Available online at www.sciencedirect.com

ScienceDirect

Procedia IUTAM 12 (2015) 31 – 41

Procedia IUTAM

www.elsevier.com/locate/procedia

IUTAM Symposium on Mechanics of Soft Active Materials

Solid Flow Fields and Growth of Soft Solid Mass

David Durban[a], Tal Cohen[b]*, Yannis Dafalias[c,d]

[a]Faculty of Aerospace Engineering, Technion, Haifa, 32000, Israel
[b]Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
[c]Department of Civil and Environmental Engineering, University of California, Davis, CA 95616, U.S.A.
[d] Department of Mechanics, National Technical University of Athens, Greece

Abstract

Recent work[1,2] on growth of actin gel, and earlier studies[3,4] on radial plastic forming processes, while seemingly distinct have in fact much in common by adopting the underlying view of source flow of solid materials. That conceptual framework, of Eulerian formulation of solid flow fields, is examined in the present contribution. We focus on radial patterns, with spherical symmetry in steady state conditions, to model kinematics of growth on a spherical bead. Constitutive response includes the Blatz-Ko hyperelastic solid, the Cauchy-Hookean elastic solid and a simple hypoelastic incompressible material. Useful analytical relations are derived for radial velocity profile, stretches and strains. High circumferential stresses at the external layer, in agreement with findings reported by Dafalias et al.[1,2] using different constitutive models, can possibly induce symmetry breakdown. Growth driving parameters are discussed, including a thermodynamic growth driving force, and thin shell asymptotic formulae are given.

Peer-review under responsibility of Konstantin Volokh and Mahmood Jabareen.

Keywords: Finite elasticity; Growth condition; Actin Gel

1. Background and Motivation

Recent studies[1,2] on the mechanics of growth of actin gel on spherical and cylindrical surfaces (beads) have employed an unorthodox growth condition for kinematic fields that do not admit continuous mapping of initial configuration into current deformed configuration, because the latter consists of mass parts which appeared at different times.

* Corresponding author. Tel.: +1-781-492-1608.
E-mail address: talco@mit.edu

Peer-review under responsibility of Konstantin Volokh and Mahmood Jabareen.
doi:10.1016/j.piutam.2014.12.005

Specifically, actin gel is generated over the bead surface (Fig. 1) uniformly and continuously to form a shell of finite thickness. Material response has been modeled[1,2] as non-linear elastic with complete solutions of the stress field for the Kirchhoff-Saint Venant and Moony-Rivlin hyperelastic solids, including reduction to small strains. These solutions predict very high levels of circumferential stresses at the free outer surface leading to symmetry breaking[5,6] observed during growth of actin gel. It should be mentioned that these solutions are also valid for inwards in regards to the bead surface actin gel growth, assuming of course the bead is hollow[2]. This inwards mass growth is more in-line with the actual phenomenon of actin gel growth in the interior surface of a cell membrane that is subsequently used for membrane deformation and associated cell motility[1,2].

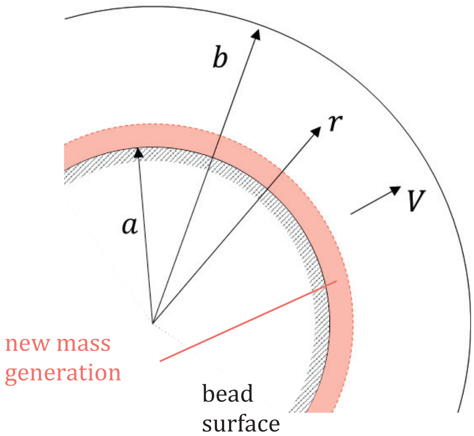

Fig. 1. Growth process over spherical bead $r = a$. New mass is uniformly and continuously generated to form a layer of thickness $h = b - a$, V - radial velocity, r - radial coordinate.

In this work we address the growth process of soft solid mass, like actin gel, within the framework of solid flow fields in steady-state conditions. We follow previous studies[3,4] on radial forming processes of plastic solids and adopt the Eulerian frame of reference to describe material kinematics. Thus, with notation of Fig. 1, mass is generated on the spherical bead $r = a$ and flows radially outward with velocity profile $V(r)$. In steady state condition we have the basic relation, transforming from Lagrangian to Eulerian description,

$$\frac{d()}{dt} = V\frac{d()}{dr} \tag{1}$$

where t stands for the time coordinate.

