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Colloquium: Zoo of quantum-topological phases of matter

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What are topological phases of matter? First, they are phases of matter at zero temperature. Second, they have a nonzero energy gap for the excitations above the ground state. Third, they are disordered liquids that seem to have no feature. But those disordered liquids actually can have rich patterns of many-body entanglement representing new kinds of order. This Colloquium gives a simple introduction and a brief survey of topological phases of matter. First topological phases with topological order (i.e., with long-range entanglement) are discussed. Then topological phases without topological order (i.e., with short-range entanglement) are covered.

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I. ORDERS AND SYMMETRIES

Condensed matter physics is a branch of science that studies various properties of all kinds of materials. Usually for each kind of material, we need a different theory (or model) to explain its properties. After seeing many different types of theories and models for condensed matter systems, a common theme starts to emerge. The common theme is the *principle of emergence*, which states that the properties of a material are mainly determined by how particles are organized in the material. This is quite different from the point of view that the properties of a material should be determined by the components that form the material. In fact, all materials are made of the same three components: electrons, protons, and neutrons. So we cannot use the richness of the components to understand the richness of the materials. The various properties of different materials originate from the various ways in which the particles are organized. The organizations of the particles are called orders. Different orders lead to different phases of matter, which in turn leads to the different properties of materials.

Therefore, according to the principle of emergence, the key to understanding a material is to understand how electrons, protons, and neutrons are organized in the material. Based on a deep insight into phase and phase transition, Landau (1937) developed a general theory of orders as well as transitions between different phases of matter. He pointed out that the reason that different phases (or orders) are different is because they have different symmetries. A phase transition is simply a transition that changes the symmetry. Introducing order parameters that transform nontrivially under the symmetry transformations, Ginzburg and Landau (1950) developed the standard theory for phases and phase transitions, where different phases of matter are classified by a pair of groups ($G_\Psi \subset G_H$). Here G_H is the symmetry group of the system and G_Ψ is the unbroken symmetry group of the equilibrium state.

Landau's theory is very successful. Using symmetry and the related group theory, we can classify all of the 230 different kinds of crystals that can exist in three dimensions. By determining how symmetry changes across a continuous phase transition, we can obtain the critical properties of the phase transition. The symmetry breaking also provides the origin of many gapless excitations, such as phonons, spin waves, etc., which determine the low-energy properties of many systems (Nambu, 1960; Goldstone, 1961). Many of the properties of those excitations, including their gaplessness, are directly determined by the symmetry.

As Landau's symmetry-breaking theory has such a broad and fundamental impact on our understanding of matter, it became a cornerstone of condensed matter theory. The picture painted by Landau's theory is so satisfactory that one starts to have a feeling that we understand, at least in principle, all kinds of orders that matter can have. One feels that we start to see the beginning of the end of condensed matter theory.

II. NEW WORLD OF CONDENSED MATTER PHYSICS

However, through the research in the last 30 years, a different picture starts to emerge. It appears that what we have seen is just the end of the beginning. There is a whole new world ahead of us waiting to be explored. A peek into the new world is offered by the discovery of the fractional quantum Hall (FQH) effect (Tsui, Stormer, and Gossard, 1982). Another peek is offered by the discovery of high T_c superconductors (Bednorz and Mueller, 1986). Both phenomena are completely beyond the paradigm of Landau's symmetry-breaking theory. Rapid and exciting developments in the FQH effect and in high T_c superconductivity resulted in many new ideas and new concepts. Looking back at those new developments, it becomes more and more clear that, in the last 30 years, we were actually witnessing an emergence of a new theme in condensed matter physics. The new theme is associated with new kinds of orders, new states of matter, and a new class of materials beyond Landau's symmetry-breaking theory. This is an exciting time for condensed matter physics. The new paradigm may even have an impact on our understanding of fundamental questions of nature—the emergence of elementary particles and the four fundamental interactions, which leads to a unification of matter and quantum information.[1]

The emergent new field of quantum-topological matter has developed very fast. Many new terms are introduced, but some of them can be very confusing.

(1) Some Haldane phases are topological, while some other Haldane phases are not. Although, the Haldane phase for a spin-1 chain is topological, it is actually a product state with no topological order.

(2) Topological insulators and superconductors [i.e., with $T^2 = (-)^{N_F}$ time-reversal symmetry and weak interactions] have no topological order. It is wrong to characterize topological insulators as insulators with a conducting surface.

(3) What is the difference between the quantum spin Hall state and the spin quantum Hall state? Are they topological insulators?

(4) "SPT state" is the abbreviation for both the symmetry protected trivial state and the symmetry protected topological state. The two mean the same.

(5) $(3+1)$D textbook s-wave superconductors have no topological order, while $(3+1)$D real-life s-wave superconductors have a Z_2-topological order.

(6) The $(2+1)$D $p+ip$ fermion paired state and the integer quantum Hall states (IQH) do not have any fractionalized topological excitations. Some people regard them as long-range entangled (i.e., topologically ordered) while others regard them as short-range entangled.

(7) What are the differences between a Chern insulator, a quantum anomalous Hall state, and an integer quantum Hall state? What are the differences between a fractionalized topological insulator and topological order?

(8) There is a very active search for Majorana fermions with non-Abelian statistics. But should a Majorana fermion be a fermion that carries Fermi statistics? Is a Majorana fermion the Bogoliubov quasiparticle in a superconductor?

In this Colloquium, we will clarify those notions.

III. TOPOLOGICALLY ORDERED PHASES

A. Chiral spin liquids and topological order

After the discovery of high T_c superconductors by Bednorz and Mueller (1986), some theorists believed that quantum spin liquids play a key role in understanding high T_c superconductors (Anderson, 1987). This is because spin liquids can lead to a so-called spin-charge separation: an electron disintegrates into two quasiparticles—a spinon (spin-$1/2$ charge 0) and a holon (spin-0 charge e). Since a holon is not a fermion, its condensation can lead to superconductivity—a novel mechanism of high T_c superconductors. Thus many people started to construct and study various spin liquids.[2]

However, despite the success of the Landau symmetry-breaking theory in describing all kinds of states, the theory cannot explain and does not even allow the existence of spin liquids (with an odd number of electrons per unit cell). This leads many theorists to doubt the very existence of spin liquids. In early proposals of spin liquids, the spinons are gapless and are confined at long distances by the emergent gauge field (Baskaran and Anderson, 1988), adding support to the opinion that a spin liquid is just fiction and does not actually exist.[3]

Kalmeyer and Laughlin (1987) introduced a special kind of spin liquid—the chiral spin liquid in an attempt to explain high-temperature superconductivity. In contrast to many other proposed spin liquids at that time, the chiral spin liquid was shown to have deconfined spinons (as well as deconfined holons) and corresponds to a stable zero-temperature phase.[4] At first, not believing Landau symmetry-breaking theory fails to describe spin liquids, people still wanted to use symmetry breaking to characterize the chiral spin liquid. They identified the chiral spin liquid as a state that breaks the time reversal and parity symmetries, but not the spin rotation and translation symmetries (Wen, Wilczek, and Zee, 1989). The chiral spin

[1] See Foerster, Nielsen, and Ninomiya (1980) and Baskaran and Anderson (1988) for the emergence of gauge interactions, Wen (2002a, 2003) and Levin and Wen (2006b) for the unification of gauge interactions and Fermi statistics, and Wen (2013b), You, BenTov, and Xu (2014), and You and Xu (2015) for the emergence of chiral fermions.

[2] See Baskaran, Zou, and Anderson (1987), Affleck and Marston (1988), Affleck, Zou *et al.* (1988), Dagotto, Fradkin, and Moreo (1988), and Rokhsar and Kivelson (1988).

[3] Now we realize that even those gapless spin liquids can exist as algebraic spin liquids without quasiparticles (Chung, Marston, and McKenzie, 2001; Rantner and Wen, 2001, 2002; Fradkin *et al.*, 2003; Hermele *et al.*, 2004; Senthil *et al.*, 2004).

[4] Recently, a chiral spin liquid was shown to exist in the Heisenberg model on the kagome lattice with J_1-J_2-J_3 coupling (Gong *et al.*, 2015; He and Chen, 2015).

liquid is also characterized by its perfect heat-conducting edge and quantized spin Hall conductance.

However, Wen (1989) quickly realized that there are many different chiral spin liquids (with different spinon statistics and spin Hall conductances) that have exactly the same symmetry. So symmetry alone is not enough to characterize different chiral spin liquids. This means that the chiral spin liquids contain a new kind of order that is beyond symmetry description. This new kind of order was named topological order.

Just like any concept in physics, the concept of topological order is also required to be defined via measurable quantities, which are called topological invariants. The first discovered topological invariants (Wen, 1990b) that define topological order were (1) the robust ground state degeneracy on a torus and other closed space manifolds (i.e., with no boundary), (2) the non-Abelian geometric phases (the modular matrices) of the degenerate ground states, and (3) the chiral central charge c of the edge states.[5] It was conjectured that those macroscopic topological invariants or, more generally, “the total gauge structures (the Abelian one plus the non-Abelian one) on the moduli spaces of the models defined on generic Riemann surfaces Σ_g completely characterize (or classify) the topological orders in $1+2$ dimensions” (Wen, 1990b).

