Implementation and Comparison of H_∞ Observers for Time-Delay Systems

by

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ABSTRACT

In this thesis, different H_∞ observers for time-delay systems are implemented and their performances are compared. Equations that can be used to calculate observer gains are mentioned. Different methods that can be used to implement observers for time-delay systems are illustrated. Various stable and unstable systems are used and H_∞ bounds are calculated using these observer designing methods. Delays are assumed to be known constants for all systems. H_∞ gains are calculated numerically using disturbance signals and performances of observers are compared.

The primary goal of this thesis is to implement the observer for Time Delay Systems designed using SOS and compare its performance with existing H_∞ optimal observers. These observers are more general than other observers for time-delay systems as they make corrections to the delayed state as well along with the present state. The observer dynamics can be represented by an ODE coupled with a PDE. Results shown in this thesis show that this type of observers performs better than other H_∞ observers. Sub-optimal observer-based state feedback system is also generated and simulated using the SOS observer. The simulation results show that the closed loop system converges very quickly, and the observer can be used to design full state-feedback closed loop system.

DEDICATION

To my parents.

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INTRODUCTION

1.1 State Observers

In many systems, it is necessary for system states to be known in order to implement the control action. However, internal states of the system can not always be measured directly using sensors. A state observer can be used in such cases to get an estimate of the states. State observer uses the system dynamics and the knowledge of the output to precisely estimate the states of the system. Thus, it can be used to design a full state feedback controller for the physical systems whose internal states can not be measured.

1.1.1 Observability

Weather all the internal states of a system are observable or not depends on a property of the system called observability.

Consider a Linear Time Invarient(LTI) system,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

where $x(t)$ is the state, $u(t)$ is the input and $y(t)$ is the output of the system.

Observability matrix for the n^{th} order system shown in (1.1) can be written as [23],

$$O = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{n-1} \end{bmatrix}. \tag{1.2}$$

A full state observer for the system (1.1) exists if the matrix O is a full rank matrix. Otherwise, the system is not observable which means that some of the internal states cannot be determined from the measurement of input and output.

1.1.2 General Forms of Observers for Linear Systems

Observer theory for LTI systems was developed by Luenberger [1, 2, 3] in 1965 and the observer that he designed is called the Luenberger observer. The observer dynamics of the Luenberger observer can be represented by the system dynamics along with a correction term proportional to the difference between the actual output and the estimated output. Observer gains are designed such that the estimation error dynamics are stable.

For the system shown in (1.1), the Luenberger observer is the observer which has the following dynamics [3],

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y - C\hat{x}(t)), \quad \hat{x}(0) = \hat{x}_0, \tag{1.3}$$

where $\hat{x}(t)$ is the state estimation and $\hat{y}(t)$ is the estimated output. If the estimation error for the above system is defined as,

$$e(t) = x(t) - \hat{x}(t). \tag{1.4}$$

So the error dynamics become [3],

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A - LC)(x(t) - \hat{x}(t)) = (A - LC)e(t). \tag{1.5}$$

If L is designed such that the eigenvalues of matrix $(A - LC)$ are negative, then the error dynamics are stable and correct value of system states can be estimated.

A more general form of the observer can be written as [24],

$$\begin{aligned} \dot{z}(t) &= Mz(t) + Ny(t) + Su(t), \\ \hat{x}(t) &= z(t) + Hy(t) \end{aligned} \tag{1.6}$$

where $z(t)$ is the observer state and $\hat{x}(t)$ is the estimated state. Matrices M, N, S and H are the observer gains. The same analysis can be done and the gains can be designed such that the error dynamics are stable.

A significant amount of research has been conducted in the field of state observer in last five decades. Theory of state observers for linear system models and their applications are discussed in detail by Oreilly [35]. Frank [36] discussed the application of the observer theory in fault diagnosis of dynamic systems. For methods to design robust H_∞ and H_2 observers for linear systems, refer [38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. Method of linear filtering of dynamic systems was developed by Kalman [22] and this approach was extended for H_∞ filtering in [48, 49, 50, 51].

1.2 Time-Delay Systems

Time-delays are present in almost all the physical systems. There can be measurement delays in manufacturing processes or transport delays in communication networks. Calculating and implementing control decisions can generate delays in actuation. Not considering the effect of delays while modeling can result in undesired system response and because of that reason, some systems are modeled as time-delay systems. Systems such as Sampled-data control, Drilling machines, Tunnel diodes, Lasers, Vehicular traffic flows and Neural networks can be modeled as time-delay systems [6].

Various researchers have contributed in the field of time-delay systems. Existence of the solution and stability conditions for Delay differential equations are discussed in [26, 27]. Razumukhin [9] developed stability criterions for time-delay systems. Other works on the stability of linear systems with delays can be found in [10, 11, 12, 13, 25, 66]. Morse [14] explained the ring models for time-delay systems and definitions of controllability and observability were introduced. Modeling, analysis, applications and open problems in the field of time-delay systems are discussed in [6, 7, 18, 68].

Time-delay systems are mathematically represented by delay differential equations(DDE). A time-delay system with discrete delays in state can be written mathematically as shown below [75],

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i) \tag{1.7}$$

where A_i are known system matrices, $x(t)$ is the current state and $x(t - \tau_i)$ are the delayed states of the system.

Time-delay systems can be modeled as a transport problem and it can be formalized using an ODE(Ordinary Differential Equation) coupled with a PDE(Partial Differential Equation) [6].

A linear DDE,

$$\begin{aligned} \tfrac{d}{dt}x(t) &= Ax(t-\tau), \quad t \geq 0, \\ x(0) &= x_0 \end{aligned} \tag{1.8}$$

can be solved by solving an equivalent ODE coupled with a PDE [6],

$$\begin{aligned} \tfrac{\partial}{\partial t}\phi(t,\theta) &= \tfrac{\partial}{\partial \theta}\phi(t,\theta), \quad t \geq 0, \quad \theta \in [-\tau, 0), \\ \tfrac{\partial}{\partial t}\phi(t,\theta)|_{\theta=0} &= A\phi(t,-\tau), \quad t \geq 0, \end{aligned} \tag{1.9}$$

Here, the first equation in (1.9) solves for the delayed state of the system and the second equation solves for the current state of the system. DDEs can be solved using various numerical algorithms and numerical methods for ODEs can be extended to solve DDEs. DDEs can be solved in Matlab using dde23 command.

Chapter 2

DEFINITIONS AND PRELIMINARIES

2.1 Signal and System Norms

Disturbances are present in almost all systems. The stability of the system can be affected due to the disturbances. The field of optimal and robust control theory deals with designing controllers and observers such that the effect of external disturbances on the system output is minimized. Such control problems are formulated in terms of norms of input and output signals.

Norm of a Vector Space: [29] A norm on a vector space V is defined as a non-negative function $||.||$ which satisfies the following equations,

$$\begin{gathered} ||x|| = 0 \quad \text{if and only if} \quad x = 0, \\ ||\alpha x|| = |\alpha|.||x||, \\ ||x+y|| \geq ||x|| + ||y||, \end{gathered} \tag{2.1}$$

for all $\alpha \in \mathbb{R}$ and $x, y \in V$.

The concept of vector norm can be extended to signals. Some important signal norms are described below.

***L_1*-Norm**: [29] The L_1-norm of a signal is defined as the integration of the absolute value of the signal,

$$||u||_1 = \int_0^\infty |u(t)|dt. \tag{2.2}$$

***L_2*-Norm**: [29] The L_2-norm of a signal is defined as the integration of squared value of the signal,

$$||u||_2 = \int_0^\infty u(t)^2 dt. \tag{2.3}$$

***L_∞*-Norm**: [29] The L_∞-norm of a signal is defined as the maximum absolute value of the signal,

$$||u||_\infty = \sup_{t\geq 0} |u(t)|. \tag{2.4}$$

Systems norm can be defined using the definitions of signal norms. One of the important concepts used in this thesis is H_∞ norm of the system which is explained below.

***H_∞*-Norm of a System**: [29] H_∞ gain of a system is defined as the ratio of the L_2-norms of output signal and input signal,

$$||G||_\infty = \sup_{u\in \boldsymbol{L_2}, ||u||_2 \neq 0} \frac{||Gu||_2}{||u||_2}. \tag{2.5}$$

Here, G is the system transfer function. Observers for time-delay systems that are designed to minimize the H_∞ gain of the observation error are discussed in chapter 3.

2.2 Linear Matrix Inequalities

Many problems in control theory such as stabilty analysis of the linear systems and H_2/H_∞ optimal controller/observer design problems can be formulated in form of Linear Matrix Inequalities. Lyapunov's direct method to determine the stability of LTI systems can be written as an LMI. Consider the linear system,

$$\dot{x} = Ax. \tag{2.6}$$

The stability condition for this system using the Lyapunov's direct method can be written as the following inequality [6],

$$PA + AP^T < 0. \tag{2.7}$$

The system shown in (2.6) is stable if there exists a matrix $P > 0$ such that the above LMI is satisfied. The same approach has been extended to get an LMI form of various problems in control theory.

In general, LMIs can be written as a constraint shown below [6],

$$\sum_{i=1}^{n} A_i x_i + B < 0 \tag{2.8}$$

where A_i and B are known matrices and x_i are decision variables. Applications of LMIs in control theory are discussed in detail in [30, 32]. Applications of LMIs in the field of time-delay systems are described by Fridman [6].

LMIs as Convex Optimization Problem

LMIs can be formulated as a convex optimization problem can be solved numerically using various algorithms. In general, LMI optimization problem can be written in the form below [54],

$$\begin{aligned} \min \quad & c^T x, \\ \text{such that} \quad & A_i(x) \geq 0 \end{aligned} \tag{2.9}$$

where $c^T x$ is called the cost function, x is the decision variable and $A_i(x)$ are the LMI constraints. Interior point algorithms can be used to solve LMI optimization problems and are discussed in detail in [28]. Various numerical solvers such as SeDuMi, MOSEK and LMI Lab are available online and are capable of solving LMI problems in polynomial time.

LMI problems in control theory can be differentiated as,

Feasibility Problem: This type of LMI problem aims at finding a feasible x such that the inequality constraints are satisfied [6].

$$\text{Find} \quad x$$
$$\text{such that} \quad A_i(x) < 0.$$

LMIs for the stability of linear systems such as the one shown in (2.7) can be classified as a feasibility problem as it aims at determining a feasible $P > 0$ that satisfies the inequality.

