such that the graph induced by its internal vertices is a path and every internal vertex has exactly p leaves. The *spine* of a caterpillar is the graph induced by its internal vertices. The length of a caterpillar is the length of its spine. We recall Theorem 1.0.4, which gives the motivation of the research in this chapter.

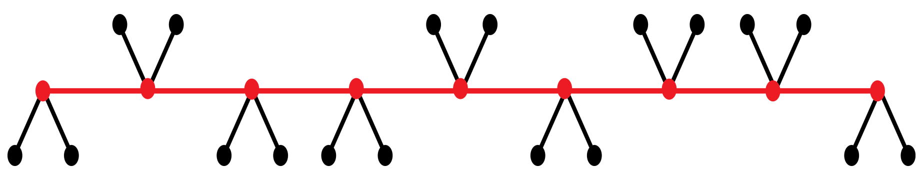

Figure 3.1: 2-caterpillar

Theorem 1.0.4. *(Faudree* et al.*, 2017) For $p \in Z^+$ there exists n_0 such that for every $n \in (p+1)Z$ such that $n \geq n_0$ the following holds. If G is a graph on n vertices such that $\delta(G) \geq \left(1 - \frac{p}{(p+1)^2}\right) n$, then G contains a spanning p-caterpillar.*

The authors of (Faudree *et al.*, 2017) ask for the tight minimum degree condition which implies that G has a spanning 1-caterpillar. In addition, they ask for a tight minimum degree condition which gives a nearly balanced p-caterpillar (every vertex on the spine has p or $p+1$ leaf neighbors). We will settle the first problem and answer the second question in the case when n is divisible by $p+1$. In this chapter we will substantially improve the minimum degree bound from Theorem 1.0.4 and give a tight minimum degree condition which guarantees existence of a spanning $p-$caterpillar. We recall the main result of this chapter.

Theorem 1.0.5. *For $p \in Z^+$, there exists n_0 such that for every $n \in (p+1)Z$ with $n \geq n_0$ the following holds. If G is a graph on n vertices such that*

$$\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } n/(p+1) \text{ is even} \\ \frac{n+1}{2} & \text{if } n/(p+1) \text{ is odd and } p > 2 \\ \frac{n-1}{2} & \text{if } n/(p+1) \text{ is odd and } p \leq 2 \end{cases}$$

then G contains a spanning p-caterpillar.

It's not difficult to see that the minimum degree condition in Theorem 1.0.5 is best possible.

Example 3.1.3. *First note that $K_{n/2} \cup K_{n/2}$ in the case n is even and $K_{(n-1)/2} \cup K_{(n+1)/2}$ in the case n is odd have no spanning caterpillars. Thus the degree condition in the case $p \leq 2$ is tight. Now suppose $p \geq 3$. Let $n/(p+1)$ be even. Then $n/2$ is an integer. Consider $K_{n/2-1,n/2+1}$. Clearly $n/(2(p+1))$ of spine vertices must be in one of the partite sets, because the spine is a path and its maximum independent set is of size $n/(2(p+1))$, but then the two partite sets must have the same size. Another example is $K_{n/2} \cup K_{n/2}$. Now, suppose $n/(p+1)$ equals $2k+1$ for some $k \in Z^+$. If n is even, then consider $K_{n/2,n/2}$. Clearly one of the partite sets must have $k+1$ spine vertices and so the other set must contain $(k+1)(p+1)-1 = \frac{n+p-1}{2} > n/2$ as $p > 1$. If n is odd then consider $K_{(n-1)/2,(n+1)/2}$. Now, $k+1$ of the spine vertices must be in the partite set of size $(n-1)/2$. Consequently, the other set must have at least $\frac{n+p-1}{2} > (n+1)/2$ as $p > 2$.*

We will prove Theorem 1.0.5 using the absorbing method from (Rödl *et al.*, 2006). In this method, we first analyze the non-extremal case and then address two extremal cases, when G is "close to" $2K_{\lfloor n/2 \rfloor}$ or $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Throughout this chapter we discuss simple undirected graphs, and the notation we use here is already described in Section 1.1. We say that a graph G is β-extremal if either $V(G)$ contains a set W such that $|W| \geq (1/2 - \beta)n$ and $||G[W]|| \leq \beta n^2$ or if $V(G)$ can be partitioned into sets V_1, V_2 so that $|V_i| \geq (1/2 - \beta)n$ for $i = 1, 2$ and $||V_1, V_2|| \leq \beta n^2$. In addition, the following notation and terminology will be used. A u, v-caterpillar is a p-caterpillar where the first vertex in the spine is u and the last is v.

The rest of Chapter 3 is structured as follows. In Section 3.2 we prove the absorbing lemma which is the key to handle the non-extremal case. In Section 3.3 we prove the non-extremal case and in Section 3.4 we address the extremal cases.

3.2 Absorbing Lemma

In this section we will prove an absorbing lemma and a few additional facts which are used in the next section to complete the proof in the case a graph is not extremal. We will start with the following observation.

Lemma 3.2.1. *For $1/8 > \beta > 0$ there is $\alpha > 0$ and n_0 such that the following holds. If G is a graph on $n \geq n_0$ vertices such that $\delta(G) \geq (1/2 - \beta^2)n$ which is not β-extremal, then for any (not necessarily distinct) vertices $u, v \in G$, $||N(u), N(v)|| \geq \beta^2 n^2/32$.*

Proof. We have $||G[N(u)]|| > \beta n^2$ from the definition of a β-extremal graph. Now suppose u, v are two distinct vertices. If $\beta n/2 \leq |N(u) \cap N(v)| \leq (1/2 - \beta/2)n$, then $|N(u) \cup N(v)| \geq 2(1/2 - \beta^2)n - (1/2 - \beta/2)n \geq (1/2 + \beta/4)n$. Thus every vertex $x \in N(u) \cap N(v)$ has at least $\beta n/8$ neighbors in $N(u) \cup N(v)$. Consequently, $||N(v), N(u)|| \geq \beta^2 n^2/32$. If $|N(u) \cap N(v)| < \beta n/2$, then $|N(v) \setminus N(u)| \geq (1/2 - 2\beta/3)n$. Thus, since G is not β-extremal $||N(u), N(v)|| \geq \beta^2 n^2/32$. If $|N(u) \cap N(v)| \geq (1/2 - \beta/2)n$, then $||G[N(u) \cap N(v)]|| \geq \beta n^2$. □

Our next objective is to establish the following connecting lemma.

Lemma 3.2.2 (Connecting Lemma). *For $1/8 > \beta > 0$ there is $\alpha > 0$ and n_0 such that the following holds. If G is a graph on $n \geq n_0$ vertices such that $\delta(G) \geq (1/2 - \beta^2)n$ which is not β-extremal, then for any two vertices $u, v \in G$ there are at least αn^{4p+2} u, v-caterpillars of length three in G.*

Proof. Let u, v be two distinct vertices. By Lemma 3.2.1, $||N(u), N(v)|| \geq \beta^2 n^2/32$. Let $\{x, y\} \in E(N(u), N(v))$. Since each vertex in $\{x, y, u, v\}$ has degree at least $(1/2 - \beta^2)n$, the number of different p-caterpillars with spine u, x, y, v is at least γn^{4p} for some $\gamma > 0$. Thus the total number of u, v-caterpillars of length three in G is at least αn^{4p+2} for some $\alpha > 0$ which depends on β only. □

We will be connecting through a small subset of $V(G)$ called a reservoir set.

Lemma 3.2.3 (Reservoir Set). *For $1/64 > \beta > 0$ and $\beta^4 > \gamma > 0$ there is n_0 such that if G is a graph on $n \geq n_0$ vertices satisfying $\delta(G) \geq (1/2 - \beta^2)n$ which is not β-extremal then there is a set $Z \subset V(G)$ such that the following holds:*

(i) $|Z| = (\gamma \pm \gamma^2)n$;

(ii) *For every* $v \in V$, $|N(v) \cap Z| \geq (1/2 - 2\beta^2)\gamma n$;

(iii) *For every* $u, v \in V$, $||N(u) \cap Z, N(v) \cap Z|| \geq \beta^6 \gamma^2 n^2/4$.

Proof. Let Z be a set obtained by selecting every vertex from V independently with probability $p := \gamma$. By Theorem 1.3.4, with probability $1 - o(1)$, the following facts hold:

(a) $(\gamma - \gamma^2)n \le |Z| \le (\gamma + \gamma^2)n$;

(b) For every vertex v, $|N(v) \cap Z| \ge (1/2 - 2\beta^2)\gamma n$.

To prove the third part let $u, v \in V$ and let $X_{u,v} := \{w \in N(u) || N(w) \cap N(v)| \ge \beta^3 n\}$. Since G is not β-extremal by Lemma 3.2.1, $||N(u), N(v)|| \ge \beta^2 n^2/32$. Thus $|X_{u,v}| \ge \beta^3 n$. Indeed, if $|X_{u,v}| < \beta^3 n$, then $||N(u), N(v)|| < 2\beta^3 n^2 < \beta^2 n^2/32$. Consequently, by Chernoff's inequality, with probability $1 - o(1/n^2), |X_{u,v} \cap Z| \ge \beta^3 \gamma n/2$. Thus with probability $1 - o(1)$ for every u, v, $|X_{u,v}| \ge \beta^3 \gamma n/2$. Let $u \in V$ be arbitrary and let $w \in V$ be such that $|N(w) \cap N(u)| \ge \beta^3 n$. Then with probability at least $1 - o(1/n^2)$, $|N(w) \cap N(u) \cap Z| \ge \beta^3 \gamma n/2$. Thus with probability at least $1 - o(1)$, we have

$$||N(u) \cap Z, N(v) \cap Z|| \ge \beta^6 \gamma^2 n^2/4$$

for every u, v. Therefore there is a set Z such that (i)-(iii) hold. □

We will continue with our proof of the absorbing lemma. We shall assume that $0 < \beta < 1/64$, $G = (V, E)$ is a graph on n vertices where n is sufficiently large which is not β-extremal and which satisfies $\delta(G) \ge (n-1)/2$. In addition, we will use an auxiliary constant τ such that $0 < \tau < \frac{\beta}{10}$.

Lemma 3.2.4. *Let u, v be two vertices in G such that $|N(u) \cap N(v)| \ge 2\tau n$. Then, at least one of the following conditions holds.*

(1) At least τn vertices $x \in N(u) \cap N(v)$ are such that $|N(x) \cap N(u)| \ge \tau^2 n$.

(2) All but at most $3\tau n$ vertices $x \in N(v)$ satisfy $|N(x) \cap N(v)| \ge \tau^3 n$.