Microscopically, topological order is a property of a local quantum system whose total Hilbert space has a tensor product decomposition $\mathcal{H}^{\text{tot}} = \otimes_i \mathcal{H}_i$, where $\mathcal{H}_i$ is the Hilbert space on each site. Such a tensor product decomposition is a part of the definition of a local system, which also satisfies the condition of short-range interaction between sites. Relative to such a tensor product decomposition, a product state is defined as a state of the form $|\Psi\rangle = \otimes_i |\Psi_i\rangle$, where $|\Psi_i\rangle \in \mathcal{H}_i$. In this Colloquium, only the tensor products of on-site states $|\Psi_i\rangle$ are called product states. With such a definition of local quantum systems, topological order is defined to describe gapped quantum liquids[6] that cannot be deformed into a product state without gap-closing phase transitions. Such quantum liquids are said to have long-range entanglement (Kitaev and Preskill, 2006; Levin and Wen, 2006a; Chen, Gu, and Wen, 2010). Long-range entanglement is the microscopic origin of topological order. A gapped state that can be smoothly deformed into a product state is short-range entangled and has no topological order. In particular, a product state has no topological order.

One may wonder: why do we need such a complicated way to characterize topological order? Is the quantized Hall conductance a more direct and simpler way to characterize topological order, at least for quantum Hall states (see Sec. III.B)? In fact, quantized Hall conductance is due to a combined effect of $U(1)$ symmetry (i.e., particle-number conservation) and topological order (i.e., long-range entanglement). If we break the $U(1)$ symmetry, quantum Hall states still have topological order, even though the Hall conductance is no longer well defined. How does one characterize topological order in such a situation? This characterization based on ground state degeneracy and non-Abelian geometric phases does not require symmetries and provides a complete characterization of topological orders in two dimensions.

We mention that the term “topological” in topological order and in topological insulators and superconductors has totally different meanings. In topological order, the term is motivated by the low-energy effective theory of the chiral spin liquids, which is a $U(1)$ Chern-Simons theory—a topological quantum field theory (Witten, 1989). Here topological really means long-range entangled, which is a property of many-body wave functions. We call it quantum topology, while in topological insulators and superconductors, the term corresponds to classical topology which is a property of a continuous manifold, related to the difference between a sphere and a torus. The vortex in a superfluid, the Chern number, and the Z_2 index in topological insulators and superconductors belong to classical topology, which represents a very different phenomenon. In fact, topological in topological insulators and superconductors really means “symmetry protected” (see Sec. IV).

B. Quantum Hall states

Soon after the proposal of the chiral spin liquid, experiments indicated that high temperature superconductors do not break the time-reversal and parity symmetries and chiral spin liquids do not describe high temperature superconductors (Lawrence, Szöke, and Laughlin, 1992). Thus the concept of topological order became a concept with no experimental realization.

Long before the discovery of high T_c superconductors, Tsui, Stormer, and Gossard (1982) discovered the FQH effect, such as the filling fraction $\nu = 1/m$ (Laughlin, 1983) state

$$\Psi_{\nu=1/m}(\{z_i\}) = \prod (z_i - z_j)^m e^{-(1/4)\sum |z_i|^2}, \tag{1}$$

where $z_i = x_i + iy_i$. It was realized that the FQH states are new states of matter. At first, influenced by the previous success of Landau’s symmetry-breaking theory, order parameters were used and long-range correlations to describe the FQH states (Girvin and MacDonald, 1987; Read, 1989; Zhang, Hansson, and Kivelson, 1989), which result in the Ginzburg-Landau-Chern-Simons effective theory of quantum Hall states. But in quantum Hall states, there is no off-diagonal long-range order in any local operators, and thinking about it can lead some in the wrong direction, such as looking for the Josephson effect in quantum Hall states.

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[5]The central charge c of the edge states is related to a gravitational response of the system described by a gravitational Chern-Simons 3-form ω_3: $\mathcal{L} = (2\pi c/24)\omega_3$, where $d\omega_3 = p_1$ is the first Pontryagin class (Abanov and Gromov, 2014; Bradlyn and Read, 2015; Gromov *et al.*, 2015). c can be measured via the thermal Hall conductivity $K_H = c(\pi k_B^2/6\hbar)T$ (Kane and Fisher, 1997).

[6]Zeng and Wen (2015) and Swingle and McGreevy (2016) introduced the notion of *gapped quantum liquids* to describe a simple kind of gapped state: the states that can enlarge themselves by dissolving product states. Only gapped quantum liquids have quantum field theory descriptions at long distances. 3D gapped states obtained by stacking 2D quantum Hall states and cubic code (Haah, 2011) are examples of gapped nonquantum liquids.

If we concentrate on physical measurable quantities, we see that all those different FQH states have exactly the same symmetry and conclude that we cannot use the Landau symmetry-breaking theory and local order parameters to describe different orders in FQH states. In fact, just like chiral spin liquids, FQH states also contain a new kind of order beyond Landau's symmetry-breaking theory. Different FQH states are also described by different topological orders (Wen and Niu, 1990). A better way to see the essence of FQH states is via topological invariants such as robust ground state degeneracy and modular matrices, as well as the nontrivial edge states (Halperin, 1982; Wen, 1990a). Thus the concept of topological order does have experimental realizations in FQH systems.

One of the most striking properties of FQH states is their fractionalized excitations that can carry fractional statistics (Arovas, Schrieffer, and Wilczek, 1984; Halperin, 1984)[7] and, if the particle number conserves, fractional charges (Tsui, Stormer, and Gossard, 1982; Laughlin, 1983).[8]

We know that a pointlike excitation above the ground state is something that can be trapped by a local change of the Hamiltonian near a spatial point $\boldsymbol{x}$. But sometimes the local change of the ground state near $\boldsymbol{x}$ cannot be created by local operators. In this case, we refer to the corresponding local change of the ground state as a topological excitation. It is those topological excitations that can carry fractional statistics and/or fractional charge.

We note that the presence of any topological excitations implies a presence of topological order in the ground state. But the reverse is not true; the absence of any topological excitations does not imply the absence of topological order in the ground state. In fact, the E_8 bosonic state and the IQH states are states with topological order but no topological excitations.

Regarding point 6 in Sec. II, some define those states with no topological excitations as short-range entangled (Kitaev, 2011). However, since those states have nonzero *chiral central charges* c for the edge states, they cannot smoothly change to a product state without a phase transition. Thus, they are topologically ordered states distinct from the trivial product states. Those topological orders with no topological excitations are called invertible topological orders,[9] and some refer to them as long-rang entangled (Chen, Gu, and Wen, 2010). Regarding point 7, the IQH state (von Klitzing, Dorda, and Pepper, 1980), the Chern insulator (Hofstadter, 1976; Thouless *et al.*, 1982), and the quantum anomalous Hall state (Haldane, 1988) are just different names for the same fermionic invertible topological order with integer chiral central charge c. Also, the fractionalized topological insulator is the same as the topological order, but may have an additional time-reversal symmetry.

C. Non-Abelian quantum Hall states

In addition to the Laughlin states, more exotic non-Abelian FQH states were proposed in 1991 by two independent works. Wen (1991b) pointed out that the FQH states described by wave functions

$$\Psi_{\nu=n/m}(\{z_i\}) = [\chi_n(\{z_i\})]^m$$
$$\text{or}\quad \Psi_{\nu=n/m+n}(\{z_i\}) = \chi_1(\{z_i\})[\chi_n(\{z_i\})]^m \qquad (2)$$

have topological excitations with non-Abelian statistics[10] of type $SU(n)_m$ [which is denoted as $A(n-1)_m$ in https://www.math.ksu.edu/~gerald/voas/] (Lan and Wen, 2017). This result was obtained via the low-energy $SU(m)_n$ effective Chern-Simons theory of these states, plus the level-rank duality. Here χ_n is the fermion wave function of n-filled Landau levels. Note that the $\nu = 1/3$ Laughlin state is given by

$$\Psi_{\nu=1/3}(\{z_i\}) = [\chi_1(\{z_i\})]^3. \qquad (3)$$

So $[\chi_n(\{z_i\})]^m$ and $\chi_1(\{z_i\})[\chi_n(\{z_i\})]^m$ are generalizations of the Laughlin state (Jain, 1991). They both have nontrivial edge states described by $U(1)\times SU(n)_m$ Kac-Moody current algebra (Blok and Wen, 1992).

In the same year, Moore and Read (1991) proposed that the FQH state described by the Pfaffian wave function

$$\Psi_{\nu=1/2} = \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right] e^{-1/4\sum |z_i|^2} \prod (z_i - z_j)^2 \qquad (4)$$

has excitations with non-Abelian statistics of the Ising type [or the $SU(2)_2$ type]. Its edge states were studied numerically (Wen, 1993) and were found to be described by a $c = 1$ chiral-boson conformal field theory (CFT) plus a $c = 1/2$ Majorana fermion CFT. Such a result about the edge states supports the proposal that the Pfaffian state is non-Abelian, since the edge for Abelian FQH states always has an integer chiral central charge c. Later, the non-Abelian statistics in the Pfaffian wave function was also confirmed by its low-energy effective $SO(5)$ level 1 Chern-Simons theory (Wen, 1999) (denoted as $B2_1$ in https://www.math.ksu.edu/~gerald/voas/), as well as a plasma analog calculation (Bonderson, Gurarie, and Nayak, 2011).

It is possible that the $SU(2)_2$ type of non-Abelian state is realized by $\nu = 5/2$ fractional quantum Hall samples (Willett *et al.*, 1987; Dolev *et al.*, 2008; Radu *et al.*, 2008).