Eigenvalue Problem: This type of LMI problem aims at finding a feasible x for minimum or maximum eigenvalue of a matrix such that inequality constraints are satisfied [6].

$$\min \quad \gamma$$
$$\text{such that} \quad A(x) < \gamma I, \quad B(x) < 0.$$

Problems such as designing H_2 or H_∞ optimal controllers and observers can be classified as eigenvalue minimization problem.

2.3 Schur Complement Lemma

The following lemma can be used to convert non-linear matrix inequalities into LMIs [6].

Lemma 1: [6] Following are equivalent for given constant matrices A, B, C,

1. $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0.$

2. $C > 0 \quad \& \quad A - BC^{-1}B^T > 0.$

2.4 Bounded Real Lemma

The bounded real lemma can be used to calculate the H_∞ norm of the system.

Lemma 2: [54] Following are equivalent for system (1.1),

1. $||G||_{H_\infty} \leq \gamma$.

2. There exists a $X = X^T > 0$ such that

$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$

2.5 Algebraic Riccati Equation

Apart from LMIs, Algebraic Riccati Equations have also been used to solve different problems in control theory. The general form of algebraic Riccati equations for continuous time systems can be written as shown below [4],

$$A^TP + PA - PBB^TP + Q = 0 \tag{2.10}$$

where, A, B, R and Q are known matrices and our goal is to find a feasible P such that the above matrix equality is satisfied.

Applications of algebraic Riccati equations in optimal control theory are described in [4]. Algorithms that can be used to solve such equations are described in [5]. In Matlab, care(A, B, Q) can be used to solve the above Riccati equation.

2.6 Polynomial Sum of Squares and SOS Optimization Problem

Analysis of nonlinear systems or PDEs using Lyapunov's method can result in stability conditions in terms of polynomials [31, 32]. To investigate whether the polynomial is positive or not is difficult. However, in some cases, sum of squares method can be used to solve such problems.

A polynomial $p(x)$ is a sum of squares if [63]

$$p(x) = \sum_i {g_i}^2(x).$$

If $p(x)$ is an SOS, then it means that $p(x)$ is a positive polynomial. Alternatively, the polynomial is called a sum of squares if the polynomial $p(x)$ can be written in the following form [63],

$$p(x) = Z(x)^T M Z(x).$$

where M is a positive definite matrix and $Z(x)$ is the vector of monomials. Thus, it can be proved that the polynomial is positive if there exists a matrix $M > 0$ and positivity of a matrix can be investigated using convex optimization.

In general, An SOS optimization problem can be written as below [32],

$$\min \quad c^T x$$

$$\text{such that} \quad \sum_{i=1}^{n} x_i A_i(y) + B_i(y) \quad \text{is an SOS.}$$

SOSTOOLS [19] is a Matlab toolbox that can be used to solve such optimization problems.

Chapter 3

OBSERVERS FOR TIME-DELAY SYSTEMS

Various methods have been designed to synthesize state observers for linear systems with delays during last four decades. Lee and Manitius [58] discussed necessary conditions for the existence of an observer for time-delay systems. Extensions of Luenberger type observers for linear time-delay systems were made in [21, 56, 57]. Lee and Olbert [16] discussed observability criterions for time-delay systems using the ring models. Some other methods were developed to design an observer for time-delay system in [23, 59, 69, 70, 71]. Observers for nonlinear time-delay systems were designed in [34, 72, 73, 74, 15, 76].

Research has been conducted in the field of H_∞ optimal observation and filtering for time-delay systems as well. An H_∞ observer for time-delay system with a single delay in state was developed using algebraic Riccati equation in [17]. H_∞ observers were developed using LMIs in [52, 53, 54]. Sum of squares method was used to develop a more general form of the observer in [33]. For H_∞ filters for time-delay systems, see [37, 52, 60, 61, 62, 63, 64, 65, 67].

In this chapter, methods to design H_∞ Observers for time-delay systems developed by Fattouh et al [17], Fridman and Shakad [52], Briat [54], Choi and Chung [55], Peet and Gu [33] are presented. Results of all the Observers listed here are reproduced and compared with each other in chapter 5.

Consider a time-delay system with single delay in state and no delay in output,

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Ew(t) + Bu(t), \\ y(t) &= C_2 x(t) + Fw(t), \\ z(t) &= C_1 x(t). \end{aligned} \tag{3.1}$$

Where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t)$ is system control input, $y(t) \in \mathbb{R}^q$ is the measured output, $z(t) \in \mathbb{R}^p$ is the state to be estimated and $w(t) \in \mathbb{R}^r$ is a disturbance vector. In the case of full state observer, $C_1 = I$.

Luenberger-type observer for this system is given by the following equation [20],

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0 \hat{x}(t) + A_1 \hat{x}(t-\tau) + Bu(t) + L(y(t) - C_2 \hat{x}(t)) \\ \hat{z}(t) &= C_1 \hat{x}(t), \quad e(t) = \hat{z}(t) - z(t) \end{aligned} \tag{3.2}$$

where, $\hat{x}(t)$ is the state estimation and e is estimation error. Error dynamics are then given by [20],

$$\dot{e}(t) = (A_0 - LC_2)e(t) + A_1 e(t-\tau) + (E - LF)w(t) \tag{3.3}$$

H_∞ observers are designed such that effect of disturbance $w(t)$ on error $e(t)$ is minimized i.e. $||e||_{L_2} \leq \gamma ||w||_{L_2}$.

3.1 Fattouh et al's Observer

Fattouh et al [17] proposed a method of designing H_∞ observer for linear systems with multiple delays in states. A simplified version of this method for a single delay as shown by Sename [20] is presented here. An algebraic Riccati equation is required to be solved in order to find the observer gain as shown below.

Theorem 1: [17, 20] Consider the time-delay system shown in (3.1). An H_∞ observer of the form shown in (3.2) can be designed for this system if for some positive contants γ and ϵ, the following algebraic Riccati equation

$$A_0{}^T + PA_0 + 2P(\gamma^2 A_1 A_1{}^T + EE^T)P - \frac{2}{\epsilon}C_2{}^T(I_p - \frac{1}{\epsilon}FF^T)C_2 + \frac{2}{\gamma^2} = 0 \qquad (3.4)$$

can be solved for a symmetric positive semidefinite matrix P. The observer gain can then be calculated using the following equation,

$$L = \frac{1}{\epsilon}P^{-1}C_2{}^T.$$

The value of γ provides an upper bound on H_∞ gain of the observation error. Examples are given in chapter 4 to illustrate this method.

3.2 Fridman and Shakad's Observer

In [52], a Luenberger type observer was designed for time-delay systems with a single delay in the state. An LMI must be solved in order to find the observer gain. The following theorem illustrates the method presented in [52].

Theorem 2: [52] Consider the system shown in (3.1). An H_∞ observer of the form shown in (3.2) can be designed for this system if for some positive γ, there exist a positive matrix Q_1, a symmetric matrix $\bar{R}$ and matrices Q_2, Q_3, all in $\mathbb{R}^{n\times n}$ and a matrix $Y \in \mathbb{R}^{n\times q}$ such that the following LMI is satisfied,

$$\begin{bmatrix} Q_2 + Q_2{}^T & * & * & * & * & * \\ Q_3^T - Q_2 + (A_0{}^T + A_1{}^T)Q_1 - C_2{}^T Y^T & -Q_3 - Q_3{}^T & * & * & * & * \\ 0 & T & -\gamma^2 I_p & * & * & * \\ 0 & \tau\bar{R}A_1 & 0 & -\tau\bar{R} & * & * \\ B^T Q_1 - D_{2,1}{}^T Y^T & 0 & 0 & 0 & -I_q & * \\ \tau Q_2 & \tau Q_3 & 0 & 0 & 0 & -\tau\bar{R} \end{bmatrix} < 0. \qquad (3.5)$$

The observer gain can then be calculated using the following equation,

$$L = {Q_1}^{-1}Y. \tag{3.6}$$

The value of γ provides a bound on H_∞ gain of the observation error. Examples are given in chapter 4 to illustrate this method. This method of designing observer was extended for systems with two and three time-varying delay in system states in [53].

3.3 Briat's Observer

In [54], an H_∞ observer was designed for LPV(Linear Parameter Varying) time-delay systems using LMI approach. A simplified version of the theorem for linear time-delay systems is presented here.

The observer form used in [54] is the extended form of observer presented in (1.6) and observer dynamics are given by,

$$\begin{aligned} \dot{\hat{x}}(t) &= M_0\hat{x}(t) + M_1\hat{x}(t-\tau) + N_0 y(t) + N_1 y(t-\tau) + Su(t) \\ \hat{z}(t) &= \hat{x}(t) + Hy(t) \end{aligned} \tag{3.7}$$

where $\hat{x} \in \mathbb{R}^r$ is the observer state, $\hat{z} \in \mathbb{R}^r$ the state estimation and the matrices M_0, M_1, N_0, N_1 and H are observer gains.