Proof. First suppose that $|N(v) \setminus N(u)| < 2\tau n + 2$. Since G is not β-extremal, $||G[N(v) \cap N(u)]|| \ge \beta n^2$ and so the first condition holds. Thus we may assume that $|N(v) \setminus N(u)| \ge 2\tau n + 2$. Since $|N(u) \cup N(v)| > (1/2 + 2\tau)n + 1$, every vertex

$x \in N(v) \cap N(u)$ has at least $2\tau n$ neighbors in $N(u) \cup N(v)$. Thus all but at most τn vertices in $N(v) \cap N(u)$ have at least $(2\tau - \tau^2) > \tau^3 n$ neighbors in $N(v)$.

Now, suppose the first condition fails and we claim that all but at most $2\tau n$ vertices $x \in N(v) \setminus N(u)$ satisfy $|N(x) \cap N(v)| \geq \tau^3 n$. Let $A := \{x \in N(u) \cap N(v) : |N(x) \cap N(u)| < \tau^2 n\}$ and note that $|A| \geq \tau n$. Therefore,

$$||A, N(v) \setminus N(u)|| \geq |A|(|N(v) \setminus N(u)| - \tau^2 n) \geq (1-\tau)|A||N(v) \setminus N(u)|.$$

Let $B := \{y \in N(v) \setminus N(u) : |N(y) \cap A| < \tau^2 |A|\}$ then $\tau^2|B||A| + (|N(v) \setminus N(u)| - |B|)|A| > ||A, N(v) \setminus N(u)||$. Hence $\tau^2|B||A| + (|N(v) \setminus N(u)| - |B|)|A| > (1-\tau)|A||N(v) \setminus N(u)|$ and so $|B| < \tau|N(v) \setminus N(u)|/(1-\tau^2) < 2\tau n$. For every vertex $x \in (N(v) \setminus N(u)) \setminus B$,

$$|N(x) \cap N(v)| \geq |N(x) \cap A| \geq \tau^2 |A| \geq \tau^3 n,$$

which completes the proof. □

Lemma 3.2.5. *Let T be a set of $p+1$ vertices in G. Then there exists a vertex $x \in T$ such that for every $y \in T$, $||N(x), N(y)|| \geq \tau^4 n^2$.*

Proof. Suppose there is a vertex $v \in T$ such that condition (2) in Lemma 3.2.4 is satisfied. Let $x := v$ and take $y \in T$. If $|N(y) \cap N(x)| \geq 5\tau n$, then $||N(x), N(y)|| \geq \frac{1}{2} \cdot (2\tau^4 n^2)$. If $|N(y) \cap N(x)| < 5\tau n$, then since G is not β-extremal, $||N(x), N(y)|| \geq \tau^4 n^2$. Therefore, we may assume that there is no such v in T. Let x be an arbitrary vertex in T. Take $y \in T$. If $|N(x) \cap N(y)| \geq 2\tau n$, then by Lemma 3.2.4 (with $u := x$, $v := y$), $||N(x), N(y)|| \geq \tau^3 n^2/2$. If $|N(x) \cap N(y)| < 2\gamma n$, then since G is not β-extremal, $||N(x), N(y)|| \geq \gamma^4 n^2$. □

We say that an x, y-caterpillar P absorbs a set T of size $p+1$, if $G[V(P) \cup T]$ contains an x, y-caterpillar on $|V(P)| + p + 1$ vertices. Let $M_q(T)$ denote the set of caterpillars of order q which absorb T. A caterpillar P is called $\gamma-$*absorbing* if P absorbs every subset $W \subset V \setminus V(P)$ with $|W| \in (p+1)Z$ and $|W| \leq \gamma n$. We will now prove our main lemma from which the absorbing lemma follows by using the deletion method.

Lemma 3.2.6. *Let $p \in Z^+$. For every $\beta > 0$ there is n_0 and $\alpha > 0$ such that the following holds. If G is a graph on $n \geq n_0$ vertices which is not β-extremal and such that $\delta(G) \geq (n-1)/2$ and $T \subset V(G)$, $|T| = p+1$, then*

$$|M_q(T)| \geq \alpha n^q$$

where $q = (3p+2)(p+1)$.

Proof. Let $T = \{x, y_1, \ldots, y_p\}$ and in view of Lemma 3.2.5 suppose that for every i, $||N(x), N(y_i)|| \geq \tau^4 n^2$. We will construct a caterpillar P which absorbs T. The counting fact follows easily from the way the construction works. To construct the caterpillar we will proceed in a few steps, selecting distinct vertices which have not been previously selected in each step. First take $v_i \in N(y_i)$ so that $v_1, \ldots, v_p$ are distinct and $|N(v_i) \cap N(x)| \geq \tau^5 n$. Now let $u_i \in N(v_i) \cap N(x)$ be such that $u_1, \ldots, u_p$ are distinct. Let x_1x_2 be an edge in $N(x)$. Use Lemma 3.2.2, to find v_i, v_{i+1} caterpillars with spines P_i for $2 \leq i \leq p-1$, all vertices distinct, and let $P := v_2P_2v_3 \ldots v_{p-1}P_{p-1}v_p$. Use Lemma 3.2.2 to find a v_2, x_2-caterpillar and denote its spine by Q_2 and a v_1, x_1-caterpillar with spine Q_1. Let $Q := v_1Q_1x_1x_2Q_2v_2Pv_p$. Then Q is a v_1, v_p-path. Disregard selected vertices not on Q. For every vertex v_i select $p-1$ distinct neighbors, so that together with u_i they give p leaves attached to v_i. For x_1, x_2 select p distinct neighbors and let S be the set containing all the vertices on Q, $u_1, \ldots, u_p$, and all the remaining neighbors. Then $G[S]$ contains a v_1, v_p- caterpillar of length $3p+1$ which contains $(3p+2)(p+1)$ vertices. In addition, $G[S \cup T]$ contains a v_1, v_p-caterpillar of length $3p+2$ obtained as follows. Insert x

between x_1 and x_2 in the spine Q, make $u_1, \ldots, u_p$ the neighbors of x, and let y_i replace u_i in the set of spikes of v_i. By Lemma 3.2.2 and in view of the construction, the number of such sets S is at least $\alpha n^{(3p+2)(p+1)}$ for some $\alpha > 0$ which depends on β and p only. □

Lemma 3.2.7. *(Absorbing Lemma) Let $p \in Z^+, q = (3p+2)(p+1), \beta > 0$ and $\alpha > 0$ be such that Lemma 3.2.6 holds. For any $\delta < \alpha/10q$, there is n_0 such that the following holds. If G is a graph on $n \geq n_0$ vertices which is not β-extremal and such that $\delta(G) \geq (n-1)/2$ then there is a caterpillar P_{abs} in G on at most δn vertices which is δ^2-absorbing.*

Proof. Let n_0 be such that Lemma 3.2.6 holds with α. Let G be a graph on $n \geq n_0$ vertices which is not $\beta-$extremal and such that $\delta(G) \geq (n-1)/2$.

Let $\mathcal{F}$ be a family obtained by selecting every set from $\binom{V}{q}$ independently with probability $\mu := \delta n / 3q\binom{n}{q}$. By Theorem 1.3.4, with probability $1 - o(1)$,

$$|\mathcal{F}| \leq 2\mu\binom{n}{q} = 2\delta n/3q$$

Now, let T be a set of size $p+1$. Again by Theorem 1.3.4, with probability $1 - o(1/n^{p+1})$,

$$|M_q(T) \cap \mathcal{F}| \geq \frac{1}{2}\mu\alpha n^q > 3\delta^2 n.$$

The expected number of pairs $\{S_1, S_2\}$ such that $S_1, S_2 \in \mathcal{F}$ and $S_1 \cap S_2 \neq \emptyset$ is at most $\binom{n}{q}\mu \cdot q\binom{n}{q-1}\mu \leq \delta^2 n$ and so by Theorem 1.3.1, with probability at least 1/2, the number of such pairs is at most $2\delta^2 n$. Therefore, with positive probability, there exists a family $\mathcal{F}$ such that $|\mathcal{F}| \leq 2\delta n/3q$, for every set T of size $p+1$, $|M_q(T) \cap \mathcal{F}| > 3\delta^2 n$, and the number of $\{S_1, S_2\}$ such that $S_1, S_2 \in \mathcal{F}$ and $S_1 \cap S_2 \neq \emptyset$ is at most $2\delta^2 n$. Let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by deleting all intersecting sets and sets that do not absorb any T. Then $|\mathcal{F}'| \leq 2\delta n/3q$, and for every set T of size $p+1$, $|M_q(T) \cap \mathcal{F}'| > \delta^2 n$.

For each $S \in \mathcal{F}'$, $G[S]$ contains a caterpillar on q vertices, so by using the minimum degree condition and Lemma 3.2.2, we can connect the endpoints of these caterpillars to obtain a new caterpillar P_{abs}. We also have that

$$|P_{abs}| \leq |\mathcal{F}'| \cdot q + 2|\mathcal{F}'| \cdot p < |\mathcal{F}'| \cdot (3q/2) \leq \delta n.$$

To show that P_{abs} is δ^2-absorbing, consider $W \subset V \setminus V(P_{abs})$ such that $(p+1)||W|$ and $|W| \leq \delta^2 n$. $\mathcal{W} = \{W_1, ..., W_m\}$ be an arbitrary partition of W into sets of size $p+1$. We have that $|M_q(W_i) \cap \mathcal{F}'| > \delta^2 n$ for every $i \in [m]$. Therefore, there exists a matching between $\mathcal{W}$ and $\mathcal{F}'$ so that every $W_i \in \mathcal{W}$ is paired with some $S_i \in M_q(W_i)$. This implies that P_{abs} absorbs W and the proof is complete. □

3.3 Non-extremal case

In this section we will finish proving the non-extremal case. The argument uses a similar approach as the proof of a corresponding fact in (Czygrinow and Molla, 2014).

Let $p \in Z^+, q = (3p+2)(p+1)$ and let ξ, β be such that $0 < \xi < 1/(4p+5), 0 < \beta < \min\{(\frac{\xi}{30p})^2, (\frac{\xi}{96})^2\}$. Now, let $\alpha > 0, n_0 \in \mathbb{N}$ be such that Lemma 3.2.7 holds. Let $\delta, \gamma > 0$ be such that $\delta < \min\{(\frac{\beta}{300})^2, \frac{\alpha}{10q}\}, \gamma < \frac{\delta^2}{4}$ and C be such that $C > \frac{80(p+1)}{\delta\gamma\beta^3}$. Let $n > \max\{n_0, \frac{4C \cdot 2^{(1+\delta)C}}{\delta^3}\}$ and G be a graph on n vertices which is not $\beta-$extremal and of minimum degree at least $(n-1)/2$. Let P_{abs} be the absorbing caterpillar obtained in the previous section and let Z be the reservoir set from Lemma 3.2.3 applied with γ which is less than β^4 because $\gamma < \frac{\delta^2}{4} < \beta^4$.