[7]The possibility of fractional statistics in $(2+1)$D was pointed out by Leinaas and Myrheim (1977) and Wilczek (1982). The relation to the braid group was discussed by Wu (1984).

[8]Fractional charge has been directly observed via quantum shot noise in the tunneling current (de Picciotto *et al.*, 1997).

[9]For every invertible topological order $\mathcal{C}$, there exists another topological order $\mathcal{D}$—the inverse, such that stacking $\mathcal{C}$ and $\mathcal{D}$ on top of each other gives us a gapped state that has no topological order, i.e., belongs to the phase of product states.

[10]Wu (1984) set up a general theory and braid group for quantum statistics in two dimensions, and Goldin, Menikoff, and Sharp (1985) pointed out that such a setup contains non-Abelian representations of the braid group, which correspond to non-Abelian statistics. A more complete description of non-Abelian statistics is given by Witten (1989) and Kitaev (2006).

D. Superconducting states (with dynamical electromagnetism)

It is interesting to point out that long before the discovery of FQH states, Onnes discovered a superconductor in 1911 (Onnes, 1911). The Ginzburg-Landau theory for symmetry-breaking phases is largely developed to explain superconductivity. However, the superconducting order, that motivates the Ginzburg-Landau theory for symmetry breaking, itself is not a symmetry-breaking order. The superconducting order [in real life with dynamical $U(1)$ gauge field] is an order that is beyond the Landau symmetry-breaking theory. The superconducting order in real life is a topological order (or more precisely a Z_2 topological order or Z_2 gauge theory) (Wen, 1991c; Hansson, Oganesyan, and Sondhi, 2004). The real-life superconductor has a stringlike topological excitation that can be trapped by modifying the Hamiltonian along a loop. Such a stringlike topological excitation is the $hc/2e$ flux loop, since the electromagnetic $U(1)$ gauge field is dynamical. The presence of a stringlike topological excitation indicates the superconductor has a topological order. The textbook superconductors usually do not contain the dynamical $U(1)$ gauge field and do not contain stringlike topological excitations that can be trapped by modifying the Hamiltonian along a loop. This explains point 5 in Sec. II.

It is quite amazing that the experimental discovery of superconducting order did not lead to a theory of topological order. But instead, it led to a theory of symmetry-breaking order that fails to describe the superconducting order itself.

E. Z_2-spin liquid in $(2+1)$D

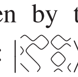
A chiral spin liquid breaks the time-reversal symmetry, while high T_c superconductors do not. So chiral spin liquids do not appear in high T_c superconductors. This motivated one to look for other spin liquids with deconfined spinons and holons that do not break the time-reversal symmetry. This leads to the theoretical discovery of a $(2+1)$D Z_2-spin liquid (Read and Sachdev, 1991; Wen, 1991a) described by effective Z_2 gauge theory (Kogut, 1979) (i.e., has a Z_2-topological order). The construction can be easily generalized to obtain a $(3+1)$D Z_2-spin liquid, which will have a Z_2 topological order identical to an s-wave superconductor. Later, an exactly solvable toric code model was constructed to realize the Z_2-topological order (Kitaev, 2003). Since then, the Z_2-topological order is also referred to as the "toric code."

The Z_2-spin liquid of spin $1/2$ on the kagome lattice may be realized by Herbertsmithite (Helton *et al.*, 2007), as suggested by recent experiments by Fu *et al.* (2015) and Han *et al.* (2016). The early numerical calculation of Yan, Huse, and White (2011) suggested that the spin-$1/2$ Heisenberg model on the kagome lattice is gapped, but details of the results are inconsistent with the Z_2-topological order, which led one to suspect that the model is gapless. A more recent numerical calculation suggests that the model has a Z_2-spin liquid ground state with long correlation length (10 unit cell length) (Mei *et al.*, 2017), while several other calculations suggest gapless $U(1)$ spin liquid ground states (He *et al.*, 2016; Jiang *et al.*, 2016; Liao *et al.*, 2016). More experimental and theoretical studies are needed to settle the issue.

F. Quantum liquids of nonoriented strings

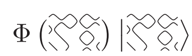
If we do not require spin rotation symmetry, one can use a string liquid to construct a state with Z_2-topological order (Kitaev, 2003). String liquids are long-range entangled (hence topologically ordered). We will see how long-range entanglement in topological order leads to fractional statistics and topological degeneracy.

1. Local "dancing" rules in string liquids

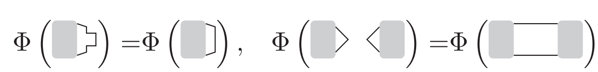
Given a spin-$1/2$ system, if we pick a particular spin-up–spin-down configuration, we get a product state. To construct a highly entangled state, one may consider a equal-weight superposition of all spin-up–spin-down configurations. But this does not work. We get a product state with all spins in the x direction. So one idea to get a highly entangled state is a partial sum. For example, we can view up spins as background and lines of down spins as the strings (see Fig. 1). The simplest topologically ordered state in such a spin-$1/2$ system is given by the equal-weight superposition of all closed strings:  (Kitaev, 2003).

To obtain other topological orders, we may consider a different superposition of strings. But those superpositions should all be determined by local rules, so that there is a local Hamiltonian that can produce a given superposition. What are those local rules that give rise to the string liquid ? The first rule is that, in the ground state, the down spins are always connected with no open ends. To describe the second rule, we need to introduce the amplitudes of closed strings in the ground state . The ground state is given by

$$\sum_{\text{all closed strings}} \Phi\left(\ \right) \left|\ \right\rangle . \tag{5}$$



Then the second rule relates to the amplitudes of closed strings in the ground state as we change the strings locally:



(6)

In other words, if we locally deform or reconnect the strings as in Fig. 2, the amplitude (or the ground state wave function) does not change.

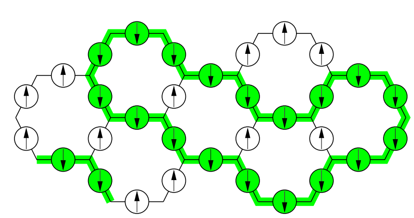


FIG. 1. The strings in a spin-$1/2$ model. In the background of up spins, the down spins form closed strings.

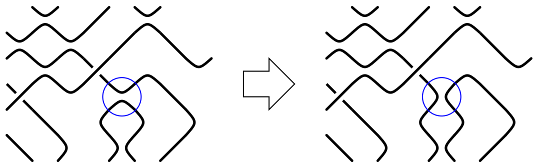


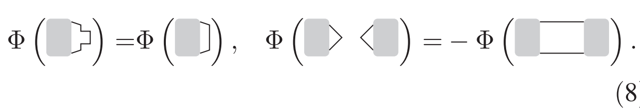
FIG. 2. In string liquid, strings can move freely, including reconnecting the strings.

The first rule tells us that the amplitude of a string configuration depends only on the topology of the string configuration. Starting from a single loop, using the local deformation and the local reconnection in Fig. 2, we can generate all closed string configurations with any number of loops. So all those closed string configurations have the same amplitude. Therefore, the local dancing rule fixes the wave function to be the equal-weight superposition of all closed strings:

$$|\Phi_{Z_2}\rangle = \sum_{\text{all closed strings}} |\;\rangle . \qquad (7)$$

In other words, the local dancing rule fixes the global dancing pattern.

If we choose another local dancing rule, then we will get a different global dancing pattern that corresponds to a different topological order. One of the new choices is obtained by just modifying the sign in Eq. (6):



We note that each local reconnection operation changes the number of loops by 1. Thus the new local dancing rules give rise to a wave function which has a form

$$|\Phi_{\text{Semi}}\rangle = \sum_{\text{all closed strings}} (-)^{N_{\text{loops}}} |\;\rangle . \qquad (9)$$

where N_{loops} is the number of loops. The wave function $|\Phi_{\text{Semi}}\rangle$ corresponds to a different global dance and a different topological order.

2. Emergence of Fermi and fractional statistics

Why do the two wave functions of nonoriented strings $|\Phi_{Z_2}\rangle$ and $|\Phi_{\text{Semi}}\rangle$ [see Eqs. (7) and (9)] have topological orders? Because the two wave functions give rise to nontrivial topological properties. The two wave functions correspond to different topological orders since they give rise to different topological properties. In this section, we discuss two topological properties: emergence of fractional statistics and, in the next section, topological degeneracy on a torus.

The two topological states in two dimensions contain only closed strings, which represent the ground states. If the wave functions contain open strings (i.e., have nonzero amplitudes for open string states), then the ends of the open strings will correspond to pointlike topological excitations above the ground states. Although an open string is an extended object, its middle part merges with the strings already in the ground states and is unobservable. Only its two ends carry energies and correspond to two pointlike particles.

We note that such a pointlike particle from the end of a string cannot be created alone. Thus the end of a string corresponds to a topological point defect, which may carry fractional quantum numbers. This is because an open string as a whole always carries nonfractionalized quantum numbers. But an open string corresponds to two topological point defects from the two ends. So we cannot say that each end of the string also carries nonfractionalized quantum numbers. Sometimes they do carry fractionalized quantum numbers.

Let us first consider the defects in the $|\Phi_{Z_2}\rangle$ state. To understand the fractionalization, consider the spin of such a defect to see if the spin is fractionalized or not (Fidkowski *et al.*, 2009). Note that here the spin is not the spin of the spin 1/2 that forms our model. The spin is the orbital angular momentum of an end. We use different fonts to distinguish them. The end of a string can be represented by

$$|\;\rangle_{\text{def}} = |\;\rangle + |\;\rangle + |\;\rangle + \dots . \qquad (10)$$

which is an equal-weight superposition of all string states obtained from the deformations and the reconnections of the string end.