Theorem 3: [54] Consider the time-delay system shown in (3.1). An H_∞ observer of the form shown in (3.7) can be designed for this system if for some positive γ, there exist positive definite symmetric matrices $P, Q, R \in \mathbb{R}^{p\times p}$, and matrices $Z \in \mathbb{R}^{p\times(2p+3q)}$, $X \in \mathbb{R}^{p\times p}$, $\bar{H} \in \mathbb{R}^{p\times q}$ such that the following LMI is satisfied,

$$\begin{bmatrix} -(X+X^T) & * & * & * & * & * & * \\ U_{21} & U_{22} & * & * & * & * & * \\ U_{31} & R & -Q-R & * & * & * & * \\ U_{41} & 0 & 0 & -\gamma I_r & * & * & * \\ 0 & I_n & 0 & 0 & -\gamma I_p & * & * \\ X & 0 & 0 & 0 & 0 & -P & * \\ h_{max}R & 0 & 0 & 0 & 0 & -h_{max}R & -R \end{bmatrix} < 0. \qquad (3.8)$$

The matrices used in the LMI are explained below.

$$U_{21} = \Theta^T X - \Xi^T \bar{L} C_1 + P, \quad U_{31} = \Upsilon^T X - \Omega^T \bar{L}^T, \quad U_{22} = -P + Q - R,$$

$$U_{41} = E^T ({C_1}^T X - {C_2}^T \bar{H}^T), \quad \bar{L} = (X^T \Phi - \bar{H})\Psi^+ + Z(I - \Psi\Psi^+),$$

$$\Theta = C_1 A_0 U - \Lambda\Gamma^+\Delta_0 \begin{bmatrix} C_2 \\ C_2 A_0 \end{bmatrix} U, \quad \Xi = -(I - \Gamma\Gamma^+)\Delta_0 \begin{bmatrix} C_2 \\ C_2 A_0 \end{bmatrix} U,$$

$$\Upsilon = C_1 A_1 U - \Lambda\Gamma^+\Delta_1 \begin{bmatrix} C_2 \\ C_2 A_1 \end{bmatrix} U, \quad \Omega = -(I - \Gamma\Gamma^+)\Delta_1 \begin{bmatrix} C_2 \\ C_2 A_1 \end{bmatrix} U,$$

$$\Phi = \Lambda\Gamma^+\Delta_H, \quad \Psi = (I - \Gamma\Gamma^+)\Delta_H, \quad S = FB, \quad F = C_1 - HC_2,$$

$$U \text{ is defined such that} \begin{bmatrix} C_1 \\ \bar{C}_1 \end{bmatrix}^{-1} = [U \quad V],$$

where $\bar{C}_1$ is a full column rank matrix such that $\begin{bmatrix} C_1 \\ \bar{C}_1 \end{bmatrix}$ is nonsingular and

$$\Gamma = \begin{bmatrix} C_1 & 0 \\ 0 & T \\ C_2 & 0 \\ 0 & C_2 \\ C_2A_0 & C_2A_1 \end{bmatrix}, \quad \Lambda = [C_1A_0 \quad C_1A_1],$$

$$\Delta_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_r & 0 \\ 0 & 0 \\ 0 & I_r \end{bmatrix}, \quad \Delta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_r & 0 \\ 0 & I_r \end{bmatrix}, \quad \Delta_H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_r \end{bmatrix}.$$

The gain can be calculated using $L = X^{-T}\bar{L}$. The value of γ provides an upper bound on H_∞ gain of the observeration error. The Observer gains can then be calculated using the following equations [54],

$$M_0 = \Theta - L\Xi,$$

$$M_1 = \Upsilon - L\Omega, H = \Phi - L\Omega,$$

$$N_0 = K_0 + M_0H,$$

$$N_1 = K_1 + M_1H.$$

where, K_0, K_1 and H is defined such that

$$[K_0 \quad H] = \Lambda\Gamma^+\Delta_0,$$

$$[K_1 \quad H] = \Lambda\Gamma^+\Delta_1.$$

3.4 Choi and Chung's Observer

A method to design an H_∞ optimal observer-based controller was presented in [55]. The observer-based controller considered is of the following form,

$$\begin{aligned}\dot{\hat{x}}(t) = A_0\hat{x}(t) + A_1\hat{x}(t-\tau) + Bu(t) - L(C\hat{x}(t) - y(t)) + EG\hat{x}(t)\\ u(t) = K\hat{x}(t)\end{aligned} \tag{3.9}$$

Two algebraic Riccati equations must be solved in order to calculate the observer and controller gain. The first Riccati equation solves for the controller gain and the second Riccati equation solves for the observer gain. The following theorem explains the method presented in [55].

Theorem 4: [55, 20] Consider the time-delay system shown in (3.1). The observer-based controller shown in (3.9) can be designed for this system if for some positive constants $\gamma, \epsilon_c, \delta_c, \epsilon_0, \delta_0$ and matrices Q_c and Q_0, the following algebraic Riccati equations

$$A_0{}^T P_c + P_c A_0 - \frac{1}{\epsilon_c} P_c (BB^T - \frac{1}{\delta_c} A_1 A_1{}^T - \frac{1}{\gamma^2} EE^T) P_c + \epsilon_c(\delta_c I_n + C^T C + Q_c) = 0 \tag{3.10}$$

$$\begin{aligned}(A_0 + EG)P_0 + P_0(A_0 + EG)^T - \tfrac{1}{\epsilon_0} P_0 (C^T C - \tfrac{1}{\gamma^2} K^T K - \tfrac{\delta_0}{\gamma^2} I_n) P_0\\ + \epsilon_0 (\tfrac{\gamma^2}{\delta_0} A_1 A_1{}^T + EE^T + Q_0) = 0\end{aligned} \tag{3.11}$$

can be solved for symmetric positive semidefinite matrices P_c and P_0. The controller and observer gains can be calculated using following equations,

$$K = \frac{1}{\epsilon_c} B^T P_c, \quad G = \frac{1}{\gamma^2 \epsilon_c} E^T P_c, \quad L = \frac{1}{\epsilon_0} P_0 C^T.$$

The value of γ provides an upper bound on H_∞ gain of the closed loop system. Examples are given in chapter 4 to illustrate this method.

3.5 Observers for Time-delay Systems using SOS Approach

In this section, a method to design an observer for time-delay systems using SOS developed by Peet and Gu [33] is presented. The observer dynamics consists of an ODE coupled with an PDE as shown below. Consider the following time-delay system,

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Ew(t) + Bu(t), \\ y(t) &= C_2 x(t), \\ z(t) &= C_1 x(t). \end{aligned} \tag{3.12}$$

where $x(t) \in \mathbb{R}^n$ is system state, $y(t) \in \mathbb{R}^q$ is system output, $u(t) \in \mathbb{R}^m$ is system input, $w(t) \in \mathbb{R}^r$ is the external disturbance and $z(t) \in \mathbb{R}^p$ is the states to be estimated.

The observer designed in [33] has the following form,

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0 \hat{x}(t) + A_1 \hat{\phi}(t,-\tau) + Bu(t) + L_1(C_2\hat{x}(t) - y(t)) + L_2(C_2\phi(t,-\tau) - y(t-\tau)) \\ &\quad + \textstyle\int_{-\tau}^{0} L_3(\theta)(C_2\hat{\phi}(t,\theta) - y(t+\theta))d\theta, \\ \partial_t \hat{\phi}(t,s) &= \partial_s \hat{\phi}(t,s) + L_4(s)(C_2\hat{x}(t) - y(t)) + L_5(s)(C_2\hat{\phi}(t,-\tau) - y(t-\tau)) \\ &\quad + L_6(s)(C_2\phi(t,-\tau) - y(t-\tau)) \\ &\quad + \textstyle\int_{-\tau}^{0} L_7(s,\theta)(C_2\hat{\phi}(t,\theta) - y(t+\theta))d\theta. \end{aligned} \tag{3.13}$$

The ODE in the above equation makes correction to the current state of the system while the PDE makes correction to the previous states of the system. Because of this reason, smaller H_∞ bound can be achieved compared to other observers which makes correction to only the current state of the system.

Theorem 6: [33] Consider the system shown in (3.12). An H_∞ observer of the form shown in (3.13) can be designed for this system if for some positive constant γ, there exist a matrix $P \in \mathbb{R}^{n\times n}$ and polynomials $Q : [-\tau, 0] \to \mathbb{R}^{n\times n}$, $S : [-\tau, 0] \to \mathbb{R}^n$

and $R : [-\tau, 0]^2 \to \mathbb{R}^{n\times n}$, matrices $Z_1, Z_2 \in \mathbb{R}^{n\times q}$ and $Z_3, Z_4, Z_5, Z_6 : [-\tau, 0] \to \mathbb{R}^{n\times q}$ and $Z_7 : [-\tau, 0] \times [-\tau, 0] \to \mathbb{R}^{n\times q}$ such that the following Linear Operator Inequalities are satisfied,

$$\mathcal{P}_{\{P-\epsilon I,Q,R,S-\epsilon I\}} > 0 \quad and \quad \mathcal{P}_{\{T,U,V,W\}} < 0.$$

where operators $\mathcal{P}$ and $\mathcal{Z}$ are defined as below,

$$\mathcal{P} = (P_{\{P,Q,R,S\}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix})(s) = \begin{bmatrix} Px_1 + \int_{-\tau}^{0} Q(\theta)x_2(\theta)d\theta \\ \tau(Q(s)^T x_1 + S(s)x_2(s) + \int_{-\tau}^{0} R(s,\theta)x_2(\theta)d\theta) \end{bmatrix} \tag{3.14}$$

$$(\mathcal{Z}\mathrm{y})(s) = \begin{bmatrix} Z_1 y_1 + Z_2 y_2(-\tau) + \int_{-\tau}^{0} Z_3(\theta)y_2(\theta)d\theta \\ Z_4(s)y_1 + Z_5(s)y_2(-\tau) + Z_6(s)y_2(s) + \int_{-\tau}^{0} Z_7(s,\theta)y_2(\theta)d\theta) \end{bmatrix} \tag{3.15}$$

and the matrix T and polynomials U, V and W are given by following equations,

$$T = \begin{bmatrix} 0 & * & * \\ 0 & PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & * \\ 0 & {A_1}^T P - Q(-\tau)^T & -S(-\tau) \end{bmatrix} + \begin{bmatrix} -\frac{\gamma}{\tau} I & * & * \\ -PB & \frac{1}{\gamma\tau}{C_1}^T C_1 + Z_1 C_2 + {C_2}^T {Z_1}^T + \epsilon I & * \\ 0 & {C_2}^T {Z_2}^T & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} -B^T Q(s) \\ {C_2}^T Z_4(s)^T + Z_3(s)C_2 \\ {C_2}^T Z_5(s)^T \end{bmatrix} + \begin{bmatrix} 0 \\ {A_0}^T Q(s) + R(s,0)^T - \dot{Q}(s) \\ {A_1}^T Q(s) - R(s,-\tau)^T \end{bmatrix}$$

$$V = -\dot{S}(s) + Z_6(s)C_2 + {C_2}^T Z_6(s)^T + \frac{\epsilon}{\tau} I$$

$$W = -R_\theta(s,\theta) - R_s(\theta,s)^T + Z_7(s,\theta)C_2 + {C_2}^T Z_7(\theta,s)^T.$$

The value of γ provides an upper bound on H_∞ gain of the observation error.

If the above theorem is satisfied, the observer gain $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ can be calculated which is defined as below [33],

$$(\mathcal{L}\mathrm{y})(s) = \begin{bmatrix} L_1 y_1 + L_2 y_2(-\tau) + \int_{-\tau}^{0} L_3(\theta) y_2(\theta) d\theta \\ L_4(s) y_1 + L_5(s) y_2(-\tau) + L_6(s) y_2(s) + \int_{-\tau}^{0} L_7(s,\theta) y_2(\theta) d\theta) \end{bmatrix} \tag{3.16}$$

where $L_1, L_2 \in \mathbb{R}^{n\times q}$ and $L_3, L_4, L_5, L_6 : [-\tau, 0] \to \mathbb{R}^{n\times q}$ and $L_7 : [-\tau, 0]^2 \to \mathbb{R}^{n\times q}$.