Claim 3.3.1. *Let P_1, P_2 be disjoint caterpillars in G such that $|Z \cap V(P_1)|, |Z \cap V(P_2)| < \frac{\beta^3\gamma n}{4}$ and the endpoints of P_1 and P_2 are not in Z. Then there is a caterpillar P containing $V(P_1) \cup V(P_2)$ which has at most $2(p+1)$ additional vertices in Z and such that its endpoints are not in Z.*

Proof. Let u_1, u_2 be the endpoints of P_1, P_2, respectively. Since $\frac{\beta^3}{2} < 1/2 - 2\beta^2$ and

$|(N(u_1) \cap Z) \cap (V(P_1) \cup V(P_2))| \cdot |(N(u_2) \cap Z) \cap (V(P_1) \cup V(P_2))| < \beta^6\gamma^2 n^2/16$, by Lemma 3.2.3, there exists $x_1 \in (N(u_1) \cap Z) - (V(P_1) \cup V(P_2)), x_2 \in (N(u_2) \cap Z) - (V(P_1) \cup V(P_2))$ such that $\{x_1, x_2\} \in E(G)$. Then we can construct new caterpillar P using $\{u_1, x_1\}, \{x_1, x_2\}, \{u_2, x_2\} \in E(G)$ and adding p vertices from $N(x_1) \cap Z - (V(P_1) \cup V(P_2))$ and another p vertices from $N(x_2) \cap Z - (V(P_1) \cup V(P_2))$ as leaf vertices of x_1, x_2. □

Now, let $G' := G[V \setminus (Z \cup V(P_{abs})]$ and let P be a longest caterpillar in G'. Starting with P we will extend P iteratively, adding at least $\delta C/2$ vertices by using at most $10(p+1)$ vertices from Z in each step, until the number of vertices left is at most $\frac{\delta^2 n}{2}$. Since the number of iterations is at most $2n/(\delta C)$, and so the number of vertices used to construct P in Z is at most $\frac{2n}{\delta C} \cdot 10(p+1) < \frac{\beta^3\gamma}{4}n$, by Claim 3.3.1, the process can be completed. Moreover, P_{abs} can be connected with P using Z and the number of remaining vertices which are not on the caterpillar is at most $|Z| + \frac{\delta^2 n}{2} \leq \delta^2 n$ and so they can be absorbed by P_{abs}. For the general step, let $W := V(G') \setminus V(P)$ and suppose $|W| > \frac{\delta^2 n}{2}$. We partition P into l blocks $B_1, \ldots, B_l$ of consecutive caterpillars so that $C \leq |B_i| \leq (1+\delta)C$.

Claim 3.3.2. *If $||G[W]|| \geq \gamma|W|^2$, then there is a caterpillar in $G[W]$ with at least $\gamma|W| - p$ vertices.*

Proof. $G[W]$ contains a subgraph H such that $\delta(H) > \gamma|W|$. Let Q be a longest caterpillar in H. If $|Q| \leq \gamma|W| - p$, then each endpoint of Q has a neighbor $x \in V(H) \setminus Q$, and every vertex not on Q has at least p neighbors outside Q. □

Case 1. $||G[W]|| \geq \gamma|W|^2$.

By Claim 3.3.2 there is a caterpillar Q in $G[W]$ on at least $\delta C/2$ vertices. Since $Q \cap Z, P \cap Z = \emptyset$, by Claim 3.3.1, we can construct a caterpillar containing both of them.

Case 2. There is a block B_i such that $||B_i, W|| \geq \left(\frac{1}{2} + \delta\right)|B_i||W|$.

Let $W' := \{w \in W||N(w) \cap B_i| \geq \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|\}$. Then $|W'| \geq \delta|W| \geq \frac{\delta^3 n}{2}$. Consider $H := G[W', B_i]$. Since there are less than $2^{(1+\delta)C}$ subsets of B_i of size $\left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$, there is a set $X \subset B_i$ such that $|X| = \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$ and for at least $|W'|/2^{(1+\delta)C}$ vertices $w \in W'$, $X \subseteq N(w) \cap B_i$. Since $\frac{|W'|}{2^{(1+\delta)C}} \geq 2C \geq (\frac{1}{2} + \frac{\delta}{2})|B_i|$, H contains $K_{D,D}$ where $D = \left(\frac{1}{2} + \frac{\delta}{2}\right)|B_i|$ which in turn contains a caterpillar on $2D - p > (\frac{1}{2} + \frac{\delta}{2})|B_i|$ vertices. By Claim 3.3.1, using at most $4(p + 1)$ vertices in Z we can connect the endpoints of this caterpillar with the endpoints of B_{i-1} and B_{i+1}.

Case 3. For every block B_i, $||B_i, W|| < \left(\frac{1}{2} + \delta\right)|B_i||W|$.

Since we are not in Case 1, $\sum_{v \in W} |N(v) \cap W| < 2\gamma|W|^2$ and so $\sum_{v \in W} |N(v) \cap P| > (1/2 - \delta - 2\gamma)n|W| - 2\gamma|W|^2$, so

$$||P, W|| \geq \left(\frac{1}{2} - 2\delta\right) n|W|.$$

A block B is called *good* if $||B, W|| \geq \left(\frac{1}{2} - 2\sqrt{\delta}\right)|W||B|$. Let q denote the number of good blocks. We have $q \geq (1 - 3\sqrt{\delta})\frac{n}{C}$ as otherwise

$$||P, W|| \leq q\left(\frac{1}{2} + \delta\right)(1 + \delta)C|W| + (l - q)\left(\frac{1}{2} - 2\sqrt{\delta}\right)(1 + \delta)C|W|$$

which is less than $\left(\frac{1}{2} - 2\delta\right) n|W|$. Using the same argument as in Case 2, for a good block B_i we can find set $C_i \subset B_i$ and $D_i \subseteq W$ such that $G[C_i, D_i] = K_{|C_i|,|D_i|}$, $|C_i| = \left(\frac{1}{2} - 3\sqrt{\delta}\right) C$ and $|D_i| \geq C$. Let $U := \bigcup(B_i \setminus C_i)$ where the union is taken over good blocks. We have

$$|U| \geq (1 - 3\sqrt{\delta})\frac{n}{C} \cdot C - \left(\frac{1}{2} - 3\sqrt{\delta}\right) C\frac{n}{C} = \frac{n}{2}.$$

Thus, since G is not β-extremal, $||G[U]|| \geq \beta n^2$, and so there exist two good blocks B_s, B_t with $s < t$ such that $||G[(B_s \setminus C_s) \cup (B_t \setminus C_t)]|| \geq \beta C^2/2$. Thus by Claim 3.3.2 $G[(B_s \setminus C_s) \cup (B_t \setminus C_t)]$ contains a caterpillar S on $\beta C/4$ vertices. In addition

$G[C_s \cup C_t, W]$ contains two disjoint caterpillars S_1, S_2, each on $(1-7\sqrt{\delta})C$ vertices. Thus $|S \cup S_s \cup S_t| - |B_s \cup B_t| \geq 2(1-7\sqrt{\delta})C + \frac{\beta C}{4} - 2(1+\delta)C \geq \delta C$. By using at most $10(p+1)$ vertices in Z, we can connect the endpoint of B_{s-1} to the endpoint of $S \cap B_s$, connect the endpoint of $S \cap B_t$ to the endpoint of B_{t-1}, connect the endpoint of B_{s+1} to the endpoint of S_1, and connect the endpoint of S_1 to the endpoint of S_2. Finally, by connecting the endpoint of S_2 to the endpoint of B_{t+1}, we form a longer caterpillar having more than at least δC vertices than previous caterpillar. □

3.4 Extremal case

In this section we will address the extremal cases. First we will deal the the case when vertices of G can be partitioned into two sets V_1, V_2 such that $||V_1, V_2|| \leq \beta n^2$ and so, G is close to a union of two complete graphs and then we address the case when G has a large almost independent set.

We will start with the following lemma.

Lemma 3.4.1. *Let $p \in Z^+$. For any $\xi < 1/(4p+5)$ there is $n_0 \in \mathbb{N}$ such that the following holds. Let H be a graph on $n \geq n_0$ vertices such that $(p+1)||H|$ and $\delta(H) \geq (1-\xi)n$. Let $x, y \in V(H)$. Then there is a spanning p-caterpillar in H connecting x and y.*

Proof. Let P be a longest $p-$caterpillar in G connecting x and y. Let $S = (x =)u_1 \dots u_q (= y)$ denote the spine of P and let $C[i]$ denote the set of spikes of u_i. For any two $u, v \in G$, $|N(u) \cap N(v)| \geq (1-2\xi)n$ and so $q \geq (1-2\xi)n/(p+1)$. Indeed, if $q < (1-2\xi)n/(p+1)$ then $|V(P)| < (1-2\xi)n$ and then there exists $u_1' \in (N(u_1) \cap N(u_2)) \setminus V(P)$, $|N(u_1') - V(P)| \geq (1-\xi)n - (1-2\xi)n > p$. If $V(P) = V(H)$ then we are done, so assume that there exists $\{v', y_1, ..., y_p\} \subset V(H) \setminus V(P)$. Since $d(v') \geq (1-\xi)n$ there exists $i \in [q]$ such that $u_i, u_{i+1} \in N(v')$. Otherwise, since

$\xi < 1/(4p+5)$,

$$(1-\xi)n \le d(v') \le (n-q/2) \le (1-\frac{1-2\xi}{2(p+1)})n \le (1-\frac{1-\xi}{4(p+1)})n < (1-\xi)n,$$

a contradiction. Moreover, there are p distinct vertices $f_1, ..., f_p \in [q] \setminus \{i, i+1\}$ such that for each $j \in [p]$, $|N(v') \cap C[f_j]| > 0$ and $f_j y_j \in E(H)$, which gives us the caterpillar P' such that $V(P') = V(P) \cup \{v', y_1, ..., v_p\}$ and P' still connects x and y. □

A p-star is a star which has exactly p leaves.