Assuming spherical symmetry of the deformed field $(a \le r \le b)$ with finite stretches $(\lambda_r, \lambda_\theta)$, the Eulerian strain rates are

$$\frac{\dot{\lambda}_r}{\lambda_r} = \frac{dV}{dr} = V', \quad \frac{\dot{\lambda}_\theta}{\lambda_\theta} = \frac{V}{r} \tag{2}$$

where the superposed dot denotes differentiation with respect to time and the prime represents differentiation with respect to the radial coordinate r. However, in view of (1), relations (2) can be rewritten as

$$V\frac{\lambda_r'}{\lambda_r} = V', \quad V\frac{\lambda_\theta'}{\lambda_\theta} = \frac{V}{r} \tag{3}$$

implying the stretches (with $\lambda_\theta = 1$ at $r = a$)

$$\lambda_r = J_a\frac{V}{V_a}, \quad \lambda_\theta = \frac{r}{a} \tag{4}$$

where $V_a = V(r = a)$ is the onset of growth velocity at the bead surface and $J_a = J(r = a)$ with J defined by

$$J = \lambda_r\lambda_\theta^2 \tag{5}$$

Relations (4) are universally valid for all continuous growth processes, over a spherical bead, regardless of constitutive nature. Useful expressions are provided by the rate of volumetric growth (flux) at $r = a$, which controls the rate of generation of new mass,

$$Q = 4\pi a^2 V_a \tag{6}$$

and by the rate of deformation energy (power) consumed at a generic stage of growth

$$P = 4\pi\int_a^b \boldsymbol{\sigma}\cdot\cdot\boldsymbol{D}r^2 dr \tag{7}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{D}$ is the Eulerian strain rate and $\cdot\cdot$ implies the trace operation.

Radial equilibrium requires that

$$\sigma_r' + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0 \tag{8}$$

where σ_r and σ_θ are the stress components, along with the stress free external boundary

$$\sigma_r(b) = 0 \tag{9}$$

Assuming hyperelastic response with strain energy density function $W(\lambda_1, \lambda_2, \lambda_3)$ where λ_i $(i = 1,2,3)$ are the principle stretches $(\lambda_1 = \lambda_r,\ \lambda_2 = \lambda_3 = \lambda_\theta)$, we have stress components

$$\sigma_r = \frac{1}{\lambda_\theta^2}\left(\frac{\partial W}{\partial \lambda_r}\right) \qquad \sigma_\theta = \frac{1}{\lambda_r \lambda_\theta}\left(\frac{\partial W}{\partial \lambda_\theta}\right) \tag{10}$$

Thus, by (4), for hyperelastic solids both stress components are known functions of the velocity profile $V(r)$, namely

$$\sigma_r = \sigma_r\left(J_a \frac{V}{V_a}, \frac{r}{a}\right) \qquad \sigma_\theta = \sigma_\theta\left(J_a \frac{V}{V_a}, \frac{r}{a}\right) \tag{11}$$

It follows that (8), when combined with (11), provides a single first order nonlinear differential equation for $V(r)$ which should be solved in conjunction with boundary condition (9). For given velocity V_a we find that growth is driven by J_a which is in fact the growth parameter.

Relations similar to (11) apply also to Cauchy elastic solids, where stress components depend on stretches, though no strain energy function exists. Solutions of (8) with (11) can be obtained in usual framework of finite elasticity with λ_r as unknown, via $(4)_2$, and this was the approach followed in[1,2] employing various specific forms of equations (10). Here however we adopt the Eulerian steady flow point of view with the velocity profile *V(r)* instead of λ_r as dependent variable. This formulation paves the way to growth analysis of soft solids that do not admit a reference configuration like hypoelastic materials or solids with a viscous branch. Dynamic effects can be incorporated as well, as with the steady source flow analysis of dynamic cavitation[7].

The view suggested here is illustrated in detail in the next section for the Blatz-Ko material. Elegant analytical relations are derived for the velocity profile and for key parameters. The high level of circumferential stress at the outer surface is discussed with approximation for thin layers. The growth process for a Cauchy-Hookean solid is examined in section 3 followed by a brief discussion on hypoelastic growth with concluding comments in section 4. We have adopted a purely mechanical approach, concentrating on representative constitutive relations to model steady flow of soft solid materials.