In order to calculate the observer gains, we should calculate the inverse of the operator $\mathcal{P}$. The expression for $\mathcal{P}^{-1}$ can be written using the following lemma,

Lemma 3: [33] Consider the operator $\mathcal{P}$ in (3.14). For some Z, if $Q(s) = HZ(s)$ and $R(s,\theta) = Z(s)^T \Gamma Z(\theta)$ then the inverse of the operator $\mathcal{P}$ can be written as,

$$\mathcal{P}^{-1} = \begin{bmatrix} \hat{P} x_1 + \frac{1}{\tau} \int_{-\tau}^{0} \hat{Q}(\theta) x_2(\theta) d\theta \\ \hat{Q}(s)^T x_1 + \frac{1}{\tau} \hat{S}(s) x_2(s) + \frac{1}{\tau} \int_{-\tau}^{0} \hat{R}(s,\theta) x_2(\theta) d\theta \end{bmatrix} \tag{3.17}$$

where

$$K = \int_{-\tau}^{0} Z(s) S^{-1}(s) Z(s)^T ds,$$

$$T = (I + K\Gamma - KH^T P^{-1} H)^{-1},$$

$$\hat{H} = -P^{-1} HT, \quad \hat{P} = [I + P^{-1} HTKH^T] P^{-1},$$

$$\hat{\Gamma} = [T^T H^T P^{-1} H - \Gamma](I + K\Gamma)^{-1},$$

$$\hat{Z}(s) = Z(s) S^{-1}(s), \quad \hat{Q} = \hat{H} \hat{Z}(\theta)$$

$$\hat{S}(s) = S(s)^{-1}, \quad \hat{R}(s,\theta) = \hat{Z}(s) \hat{\Gamma} \hat{Z}(\theta)$$

Using Equations (3.17) and (3.15), observer gains in (3.16) can be written as [33],

$$L_1 = \hat{P}Z_1 + \int_{-\tau}^{0} \hat{Q}(\theta)Z_4(\theta)d\theta, \quad L_2 = \hat{P}Z_2 + \int_{-\tau}^{0} \hat{Q}(\theta)Z_5(\theta)d\theta,$$

$$L_3(\theta) = \hat{P}Z_3(\theta) + \hat{Q}Z_6(\theta) + \int_{-\tau}^{0} \hat{Q}(s)Z_7(s,\theta)d\theta,$$

$$L_4(s) = \hat{Q}(s)Z_1 + \hat{S}(s)Z_4(s) + \int_{-\tau}^{0} \hat{R}(s,\theta)Z_4(\theta)d\theta,$$

$$L_5(s) = \hat{Q}(s)Z_2 + \hat{S}(s)Z_5(s) + \int_{-\tau}^{0} \hat{R}(s,\theta)Z_5(\theta)d\theta, \quad L_6(s) = \hat{S}(s)Z_6(s),$$

$$L_7(s,\theta) = \hat{Q}(s)Z_3(\theta) + \hat{S}(s)Z_7(s,\theta) + \hat{R}(s,\theta)Z_6(\theta) + \int_{-\tau}^{0} \hat{R}(s,\xi)Z_7(\xi,\theta)d\xi.$$

Chapter 4

OBSERVER IMPLEMENTATION METHODS

In this chapter, various methods that can be used to implement observers for time-delay systems are proposed. A simple time-delay system is used to explain the methods but they can easily be extended to complex systems such as the one shown in the equation (3.13).

Consider a linear time-delay system,

$$\tfrac{d}{dt}x(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t), \quad t \geq 0, \tag{4.1}$$

As shown in section (1.2), equivalent ODE coupled PDE for the above system can be written as,

$$\begin{aligned} \tfrac{d}{dt}\phi(t,0) &= A_0\phi(t,0) + A_1\phi(t,-\tau) + Bu(t), \quad t \geq 0 \\ \tfrac{\partial}{\partial t}\phi(t,s) &= \tfrac{\partial}{\partial s}\phi(t,s), \quad t \geq 0 \\ \phi(t,0) &= x(t), \quad s \in [-\tau, 0]. \end{aligned} \tag{4.2}$$

The simplest method to solve these equations numerically is by using Forward Time-Backword Space (FTBS) method to solve the PDE [8],

$$\begin{aligned} \frac{\phi(t_{i+1}, s_j) - \phi(t_i, s_j)}{\delta t} &= \frac{\phi(t_i, s_j) - \phi(t_i, s_j - 1)}{\delta s} \\ \phi(t_{i+1}, s_j) &= \phi(t_i, s_j) + \frac{\delta t}{\delta s}(\phi(t_i, s_j) - \phi(t_i, s_j - 1)). \end{aligned} \tag{4.3}$$

Solving the PDE using this method is computationally very fast. However, numerical errors will be high as we are using first order approximation for both spatial and temporal derivative. One method to increase the numerical accuracy is to use higher order approximation in time as shown below.

The PDE in above equation can be discretized at N spatial points as shown below.

$$\tfrac{d}{dt}\phi(t,s)|_{s=s_j} = \tfrac{\phi(t,s_j)-\phi(t,s_j-1)}{\delta s}, \quad j = 1,2,3,...,N-1, \tag{4.4}$$

$$\tfrac{d}{dt}\phi(t,s)|_{s=s_N} = A_0\phi(t,0) + A_1\phi(t,-\tau) + Bu(t).$$

where $s \in [0,-\tau]$ and $j = 1$ corresponds to $s = 0$ and $j = N$ corresponds to $s = -\tau$. The system above can be reformulated as,

$$\frac{d}{dt}y(t) = f(y(t),t). \tag{4.5}$$

Where,

$$y(t) = \begin{bmatrix} \phi(t,s_N) \\ \phi(t,s_{N-1}) \\ . \\ . \\ \phi(t,s_1) \end{bmatrix}, \quad f(y(t),t) = \begin{bmatrix} \frac{\phi(t,s_N)-\phi(t,s_{N-1})}{\delta s} \\ \frac{\phi(t,s_{N-1})-\phi(t,s_{N-2})}{\delta s} \\ . \\ . \\ A_0\phi(t,s_1) + A_1\phi(t,s_N) \end{bmatrix}.$$

This system of equations can be solved using 4^{th} order Runge-Kutta method which is explained below [8].

$$k_1 = f(y(t_i),t_i)$$

$$k_2 = f(y(t_i) + k_1\frac{h}{2}, t_i + \frac{h}{2})$$

$$k_3 = f(y(t_i) + k_2\frac{h}{2}, t_i + \frac{h}{2})$$

$$k_4 = f(y(t_i) + k_3 h, t_i + h)$$

$$y(t_i + h) = y(t_i) + (\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4)h \tag{4.6}$$

Here, h is the time-step. The accuracy of this method is higher as we are using 4^{th} order approximation in time. The same method can be extended to the estimator dynamics shown in (3.13).

Another implementation method is to convert the TDS into an equivalent LTI system. The method is explained below.

Consider a linear TDS,

$$\tfrac{d}{dt}x(t) = A_0 x(t) + x(t-\tau) + Bu(t), \quad t \geq 0, \tag{4.7}$$

The equation above can be discretized by using the same method used in (4.4) to generate the system of N ODEs. The system shown in (4.4) can be written as a simple matrix differential equation,

$$\frac{d}{dt}\begin{bmatrix} \phi_N \\ \phi_{N-1} \\ . \\ . \\ \phi_2 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{ds}I_n & -\frac{1}{ds}I_n & 0 & 0 & . & 0 \\ 0 & \frac{1}{ds}I_n & -\frac{1}{ds}I_n & 0 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & \frac{1}{ds}I_n & -\frac{1}{ds}I_n \\ A_1 & 0 & . & . & 0 & A_0 \end{bmatrix} \begin{bmatrix} \phi_N \\ \phi_{N-1} \\ . \\ . \\ \phi_2 \\ \phi_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ B \end{bmatrix} u(t) \tag{4.8}$$

Here, ϕ_1 corresponds to $x(t)$ and ϕ_N corresponds to $x(t-\tau)$. This form is equivalent to simple LTI state-space form $\dot{x}(t) = A_{sys}x(t) + B_{sys}u(t)$. The same approach can be extended to observer shown in (3.13). We can also combine LTI form of system and estimator dynamics which can be extended to generate a similar LTI form for the Observer based Controller.

Chapter 5

SIMULATION RESULTS AND COMPARISON OF THE OBSERVERS

In this chapter, theorems explained earlier have been used to design observers for various stable and unstable time-delay systems. H_∞ gains in estimation error are calculated numerically and performances of observers are compared.

Example 1: In this example, we consider an unstable system

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-0.3) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = [0 \quad 7]x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Using Theorem 1, we get $\gamma = 0.021$ for $\epsilon = 0.01$. Applying LMI approach in Theorem 2, we get $\gamma = 0.349$. Applying SOS approach in Theorem 5, we get $\gamma = 0.0062$. The input $w(t) = 10sinc(5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0026 while calculated H_∞ gains for observers in Theorem 1 and Theorem 2 are 0.0061 and 0.1758 respectively. Simulations of the error dynamics for these observers are shown in Figure 5.1-5.3. Comparison of State estimation errors of all the observers is shown in Figure 5.4.

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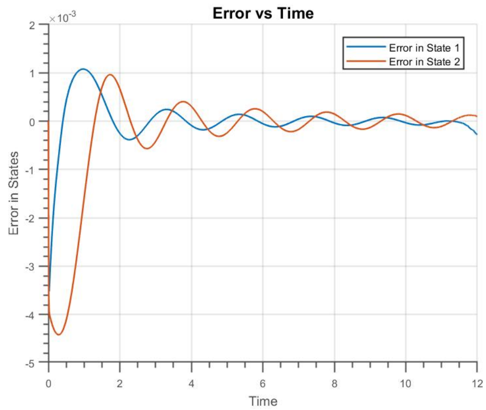

Figure 5.1: State Estimation Error for System in Example 1 Using Our Method.