Lemma 3.4.2. *Let $p \in Z^+$. There is $\beta > 0$ and n_0 such that if G is a graph on $n \ge n_0$ vertices such that $(p+1)|n$, $\delta(G) \ge \frac{n-1}{2}$, and for some partition V_1, V_2 of $V(G)$ with $|V_i| \ge (1/2-\beta)n$, $||G[V_1, V_2]|| \le \beta n^2$, then G contains a spanning p-caterpillar.*

Proof. Let ξ and β be such that $0 < \xi < 1/(4p+5)$ and $0 < \beta \le (\frac{\xi}{30p})^2$. Let $W_i := \{v \in V_i : |N(v) \cap V_i| < (1/2 - 5\sqrt{\beta})n\}$. We have $\sum_{v \in V_i} |N(v) \cap V_i| \ge (1/4 - \beta/2)n(n-1) - \beta n^2 \ge (1/4 - 2\beta)n^2$ and

$$\sum_{v \in V_i} |N(v) \cap V_i| < |W_i|(1/2 - 5\sqrt{\beta})n + (|V_i| - |W_i|)|V_i|$$

and so $|W_i| \le \sqrt{\beta}n - 1$. In addition, for every $v \in W_i$, $|N(v) \cap (V_{3-i} \setminus W_{3-i})| \ge 4\sqrt{\beta}n$. Let $U_i := V_i \setminus W_i$ and $X_i := W_{3-i}$. Then

- for every $v \in U_i$, $|N(v) \cap U_i| \ge (1/2 - 6\sqrt{\beta})n$,
- for every $v \in X_i$, $|N(v) \cap U_i| \ge 4\sqrt{\beta}n$.

Without loss of generality, suppose $|U_1 \cup X_1| \le |U_2 \cup X_2|$. Then for every $v \in U_1 \cup X_1$, $|N(v) \cap (U_2 \cup X_2)| \ge 1$. Let $r_i := |U_i \cup X_i| \bmod (p+1)$. Since very vertex in $U_1 \cup X_1$ has at least one neighbor in $U_2 \cup X_2$, we pick r_1 vertices $u_1, \ldots, u_{r_1}$ in $U_1 \cup X_1$ and choose one neighbor in $U_2 \cup X_2$ for each. Note that clearly these neighbors do not need to be distinct. Let $w_1, \ldots, w_l$ denote distinct vertices in $U_2 \cup X_2$ chosen in this way. We have $l \leq p$ and each w_i was chosen by at most $r_1 \leq p$ vertices. We will construct a spanning caterpillar by starting with $(U_1 \cup X_1) \setminus \{u_1, \ldots, u_{r_1}\}$. Since $|X_1| \leq \sqrt{\beta}n - 1$, there is a matching from $X_1 \setminus \{u_1, \ldots, u_{r_1}\}$ to $U_1 \setminus \{u_1, \ldots, u_{r_1}\}$. The matching can be easily extended to a caterpillar P in $G[(U_1 \cup X_1) \setminus \{u_1, \ldots, u_{r_1}\}]$ on at most $2(p+1)\sqrt{\beta}n$ vertices. Let b be the starting point of P. Let $G' = G[(U_1 \cup X_1) \setminus (\{u_1, \ldots, u_{r_1}\} \cup V(P))$ and $b' \in N(b) \cap V(G')$. Since $\delta(G') \geq (1/2 - (8 + 2p)\sqrt{\beta})n \geq \frac{(1-\xi)n}{2} \geq (1 - \xi)|G'|$, by Lemma 3.4.1 G' contains a spanning $p-$caterpillar P' starting at b'.

Denote by x the another starting point of P', i.e P' is b', x-caterpillar. Now, pick $y \in N(x) \cap (U_2 \cup X_2)$. If $y \in X_2$ then construct a star Y_0 centered at y such that $Y_0 \subset (X_2 \cup Y_2) \setminus \{w_1, \ldots, w_l\}$ and choose $y' \in N(y) \cap (U_2 \setminus (\{w_1, \ldots, w_l\} \cup Y_0))$, otherwise $y' = y$. We will now construct a caterpillar in $G[U_2 \cup X_2 \cup \{u_1, \ldots, u_{r_1}\}]$. If a_i denotes the number of vertices which choose w_i, then select $p - a_i$ neighbors of w_i in $U_2 \cup X_2 \setminus \{y, w_1, \ldots, w_l\}$, all vertices distinct for different values of i. Let S_i denote the p-star with center at w_i. Note that y can be among $w_1, \ldots, w_l$ but it cannot be among the remaining vertices of $S_1, \ldots, S_l$. Since $|X_2| \leq \sqrt{\beta}n$, there is a matching from $X_2 \setminus (\{y, y'\} \cup Y_0 \cup \bigcup_{i \in [l]} S_i)$ to $U_2 \setminus (\{y, y'\} \cup Y_0 \cup \bigcup_{i \in [l]} S_i)$. The matching and $S_1, ..., S_l$ also can be extended to a caterpillar P'' in $G[X_2 \cup U_2 \cup \{u_1, \ldots, u_{r_1}\}]$. Denote by y'' the other endpoint of the spine of P'' and let $y''' \in (N(y'') \setminus (V(P'') \cup Y_0)) \cap U_2$. Let $G'' = G[U_2 \cup X_2 \setminus (V(P'') \cup \{w_1, \ldots, w_l\})]$. Since $\delta(G'') \geq (1 - \xi)|G''|$, again by Lemma 3.4.1, there exists a spanning p-caterpillar P''' connecting y' and y'''. Then we get a spanning p-caterpillar of G by linking P', P'' and P'''. $\square$

We will now proceed to prove the other extremal case. We have the following lemma.

Lemma 3.4.3. *Let $p \in Z^+$. For any $\xi < 1/(4p + 5)$ there is $n_0 \in \mathbb{N}$ such that* *the following holds. Let $H = (A_1, A_2)$ be a bipartite graph on $n \geq n_0$ vertices with $(p+1)|n$ such that $|A_1| = |A_2| = \frac{n}{2}$ if $n/(p+1)$ is even and $|A_2| = \frac{n+p-1}{2}$ if $n/(p+1)$ is odd. Suppose that for any $v \in A_i$, $d(v) \geq (1-\xi)|A_{3-i}|$. For any $x \in A_1$, there exists a spanning p-caterpillar starting at x in H.*

Proof. First suppose $n/(p+1)$ is even. Let B_i be an arbitrary set of $n/(2(p+1))$ vertices in A_i such that $x \in B_1$. For any vertex $v \in B_i$ and any set $C \subseteq A_{3-i}$ of size $n/(2(p+1))$, $|N(v) \cap C| \geq |C| - \xi n > |C|/2$. Consequently, by Hall's theorem, there is a set of pairwise disjoint p-stars with centers in B_i and leaves in $A_{3-i} \setminus B_{3-i}$. In addition, $G[B_1, B_2]$ has a Hamilton cycle and so a spanning path which starts at x. The path, in connection with stars, gives a p-caterpillar starting at x. Now suppose $|A_2| = \frac{n+p-1}{2}$. Let B_2 be a subset of A_2 of size $(n-p-1)/(2(p+1))$ and let B_1 be a subset of A_1 of size $(n+p+1)/(2(p+1))$. Note that $|B_i|p = |A_{3-i}| - |B_{3-i}|$. As before, by Hall's theorem there are pairwise disjoint p-stars with centers in B_i and leaves in $A_{3-i} \setminus B_{3-i}$ and $G[B_1, B_2]$ has a spanning path. □

Lemma 3.4.4. *Let $p \in Z^+$. There is $\beta > 0$ and n_0 such that if G is a graph on $n \geq n_0$ vertices such that $(p+1)|n$, for some set S of $V(G)$ with $|S| \geq (1/2 - \beta)n$, $||G[S]|| \leq \beta n^2$, and*

$$\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } n/(p+1) \text{ is even} \\ \frac{n+1}{2} & \text{if } n/(p+1) \text{ is odd and } p > 2 \\ \frac{n-1}{2} & \text{if } n/(p+1) \text{ is odd and } p \leq 2 \end{cases}$$

then G contains a spanning p-caterpillar.

Proof. Let ξ and β be such that $0 < \xi < 1/(4p+5)$ and $0 < \beta \leq \min\{(\frac{\xi}{96})^2, (\frac{\xi}{10+3p})^2\}$. We may assume that $|S| \leq n/2$. Let $U_1 := S$ and $U_2 := V \setminus S$. We have

$$||G[U_1, U_2]|| \geq (1/2 - \beta)n^2/2 - 2\beta n^2 \geq (1 - 10\beta)|U_1||U_2|.$$

Let $W_i := \{u \in U_i || N(u) \cap U_{3-i}| \leq (1 - 10\sqrt{\beta})|U_{3-i}|\}$. Then

$$||G[U_1, U_2]|| \leq |W_1||U_2|(1 - 10\sqrt{\beta}) + (|U_1| - |W_1|)|U_2|$$

and so $|W_1| \leq \sqrt{\beta}|U_1|$ and similarly $|W_2| \leq \sqrt{\beta}|U_2|$.

We define s to be $n/2$ when $n/(p+1)$ is even and $(n-p+1)/2$ when $n/(p+1)$ is odd. Let $W := W_1 \cup W_2$. Distribute vertices from W to X_1, X_2 so that the following holds.

(a) If $x \in X_i$, then $|N(x) \cap U_{3-i}| \geq 10\sqrt{\beta}n$.

(b) $|\min\{|X_1 \cup (U_1 \setminus W_1)|, |X_2 \cup (U_2 \setminus W_2)|\} - s|$ is the least possible.

If the quantity in the second condition is positive, we further move vertices from $U_i \setminus W_i$ to X_{3-i} which satisfy (a) to make $|\min\{|X_1 \cup (U_1 \setminus W_1)|, |X_2 \cup (U_2 \setminus W_2)|\} - s|$ as small as possible. Let $Y_i := X_i \cup (U_i \setminus W_i)$ and suppose $|Y_1| \leq |Y_2|$.

First, assume that $|Y_1| = s$. Since for each $v \in W_1 \cup W_2$, $d(v) \geq 10\sqrt{\beta}n > |W_1| + |W_2|$, there is a matching M such that for any $e \in M$, $|e \cap W_1| + |e \cap W_2| = 1$. Then we extend this matching to $p-$caterpillar P so that for any $e \in M$, $e \cap W_i$ is a vertex of spike. Let $G' = G[V \setminus V(P)] = (V', E')$ and note that $V' \cap W = \emptyset$. Let x be a last vertex of P and $x' \in N(x) \cap V'$. Let $Y'_i = Y_i \cap V'$. For any $v \in Y'_i$,

$$|N(v) \cap Y'_{3-i}| \geq (1 - 10\sqrt{\beta})|U_{3-i}| - 3p\sqrt{\beta}|U_{3-i}| \geq (1 - \xi)|Y'_{3-i}|.$$

By Lemma 3.4.3, there exists a spanning caterpillar P' starting at x' of G', then we get a spanning caterpillar of G by attaching P to P'.