Under a 360° rotation, the end of a string is changed to $|\;\rangle_{\text{def}}$, which is an equal-weight superposition of all string states obtained from the deformations and the reconnections of the rotated string end. Since $|\;\rangle_{\text{def}}$ and $|\;\rangle_{\text{def}}$ are always different, $|\;\rangle_{\text{def}}$ is not an eigenstate of 360° rotation and does not carry a definite spin.

To construct the eigenstates of 360° rotation, let us make a 360° rotation to $|\;\rangle_{\text{def}}$. To do that, we first use the string reconnection move in Fig. 2 to show that $|\;\rangle_{\text{def}} = |\;\rangle_{\text{def}}$. A 360° rotation on $|\;\rangle_{\text{def}}$ gives us $|\;\rangle_{\text{def}}$.

We see that the 360° rotation exchanges $|\;\rangle_{\text{def}}$ and $|\;\rangle_{\text{def}}$. Thus the eigenstates of 360° rotation are given by $|\;\rangle_{\text{def}} + |\;\rangle_{\text{def}}$ with eigenvalue 1, and by $|\;\rangle_{\text{def}} - |\;\rangle_{\text{def}}$ with eigenvalue -1. So the particle $|\;\rangle_{\text{def}} + |\;\rangle$ has a spin 0 (mod 1), and the particle $|\;\rangle_{\text{def}} - |\;\rangle_{\text{def}}$ has a spin 1/2 (mod 1).

If one believes in the spin-statistics theorem, one can guess that the particle $|\;\rangle_{\text{def}} + |\;\rangle_{\text{def}}$ is a boson and the particle $|\;\rangle_{\text{def}} - |\;\rangle_{\text{def}}$ is a fermion. This guess is indeed correct. From Fig. 3 we see that we can use deformation of strings and two reconnection moves to generate an exchange of two ends of strings and a 360° rotation of one of the ends of a string. Such operations allow us to show that Figs. 3(a) and 3(e) have

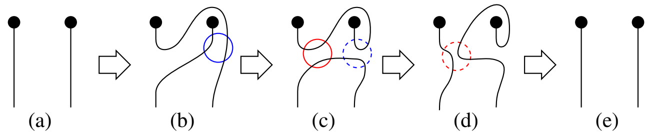


FIG. 3. Deformation of strings and two reconnection moves, plus an exchange of two ends of strings and a 360° rotation of one of the ends of a string, change the configuration (a) back to itself. Note that from (a) to (b) we exchange the two ends of srings, and from (d) to (e) we rotate one of the ends of a string by 360°. The combination of those moves does not generate any phase.

the same amplitude, which means that an exchange of two ends of strings followed by a 360° rotation of one of the ends of a string do not generate any phase. This is nothing but the spin-statistics theorem.

The emergence of Fermi statistics in the $|\Phi_{Z_2}\rangle$ state of a purely bosonic spin-1/2 model indicates that the state is a topologically ordered state. We also see that the $|\Phi_{Z_2}\rangle$ state has a bosonic quasiparticle $|\uparrow\rangle_{\text{def}} + |\circlearrowleft\rangle_{\text{def}}$ and a fermionic quasiparticle $|\uparrow\rangle_{\text{def}} - |\circlearrowleft\rangle_{\text{def}}$. The bound state of these two particles is a boson (not a fermion) due to their mutual semion statistics. Such quasiparticle content agrees exactly with the Z_2 gauge theory which also has three types of topological excitations, two bosons and one fermion. In fact, the low-energy effective theory of the topologically ordered state $|\Phi_{Z_2}\rangle$ is the Z_2 gauge theory and we call $|\Phi_{Z_2}\rangle$ a Z_2-topologically ordered state (Read and Sachdev, 1991; Wen, 1991a).

Next, let us consider the defects in the $|\Phi_{\text{Semi}}\rangle$ state. Now

$$|\uparrow\rangle_{\text{def}} = |\uparrow\rangle + |\uparrow\!\supset\rangle - |\uparrow\!\circ\rangle + \ldots. \tag{11}$$

and a similar expression for $|\circlearrowleft\rangle_{\text{def}}$, due to a change of the local rule for reconnecting the strings [see Eq. (8)]. Using the string reconnection move in Fig. 2, we find that $|\circlearrowleft\rangle_{\text{def}} = -|\circlearrowleft\rangle_{\text{def}}$. So a 360° rotation changes $(|\uparrow\rangle_{\text{def}}, |\circlearrowleft\rangle_{\text{def}})$ to $(|\circlearrowleft\rangle_{\text{def}}, -|\uparrow\rangle_{\text{def}})$. We find that $|\uparrow\rangle_{\text{def}} + i|\circlearrowleft\rangle_{\text{def}}$ is the eigenstate of the 360° rotation with eigenvalue $-i$, and $|\uparrow\rangle_{\text{def}} - i|\circlearrowleft\rangle_{\text{def}}$ is the other eigenstate of the 360° rotation with eigenvalue i. So the particle $|\uparrow\rangle_{\text{def}} + i|\circlearrowleft\rangle_{\text{def}}$ has a spin $-1/4$, and the particle $|\uparrow\rangle_{\text{def}} - i|\circlearrowleft\rangle_{\text{def}}$ has a spin 1/4. The spin-statistics theorem is still valid for $|\Phi_{\text{Semi}}\rangle_{\text{def}}$ state, as one can see from Fig. 3. So, the particle $|\uparrow\rangle_{\text{def}} + i|\circlearrowleft\rangle_{\text{def}}$ and particle $|\uparrow\rangle_{\text{def}} - i|\circlearrowleft\rangle_{\text{def}}$ have fractional statistics with statistical angles of semion $\pm\pi/2$. Thus the $|\Phi_{\text{Semi}}\rangle$ state contains a topological order. We call such a topological order a double-semion topological order (Freedman *et al.*, 2004; Levin and Wen, 2005).

It is amazing to see that the long-range quantum entanglement in string liquid can gives rise to fractional spin and fractional statistics, even from a purely bosonic model. Fractional spin and Fermi statistics are two of the most mysterious phenomena in nature. Now we understand them as merely a phenomenon of long-range quantum entanglement. They are no longer mysterious.

3. Topological degeneracy

The Z_2-topological order has another important topological property: topological degeneracy (Read and Chakraborty, 1989; Wen, 1991a). Topological degeneracy is the ground state degeneracy of a gapped many-body system that is robust against any local perturbations as long as the system size is large (Wen and Niu, 1990). It implies the presence of topological order.

Topological degeneracy can be used as protected qubits which allows us to perform topological quantum computation (Kitaev, 2003). It is believed that the appearance of topological degeneracy implies the topological order (or long-range entanglement) in the ground state. Many-body states with topological degeneracy are described by topological quantum field theory at low energies.

The simplest topological degeneracy appears when we put topologically ordered states on compact spaces with no boundary. We can use the global entanglement pattern to understand the topological degeneracy. We know that the local rules determine the global entanglement pattern. On a sphere, the local rules determine a unique global entanglement pattern. So the ground state is nondegenerate. However, on other compact spaces, there can be several global entanglement patterns that all satisfy the same local rules. In this case, the ground state is degenerate.

For the Z_2-topological state on a torus, the local rule relates the amplitudes of the string configurations that differ by a string reconnection operation in Fig. 2. On a torus, the closed string configurations can be divided into four sectors (see Fig. 4), depending on even or odd numbers of strings crossing the x or y axis. The string reconnection move connects only the string configurations among each sector. So the superposition of the string configurations in each sector represents a different many-body wave function. Since those many-body wave functions are locally indistinguishable, they correspond to different degenerate ground states. Therefore, the local rule for the Z_2-topological order gives rise to fourfold degenerate ground states on a torus.

Similarly, the double-semion topological order also gives rise to fourfold degenerate ground states on a torus.

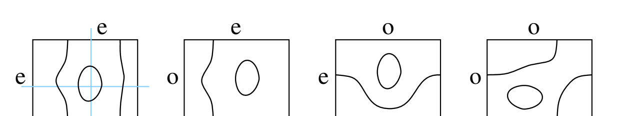


FIG. 4. On a torus, the closed string configurations can be divided into four sectors, depending on even or odd numbers of strings crossing the x or y axis.

TABLE I. Topologically ordered states with long-range entanglement. Here 1B refers to the one-dimensional bosonic system, 2F to the two-dimensional fermionic system, etc. The second column indicates the presence of fractionalized pointlike excitations. The third column indicates the presence of non-Abelian statistics. The fourth column indicates whether the boundary must be gapless, can be gapped, or for some must be gapless and for others can be gapped. TQC in the last column means topological quantum computation.