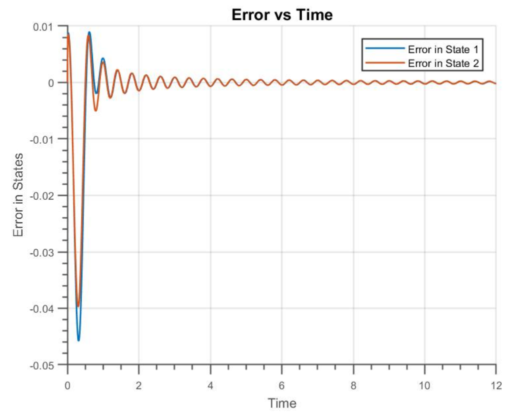

Figure 5.2: State Estimation Error for System in Example 1 Using Fattouh's Method.

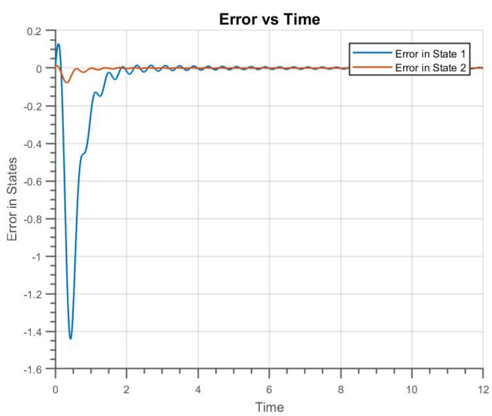

Figure 5.3: State Estimation Error for System in Example 1 Using Fridman's Method.

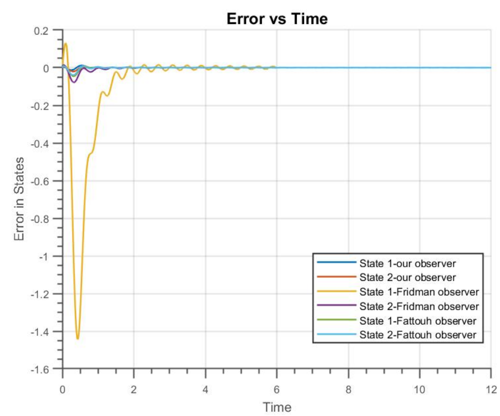

Figure 5.4: Comparison of State Estimation Errors for the System in Example 1.

Example 2: In this example, we consider an unstable system

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-0.3) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t)$$

$$y(t) = [0 \quad 7]x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

In the previous example, $E = [1 \quad 1]^T$, which means that both the states are affected equally due to the disturbance. However, this approach is not realistic. In this example, E is chosen such that both the states are affected differently due to the disturbances. Using Theorem 1, we get $\gamma = 0.58$ for $\epsilon = 0.01$. Applying SOS approach in Theorem 5, we get $\gamma = 0.236$. The input $w(t) = 10sinc(5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0474 while calculated H_∞ gain for the observer in Theorem 1 is 0.0617. Simulations of the error dynamics for these observers are shown in Figure 5.5-5.6.

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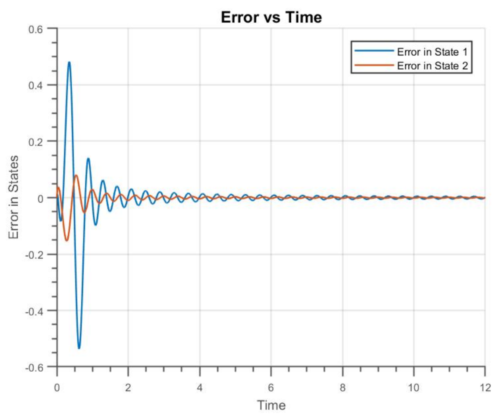

Figure 5.5: State Estimation Error for System in Example 2 Using Our Method.

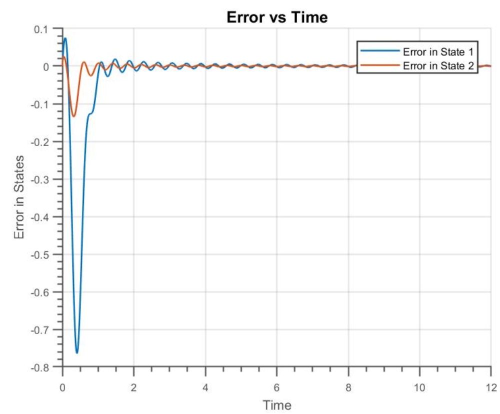

Figure 5.6: State Estimation Error for System in Example 2 Using Fattouh's Method.

Example 3: Consider an unstable system

$$\dot{x}(t) = \begin{bmatrix} -10 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x(t-0.3) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = [0 \quad 10]x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Using Theorem 1, we get $\gamma = 0.025$ for $\epsilon = 0.01$. Applying LMI approach in Theorem 2, we get $\gamma = 0.1275$. Applying SOS approach in Theorem 5, we get $\gamma = 0.0065$. The input $w(t) = 10sinc(5(t - \tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0024 while calculated H_∞ gains for observers in Theorem 1 and Theorem 2 are 0.0115 and 0.0792 respectively. Simulations of the error dynamics for these observers are shown in Figure 5.7-5.9.

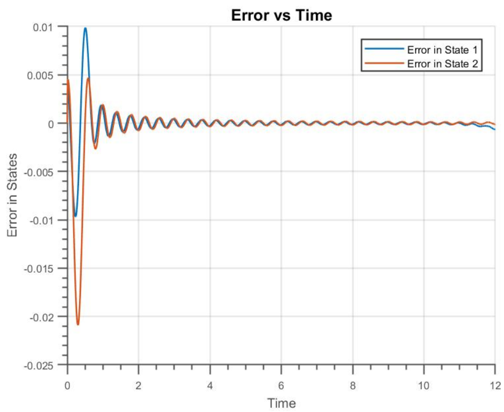

Figure 5.7: State Estimation Error for System in Example 3 Using Our Method.

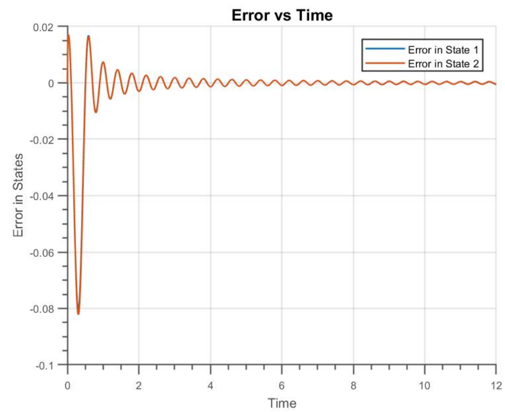

Figure 5.8: State Estimation Error for System in Example 3 Using Fattouh's Method.

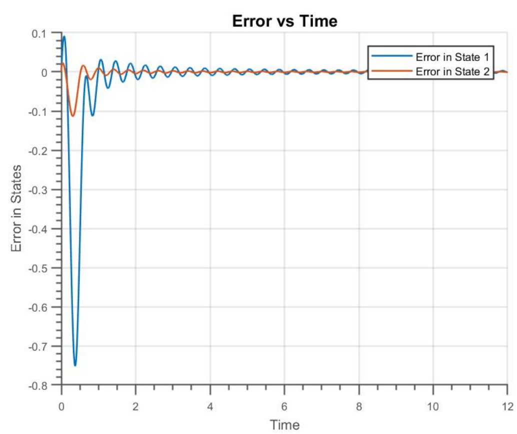

Figure 5.9: State Estimation Error for System in Example 3 Using Fridman's Method.

Example 4: Consider a stable system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.1 \\ -0.2 & -0.3 \end{bmatrix} x(t-0.5) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Riccati approach in Theorem 1 and LMI approach in Theorem 2 were infeasible for any value of gain. Briat's method (Theorem 3) obtained a stable observer for $\gamma = 0.2235$. Applying conditions in Theorem 5, we get $\gamma = 0.125$. The input $w(t) = 10sinc(5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0955 while calculated H_∞ gains for observer in Theorem 3 is 0.145. Simulation of the error dynamics for this observer is shown in Figure 5.10-5.11.

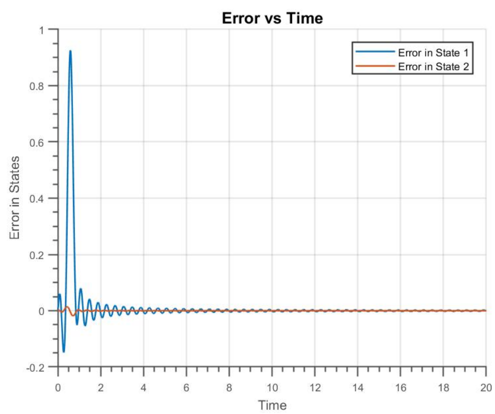

Figure 5.10: State Estimation Error for System in Example 4 Using Our Method.

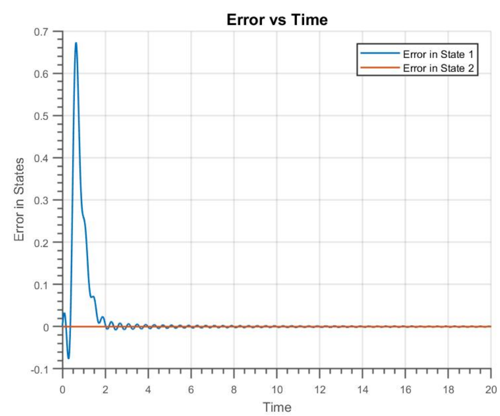

Figure 5.11: State Estimation Error for System in Example 4 Using Briat's Method.

Example 5: Consider an unstable system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -\frac{\pi}{2} & 0 \\ 0 & 0 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = [0 \quad 1]x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Using Theorem 1, we get $\gamma = 1.8995$ for $\epsilon = 0.01$. LMI approach in Theorem 2 was infeasible for any γ. Applying SOS approach in Theorem 5, we get an $\gamma = 0.003$. The input $w(t) = 10sinc(5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.000971 while calculated H_∞ gain for observer in Theorem 1 is 0.8092. Simulations of the error dynamics for these observers are shown in Figure 5.12-5.13.