Now, we assume that $|Y_1| \neq s$. If $|Y_1| > s$ then $n/(p+1)$ is odd and since $|Y_1| \leq |Y_2|$, $p \geq 3$,i.e $\delta(G) \geq \frac{n+1}{2}$. We have $\delta(G[Y_2]) \geq \delta(G) - |Y_1| \geq 1$. If $|Y_1| < s$ (and so $|Y_1| < n/2$), then $\delta(G[Y_2]) \geq \delta(G) - |Y_1| \geq 1$.

In the first case we proceed as follows. Let $l = \frac{n+p-1}{2} - |Y_1|$. Since $|Y_1| > \frac{n-p+1}{2}$, $l < p - 1$. If there is a vertex $y \in Y_2$ such that $|N(y) \cap Y_2| \geq p - 1$, then pick $p - l$

neighbors of y from Y_1, l from Y_2 to form a p star S centered at y and let x be one more neighbor of y in Y_2. Deleting x and all vertices in S gives Y_1', Y_2' such that $|Y_i'| = \frac{n-p-1}{2}$, and so by Lemma 3.4.3 there is a spanning p-caterpillar in $G[Y_1', Y_2']$ starting at x. If no such vertex exists, then $\Delta(G[Y_2]) \leq p-2$. Since $\delta(G[Y_2]) \geq 1$, there is a matching in $G[Y_2]$ of size at least $n/2(p-1)(> p+1)$. Let $y \in Y_2$ be arbitrary and let x be a neighbor of y in Y_2. Let $M = \{a_1b_1, \ldots, a_lb_l\}$ be a matching in $G[Y_2]$ such that $x, y \notin V(M)$. We construct caterpillar Q as follows. Start with x and pick p neighbors of x in Y_1. We will use y as a vertex on spine of Q. Pick a neighbor new vertex $y' \in N(y) \cap Y_1$ and a $a_1' \in N(a_1) \cap Y_1$. Note that $\Delta(G[Y_1]) \leq 20\sqrt{\beta}n$ as we can't move any vertices from Y_1 and so any two vertices in Y_1 have at least $n/4$ common neighbors in $Y_2 \setminus V(M)$. Select one such unused vertex z which gives a y, a_1-path of length four which will be added to the spine of Q. Now select p neighbors from the opposite set for each vertices except a_1. In the case of a_1, pick $p-1$ neighbors from Y_1 and make b_1 one of the spikes. Now continue to add additional vertices. Then Q has $2l+2$ spine vertices in Y_2, $2l$ spine vertices in Y_1 and $|V(Q) \cap Y_2| = (2+2l) + 2lp + l$, $|V(Q) \cap Y_1| = (2l+2)p + 2l - l$. This concludes the construction of Q. Let x' be one new neighbor of a_l in Y_1. Note that $|Y_2 \setminus V(Q)| = \frac{n-p+1}{2} + l - (2+2l+2lp+l) = \frac{n'+p-1}{2}$ where $n' = n - (4l+2)(p+1) = |V - V(Q)|$. Thus by Lemma 3.4.3 we can extend Q to get a spanning caterpillar in G.

In the second case, we have $\delta(G[Y_2]) \geq s - |Y_1| \geq 1$ and because no vertex can be moved from Y_2 to Y_1, $\Delta(G[Y_2]) \leq 20\sqrt{\beta}n$. Let M be a maximum matching in $G[Y_2]$ and suppose $|M| < s - |Y_1|$. Then the number of edges in $G[Y_2]$ incident to $V(M)$ is at most $40\sqrt{\beta}n|M| < 40\sqrt{\beta}n(s-|Y_1|)$, but $||G[Y_2]|| \geq \frac{|Y_2|}{2}(s - |Y_1|)$, and $|Y_2| \geq 80\sqrt{\beta}n$.

The rest of the argument is similar to the previous case. For every $y \in Y_2$, we have $|N(y) \cap Y_1| \geq (1/2 - 20\sqrt{\beta})n$. Let $M = \{a_1b_1, \ldots, a_qb_q\}$. We move $b_1, \ldots, b_q$ from Y_2 to

Y_1 so that after moving $|Y_1| = s$. Note that $|Y_1| = |Y_2|$ or $|Y_1| = \frac{n-p+1}{2}, |Y_2| = \frac{n+p-1}{2}$. Then we extend this matching to a $p-$caterpillar P so that for any $i \in [q]$, b_i is a spike in P. Let $G' = G[V \setminus V(P)] = (V', E')$. Let x be the last vertex of P in Y_2 and $x' \in N(x) \cap V'$. Let $Y_i' = Y_i \cap V'$. For any $v \in Y_i'$, since $q \leq 4\sqrt{\beta}n$,

$$|N(v) \cap Y_{3-i}'| \geq (1/2 - 24\sqrt{\beta})n \geq (1-\xi)|Y_{3-i}'|,$$

By Lemma 3.4.3, there exists a spanning caterpillar P' starting at x' of G', then we get a spanning caterpillar of G by attaching P to P'. □

Chapter 4

TURÁN-TYPE RESULT AND MULTI-COLOR RAMSEY NUMBER FOR A LOOSE 3 UNIFORM PATH OF LENGTH 3

4.1 Introduction

One of the most important problems in combinatorics and graph theory is determining or estimating the Ramsey numbers. In contrast to the graph case, there are few known results about the Ramsey numbers of hypergraphs. We denote by $R(F;r)$ the least integer n such that in every coloring of the edges of complete graph of order n by r colors there is a monochromatic copy of F.

The r-uniform loose cycle C_n^r is the r-graph with vertex set $\{v_1, v_2, \ldots, v_{n(r-1)}\}$ and with the set of n edges $e_i = \{v_1, \ldots, v_r\} + i(r-1)$, $i = 0, 1, \ldots, n-1$, where we use mod $n(r-1)$ arithmetic and adding a number t to a set $H = \{v_1, \ldots, v_r\}$ means a shift, i.e. the set obtained by adding t to subscripts of each element of H. Similarly, the r-uniform loose path P_n^r is the r-graph with vertex set $\{v_1, v_2, \ldots, v_{n(r-1)+1}\}$ and with the set of n edges $e_i = \{v_1, \ldots, v_r\} + i(r-1), i = 0, 1, \ldots, n-1$. The complete r-graph K_n^r is a r-graph on n vertices in which every $r-$element subset of the vertex set forms an edge.

It was proved in (Haxell *et al.*, 2006) that $R(P_n^3; 2)$ and $R(C_n^3; 2)$ are asymptotically equal to $\frac{5n}{2}$. Subsequently, Omidi and Shahsiah in (Omidi and Shahsiah, 2014) proved that

Theorem 4.1.1. *(Omidi and Shahsiah, 2014)*

$$R(P_n^3; 2) = R(C_n^3; 2) + 1 = \lceil \frac{5n+1}{2} \rceil.$$

Gyárfás and Raeisi (Gyárfás and Raeisi, 2012) found the values of $R(P_n^k; 2)$ and $R(C_n^k; 2)$ for $n \leq 4$ and $k \geq 3$. They also determined the 3-color Ramsey number of C_3^3,

Theorem 4.1.2. *(Gyárfás and Raeisi, 2012)*

$$R(C_3^3; 3) = 8.$$

Recently, Jackowska, Polcyn, Ruciński (Ruciński *et al.*, 2017) determined the $r-$color Ramsey number for $P := P_3^3$ and showed the following.

Theorem 4.1.3. *(Ruciński* et al.*, 2017)*

$$R(P; r) = r + 6 \textit{ for } r \in [7].$$

As a part of their argument, they also determined the Turán graph for P and proved an useful lemma which we will use in our proof. Denote by $\mathrm{ex}(n; P)$, $\mathrm{Ex}(n; P)$ the Turán number and graph for P, respectively.

Theorem 4.1.4. *(Jackowska* et al.*, 2016)*

$$\mathrm{ex}(n; P) = \begin{cases} \binom{n}{3} \textit{ and } \mathrm{Ex}(n; P) = \{K_n^3\} \textit{ for } n \leq 6 \\ 20 \textit{ and } \mathrm{Ex}(n; P) = \{K_6^3 \cup K_1^3\} \textit{ for } n = 7 \\ \binom{n-1}{2} \textit{ and } \mathrm{Ex}(n; P) = \{S_n^3\} \textit{ for } n \geq 8 \end{cases}$$

where $V(S_n^3) = [n]$, $E(S_n^3) = \{e \in \binom{S_n^3}{3} : 1 \in e\}$.

Lemma 4.1.5. *(Ruciński* et al.*, 2017) If H is a connected P-free 3-graph with $n \geq 7$ vertices and $H \supset C_3^3$, then*

$$|E(H)| \leq 3n - 8.$$

Subsequently, Polcyn, Ruciński extended the result from (Polcyn, 2017; Polcyn and Ruciński, 2017) by showing that the formula also holds in the case $r \in \{8, 9, 10\}$.

Theorem 4.1.6. *(Polcyn, 2017; Polcyn and Ruciński, 2017)*

$$R(P;r) = r + 6 \textit{ for } r \in [10].$$

Moreover, Łuczak and Polcyn showed the general upper bound of multi-color Ramsey number for P. We recall the result of Łuczak and Polcyn.

Theorem 1.0.8. *(Łuczak and Polcyn, 2017)*

$$R(P;r) \leq 2r + \sqrt{18r+1} + 2 \textit{ for } r \in \mathbb{N}.$$

In this chapter, we will prove better bounds for $R(P,r)$ by analyzing critical vertices which we define in the following section. We recall the other main result of this chapter.

Theorem 1.0.7. $r + 6 \leq R(P;r) \leq 2r$ *for* $r \geq 6$.

After the paper was submitted Polcyn and Łuczak (Łuczak and Polcyn, 2018) minimally improved the bound and showed that the upper bound is at most $\lambda_0 r + 7\sqrt{r}$ where $\lambda_0 = 1.97466..$.

Since we mainly handle 3-graphs in this chapter, there are some notations which are not described in Chapter 1 but we use in this chapter.

Given a 3-graph $H = (V, E)$, the neighborhood of $\{u, v\} \in \binom{V}{2}$, i.e. the set of vertices form an edge with $\{u, v\}$ is denoted by $N_H(\{u, v\})$ or $N(\{u, v\})$ for short , the neighborhood of $v \in V$, i.e. the set of pairs of vertices form an edge with v is denoted by $N_H(v)$ or $N(v)$ for short, and the union of all pairs in $N(v)$ is denoted by $V(N(v))$. For any $A, B, C \subset V$, $E(A, B, C) = \{e = \{a, b, c\} \in E | a \in A, b \in B, c \in C\}$, and $e(A, B, C) = |E(A, B, C)|$. For any r-edge coloring, let H_i be the 3-graph colored by color $i \in [r]$.