2. The Blatz-Ko Material

This is a compressible hyperelastic solid with the strain energy function

$$W = \frac{\mu}{2}\left[\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right) + 2\left(\lambda_1 \lambda_2 \lambda_3 - 1\right)\right] \tag{12}$$

where μ denotes the shear modulus. The stress components (10)-(11) follow as

$$\sigma_r = \mu\left[1-\left(J_a\frac{V}{V_a}\right)^{-3}\left(\frac{a}{r}\right)^2\right] \tag{13a}$$

$$\sigma_\theta = \mu\left[1-\left(J_a\frac{V}{V_a}\right)^{-1}\left(\frac{a}{r}\right)^4\right] \tag{13b}$$

A further substitution of (13) in (8) yields the differential relation

$$\left(J_a\frac{V}{V_a}\right)^{-1} d\left(J_a\frac{V}{V_a}\right)^{-1} = \frac{2}{3}\left(\frac{r}{a}\right)^{-3} d\left(\frac{r}{a}\right)$$

with the solution

$$\frac{V}{V_a} = \left[1+\frac{2}{3}J_a^2\left(1-\frac{a^2}{r^2}\right)\right]^{-1/2} \tag{14}$$

on account of bead condition $V(r=a)=V_a$.

The stress free condition (9) can be rewritten, by (13a), as

$$J_a\frac{V_b}{V_a} = \left(\frac{a}{b}\right)^{2/3} \tag{15}$$

hence, with the aid of (14),

$$J_a = \left[\left(\frac{b}{a}\right)^{4/3} - \frac{2}{3}\left(1-\frac{a^2}{b^2}\right)\right]^{-1/2} \tag{16}$$

which determines the growth relation between J_a and the thickness ratio b/a. Likewise, the strain energy density at onset of growth follows from (12) as

$$W_a = W(r=a) = \frac{\mu}{2}\left(J_a^{-2}+2J_a^{-1}-3\right) \tag{17}$$

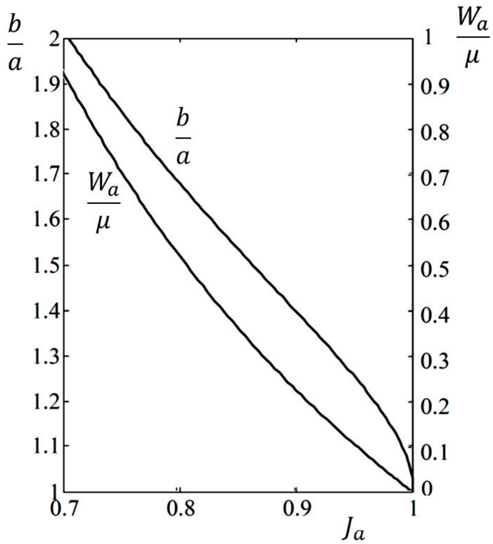

Fig. 2. Variation of thickness ratio b/a and strain energy density W_a at onset of growth with J_a. Blatz-Ko material.

Fig. 2 shows the variation of b/a and W_a with J_a, for the Blatz-Ko hyperelastic material.

For thin layers (Fig.3), of thickness $h=b-a$, we can use the approximations

$$J_a \approx 1-\frac{10}{9}\left(\frac{h}{a}\right)^2 \tag{18a}$$

$$W_a \approx \frac{5}{3}\mu\left(\frac{h}{a}\right)^2 \tag{18b}$$

reflecting the thin layer connection

$$J_a \approx 1-\frac{2W_a}{3\mu} \tag{19}$$

The stress components at $r=a$ are, by (13), approximated as

$$\sigma_r(r=a)=\mu\left(1-J_a^{-3}\right)\approx -\frac{10}{3}\mu\left(\frac{h}{a}\right)^2 \tag{20a}$$

$$\sigma_\theta(r=a)=\mu\left(1-J_a^{-1}\right)\approx -\frac{10}{9}\mu\left(\frac{h}{a}\right)^2 \tag{20b}$$

while at the outer surface $(r = b)$

$$\sigma_r(r = b) = 0, \qquad \sigma_\theta(r = b) \approx \frac{10}{3}\mu\left(\frac{h}{a}\right) \tag{21}$$

It follows that while the radial stress decreases (in absolute value) with distance from the bead surface to $r = b$, the circumferential stress increases considerably, viz

$$\frac{\sigma_\theta(r = b)}{\left|\sigma_\theta(r = a)\right|} \approx 3\frac{a}{h} \tag{22}$$

with a representative value of $h/a = 0.1$ the ratio (22) is 30, indicating the high level of circumferential stress at $r = b$. This high rise of circumferential tension stresses near the free surface, observed already in[3,4], can explain the emergence of symmetry breaking[5,6] in actin gel growth processes.

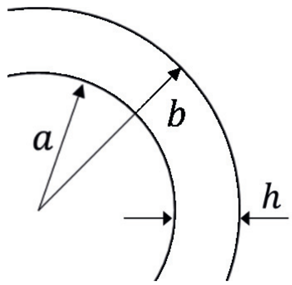

Fig. 3. Thin layer notation $h/a \ll 1$, a is bead radius.