Topological order	Frac. exc.	Non-Ab. sta.	Boundary	Classification/comment
1F Majorana chain	No	Not any	Majorana zero mode	$\mathbb{Z}_2$ (Z_2^f symmetry breaking)
2B bosonic E_8 state	No	No	Gapless	Invertible topological order
2B chiral spin liquid	Semion	No	Gapless	Spin quantum Hall state
2B Z_2-spin liquid	Fermion	No	Gapped	Z_2 gauge/toric code
2B double-semion state	Fermion	No	Gapped	Z_2 Dijkgraaf-Witten
2B string-net liquids	Yes	Yes	Gapped	Unitary fusion category
2F $p+ip$ fermion paired state	No	No	Gapless	Invertible topological order
2F integer quantum Hall states	No	No	Gapless	$\mathbb{Z}$ (invertible topological order)
2F Laughlin states, 2F Halperin states	Yes	No	Gapped/gapless	K matrix (symmetric, integral)
2F $\chi_1\chi_2^2$ state	Yes	$SU(2)_2$	Gapless	Cannot do universal TQC
2F χ_2^3 state	Yes	$SU(3)_2$	Gapless	Can do universal TQC
2F Pfaffian state	Yes	$SU(2)_2$	Gapless	Cannot do universal TQC
2F Z_3 parafermion state	Yes	$SU(2)_3$	Gapless	Can do universal TQC
2F string-net liquids	Yes	Yes	Gapped	Unitary superfusion category
$(3+1)$D superconductor	Fermion	Not any	Gapped	With dynamical $U(1)$ gauge field
3B string-net liquids	Fermion	Not any	Gapped	Symmetric fusion category
3B Walker-Wang model	Fermion	Not any	Gapped	Premodular tensor category
3B all-boson topological order	Boson	Not any	Gapped	Pointed fusion 2-category

G. Table of some topological orders

In Table I we list some topological orders in bosonic and fermionic systems in various dimensions. The simplest one in the table is the $(2+1)$D IQH states (von Klitzing, Dorda, and Pepper, 1980). Some entries in Table I have not been discussed. In particular, the string-net liquids for bosonic systems (Levin and Wen, 2005) and fermionic systems (Gu, Wang, and Wen, 2015; Bhardwaj, Gaiotto, and Kapustin, 2016) allow us to obtain all $(2+1)$D topological orders with a gappable boundary (Kitaev and Kong, 2012; Lan and Wen, 2014). It reveals that $(2+1)$D bosonic topological orders are classified by unitary fusion categories (Etingof, Nikshych, and Ostrik, 2005), while $(2+1)$D fermionic topological orders are classified by unitary superfusion categories. For more general $(2+1)$D bosonic topological orders, it was conjectured (Wen, 1990b), and became more and more clear (Keski-Vakkuri and Wen, 1993; Kitaev, 2006; Rowell, Stong, and Wang, 2009; Wen, 2016), that they are classified by the modular matrices S and T [which encode unitary modular tensor categories (MTC) (Moore and Seiberg, 1989)] plus the chiral central charge c of the edge states. Physically, the so-called MTC can be viewed as a set of topological excitations, together with the data that describe the fusion and braiding of those excitations.

Many topological orders have fractionalized excitations (see the second column of Table I), and some $(2+1)$D topological orders even have non-Abelian excitations (see the third column of Table I). In $(1+1)$D fermion systems and $(2+1)$D boson or fermion systems, there are even topological orders that have no fractionalized excitations (the second column with an "No" entry). Those topological orders are called invertible topological orders (Freed, 2014; Kapustin, 2014a; Kong and Wen, 2014), and their nontriviality is reflected in their nontrivial boundary states which have a gravitational anomaly (Wen, 2013a; Kong and Wen, 2014).

Regarding point 8 in Sec. II, we note that the fermions are fractionalized topological excitations in bosonic systems. But they are local nontopological excitations in fermionic systems. For example, Majorana fermions are local nontopological excitations in fermionic superconductors [with spin-orbital coupling and no dynamical $U(1)$ gauge field], since they are antiparticles of themselves. Therefore, Majorana fermions are indeed fermions with Fermi statistics. They are not particles with non-Abelian statistics. In fact, Majorana fermions are the familiar Bogoliubov quasiparticles in superconductors which were discovered a long time ago. So what one is looking for, in the intensive experimental search, is not the Majorana fermion first introduced by Majorana, but instead a Majorana zero mode that can appear, for example, at the end of a 1D p-wave superconductor (Kitaev, 2001), or at the center of a vortex in a 2D $p+ip$ fermion paired state (Senthil, Marston, and Fisher, 1999; Read and Green, 2000). A Majorana zero mode is not a Majorana fermion. In fact, it is not even a particle. It is a *property* of a particle, just like the mass is a property of a particle. If the mobile particle carries a Majorana zero mode, then the particle will have a non-Abelian statistics (Ivanov, 2001). So one should not mix a Majorana zero mode with a Majorana fermion.

We also mention that the $SU(2)_2$ types of non-Abelian statistics in the $\chi_1\chi_2^2$ FQH and the Pfaffian states contain a non-Abelian quasiparticle that carries a Majorana zero mode. Such a particle has an internal degrees of freedom of half of a qubit (i.e., quantum dimension $d=\sqrt{2}$).[11]

Last, this Colloquium discusses only topological phases at zero temperature. Phases beyond Landau symmetry-breaking order also exist for $T\neq 0$, which are not reviewed here since they require a different theoretical framework.

[11]A physical explanation of quantum dimension can be found in Kitaev (2006) and Wen (2016).

IV. SPT STATES: NONTRIVIAL SYMMETRIC PRODUCT STATES

One expects gapped product states that have neither symmetry-breaking order nor topological order to be trivial, in the sense that all those states belong to one single phase. In this section, we see that in fact those states can belong to several different phases if there is a symmetry and thus can be nontrivial.

A. Gapped integer-spin chain: Haldane phases

The ground state of the $SO(3)$ symmetric antiferromagnetic spin-$1/2$ Heisenberg chain

$$H = \sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} \tag{12}$$

cannot break the $SO(3)$ spin rotation symmetry due to quantum fluctuations (Mermin and Wagner, 1966). What is the nature of this symmetric ground state? The Bethe ansatz approach, bosonization, and Lie-Schultz-Mattis theorem (Lieb, Schultz, and Mattis, 1961) all indicate the ground state of the spin-$1/2$ Heisenberg chain behaves almost like a spontaneous $SO(3)$ symmetry-breaking state: the spin-spin correlation has a slow algebraic decay (in contrast to exponential decay for a typical disordered system) and the chain is gapless [as if having a Goldstone mode (Goldstone, 1961)]. This result led to the belief that all spin-S chains are also gapless and have algebraic decaying spin-spin correlations, since for $S > 1/2$ the spins have even weaker quantum fluctuations than the spin-$1/2$ chain.

In 1983, Haldane considered spin fluctuations in $(1+1)$D space-time that have a nontrivial "winding" number in $\pi_2(S^2)$. He realized that the spin configuration with winding number ± 1 has a phase factor -1 if the spin is a half integer and a phase factor 1 if the spin is an integer. So the half-integer spin chain and integer-spin chain may have different dynamics. Haldane (1983) conjectured that the spin chain is gapped if the spin is an integer, despite it having weaker quantum fluctuations than the spin-$1/2$ chain. If the spin is a half integer, then the spin chain is gapless. The gapped ground state of an integer-spin chain is called a Haldane phase. At that time, it was believed that the Haldane phase to be a trivial disordered phase, just like the product state formed by spin 0 on each site.

However, such an opinion was put in doubt by an exactly solvable integer-spin chain. It was shown that, for the exactly solvable model (Affleck, Kennedy *et al.*, 1988), the boundary of the integer spin-S chain carries degenerate degrees of freedom of spin $S/2$. Since the gapless edge excitations for $(2+1)$D FQH states implies a bulk topological order, people started to wonder that maybe a similar picture applies to one lower dimension: the gapped $(1+1)$D ground states of integer-spin chains also have topological orders due to the gapless spin-$S/2$ boundary.

But this point of view seems incorrect. The gapless boundary of a $(2+1)$D chiral topological order is actually a bulk property, since gaplessness is robust against any modifications on the boundary. This is why the gapless boundary reflects a bulk topological order. However, a gapless spin-$S/2$ boundary of a spin-S chain can be easily gapped by applying a Zeeman field at the boundary. This seems to suggest that the gapped ground state of the integer-spin chain is trivial.

B. Gapped Haldane phases: Topological or not topological?

What is the nature of the Haldane phase for the integer spin-S chain? Topological or not topological? This question bothered us for 15 years, until we used the tensor-entanglement-filtering renormalization (TEFR) approach [see Fig. 5(a)] to study the spin-1 XXZ chain (Gu and Wen, 2009):

$$H = \sum_i J \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2. \tag{13}$$

Unlike the density matrix renormalization group approach (White, 1992), the TEFR approach gives us a simple fixed-point tensor. We found that the fixed-point tensor has a corner-double-line structure (with degenerate weights) when $U \approx 0$ [see Fig. 5(b)], and the fixed-point tensor becomes a dimension-1 trivial tensor when $U \gg J$ [see Fig. 5(a) where the indices of T are all equal to 1].