The rest of Chapter 4 is organized as follows. In Section 4.2, we state the definition of a *critical vertex*, Theorem 1.0.6 and give some corollaries. In Section 4.3, we prove

Theorem 1.0.7 by making use of theorems stated in Section 1.0.6. In Section 4.4, we present a proof of Theorem 1.0.6.

4.2 k-centric Turán number

In this section, we define the k-centric Turán number for P and state the main result of the paper. First, we define *critical vertex.*

Definition 4.2.1. *Let $H = (V, E)$ be a 3-graph. If there exists $v \in V$ and a non empty set $D_v \subset V \backslash \{v\}$ such that*

$$\forall u \in D_v, d(u) > 0$$

$$E(D_v, V, V) = E(\{v\}, D_v, D_v), e(\{v\}, D_v, D_v) \geq |D_v| \geq 4,$$

then by choosing D_v as big as possible, we call v a critical vertex *with the subordinate set D_v and we call $|D_v|$ the size of v. If there exists $u \in V - D_v$ such that $e(\{v\}, \{u\}, V) > 0$ then we call such u a* trivial vertex *of v.*

Now, we employ the concept of *center* to classify P-free 3-graph having critical vertices.

Definition 4.2.2. *Let $H = (V, E)$ be a P-free 3-graph with $|H| = n$. Let C be a set of critical vertices in H. If $C \neq \emptyset$ then we call $v \in C$ the* center *of H if $|D_v| = \max_{u \in C} |D_u|$, and H is called the k-*centric *Turán graph for P if $|D_v| = n - k$.*

Note two simple facts:

Fact 4.2.3. *If $u \in D_v$ for some $v \in V$ then*

$$d(u) \leq |D_v| - 1.$$

Fact 4.2.4. *If v is a critical vertex of H with the subordinate set D_v then for any $v' \in D_v$, v' is not a critical vertex. Moreover, for any two critical vertices c_1, c_2, $D_{c_1} \cap D_{c_2} = \emptyset$.*

Proof. Suppose that v' is a critical vertex with $D_{v'}$. Since $v' \in D_v$, $D'_v \subset D_v \cup \{v\}$ and every edge containing v' must contain v. Since $e(\{v'\}, D_{v'}, D_{v'}) \geq |D_{v'}|$, there exists $e'' \in E(\{v'\}, D_{v'}, D_{v'})$ such that $v \notin e''$, a contradiction.

Let c_1, c_2 be arbitrary two critical vertices. We may assume that $c_1 \notin D_{c_2}, c_2 \notin D_{c_1}$. Now, if there exists $u \in D_{c_1} \cap D_{c_2}$ then, without loss of generality, there is $u' \in D_{c_1}$ such that $\{u, u', c_1\} \in E(H)$. Since $c_1 \notin D_{c_2}$, it is a contradiction to the fact that $u \in D_{c_2}$. $\square$

Theorem 1.0.6 is one of the main results in Chapter 4. For this theorem, we clarify the notion of connectivity of a hypergraph.

Definition 4.2.5. *A hypergraph H is connected if for any two vertices $u, v \in V(H)$ there exists a sequence of edges $P = e_1 ... e_s$ such that $u \in e_1, v \in e_s$ and for any $i \in [s-1]$, $|e_i \cap e_{i+1}| > 0$.*

We recall Theorem 1.0.6.

Theorem 1.0.6. *Let $H = (V, E)$ be a connected 3-graph with $|H| = n \geq 7$ and $\Delta(H) \geq n-2$. If $||H|| > 3n-8$ then either H contains P or a critical vertex.*

Corollary 4.2.6. *Let $H = (V, E)$ be a P-free 3-graph with $|H| = n$. If H has no critical vertex, then for any $S \subset V$,*

$$\sum_{u \in S} d(u) \leq \max\{|S|(n-3), 9n-24, 10|S|\}.$$

Proof. If $\Delta(H) \leq \max\{n-3, 10\}$ then it is obvious, so we may assume that $\Delta(H) > \max\{n-3, 10\}$, and so $|H| \geq 7$. If H is connected, then by Theorem 1.0.6, $||H|| \leq 3n-8$, and therefore,

$$\sum_{u \in S} d(u) \leq 3||H|| \leq 9n-24.$$

Hence we may assume that H is disconnected and let $V = \bigcup_i V_i$ where each $H[V_i]$ is a component. If for all i, $\Delta(H[V_i]) \leq \max\{10, |V_i| - 3\}$, then we are done, so there exists $H[V_i]$ such that $\Delta(H[V_i]) > \max\{10, |V_i| - 3\}$.

Suppose the inequality is not true and choose $S \subset V$ such that S is a counter example, and subject to this, $|S|$ is as small as possible. If H has a component V_i such that $\Delta(H[V_i]) \leq \max\{10, |V_i| - 3\}$, then by the choice of S, $S \cap V_i = \emptyset$.

Hence for any V_i such that $V_i \cap S \neq \emptyset$, $\Delta(H[V_i]) > \max\{10, |V_i| - 3\}$, so $|V_i| \geq 7$ and by Theorem 1.0.6, $||H[V_i]|| \leq 3|V_i| - 8$. Therefore,

$$\sum_{u \in S} d(u) \leq 3(\sum_i (3|H_i| - 8)) \leq 9n - 24,$$

a contradiction. $\square$

We have the following lemma.

Lemma 4.2.7. *Let $H = (V, E)$ be a P-free 3-graph and let v be a critical vertex with subordinate set D_v. If there exists a trivial vertex u of v, then either $H[\{u\} \cup V(N(u))] \subset K_4^3$ or there exists another trivial vertex u' such that*

$$E(\{u, u'\}, V, V) \subset E(\{u\}, \{u'\}, V - D_v).$$

The proof of this lemma appears in section 4.4 and it yields the following lemma.

Lemma 4.2.8. *Let $H = (V, E)$ be a P-free 3-graph. Let C be the set of critical vertices. For any $v \in C$, denote by D_v the subordinate set of v and let $D = \cup_{v \in C} D_v$. Then $H' = H[V - D] = (V', E')$ does not contain a critical vertex.*

Proof. If C is empty then the statement is vacuously true. Suppose for a contrary that H' has a critical vertex, say u. By the construction, $u \notin C$. If there is no $v \in C$ such that $e(\{u\}, \{v\}, V) > 0$, then u is a critical vertex of H, a contradiction. So we may assume that there is $v \in C$ such that $e(\{u\}, \{v\}, V) > 0$, i.e u is a trivial vertex of v in H. Lemma 4.2.7 implies that either $H[\{u\} \cup N(u)] \subset K_4^3$ or there is $u' \in V'$ such that $E(\{u\}, V', V') = E(\{u\}, \{u'\}, V - D_v)$. If $H[\{u\} \cup N(u)] \subset K_4^3$, then $|D_u| \leq 3$, a contradiction to that u is a critical vertex of H'. Otherwise, $e(\{u\}, D_u, D_u) = e(\{u\}, \{u'\}, D_u) \leq |D_u| - 1$, which is also a contradiction to that u is a critical vertex of H'. □

4.3 Proof of Theorem 1.0.7

The proof of Theorem 1.0.7 entirely relies on Theorem 1.0.6. We start with one lemma.

Lemma 4.3.1. *Let H be a $k-$centric Turán graph for P with $|H| = n \geq 22$ and $k \geq 2$. Denote by c the center of H. For any $S \subset V - \{c\}$ such that $|S| \geq \frac{n}{2}$,*

$$\sum_{u \in S} d(u) \leq |S|(n-3).$$

Proof. Let S be an arbitrary subset of $V - \{c\}$ such that $|S| \geq \frac{n}{2}$. If $|D_c| = n - k \geq n - 6$ then by Fact 4.2.3, every vertex in D_c has degree at most $n - k - 1$ and every vertex but c in $V - D_c$ has degree at most $\binom{5}{2}$, and therefore,

$$\sum_{u \in S} d(u) \leq |S| \cdot \max\{10, (n-k-1)\} \leq |S|(n-3).$$

Thus we may assume $|D_c| \leq n - 7$, i.e $k \geq 7$.

Denote by C the set of critical vertices and let $C' = C - \{c\}$. For any $v \in C$, denote by D_v the subordinate set of v. Set $D := \cup_{v \in C} D_v$, $H' := H[V - D]$. Lemma 4.2.8 implies that H' does not contain a critical vertex. Set $k_2 := |V - D|$, $k_1 := \sum_{v \in C'} |D_v|$. By Fact 4.2.4, $k_1 = |\cup_{v \in C'} D_v|$, and so,

$$k = k_1 + k_2.$$

Note that for any $t \leq n$,

$$t|S| \geq \frac{nt}{2} > \binom{t}{2}.$$

Note that if $n - k = q$, then for any $u \in C'$, $|D_u| \leq q$, and so,

$$\sum_{u \in C'} |\{\{x, y\} \in N(u) : \{x, y\} \subset D_u\}| \leq k_1/q \cdot \binom{q}{2} = \frac{k_1(q-1)}{2}.$$

Set $S_1 := S \cap D$, $S_2 := S - S_1$. Note that

$$\sum_{v \in S} d(v) \leq |S_1|(|D_c| - 1) + \sum_{u \in S \cap C'} |N(u) \cap D_u| + \sum_{u \in S_2} d_{H'}(u).$$

By Corollary 4.2.6,

$$\sum_{u \in S_2} d_{H'}(u) \leq \max\{|S_2|(k_2 - 3), 9k_2 - 24, 10|S_2|\}.$$

Case 1. $\max\{|S_2|(k_2 - 3), 9k_2 - 24, 10|S_2|\} = 9k_2 - 24.$

Note that $(k_2 - 2)|S| \geq 11(k_2 - 2) \geq 9k_2 - 24$. Thus

$$\begin{aligned}
\sum_{v \in S} d(v) &\leq |S_1|(n - k - 1) + \binom{k_1}{2} + 9k_2 - 24 \\
&\leq |S|(n-3) - |S|(k_1 + k_2 - 2) + \binom{k_1}{2} + 9k_2 - 24 \\
&\leq |S|(n-3) - |S|k_1 + \binom{k_1}{2} \\
&\leq |S|(n-3).
\end{aligned}$$

Case 2. $\max\{|S_2|(k_2 - 3), 9k_2 - 24, 10|S_2|\} = |S_2|(k_2 - 3).$

Note that $k_2 - 3 \geq 10$, $|S_2| \geq 9$. If $n - k \leq k_2 - 2$, then $\sum_{u \in C'} |N(u) \cap D_u| \leq \frac{k_1 k_2}{2}$, and then

$$\begin{aligned}
\sum_{v \in S} d(v) &\leq |S_1|(n - k - 1) + \frac{k_1 k_2}{2} + |S_2|(k_2 - 3) \\
&\leq |S|(n-3) - |S|(k_1 + k_2 - 2) + \frac{k_1 k_2}{2} + |S_2|(k_2 - 3) \\
&\leq |S|(n-3) - (|S|k_1 - \frac{k_1 k_2}{2}) - (|S|(k_2 - 2) - |S_2|(k_2 - 3)) \\
&\leq |S|(n-3).
\end{aligned}$$

Otherwise,

$$\begin{aligned}
\sum_{v\in S} d(v) &\leq |S_1|(n-k-1) + \binom{k_1}{2} + |S_2|(k_2-3) \\
&= |S|(n-k-1) - |S_2|(n-k-1) + \binom{k_1}{2} + |S_2|(k_2-3) \\
&\leq |S|(n-k-1) + \binom{k_1}{2} \\
&\leq |S|(n-3) - \frac{k_1 n}{2} + \binom{k_1}{2} \\
&\leq |S|(n-3).
\end{aligned}$$

Case 3. $\max\{|S_2|(k_2-3), 9k_2-24, 10|S_2|\} = 10|S_2|$.