For thick layers we deduce from (13b) and (15) the relation

$$\sigma_\theta(r = b) = \mu\left[1 - \left(\frac{a}{b}\right)^{\frac{10}{3}}\right] \tag{23}$$

which approaches the asymptotic level of shear modulus μ as b/a increases.

3. Cauchy-Hookean Solid

Our next example is that of a finite strain version of a finite strain linear elastic material, namely

$$\boldsymbol{\sigma} = 2\mu \boldsymbol{E}_L + \lambda \ln J \boldsymbol{I} \tag{24}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{E}_L$ is the finite logarithmic strain tensor and

$$\mu = \frac{E}{2(1+\upsilon)} \qquad \lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} \tag{25}$$

are the Lamé type constants with (E, υ) denoting the elastic modulus and Poisson ratio, respectively. Relation (24) describes an elastic material which does not admit a strain energy function. The hyperelastic version of (24), known as the Kirchhoff-Saint Venant material, was used by Dafalias et al.[1,2] to model growth of actin gel.

For spherical fields the logarithmic strains are $(\ln\lambda_r, \ln\lambda_\theta)$ and the constitutive relations are reduced to

$$\sigma_r = 2\mu \ln\left(J_a \frac{V}{V_a}\right) + \lambda \ln\left[J_a \frac{V}{V_a}\left(\frac{r}{a}\right)^2\right] \tag{26a}$$

$$\sigma_\theta = 2\mu \ln\left(\frac{r}{a}\right) + \lambda \ln\left[J_a \frac{V}{V_a}\left(\frac{r}{a}\right)^2\right] \tag{26b}$$

Substituting the stresses (26) in equilibrium equation (8) and solving for the velocity profile, so that (26a) satisfies the stress free condition (9), we find

$$\ln\left(J_a \frac{V}{V_a}\right) = \left(\frac{3\kappa}{4\mu} - \frac{3\kappa}{\kappa + \frac{4}{3}\mu}\ln\frac{b}{a}\right)\left(\frac{b}{r}\right)^{\frac{4\mu}{\kappa+\frac{4}{3}\mu}} \ln\frac{r}{a} - \frac{3\kappa}{4\mu} \tag{27}$$

where

$$\kappa = \frac{E}{3(1-2\upsilon)}$$

is the bulk modulus. The growth relation follows from (27), with $r = a$, in the form

$$\ln J_a = \frac{1+\upsilon}{2(1-2\upsilon)}\left[\left(1-\frac{2(1-2\upsilon)}{1-\upsilon}\ln\frac{b}{a}\right)\left(\frac{b}{a}\right)^{\frac{2(1-2\upsilon)}{1-\upsilon}} - 1\right] \tag{28}$$

The influence of compressibility on the growth relation (28) is apparent from Fig. 4 which shows the variation of J_a with b/a.

The circumferential stress at the outer surface, obtained from (26b) with (28), is given by

$$\sigma_\theta(r=b) = \frac{E}{1-\upsilon}\ln\frac{b}{a} \tag{29}$$

Thus, in common with previous studies[3,4], with increasing thickness, high levels of circumferential stress are likely to cause breaking of spherical symmetry (compare with (23)).

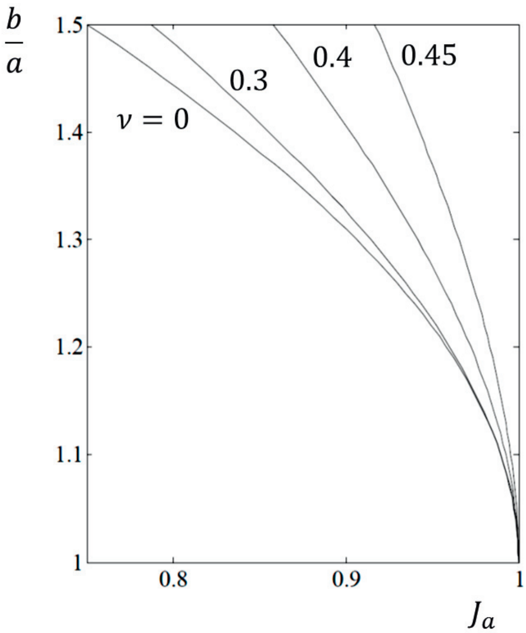

Fig. 4. Growth relations for Cauchy-Hookean solid for different values of Poisson ratio υ.

4. Concluding Remarks

Radial growth of soft solid mass has been analyzed within the Eulerian frame of steady flow. The unorthodox growth condition suggested by Dafalias et al.[1,2] is implemented in the mathematical model with examples given for spherical growth of Blatz-Ko hyperelastic solids and for Cauchy-Hookean elastic solids. Results reveal the parameters which control the process and the sensitivity to material compressibility. Circumferential stresses at the outer surface are very high and can induce symmetry braking.