The ground state for $U \gg J$ is a product state of $|S_i^z = 0\rangle$ which is consistent with the trivial dimension-1 fixed-point tensor. The corner-double-line fixed-point tensor for $U = 0$ corresponds to a fixed-point wave function that contains four states per site [increased from three states of spin 1, see Fig. 5(b)]. The four states form the $(3 \oplus 1)$-dimensional representation of $SO(3)$, which can be viewed as two spin-$1/2$ representations [the projective representations of $SO(3)$]

$$3 \oplus 1 = 2 \otimes 2. \tag{14}$$

In such a fixed-point wave function, the two spin $1/2$ on neighboring sites form a spin singlet. The total fixed-point

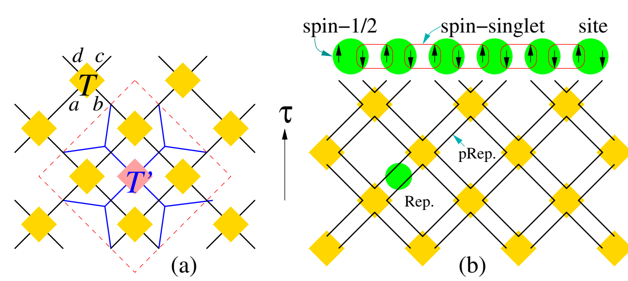


FIG. 5. (a) A tensor network representation of the partition function $Z = \mathrm{Tr}e^{-\tau H}$ obtained from a path integral for a $(1+1)$D quantum system. Each vertex is a rank-4 tensor T_{abcd} where each leg corresponds to an index. The range of the index is the dimension of the tensor T. The partition function Z is obtained as a product of all tensors, with the common indices on the edges linking two vertices summed over (which corresponds to the path integral). We can combine four tensors T to form a new tensor T' and obtain a new coarse-grained tensor network that produces the same partition function Z. After many coarse-graining iterations, we obtain a fixed-point tensor T^{fix} that characterizes a quantum phase. (b) The fixed-point tensor of a spin-1 Heisenberg chain has a corner-double-line structure. It gives rise to the fixed-point wave function of an ideal $SO(3)$-SPT state.

wave function is the product state of those spin singlets [see Fig. 5(b)]. We discovered that, just like the $U \gg J$ limit, the spin-1 Haldane phase is also a short-range entangled state equivalent to a product state. It is not topological despite the fractionalized spin-$1/2$ boundary.

However, nontopological does not mean trivial. We found that, for the spin-1 chain, the corner-double-line structure even appears for the following generic Hamiltonian:

$$H = \sum_i [J\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + U(S_i^z)^2] + \sum_i B_x S_i^x + B_z S_i^z + B'_x[S_i^x(S_{i+1}^z)^2 + S_{i+1}^x(S_i^z)^2] \quad (15)$$

when $U, B_{x,z}, B'_x \approx 0$. This suggests that the corner-double-line structure is stable against any perturbations with time-reversal symmetry T^* (which is the usual time reversal plus a 180° spin-S^y rotation) and spatial reflection symmetry.[12] On the other hand, the corner-double-line structure can be destroyed by perturbations that break those symmetries. This suggests that the spin-1 Haldane phase characterized by the corner-double-line tensor (or the dimmerized fixed-point wave function) is a stable phase, distinct from the product state of $|S_z = 0\rangle$, as long as we do not break those symmetries. We conclude that the Haldane phase of the spin-1 chain is nontrivial despite the fact it is a product state that does not spontaneously break any symmetry. This is a new state of matter and we propose the concept of symmetry protected trivial order to describe this new state of matter. SPT orders are characterized by the corner-double-line fixed-point tensors with degenerate weights (or the dimmerized fixed-point wave function). Later, Pollmann *et al.* (2010) showed that SPT orders can also be characterized via the entanglement spectrum. It is interesting to see that even product states without spontaneous symmetry breaking can be nontrivial. However, the spin-1 Haldane phase at that time was already widely referred to as a topological phase. So we gave the term "SPT order" another representation of "symmetry protected topological order."[13]

It is important to regard SPT states as short-range entangled, not topological (in the sense of orange versus donut). This correct way of thinking leads to a complete classification of all 1D gapped interacting phases (Chen, Gu, and Wen, 2011a; Schuch, Perez-Garcia, and Cirac, 2011), in terms of projective representations of the symmetry group one year later (Pollmann *et al.*, 2010) and the systematic group cohomology theory of SPT phases in higher dimensions two years later (Chen, Gu *et al.*, 2013). In particular, the projective-representation classification of $(1+1)$D SPT phases indicates that only the odd-integer-spin Haldane phases are the $SO(3)$-SPT phases, while the even-integer-spin Haldane phases are not the $SO(3)$-SPT phase just like the product state of spin 0s (Pollmann *et al.*, 2012). So Haldane phases can be topological or nontopological depending on the spin being an odd or even integer. This explains point 1 in Sec. II.

C. A Z_2-SPT state in $(2+1)$D

After realizing SPT states to be product states, it becomes easy to construct SPT states in any dimension. We just need to write a product state in some complicated form, and then try to find all the twisted ways to implement the symmetry.

First, we need to introduce the concept of on-site symmetry, which is usually referred to as global symmetry. Relative to the tensor product decomposition $\mathcal{H}^{\text{tot}} = \otimes_i \mathcal{H}_i$ of the total Hilbert space, a symmetry transformation is on site if it has a tensor product decomposition $U = \prod_i U_i$, where U_i is the symmetry transformation acting on $\mathcal{H}_i$. The notion of on-site symmetry is stressed in Chen, Gu, and Wen (2011a) and Chen, Liu, and Wen (2011c), which is a key to understanding SPT states.

The first lattice model that realizes (Chen, Liu, and Wen, 2011c) a $(2+1)$D SPT state has four qubits (or spin-$1/2$ spins) on each site (see Fig. 6). A complicated product state is given by

$$|\Psi_0\rangle = \bigotimes_{\text{plaquette}} \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle), \quad (16)$$

where $(1/\sqrt{2})(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$ is the wave function for the four spins in the plaquette (see Fig. 6). Note that the four spins in $(1/\sqrt{2})(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$ are on four different sites.

One way to introduce a Z_2 symmetry is to define the transformation on each site to be the spin flipping:

$$U_X = \sigma_1^x\sigma_2^x\sigma_3^x\sigma_4^x, \qquad U_X^2 = 1. \quad (17)$$

Obviously, $|\Psi_0\rangle$ is invariant under such a spin flipping Z_2 transformation. But for such a Z_2 symmetry, $|\Psi_0\rangle$ is not an SPT state.

There is another way to define Z_2 symmetry (on each site, see Fig. 6), but this time with a twist:

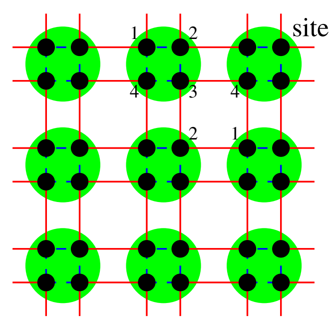


FIG. 6. The solid dots are qubits (or spin $1/2$). A large disk (with four dots inside) represents a site. The dashed line connecting dots 1 and 2 represents the phase factor CZ_{12} in the Z_2 global symmetry transformation. In the Z_2-SPT state, the four spins in a plaquette (connected by solid lines that form a square) are described by $(1/\sqrt{2})(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$.

[12]In fact, the corner-double-line structure is stable against any perturbations with time-reversal symmetry T^* or spatial reflection symmetry (Pollmann *et al.*, 2012).

[13]After long debates, Gu and Wen (2009) eventually used the second less-accurate representation.

$$U_{CZX} = U_X U_{CZ}, \tag{18}$$

where the ± 1 phase twist U_{CZ} is a product of CZ_{ij} that acts on the two spins at i and j: $CZ_{ij} = -1$ when it acts on $|\downarrow\downarrow\rangle$ and $CZ_{ij} = 1$ otherwise. More specifically

$$U_{CZ} = \prod_{j=1,2,3,4} CZ_{j,j+1} = \prod_{j=1,2,3,4} \frac{1 + \sigma^z_{j+1} + \sigma^z_j - \sigma^z_{j+1}\sigma^z_j}{2}, \tag{19}$$

where $j = 5$ is the same as $j = 1$. It is a nontrivial exercise but one can indeed check that $U^2_{CZX} = 1$. $|\Psi_0\rangle$ is invariant under such a twisted spin flipping Z_2 transformation since all the ± 1 CZ_{ij} factors cancel each other. For the new Z_2 symmetry, $|\Psi_0\rangle$ is an SPT state (Chen, Liu, and Wen, 2011c). In fact, one can construct an exactly soluble lattice Hamiltonian, which is symmetric under the new symmetry and has $|\Psi_0\rangle$ as its unique gapped group state.

This construction has been generalized to higher dimensions and an arbitrary compact symmetry group via group cohomology theory: for each element in $\mathcal{H}^{d+1}(G;\mathbb{R}/\mathbb{Z})$, we can construct a $(d+1)$D SPT state protected by G symmetry. But one thing is still unclear: how to see that such constructed state is actually a G-SPT state?

D. Probing SPT orders

An SPT state is almost trivial. For example, all the correlations are short ranged and featureless, as well as all the bulk excitations are local excitations without fractionalization. So, it is not easy to see the nontriviality of an SPT state. One way to reveal the nontriviality is to probe the boundary (Chen, Liu, and Wen, 2011c): The boundary of an SPT state cannot be gapped and is nondegenerate if the symmetry is not broken explicitly. This is because the effective symmetry on the low-energy boundary degrees of freedom must be non on site, and the non-on-site property for the boundary theory exactly corresponds to and classifies the anomaly in global symmetry (Wen, 2013a). This implies the boundary of an SPT state is symmetry breaking, gapless, and/or topologically ordered.

Another way to detect the nontriviality of an SPT state is to twist the symmetry and measure the ground state response under the twisted symmetry (Levin and Gu, 2012). To understand how to twist the symmetry, let us assume that a 2D lattice Hamiltonian for an SPT state with symmetry G has a form (see Fig. 7) $H = \sum_{(ijk)} H_{ijk}$, where $\sum_{(ijk)}$ sums over all the triangles (ijk) in Fig. 7 and H_{ijk} acts on the states on site i, site j, and site k. H and H_{ijk} are invariant under the global G transformations.