If $n-k \leq 10$ then $\sum_{u\in C'} |N(u)\cap D_u| \leq 5k_1$, and then

$$\begin{aligned}
\sum_{v\in S} d(v) &\leq |S_1|(n-k-1) + 5k_1 + 10|S_2| \\
&= |S|(n-3) - |S_2|(n-3) - |S_1|(k-2) + 5k_1 + 10|S_2| \\
&\leq |S|(n-3) - (|S_2|(n-13) + |S_1|(n-12) - 5k_1) \\
&= |S|(n-3) - (|S|(n-12) - |S_2| - 5k_1) \\
&\leq |S|(n-3) - (10|S| - |S_2| - 5k_1) \\
&\leq |S|(n-3) - (5(n-k_1) - |S_2|) \\
&\leq |S|(n-3) - (n-k_1-|S_2|) \\
&\leq |S|(n-3).
\end{aligned}$$

Otherwise,

$$\begin{aligned}
\sum_{v\in S} d(v) &\le |S_1|(|D_c|-1) + \sum_{u\in C'} |N(u)\cap D_u| + \sum_{u\in S_2} d_{H'}(u) \\
&\le |S|(n-k-1) - |S_2|(n-k-1) + \binom{k_1}{2} + 10|S_2| \\
&\le |S|(n-3) - |S|(k-2) + \binom{k_1}{2} \\
&\le |S|(n-3).
\end{aligned}$$

□

Now we prove Theorem 1.0.7.

Proof of Theorem 1.0.7. We argue by induction on r. The base step follows immediately from Theorem 4.1.6. So we may assume that $r \ge 11$ and let $n = 2r \ge 22$. Suppose to the contrary that there exists a r-coloring of K_n^3 which does not contain a monochromatic P. If one of the colors is the subgraph of S_n^3, then we remove the center of this S_n^3 together with all its incident edges, and then we get an $r-1$-coloring of K_{n-1}^3, hence we get a monochromatic P by induction hypothesis. So we may assume that there is no 1-centric Turán graph for P.

Let $H = (V, E)$ be such a r-coloring of K_n^3. For any $i \in [r]$, let $H_i = (V, E_i)$ where $E_i = \{e \in E : e \text{ is colored by color } i\}$. Let

$$R_1 = \{i \in [r] : H_i \text{ has a critical vertex }\},$$

$$R_2 = \{i \in [r] : H_i \text{ has no critical vertex }\}.$$

Define $S \subseteq V$ as follows:

$$S = \{v \in V : v \text{ is not the center for any } H_i, i \in R_1\}.$$

Since $|R_1| \le r$,

$$|S| \ge n - r = r = \frac{n}{2} \ge 11.$$

By Corollary 4.2.6 and Lemma 4.3.1,

$$\sum_{i\in[r]}\sum_{v\in S} d_{H_i}(v) \le r\cdot|S|(n-3),$$

where $d_{H_i}(v)$ is the degree of v in H_i. Therefore, there exists $c \in S$ such that

$$\binom{n-1}{2} = \sum_{i\in[r]} d_{H_i}(c) \le r(n-3) = \frac{n}{2}\cdot(n-3) < \binom{n-1}{2},$$

a contradiction. □

4.4 Proof of Theorem 1.0.6

In this section, we present the proof of Theorem 1.0.6. Defining following auxiliary graph is our first step.

Definition 4.4.1. *Let $H = (V, E)$ be a 3-graph. Fix $v \in V$ and define a graph $G = (V', E')$ as $V' = V - \{v\}$ and $E' = \{\{x, y\} \in \binom{V'}{2} : \{v, x, y\} \in E\}$. We call this G the derived graph with v on V'.*

An useful observation follows:

Observation 4.4.2. *Let $H = (V, E)$ be a P-free 3-graph. Fix $v \in V$ and let $G = (V', E')$ be the derived graph with v on V'. Let e be an edge in H such that $v \notin e$. Then there exists no pair of edges $e', e'' \in E'$ such that*

$$e' \cap e'' = \emptyset$$

$$|e' \cap e| = 1, e'' \cap e = \emptyset.$$

To avoid confusion, for any $u \in V', U \subset V'$, we denote by $N_G(u)$ neighborhood of u in G and $N_G(U) = \cup_{u\in U} N_G(u)$. In a similar way, let $d_G(u) = |N_G(u)|, d_G(U) = |N_G(U)|$. We will show two lemmas which develop Observation 4.4.2.

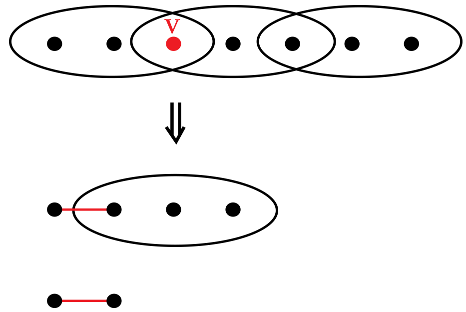

Figure 4.1: Observation 4.4.2

Lemma 4.4.3. *Let $H = (V, E)$ be a P-free 3-graph. Fix $v \in V$ and let $G = (V', E')$ be the derived graph with v on V'. Let e be an edge in E such that $v \notin e$. If $N_G(e) - e \neq \emptyset$, then for any $x \in N_G(e) - e$,*

$$||G[V' - e - x]|| = 0.$$

Proof. Suppose not. Then there exist $x \in N_G(e) - e$ and $e'' \in E(G[V' - e - x])$. Let $e' = \{x, y\} \in E'$ where $y \in e$. Then

$$e' \cap e'' = \emptyset, |e' \cap e| = |\{y\}| = 1, e'' \cap e = \emptyset,$$

a contradiction to Observation 4.4.2. □

Now we prove Lemma 4.2.7 briefly. Before proving the lemma we need to recall *vertex cover*. A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex from the set. Given a graph $G = (V, E)$, the size of a minimum vertex cover of G is denoted by $\tau(G)$.

Fact 4.4.4. *Let G be a graph. If $||G|| \geq |G|$, then $\tau(G) > 1$.*

Proof of Lemma 4.2.7. Let G be a derived graph with v on V' and D_v is a subordinate set of v. Let C be the component of G such that $u \in C$. Since u is a trivial vertex, $d_G(u) \geq 1$. If there is no edge $e' \in E$ such that $v \notin e'$ and $e' \cap C \neq \emptyset$, then $D_v \cup C$ is a subordinate set, a contradiction. So let $e' \in E$ be such that $v \notin e'$ and $e' \cap C \neq \emptyset$.

If $|N_G(e') - e'| \geq 1$, then since $\tau(G[D_v]) > 1$, by Lemma 4.4.3, H contains a P, a contradiction. Hence we may assume that $N_G(e') \subset e'$, and so $|C| \in \{2, 3\}$. If $|C| = 3$, then $e' = C$ and then $H[\{v\} \cup V(N(u))] \subset K_4^3$. If $|C| = 2$, then $d_G(u) = 1$, say $N_G(u) = \{u'\}$, and then $\{u, u'\} \subset e'$, therefore,

$$E(\{u, u'\}, V, V) \subset E(\{u\}, \{u'\}, V - D_v).$$

□

By Lemma 4.1.5, our argument will be based on the assumption that H is also C_3^3-free and we have the lemma developing the assumption. Here is an observation for that H is C_3^3-free.

Observation 4.4.5. *Let $H = (V, E)$ be a C_3^3-free 3-graph with $|H| = n$. Fix $v \in V$ and let $G = (V', E')$ be the derived graph with v on V'. Let e be an edge in E such that $v \notin e$. There exists no pair of edges $e', e'' \in E'$ such that*

$$e' \cap e'' = \emptyset, |e' \cap e|, |e'' \cap e| = 1.$$

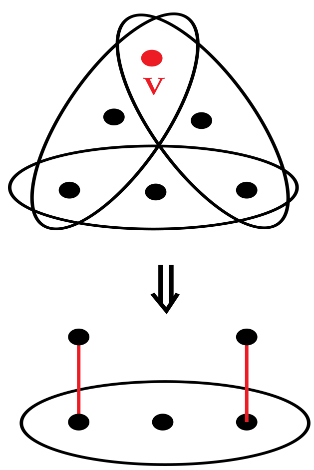

Figure 4.2: Observation 4.4.5

Lemma 4.4.6. *Let $H = (V, E)$ be a connected P, C_3^3-free 3-graph. Fix $v \in V$ and let $G = (V', E')$ be the derived graph with v on V' where $V' = V - \{v\}$. Let $E'' =$*

$\{e \in E : v \notin e\}$. *For any* $e \in E''$ *such that* $N_G(e) - e \neq \emptyset$*, the set of edges in* $E(G) \setminus E(G[e])$ *forms an intersecting family.*

Proof. Suppose by the way of contradiction that there exists $e \in E''$ and $e_1, e_2 \in E(G) \setminus E(G[e])$ such that $e_1 \cap e_2 = \emptyset$. Note that $|e_1 \cap e|, |e_2 \cap e| \leq 1$ and by Observation 4.4.2,

$$|e_1 \cap e| = |e_2 \cap e|.$$

If $|e_1 \cap e| = |e_2 \cap e| = 1$ then by Observation 4.4.5, H contains C_3^3. Hence we may assume that $|e_1 \cap e| = |e_2 \cap e| = 0$. Then there exists $x \in N_G(e) - e$, $i \in [2]$ such that $x \notin e_i$, which implies that

$$e_i \in E(G[V' - e - x]),$$

a contradiction to Lemma 4.4.3. □

Finally, we recall the result in (Keevash *et al.*, 2006).

Theorem 4.4.7. *(Keevash* et al.*, 2006) If* H *is a* k*-graph on* n *vertices with no* P_2^k *where* $k \geq 3$*, then* $||H|| \leq \binom{n}{k-2}$.