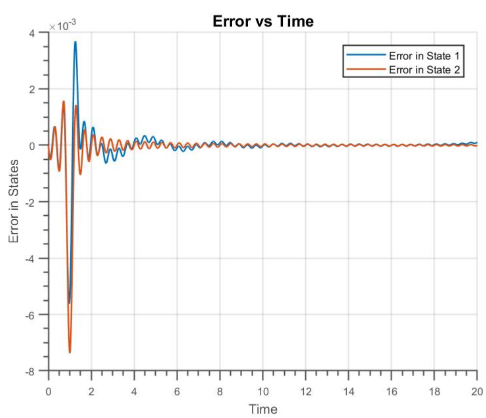

Figure 5.12: State Estimation Error for System in Example 5 Using Our Method.

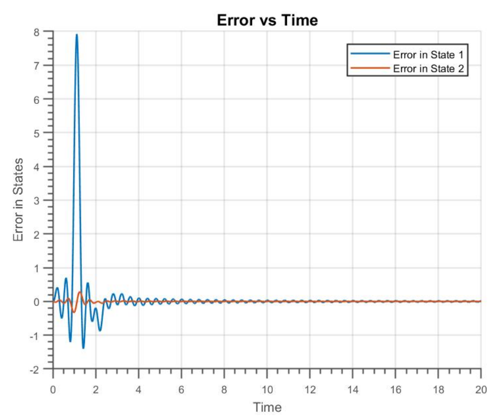

Figure 5.13: State Estimation Error for System in Example 5 Using Fattouh's Method.

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Example 6: Consider a stable system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = [1 \quad 0]x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Using Theorem 1, we get $\gamma = 0.3223$ for $\epsilon = 0.01$. Applying LMI approach in Theorem 2, we get an L_2-gain of $\gamma = 2929$. Applying SOS approach in Theorem 5, we get $\gamma = 0.0015$. The input $w(t) = 10sinc(5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.00078 while calculated H_∞ gains for observers in Theorem 1 and Theorem 2 are 0.1019 and 0.1758 respectively. Simulations of the error dynamics for these observers are shown in Figure 5.14-5.16.

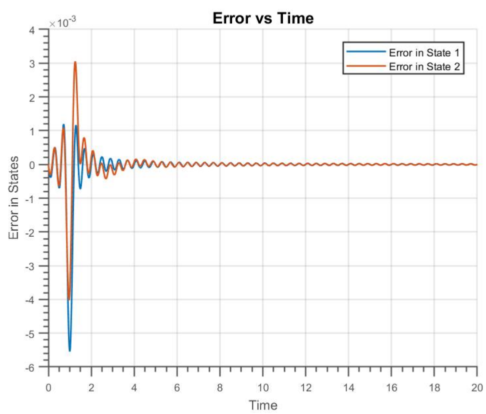

Figure 5.14: State Estimation Error for System in Example 6 Using Our Method.

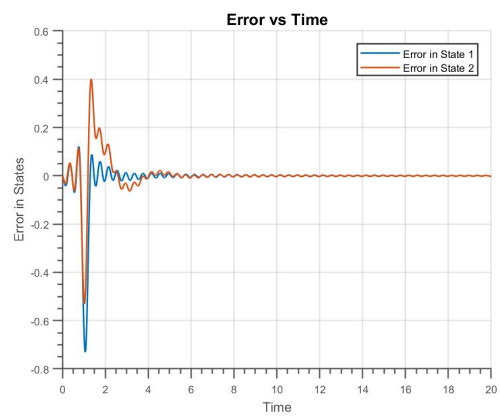

Figure 5.15: State Estimation Error for System in Example 6 Using Fattouh's Method.

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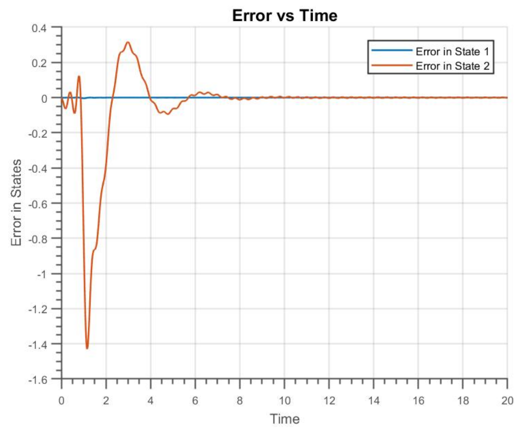

Figure 5.16: State Estimation Error for System in Example 6 Using Fridman's Method.

Example 7: Consider a stable system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} x(t-0.3) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = [1 \quad 0]x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Using Theorem 1, we get $\gamma = 1.785$ for $\epsilon = 0.01$. Applying LMI approach in Theorem 2, we get $\gamma = 0.855$. Applying SOS approach in Theorem 5, we get $\gamma = 0.006$. The input $w(t) = 10sinc(5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0017 while calculated H_∞ gains for observers in Theorem 1 and Theorem 2 are 0.5416 and 0.3402 respectively. Simulations of the error dynamics for these observers are shown in Figure 5.17-5.19.

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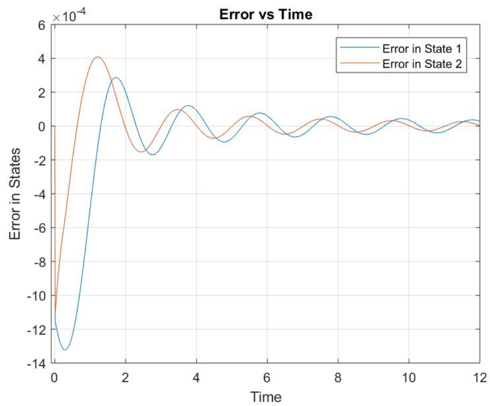

Figure 5.17: State Estimation Error for System in Example 7 Using Our Method.

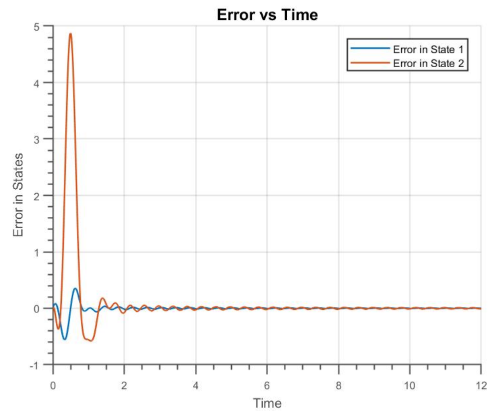

Figure 5.18: State Estimation Error for System in Example 7 Using Fattouh's Method.

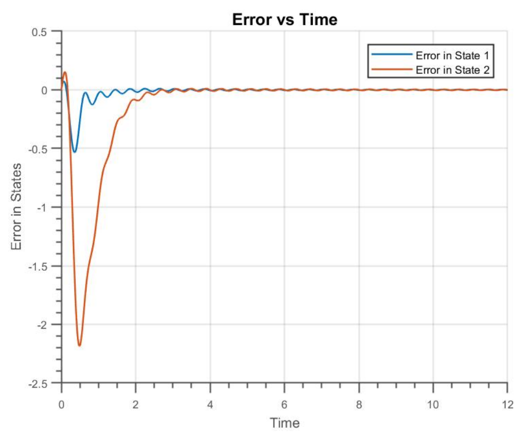

Figure 5.19: State Estimation Error for System in Example 7 Using Fridman's Method.

Example 8: Consider an unstable system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} x(t-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Riccati approach in Theorem 1 was infeasible for any γ. Applying LMI approach in Theorem 2, we get $\gamma = 22.37$. Applying SOS approach in Theorem 5, we get $\gamma = 0.0055$. The input $w(t) = 10sinc(5(t - \tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0016 while calculated H_∞ gain for observer in Theorem 2 is 0.5522. Simulations of the error dynamics for these observers are shown in Figure 5.20-5.21.

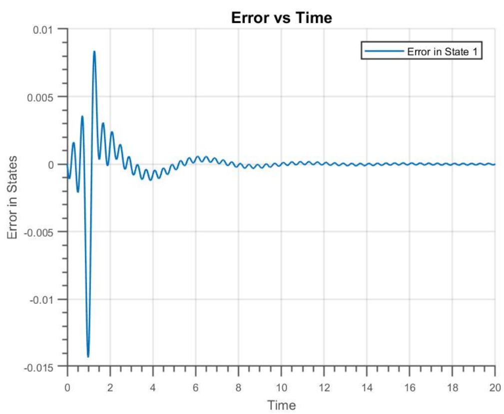

Figure 5.20: State Estimation Error for System in Example 8 Using Our Method.

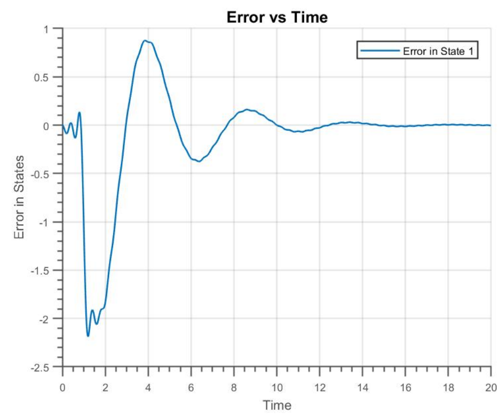

Figure 5.21: State Estimation Error for System in Example 8 Using Fridman's Method.

Simulation of Observer-based Controller: As the main objective to design an observer is to implement a full state feedback control loop, it is important to study the system behavior when the observer-based controller is implemented. The Controller is designed using equation (3.10) and combined with the observer shown in (3.13) to generate a suboptimal observer-based controller. The performance of this closed-loop system is compared with the performance of observer-based controller designed using theorem 4.

Example 9: Consider the unstable system in Example 1. Applying Theorem 4 to design an optimal observer-based controller, we get $\gamma = 1.489$. Using equation (3.10) to design the controller we get $\gamma = 0.66261$ and using Theorem 6 to design the observer we get $\gamma = 0.00625$. A suboptimal observer-based controller can be designed by combining these controller and observer gains. The input $w(t) = 10sinc(0.5(t-\tau))$ and the numerically calculated H_∞ gain of the SOS observer is 0.0023 while calculated H_∞ gain for observer in Theorem 4 is 0.0233. Simulations of the system dynamics and error dynamics for these observer-based controllers are shown in Figure 5.22-5.25.

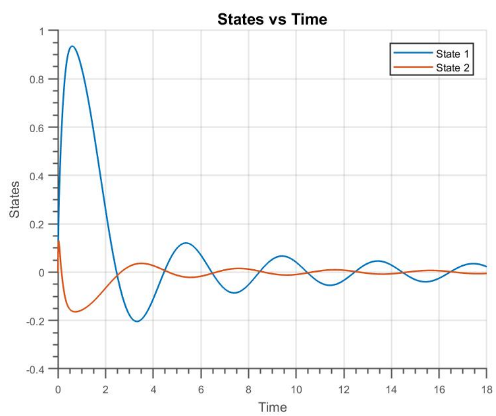

Figure 5.22: States for the Closed Loop System in Example 9 Using Our Method.