Let us perform a local $g \in G$ transformation which acts only on the sites in the shaded region in Fig. 7. Such a local transformation will change H to $\tilde{H}$. However, only the Hamiltonian terms on the triangles (ijk) across the boundary of the shaded region are changed from H_{ijk} to H^g_{ijk}. Since the G transformation is a unitary transformation, H and $\tilde{H}$ have the same energy spectrum. In other words, the boundary

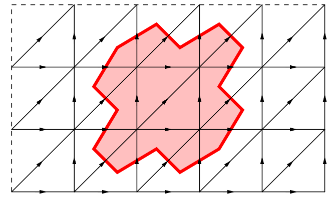


FIG. 7. A 2D lattice on a torus. A $g \in G$ transformation is performed on the sites in the shaded region. The g transformation changes the Hamiltonian term on the triangle (ijk) across the boundary from H_{ijk} to H^g_{ijk}.

(called the g cut) in Fig. 7 (described by H^g_{ijk}) does not cost any energy.

Now let us consider a Hamiltonian on a lattice with some g cuts (see Fig. 8) $\tilde{H} = \sum_{(ijk)}{}' H_{ijk} + \sum^{g\text{ cut}}_{(ijk)} H^g_{ijk}$, where $\sum'_{(ijk)}$ sums over the triangles not on the cut and $\sum^{g\text{ cut}}_{(ijk)}$ sums over the triangles that are divided into disconnected pieces by the g cut. The triangles at the ends of the cut have no Hamiltonian terms. We note that the cut carries no energy. Only the ends of the cut cost energy. So Fig. 8 corresponds to three monodromy defects. If g is a generator of G, then the end of the g cut will be called an elementary monodromy defect. We point out that dislocation in a crystal is an example of a monodromy defect of translation symmetry. It has been used to detect SPT phases protected by translation symmetry (the so-called weak topological phases) (Ran, Zhang, and Vishwanath, 2009; Teo and Kane, 2010; Slager *et al.*, 2014). We also point out that the procedure to obtain $\tilde{H}$ is actually the "gauging" of the G symmetry (Levin and Gu, 2012). $\tilde{H}$ is a gauged Hamiltonian that contains three G vortices at the ends of the cut.

Using the monodromy defects, we can detect the Z_n-SPT order (Wen, 2017): n identical elementary monodromy defects in a $(2+1)$D Z_n-SPT state on a torus always carry a total Z_n charge m, if the Z_n-SPT state is described by the mth cocycle in $\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})$.

The total Z_n charge of n identical monodromy defects allows us to completely characterize the $(2+1)$D Z_n SPT states. Another way to probe the Z_n-SPT order is to use the statistics of the monodromy defects (Levin and Gu, 2012): The statistical angle θ_M of an elementary monodromy defect

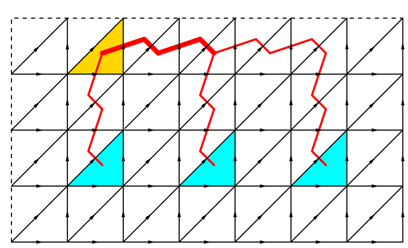


FIG. 8. Three identical monodromy defects (the three lower triangles) from the $G = Z_3 = \{0, 1, 2\}$ symmetry twist. The thin lines are 1 cuts, and the thick line is a 2 cut. The g cuts can be relocated by local Z_3 transformations as in Fig. 7. The single upper triangle can also be relocated by local Z_3 transformations. Thus it is not a monodromy defect.

TABLE II. SPT states with short-range entanglement. Here 1B refers to the one-dimensional bosonic system, 2F to the two-dimensional fermionic system, etc. Also T represents the time-reversal symmetry, which generates the group Z_2^T for bosonic systems, and Z_4^T for electron systems. This is because $T^2 = (-)^{N_F}$ is the fermion-number-parity operator for electron systems. The last column describes the degenerate state at the end of 1D SPT phases or other SPT probes for higher dimensions.

SPT order	Symmetry	Classification	Chain end/SPT probe
1B spin-1 Haldane phase	$SO(3)$	$\mathcal{H}^2(SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	Spin 1/2
1B spin-1 Haldane phase	Z_2^T	$\mathcal{H}^2(Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	Kramer doublet
1B symmetry gapped phases	G	$\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$	Projective representation of G
1F insulator w/ coplanar spin order	$U^f(1) \rtimes Z_2^T$	$\mathbb{Z}_2$	Kramers doublet
1F topological superconductor	Z_4^T	$\mathbb{Z}_2$	Charge-0 Kramers doublet
1F G^f-SPT phases	G^f	$\mathcal{H}^2(G^f, \mathbb{R}/\mathbb{Z})$	Projective representation of G^f
2B Z_n-SPT states	Z_n	$\mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$	Z_n dislocation has fractional statistics/Z_n charge
2B SPT insulator	$U(1)$	$\mathcal{H}^3(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$	Even-integer Hall conductance
2B T-symmetric SPT insulator	$U(1) \rtimes Z_2^T$	$\mathcal{H}^3(U(1) \rtimes Z_2^T, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	π flux has Kramers doublet
2B spin quantum Hall states	$SO(3)$	$\mathcal{H}^3(SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$	Quantized spin Hall conductance
2B T-symmetric SPT spin liquid	$Z_2^T \times SO(3)$	$\mathcal{H}^3(Z_2^T \times SO(3), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$	
2B G-SPT states	G	$\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$	
2F quantum spin Hall states	$U^f(1) \times U^f(1)$	$\mathbb{Z}$	Spin-charge Hall conductance
2F topological insulator	$[U^f(1) \rtimes Z_4^T]/Z_2$	$\mathbb{Z}_2$	π flux carries charge-0 Kramers doublet
2F topological superconductor	Z_4^T	$\mathbb{Z}_2$	π flux carries charge-even Kramers doublet
2F G^f-SPT states	G^f without T	Chiral central charge $c = 0$ modular extensions of sRep(G^f)	
3B T-symmetric SPT states	Z_2^T	$\mathcal{H}^4(Z_2^T, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^2$	
3B T-symmetric SPT insulator	$U(1) \rtimes Z_2^T$	$\mathcal{H}^4(U(1) \rtimes Z_2^T, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^3$	A monople is a fermion
3B T-symmetric SPT spin liquid	$Z_2^T \times SO(3)$	$\mathcal{H}^4(Z_2^T \times SO(3), \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2 = \mathbb{Z}_2^4$	
3B G-SPT states	G without T	$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$	
3B G-SPT states	G with T	$\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{Z}_2$	
3F topological insulator	$[U^f(1) \rtimes Z_4^T]/Z_2$	$\mathbb{Z}_2$	A monople carries half-integer charge
3F topological superconductor	Z_4^T	$\mathbb{Z}_{16}$	

satisfies $\mathrm{mod}(\theta_M/2\pi, 1/n) = m/n^2$ for a Z_n-SPT state characterized by $m \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$.

This way of probing an SPT state is like using the modular extensions of Rep(G) to probe the G-SPT order (Lan, Kong, and Wen, 2016b, 2017b). (The so-called *modular extension* can be viewed as including all the monodromy defects and considering their statistics.) It has been shown that the modular extensions of Rep(G) one to one correspond to the elements in $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$ (Drinfeld *et al.*, 2007; Lan, Kong, and Wen, 2016b). So the modular extensions can fully characterize $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$. In other words, measuring the Abelian and/or non-Abelian statistics among the monodromy defects and the local excitations described by Rep(G) allows us to fully detect the G-SPT order in $(2+1)$D for any unitary symmetry G. A similar idea also applies to $(3+1)$D SPT states (Wang and Levin, 2014; Lan, Kong, and Wen, 2017a). If the symmetry group contains $U(1)$, one can also use the $U(1)$ monopoles to probe the $(3+1)$D SPT states (Metlitski, Kane, and Fisher, 2013; Wen, 2014; Ye and Wen, 2014). A systematic discussion to probe all SPT orders in any dimensions can be found in Hung and Wen (2014).

E. Table of some SPT states

In Table II, we list bosonic and fermionic SPT states for various symmetries and in various dimensions. For bosonic SPT states with on-site symmetry G, a partial classification was first given by the group cohomology of the symmetry group $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$, where d is the space dimension (Chen, Gu *et al.*, 2013). Later, it was pointed out that the group cohomology description is incomplete when $d = 3$ and when G contains time-reversal symmetry (Vishwanath and Senthil, 2013; Wang and Senthil, 2013). Then it was realized that bosonic SPT states can all be classified by generalized group cohomology $\mathcal{H}^{d+1}(G \times SO_\infty, \mathbb{R}/\mathbb{Z})/\Gamma$. This implies that in $(1+1)$D and $(2+1)$D bosonic SPT states are classified by $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$ and $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$, respectively. In $(3+1)$D, bosonic SPT states are classified by $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ if the on-site symmetry G does not contain time reversal, and by $\mathcal{H}^4(G, (\mathbb{R}/\mathbb{Z})_T) \oplus \mathbb{Z}_2$ if G contains time reversal. Recent work also generalizes the cohomology classification of bosonic SPT states to translation and point-group symmetries (Hsieh *et al.*, 2014; You and Xu, 2014; Hermele and Chen, 2016; Lake, 2016; Thorngren and Else, 2016; Song *et al.*, 2017).