Especially,

Fact 4.4.8. *For* $n \geq 1$, $\mathrm{ex}(n; P_2^3) \leq n$.

Now we prove Theorem 1.0.6.

Proof of Theorem 1.0.6. If $H \supset C_3^3$, then Lemma 4.1.5 implies that

$$||H|| \leq 3n - 8.$$

Therefore, we may assume that H is C_3^3-free.

Now we choose $v \in V$ so that $d_H(v)$ is maximum and define the derived graph G_v with v on V' and denote by E'' the set of edges in E which does not contain v. By the choice of v and the given condition,

$$||G|| = \Delta(H) \geq n - 2 \geq 5.$$

Let $I = \{u \in V' : d_G(u) = 0\}$ and we classify vertices in V' as follows: For any $u \in V'$, denote by $C(u)$ the component of G containing u.

- $D_2 = \{u \in V' : |C(u)| = 2\}$.
- $D_3 = \{u \in V' : |C(u)| = 3\}$.
- $D = \{u \in V' : |C(u)| \geq 4\}$.

Let $|I| = t_0, |D_2| = 2t_2, |D_3| = 3t_3, |D| = t$. For $i \in [t_2]$, denote by p_i a component of $G[D_2]$. For $j \in [t_3]$, denote by C_j a component of $G[D_3]$.

Lemma 4.4.6 is the key of the proof. We have a following claim which is for the case that Lemma 4.4.6 is applied.

Claim 4.4.9. *If there exists $e \in E''$ such that $N_G(e) - e \neq \emptyset$ then $||H|| \leq 2n - 2$.*

Proof. By Lemma 4.4.6, the set of edges in $E(G) \setminus E(G[e])$ forms an intersecting family, and therefore, the set of edges in $E(G) \setminus E(G[e])$ forms a star or a triangle.

- The set of edges in $E(G) \setminus E(G[e])$ forms a K_3. Then $|V(K_3) \cap e| = 1$, so $|V(K_3) \cup e| = 5$. Since $|V'| \geq 6$, $I = V' \setminus (V(K_3) \cup e) \neq \emptyset$. Since $||G|| \geq 5$, $||G[e]|| \geq 2$, so $G[V(K_3) \cup e]$ is connected. Since H is connected, there exists $e' \in E''$ such that $e' \cap I \neq \emptyset$ and $(V(K_3) \cup e) \cap e' \neq \emptyset$. Then $N_G(e') - e' \neq \emptyset$. If $|e \cap e'| \geq 1$ then the set of edges in $E(G) \setminus E(G[e'])$ does not form an intersecting family, a contradiction to Lemma 4.4.6. If $|e \cap e'| = 0$ then there exists an

$e'' \in E(G)$ such that $|e'' \cap e| = |e'' \cap e'| = 1$, and then $e, e', e'' \cup \{v\}$ forms a P, a contradiction.

- The set of edges in $E(G) \setminus E(G[e])$ forms a star. Let c be the center of the star. First, assume that G is a star. Since $||G|| \geq |V'| - 1$,

$$G \cong K_{1,|V'|-1}.$$

If there is $e'' \in E''$ such that $c \in e''$, then $d(c) \geq d_G(c) + 1 = |V'| > |V'| - 1 = d(v)$, a contradiction to the choice of v. Hence for any $e'' \in E''$, $c \notin e''$. If there exists $P_2^3 \in H[V' - \{c\}]$, say $\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}$, then it forms a P with $\{v, c, v_5\}$. By Fact 4.4.8

$$|E''| \leq |V'| - 1.$$

Therefore,

$$||H|| = ||G|| + |E''| \leq (|V'| - 1) + (|V'| - 1) = 2n - 4 < 2n - 2.$$

Hence we may assume that G is not a star, which implies that there exists $e' \in E(G[e])$ such that $c \notin e'$. If $t_2 \neq 0$ then $t_2 = 1, t_3 = 0$, $p_1 \subset e$ and $G[D]$ is a star, then $||G|| = t - 1 + 1 = t \leq |V'| - 2t_2 = |V'| - 2 = n - 3$, a contradiction. Hence $t_2 = 0$. If $t_3 \neq 0$ then $t > 0$ or $t_3 \geq 2$, but then the set of edges in $E(G) \setminus E(G[e])$ can not be a star, hence $t_3 = 0$. Moreover, $|V'| - 1 \leq ||G|| \leq (t - 1) + 3 = |V'| - t_0 - 1 + 3$, which implies that

$$t_0 \leq 3.$$

Note that if there exists $e'' \in E''$ such that $|e'' \cap e'| \leq 1$, then the set of edges in $E(G) \setminus E(G[e''])$ does not form an intersecting family. If there exists $e'' \in E''$ such that $|e'' \cap e'| = 1$, then $N_G(e'') - e'' \neq \emptyset$, a contradiction to Lemma 4.4.6. So we see that for any $e'' \in E''$, $|e'' \cap e'| \in \{0, 2\}$. Now, assume that there exists

$e'' \in E''$ such that $e'' \cap e' = \emptyset$. If $e'' \nsubseteq I$ then $e'' \cap D \neq \emptyset$, so $N_G(e'') - e'' \neq \emptyset$, a contradiction. Hence $e'' \subset I$, so $e'' = I$ and $t_0 = 3$. Since H is connected, there is $e''' \in E''$ such that $e''' \cap e'' \neq \emptyset$ and $e''' \cap D \neq \emptyset$. If $|e''' \cap e''| = 2$ then $N_G(e''') - e''' \neq \emptyset$ and $|e''' \cap e'| \leq 1$, so the set of edges in $E(G) \setminus E(G[e'''])$ does not form a intersecting family, if $|e''' \cap e''| = 1$ then there is $x \in e''' \cap D$ such that $N_G(x) - e''' \neq \emptyset$, say $y \in N_G(x) - e'''$, and then $e''', e'', \{v, x, y\}$ forms a P, a contradiction. Hence we may assume that for any $e'' \in E''$,

$$e' \subset e''.$$

It implies that

$$|E''| \leq n - 3.$$

Therefore,

$$||H|| = |E''| + ||G|| \leq n - 3 + n + 1 = 2n - 2.$$

□

To finish the proof, we need the following.

Claim 4.4.10. *If for any $e \in E''$, $N_G(e) - e = \emptyset$ then $||H|| < 3n - 8$.*

Proof. Note that for any $e \in E''$, $e \cap D = \emptyset$. If $||G[D]|| \geq |D|$, then D is a sub-ordinate set, so $||G[D]|| \leq |D| - 1$, and then $t_0 + t_2 = 0$. Hence $E'' \subset \{C_i : i \in [t_3]\}$. Therefore,

$$||H|| = ||G|| + |E''| \leq 3t_3 + (t - 1) + t_3 < 3n - 8.$$

□

This completes the proof of the theorem. □

Chapter 5

CONCLUSIONS

The aim of this thesis is to provide optimal conditions for some Turán type problems in extremal graph theory.

We conclude by giving an overview of the results provided in this thesis and suggesting possible future research.

5.1 Brief Summary of Results

This section includes a brief list of the main results in this thesis.

5.1.1 Even cycles in dense graphs

In Theorem 1.0.2, the following result is proved.

For every $0 < \alpha < \frac{1}{2}$, there is a natural number $N = N(\alpha)$ such that the following holds. For any $n_1, ..., n_l \in Z^+$ such that $\sum_{i=1}^{l} n_i = \delta(G)$ and $n_i \geq 2$ for all $i \in [l]$, every 2-connected graph G of order $n \geq N$ and $\alpha n \leq \delta(G) < n/2 - 1$ contains C where C is a disjoint union of $C_{2n_1}, \ldots, C_{2n_l}$ or G is one of the graphs from Example 2.1.3 and $n_1 = n_2 = \delta$ or G is a subgraph of the graph from Example 2.1.4 and $n_i = 2$ for every i.

5.1.2 Balanced spanning caterpillar

In Theorem 1.0.5, the following result is proved.

For $p \in Z^+$, there exists n_0 such that for every $n \in (p+1)Z$ with $n \geq n_0$ the following holds. If G is a graph on n vertices such that

$$\delta(G) \geq \begin{cases} \frac{n}{2} & \text{if } n/(p+1) \text{ is even} \\ \frac{n+1}{2} & \text{if } n/(p+1) \text{ is odd and } p > 2 \\ \frac{n-1}{2} & \text{if } n/(p+1) \text{ is odd and } p \leq 2 \end{cases}$$

then G contains a spanning p-caterpillar. This result is sharp.

5.1.3 Turán-type result and multi-color Ramsey number for a loose 3 uniform path of length 3

In Theorem 1.0.7, the following result is proved. For any $r \geq 6$,

$$r + 6 \leq R(P_3^3; r) \leq 2r.$$

5.2 Future Research

Since we showed that the condition for the result in Chapter 3 is best possible, our proposal for future research only discusses topics from Chapter 2 and 4.

In Chapter 2, we only investigated spectrum of even cycles. But Conjecture 1.0.1 which was our original motivation can be approached using similar methods in the case of edd cycles. Hence it is natural to consider the odd case of Conjecture 1.0.1 in the case of that a graph is dense and its order is sufficiently large for future research. We believe that using the framework of Chapter 2, with some additional considerations, the following theorem can be established. Nevertheless the details of a possible argument are left as future work.

Theorem 5.2.1. *(Yie and Czygrinow, 2018) For every $0 < \alpha < \frac{1}{2}$, there is a natural number $N = N(\alpha)$ such that the following holds. If G is 2-connected graph such that G is not bipartite and $|G| \geq N$, then $|S_o| \geq \delta(G)$ where $S_o = \{|C| : C$ is an odd cycle contained in $G\}$.*

Our auxiliary results, Lemma 2.4.7, 2.3.6, 2.3.9, 2.3.10, 2.4.3 in Chapter 2 would give us a framework for the proof of Theorem 5.2.1.

Although our result in Chapter 4 has been minimally improved by Polcyn and Łuczak, the bound seems to leave a lot of room for improvement. We conjecture that the correct answer is as follow.

Conjecture 5.2.2. *$R(P_3^3; r) = r + 6$ for $r \geq 3$.*

The limitation of application of Theorem 1.0.6 comes from the maximum degree condition of Theorem 1.0.6. Hence the following theorem should be our next goal.

Theorem 5.2.3. *(Yie, 2018) Let $H = (V, E)$ be a 3-uniform hypergraph with $|H| = n \geq 7$. If H is connected and $||H|| > \max\{3n - 8, \frac{\Delta(H)(n-\Delta(H))}{2}, \Delta(H)(n - 2\Delta(H))\}$, then H contains either P_3^3 or a critical vertex.*

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