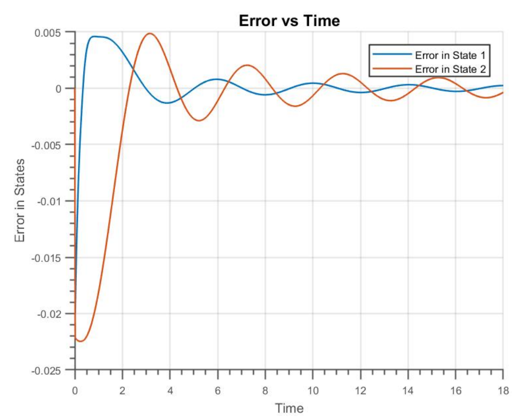

Figure 5.23: State Estimation Error for the Closed Loop System in Example 9 Using Our Method.

Example 10: Consider the unstable system in Example 3. Applying Theorem 4 to design an optimal observer-based controller, we get $\gamma = 1.41$. Using equation (3.10) to design the controller we get $\gamma = 0.59$ and using Theorem 6 to design the observer we get $\gamma = 0.004$. Suboptimal observer-based controller can be designed by combin-

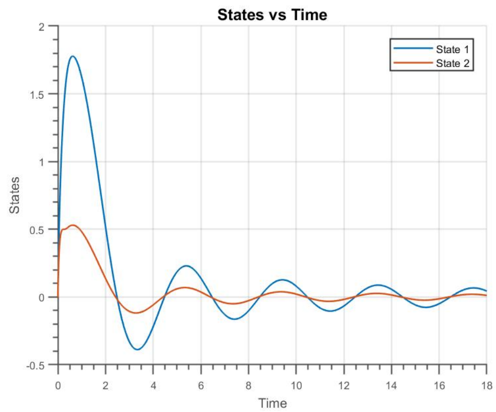

Figure 5.24: States for the Closed Loop System in Example 9 Using Choi and Chung's Method.

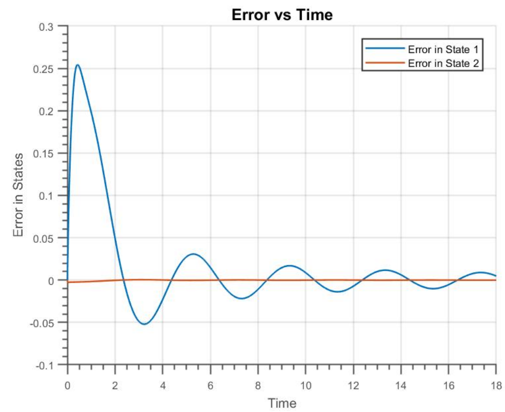

Figure 5.25: State Estimation Error for the Closed Loop System in Example 9 Using Choi and Chung's Method.

ing these controller and observer gains. The input $w(t) = 10sinc(0.5(t-\tau))$ and the numerically calculated H_∞ gain for the SOS observer is 0.0013 while calculated H_∞ gain for observer in Theorem 4 is 0.0045. Simulations of the system dynamics and error dynamics for these observer-based controllers are shown in Figure 5.26-5.29.

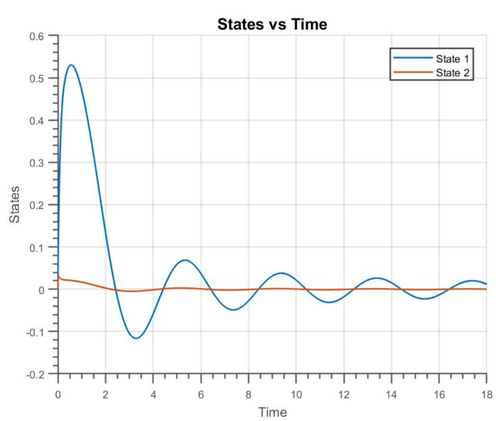

Figure 5.26: States for the Closed Loop System in Example 10 Using Our Method.

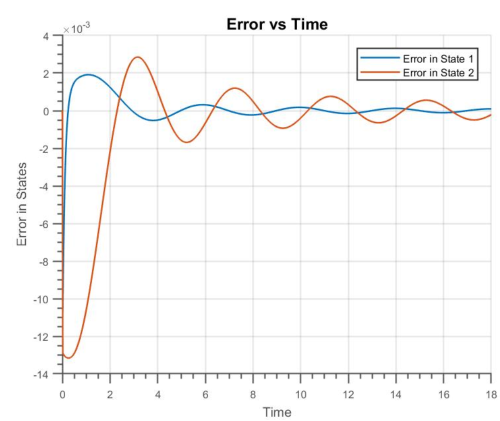

Figure 5.27: State Estimation Error for the Closed Loop System in Example 10 Using Our Method.

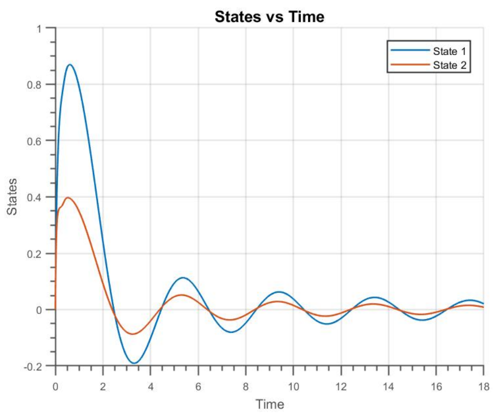

Figure 5.28: States for the Closed Loop System in Example 10 Using Choi and Chung's Method.

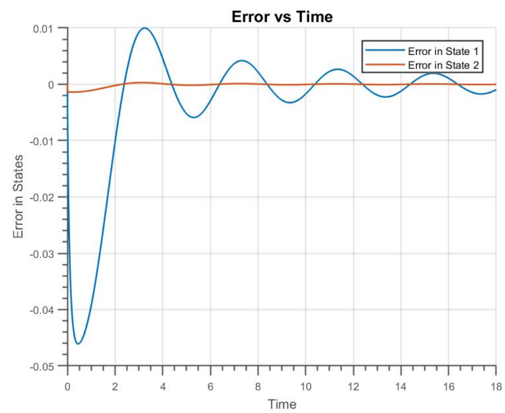

Figure 5.29: State Estimation Error for the Closed Loop System in Example 10 Using Choi and Chung's Method.

Chapter 6

SUMMARY AND CONCLUSION

In this thesis, methods to implement observers for time-delay systems are shown. Various observers for time-delay systems are implemented and compared with each-other and the results are shown.

A. Summary

Introduction to observers and time-delay systems are given in chapter 1. Definition of observability for LTI systems is listed. Delay-differential equations are briefed and a method to solve them is shown.

Brief introduction to signal and system norms is given in chapter 2. Linear Matrix Inequalities are introduced and LMI optimization problems widely used in the control theory are also summarized. Sum of squares method as a convex optimization problem is briefed.

In chapter 3, relevant literature is reviewed as an introduction to current research in the area of observation of time-delay systems. Various H_∞ optimal observers are listed and equations to calculate observer gains are given.

In chapter 4, methods to implement the observers for time-delay systems are illustrated and various observers are implemented and compared using these methods in chapter 5.

B. Conclusion

The main purpose of this thesis was to measure the performance of the observer presented in [33] with existing H_∞ optimal observers. Simulation results clearly show that very small H_∞ bounds can be achieved using this observer. Different signals are used as a disturbance and numerically calculated H_∞ gains follow the

H_∞ bound calculated using theorem 6. Simulations of error dynamics for various stable and unstable systems are presented and the data shows that the observer shown in [33] performs better than other H_∞ observers. For some of the systems we considered, other methods to calculate observer gains were infeasible. However, we were able to calculate observer gains for all the systems we tested on using the method shown in [33]. Sub-optimal observer-based state feedback system is also generated and simulated. The simulation results show that the closed loop system converges very quickly, and the observer can be used to design full state-feedback closed loop system.

References

1. D. Luenberger, “Observing the state of a linear system,” IEEE Transactions on Military Electronics, vol. 8, no. 2, pp. 74-80, 1964.

2. D. Luenberger, “Observers for multivariable systems,” IEEE Transactions on Automatic Control, vol. 11, no. 2, pp. 190-197, 1966.

3. D. Luenberger, “An introduction to observers,” IEEE Transactions on Automatic Control, vol. 16, no. 6: pp. 596-602, 1971.

4. J. Bellon, “Riccati equations in optimal control theory,” MS thesis, Georgia State University, 2008.

5. D. Bini, B. Iannazzo, B. Meini, “Numerical solution of algebraic Riccati equations,” Society for Industrial and Applied Mathematics, Philadelphia, 2012.

6. E. Fridman, “Introduction to time-delay system,” Birkhauser Basel, Boston, 2014.

7. J. Richard, “Time-delay systems: an overview of some recent advances and open problems,” Automatica, vol. 39, no. 10, 16671694, 2003.

8. J. Hoffman, “Numerical methods for engineers and scientists,” Marcel Dekker, New York, 2001.

9. B. Razumikhin, “On the stability of systems with a delay,” Prikl Math Mech, vol. 20, pp. 500-512, 1956.

10. E. Fridman, “Stability of systems with uncertain Delays: A new complete Lyapunov-Krasovskii functional,” IEEE Transactions on Automation Control, vol. 51, no. 5, pp. 885-890, 2006.

11. P. Pepe, and Z. Jiang, “A Lyapunov-Krasovskii methodology for ISS of time-delay,” Proceedings of the 44th IEEE Conference on Decision and Control, pp. 5782-5788, 2005.

12. A. Papachristodoulou, M. Peet, and S. Lall, “Constructing Lyapunov-Krasovskii functionals For linear time delay systems,” Proceedings of the American Control Conference, pp. 2845-2850, 2005,.

13. P. Liu, “On stability of neutral-type uncertain systems with multiple delays,” International Journal of Applied Mathematics and Computer Science, vol. 15, no. 2, 221-229, 2005.