It has been tacitly assumed that the hyperelastic law (17), as well as the elastic law (24), are valid at onset of growth for $r = a$, connecting stresses with finite strains even though past loading history is not known. This assumption can be relaxed for hypoelastic response which is free of any reference configuration.

Consider as a simple example the incompressible hypoelastic solid

$$\overset{\nabla}{\boldsymbol{S}} = 2\mu \boldsymbol{D} \quad \text{with} \quad \boldsymbol{I} \cdot\cdot \boldsymbol{D} = \boldsymbol{0} \tag{30}$$

where $\boldsymbol{S}$ is the stress deviator, $\overset{\nabla}{()}$ denotes the objective stress rate, $\boldsymbol{D}$ is the Eulerian strain rate and μ is the constant shear modulus. Incompressibility dictates the velocity profile

$$V = V_a \left(\frac{a}{r} \right)^2 \tag{31}$$

and, in absence of spin, the constitutive relation (30) reduces to the single equation

$$\frac{d}{dt}(\sigma_r - \sigma_\theta) = 2\mu \left(V' - \frac{V}{r} \right) \tag{32}$$

or, in view of the steady state condition (1),

$$(\sigma_r - \sigma_\theta)' = 2\mu \left(\frac{V'}{V} - \frac{1}{r} \right) = -\frac{6\mu}{r} \tag{33}$$

by (31). Integrating (33) gives

$$\sigma_r - \sigma_\theta = -6\mu \ln \frac{r}{a} - 2\tau_a \tag{34}$$

where τ_a is the extremal shear stress at $r = a$.

A further substitution of (34) in (8) and integration yields the radial stress profile that complies with (9),

$$\sigma_r = \left(6\mu \ln \frac{br}{a^2} + 4\tau_a \right) \ln \frac{r}{b} \tag{35}$$

which completes the solution for the stress field.

In this analysis, growth is controlled by τ_a regardless of any finite strain kinematics. A possible direction for future study is suggested by considering the power consumed at a generic stage (7)

$$P = 4\pi\int_a^b \boldsymbol{\sigma}\cdot\cdot\boldsymbol{D}r^2 dr = 8\pi\int_a^b (\sigma_\theta - \sigma_r)Vrdr \tag{36}$$

due to incompressibility. Inserting (31) and (34) in (36) we find

$$P = 8\pi a^2 V_a\left(3\mu\ln\frac{b}{a} + 2\tau_a\right)\ln\frac{b}{a} \tag{37}$$

Now, a thermodynamic growth driving force F_g can be defined by $P = F_g V_a$ where

$$F_g = 8\pi a^2\left(3\mu\ln\frac{b}{a} + 2\tau_a\right)\ln\frac{b}{a} \approx 8\pi a^2\left(3\mu\frac{h}{a} + 2\tau_a\right)\frac{h}{a} \tag{38}$$

where the approximation holds for $h << a$. This solution connects the parameter τ_a with the thermodynamic force F_g that drives the growth process.

Work under progress aims at enhancing the approach outlined in the present paper, covering a variety of constitutive families to model steady growth of soft solids. This Eulerian approach with no need of a global reference configuration, the lack of which characterizes the present case, will be proved very useful when inelastic constitutive relations in terms of rates of deformation are employed.

References

1. Dafalias YF, Pitouras Z. Stress field in actin gel growing on spherical substrate. *Biomech. Model. Mechanobiol.* 2009; **8**: 9-24.
2. Dafalias YF, Panayotounakos DE, Pitouras Z. Stress field due to elastic mass growth on spherical and cylindrical substrates. *Int. J. Solids Struct.* 2008; **45**: 4629-4647.
3. Durban D, Budiansky B. Plane-strain radial flow of plastic materials. *J. Mech. Phys. Solids.* 1978; **26**: 303-324.
4. Durban D. Axially symmetric radial flow of rigid/linear-hardening materials. *J. Appl. Mech.* 1979; **46**: 322-328.
5. Sekimoto K, Prost J, Julicher F, Boukellal H, Bernheim-Grosswasser A. Role of tensile stress in actin gels and symmetry breaking instability. *Eur. Phys. J. E.* 2004; **13**: 247-259.
6. Gucht JVD, Plauch E, Plastino J, Sykes C. Stress release drives symmetry breaking for actin based moving. *PNAS* 2005; **102.22**:4847-7852.
7. Cohen T, Masri R, Durban D. Shock waves in dynamic cavity expansion. *J. Appl. Mech.* 2010; **77**: 041009.