14. A. Morse, “Ring models for delay differential systems,” Automatica, vol. 12, no. 5, 529-531, 1976.

15. A. Germani, C. Manes, and P. Pepe, “A new approach to state observation of nonlinear systems with delayed-outputs,” IEEE Transactions on Automatic Control, vol. 47, no. 1, pp. 96101, 2002.

16. E. Lee and A. Olbrot, “Observability and related structural results for linear hereditary systems,” International Journal of Control, vol. 34, no. 6, 1061-1078, 1981.

17. A. Fattouh, O. Sename, and J. Dion, “Robust observer design for time-delay systems: A Riccati equation approach,” Kybernetika, vol. 35, no. 6, 753-764, 1999.

18. C. Shah, “On the modeling of time-varying delays” MS Thesis, Texas A&M University, 2004.

19. S. Prajna, A. Papachristodoulou and P. Parrilo, “Introducing sostools: a general purpose sum of squares programming solver,” Proceedings of 41^{st} of IEEE Conference on Decision and Control, vol. 1, pp. 741746, 2002.

20. O. Sename, “New trends in design of observers for time-delay systems,” Kybernetika, vol. 37, no. 4, pp. 427-458, 2001.

21. K. Bhat, H. Koivo, “An observer theory for time-delay systems,” IEEE Transactions on Automatic Control, vol. 21, no. 2, pp. 266-269, 1976.

22. R. Kalman, “A linear approach to linear filtering and prediction,” Journal of Basic Engineering(ASME), Vol. 82, pp. 35-45, 1960.

23. P. Muralidhar, “Observer synthesis for linear/nonlinear dynamical systems subject to measurement delays,” MS Thesis, The University of Texas at Arlington, 2006.

24. M. Darouach, M. Zasadzinski, and S. J. Xu, “Full-order observers for linear systems with unknown inputs,” IEEE Transactions on Automatic Control, vol. 39, no. 3, pp. 606609, 1994.

25. K. Gu, V. Kharitonov and J. Chen, “Stability of time-delay systems,” Birkhauser Basel, Boston, 2003.

26. R. Bellman and K. Cooke, “Differential-difference equations,” Academic Press, New York, 1963.

27. J. Wiener and J. Hale, “Ordinary and delay differential equations,” Longman Scientific & Technical, New York, 1992.

28. Y. Nesterov and A. Nemirovskii, “Interior-point polynomial algorithm in convex programming,” Society for Industrial and Applied Mathematics, Philadelphia, 1994.

29. R. Toscano, “Structured controllers for uncertain systems,” Springer-Verlag London, 2013.

30. S. Boyd, L. Ghaoui, E. Feron and V. Balakrishnan, “Linear matrix inequality in systems and control theory,” Society for Industrial and Applied Mathematics, Philadelphia, 1994.

31. M. Peet and A. Papachristodoulou, “A converse Sum-of-Squares Lyapunov result with a degree bound,” IEEE Transactions on Automatic Control, vol. 57, no. 9, pp. 2281-2293, 2012.

32. A. Gahlawat, “Analysis and Control of Parabolic Partial Differential Equations with Application to Tokamaks using Sum-of-Squares Polynomials,” Ph.D. Thesis, Illinois Institute of Technology, 2015.

33. M. Peet and K. Gu, “Synthesis of full-state observers for time-delay systems using SOS,” Submitted to the Mathematical Theory of Networks and Systems, 2018.

34. P. Pepe, “Approximated delayless observers for a class of nonlinear time delay systems,” In Proceedings of 40th IEEE Conference on Decision & Control, 2001.

35. J. OReilly, “Observer for linear systems,” Academic Press, New York, 1983.

36. P. Frank, “Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy: a survey and some new results,” Automatica, vol. 26, no. 3, pp. 459-474, 1990.

37. Z. Wang, B. Huang, and H. Unbehauen, “Robust H_∞ observer design of linear time-delay systems with parametric uncertainty,” Systems & Control Letters, vol. 42, no. 4, pp. 303-312, 2001.

38. B. Barmish, A. Galimidi, “Robustness of Luenberger observers: linear systems stabilized via nonlinear control,” Automatica, vol. 22, no. 4, pp. 413-423, 1986.

39. S. Bhattacharyya, “The structure of robust observers,” IEEE Transactions on Automatic Control, vol. 21, no. 4, pp. 581-588, 1976.

40. J. Bongiorno, “On the design of observer for insensitivity to plant parameter variations,” International Journal of Control, vol. 18, no. 3, pp. 597-605, 1973.

41. J. Doyle, G. Stein, “Robustness with observer,” IEEE Transactions on Automatic Control, vol. 24, no. 4, pp. 607-611, 1979.

42. K. Furuta, S. Hara, S. Mori, “A class of systems with the same observer,” IEEE Transactions on Automatic Control, vol. 21, no. 4, pp. 572-576, 1976.

43. A. Hajime, I. Hiroyuki, “Insensitive observers for discrete-time linear systems,”

Automatica, vol. 15, no. 6, pp. 641-651, 1979.

44. T. Mita, "Design of a zero-sensitive observer," International Journal of Control, vol. 22, no. 2, pp. 215-227, 1975.

45. C. Tsui, "On robust observer compensator design," Automatica, vol. 24, no. 5, 687-692, 1988.

46. P. Frank, "Enhancement of robustness in observer-based fault detection," International Journal of Control, vol. 59, no. 4, pp. 955-981, 1994.

47. H. Marquez and C. Diduch, "Sensitivity of failure detection using generalized observers," Automatica, vol. 28, no. 4, pp. 837-840, 1992.

48. H. Li, M. Fu, "Linear matrix inequality approach to robust H_∞ filtering," IEEE Transactions on Signal Processing, vol. 45, no. 9, 2338-2350, 1997.

49. A. Packard, J. Doyle, "The complex structured singular value," Automatica, vol. 29, no. 1, pp. 71-109, 1993.

50. I. Petersen and D. McFarlane, "Optimal guaranteed cost control and filtering for uncertain linear systems," IEEE Transactions on Automatic Control, vol. 39, no. 9, pp. 19711977, 1994.

51. U. Shaked, C. de Souza, "Robust minimum variance filtering," IEEE Transactions on Signal Processing, vol. 43, no. 11, pp. 2474-2483, 1995.

52. E. Fridman and U. Shaked, "A new H_∞ filter design for linear time delay systems," IEEE Transactions on Signal Processing, vol. 49, no. 11, pp. 2839-2843, 2001.

53. E. Fridman, U. Shaked and L. Xie, "Robust H_∞ filtering of linear systems with time varying delay." IEEE Transactions on Automatic Control, vol. 48, no. 1, 159-165, 2003.

54. C. Briat, "Robust control and observation of LPV time-delay systems," Ph.D. Thesis, Institute National Polytechnique de Grenoble-INPG, 2008.

55. H. Choi and M. Chung, "Observer-based H_∞ controller design for state delayed linear systems," Automatica, vol. 32, no. 7, pp. 1073-1075, 1996.

56. R. Gressang, "Observers and pseudo observers for linear time-delay systems," in Proceedings of 12^{th} Allerton Conference Circuit and System Theory, 1974

57. G. Hewer and G. Nazaroff, "Observer theory for delayed differential equations," International Journal of Control, vol. 18, no. 1, pp. 1-7, 1973.

58. E. Lee and A. Manitius, "Synthesis techniques for multivariable feedback sys-

tems with time-delays," IFAC Proceedings, vol. 7, no. 1, 303-313, Sept. 1974.

59. R. Gressang and G. Lamont, "Observers for systems characterized by semi-groups" IEEE Transactions on Automatic Control, vol. 20, no. 4, pp. 523-528, 1975

60. A. Pila, U. Shaked, and C. de Souza, "H_∞ filtering for continuous-time linear systems with delay," IEEE Transactions on Automatic Control, vol. 44, no. 7, pp. 14121417, 1999

61. C. de Souza, R. Palhares, and P. Peres, "Robust H_∞ filtering for uncertain linear continuous-time systems with multiple time-varying state delays: An LMI approach," In Proceedings of at the 38th IEEE Conference on Decision & Control, 1999, pp. 2023-2028

62. R. Palhares, P. Peres, and C. de Souza, "Robust H_∞ filtering for linear continuous-time uncertain systems with multiple delays: An LMI approach," In Proceedings of the 3rd IFAC Conference on Robust Control Design, 2000.

63. H. Zakeri and S. Ozgoli, "A sum of squares approach to robust PI controller synthesis for a class of polynomial multi-input multi-output nonlinear systems," Nonlinear Dynamics, vol. 76, no. 2, pp. 1485-1495, 2014.

64. E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," Systems & Control Letter, vol. 43, no. 4, pp 309-319, 2001.

65. E. Fridman and U. Shaked, "New bounded real lemma representations for time-delay systems and their applications," IEEE Transactions on Automatic Control, vol. 46, no. 12, pp. 1973-1979, 2001.

66. P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays," IEEE Transactions on Automatic Control, vol. 44, no. 4, pp. 876-877, 1999.

67. E. Fridman and U. Shaked, "An improved delay-dependent H_∞ filtering of linear neutral systems," IEEE Transactions on Signal Processing, vol. 52, no. 3, 2004.

68. M. Malek-Zavarei and M. Jamshidi, "Time-Delay Systems: analysis, optimization and applications," Elsevier Science, Amsterdam, 1987.

69. H. Hashemi and C. Leondes, "Observer theory for systems with time delay," International Journal Systems Science, vol. 10, no. 7, pp. 797-806, 1979.

70. B. Ogunnaike, "A new approach to observer design for time-delay systems," International Journal of Control, vol. 33, no. 3, pp. 519-542, 1981.

71. A. Pearson and Y. Fiagbedzi, "An observer for time lag systems," IEEE Trans-

actions on Automatic Control, vol. 34, no. 7, pp. 775-777, 1989.

72. A. Germani, C. Manes, and P. Pepe, "A state observer for nonlinear delay systems," In Proceedings of 37th IEEE Conference on Decision & Control, 1998, pp. 355-360.

73. A. Germani, C. Manes, and P. Pepe, "An observer for m.i.m.o nonlinear delay systems," IFAC 14th World Congress, 1999, pp. 243-248.

74. A. Germani, C. Manes, and P. Pepe, "An asymptotic state observer for a class of nonlinear delay systems," Kybernetika, vol. 37, no. 4, pp. 459-478, 2001.

75. W.Michiels and S. Niculescu, "Stability and stabilization of time-delay systems: An Eigenvalue-based approach", Society for Industrial and Applied Mathematics, 2007.