For noninteracting fermionic SPT states (Kane and Mele, 2005b; Bernevig, Hughes, and Zhang, 2006; Roy, 2006; Fu, Kane, and Mele, 2007; Moore and Balents, 2007; Qi, Hughes, and Zhang, 2008), there is a related classification of noninteracting gapped states based on K theory (Kitaev, 2009) or the nonlinear σ model of disordered fermions (Schnyder *et al.*, 2008) (see Tables III and IV). But such a classification does not apply to interacting fermions. For interacting fermionic SPT states (Wang, Potter, and Senthil, 2014; Wang and Senthil, 2014), there is a systematic understanding based on group supercohomology theory (Gu and Wen, 2014; Gaiotto and Kapustin, 2015; Kapustin and Thorngren, 2017; Wang and Gu, 2017), if the total symmetry group

TABLE III. Classification of the gapped phases of noninteracting fermions in d-dimensional space for some symmetries. The space of the gapped states is given by $C_{p+d \bmod 2}$, where p depends on the symmetry group. The distinct phases are given by $\pi_0(C_{p+d \bmod 2})$. "0" means that only trivial phases exist. $\mathbb{Z}$ means that nontrivial phases are labeled by nonzero integers and the trivial phase is labeled by 0. $U^f(1)$ means that the π rotation is $(-)^{N_F}$. Z_4^f is generated by C satisfying $C^2 = (-)^{N_F}$. Adapted from Wen, 2011.

Symmetry group	$C_p\vert_{\text{for } d=0}$	Class	$p\backslash d$	0	1	2	3	4	5	6	7	Example
$U^f(1)$ Z_4^f	$\frac{U(l+m)}{U(l)\times U(m)} \times \mathbb{Z}$	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	(Chern) insulator superconductor with collinear spin order
$U^f(1) \times Z_2^T$ $Z_4^f \times Z_2^T$	$U(n)$	AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	Superconductor with real pairing and S_z conserving spin-orbital coupling

has a form $G^f = G_b \times Z_2^f$. Here Z_2^f is the fermion-number-parity symmetry which is always present for fermion systems. Recently, a complete classification for all $(2+1)$D fermionic SPT states was found for generic on-site symmetry G^f which does not contain time reversal (Lan, Kong, and Wen, 2016b): $(2+1)$D fermionic SPT phases are classified by the modular extensions of sRep(G^f). Here sRep(G^f) is the symmetric fusion category formed representations of G^f, where the representations with nontrivial Z_2^f action are fermions. Last, we mention that, in addition to the cohomological and categorical approach, there is also a cobordism approach for bosonic or fermionic SPT states, which can lead to a classifying result for all dimensions and for some simple symmetries (Kapustin, 2014a, 2014b; Kapustin *et al.*, 2015).

Regarding point 3 in Sec. II, the quantum spin Hall effect refers to the quantized transverse S^z-spin current induced by force acting on electric charges (i.e., a quantized mixed electrospin Hall conductance) (Kane and Mele, 2005b; Bernevig and Zhang, 2006), while spin quantum Hall effect refers to quantized transverse S^z spin current induced by force acting on the "S^z charge" (i.e., a quantized spin Hall conductance). They have vanishing charge-Hall and thermo-Hall conductances. Under such definitions, the quantum spin Hall states (Kane and Mele, 2005a; Bernevig and Zhang, 2006) and topological insulators in $(2+1)$D (Kane and Mele, 2005b) (both appear in Table II) are different fermionic SPT states. They even have different symmetries: quantum spin Hall states have $[U_\uparrow(1) \times U_\downarrow(1)]^f$ symmetry, while topological insulators have $[U^f(1) \rtimes Z_4^T]/Z_2$ symmetry.[14]

Even though the topological insulator arises from the studies of the quantum spin Hall effect, it is incorrect to think the topological insulator is due to the quantum spin Hall effect. In particular, Kane and Mele (2005b), in "Z_2 Topological Order and the Quantum Spin Hall Effect," concluded that even without the quantum spin Hall effect, an insulator can still be nontrivial. This led to the notion of a topological insulator. This is a surprising discovery that started the active field of topological insulators. Despite the term "topological order" in the title, the topological insulator is a short-range entangled SPT state. It has no topological order as introduced by Wen and Niu (1990) and Wen (1990b), which involves long-range entanglement. This explains point 2 in Sec. II. Kane and Mele (2005b) deal only with noninteracting fermions in $(2+1)$D. Soon, it was shown that the $(2+1)$D topological insulator is stable against weak interactions (Wu, Bernevig, and Zhang, 2006; Xu and Moore, 2006).

With regard to the second part of point 2, many popular articles characterize topological insulators as insulators with conducting surfaces. Such a characterization is incorrect, since both a trivial insulator and a topological insulator can sometimes have conducting surfaces, and other times have insulating surfaces (for interacting electrons) (Chen, Fidkowski, and Vishwanath, 2013; Wang, Potter, and Senthil, 2013). Maybe it is more correct to say a "topological insulator is an insulator with conducting surface when electrons interact weakly." But even when electrons interact weakly, both a trivial insulator and a topological insulator can have conducting surfaces. We need to measure the Fermi surface to be sure (Hsieh *et al.*, 2008), but it does not work for $(2+1)$D topological insulators. So a more accurate characterization of $(2+1)$D topological insulators is that the charge-0 time-reversal symmetric π flux must be a Kramers doublet (Qi and Zhang, 2008; Ran, Vishwanath, and Lee, 2008).

V. TOWARD A CLASSIFICATION OF ALL GAPPED PHASES

Only for a few times in history have we completely classified some large class of matter states. The first time was the classification of all spontaneous symmetry-breaking orders, which can be classified by a pair of groups:

$$(G_\Psi \subset G_H), \tag{20}$$

where G_H is the symmetry group of the system and G_Ψ, a subgroup of G_H, is the symmetry group of the ground state.

[14]The superscript f means that the $U(1)$ groups contain Z_2^f as a subgroup. $U_{\uparrow,\downarrow}(1)$ is the symmetry of $\uparrow,\downarrow$-spin conservation, and $U^f(1)$ is the symmetry of charge conservation. Z_4^T is the group generated by time-reversal transformation T that satisfies $T^2 = (-)^{N_F}$ and $(-)^{N_F}$ is the fermion-number parity. After the discovery of the Z_2-topological invariant and the $(2+1)$D topological insulator (Kane and Mele, 2005b), the quantum spin Hall state sometimes was also defined as the $(2+1)$D topological insulator. Such a quantum spin Hall state has no quantum spin Hall effect nor spin quantum Hall effect, since even the S^z current is not conserved.

TABLE IV. Classification of gapped phases of noninteracting fermions in d spatial dimensions for some symmetries. The space of the gapped states is $R_{p-d \bmod 8}$, where p depends on the symmetry. The phases are classified by $\pi_0(R_{p-d \bmod 8})$. $\mathbb{Z}_2$ means that there is one nontrivial and one trivial phase labeled by 1 and 0. Note that $U^f(1) \rtimes Z_4^T \times Z_4^f / Z_2^2$ is the symmetry group generated by time reversal T, charge conjugation $c \to i\sigma^y c^\dagger$, and charge conservation. Adapted from Wen, 2011.

Symmetry group	$U^f(1)\rtimes Z_2^T$	$Z_2^T \times Z_2^f$	Z_2^f $Z_2 \times Z_2^f$	Z_4^T $Z_4^T \times Z_2$	$[U^f(1)\rtimes Z_4^T]/Z_2$ $[Z_4^f \rtimes Z_4^T]/Z_2$	$\frac{U^f(1)\rtimes Z_4^T \times Z_4^f}{Z_2^2}$	$SU^f(2)$	$\frac{SU^f(2)\times Z_4^T}{Z_2}$
$R_p\|_{\text{for } d=0}$	$\frac{O(l+m)}{O(l)\times O(m)} \times \mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)} \times \mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
Class	AI	BDI	D	DIII	AII	CII	C	CI
$d=0$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$d=1$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
$d=2$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
$d=3$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$d=4$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$d=5$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$d=6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$d=7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
Example	Insulator with coplanar spin order	Superconductor with coplanar spin order	Superconductor	Superconductor with time reversal	Insulator with time reversal	Insulator with time reversal and intersublattice hopping	Spin singlet superconductor	Spin singlet superconductor with time reversal

This includes the classification of all 230 crystal orders in three dimensions.

The second time was the classification of all gapped one-dimensional quantum states: gapped one-dimensional quantum states with on-site symmetry G_H can be classified by a triple (Chen, Gu, and Wen, 2011a; Schuch, Perez-Garcia, and Cirac, 2011):

$$[G_\Psi \subset G_H; \quad \mathrm{pRep}(G_\Psi)], \tag{21}$$

where $\mathrm{pRep}(G_\Psi)$ is a projective representation of G_Ψ.

The third time was the classification of all gapped quantum phases in $(2+1)$D. Since early on, it was conjectured that all $(2+1)$D bosonic topological orders (without symmetry) are classified by S and T modular matrices [plus a $U(1)$ gauge connection] (Wen, 1990b), or more precisely by a pair (Kitaev, 2006; Rowell, Stong, and Wang, 2009; Wen, 2016):

$$(\mathrm{MTC}, c), \tag{22}$$

where MTC is a unitary modular tensor category and c is the chiral central charge c of the edge states. Recently, this result was generalized to fermion systems: $(2+1)$D fermionic topological orders are classified by a triple (Lan, Kong, and Wen, 2016a)

$$[\mathrm{sRep}(Z_2^f) \subset \mathrm{BFC}; c], \tag{23}$$

where $\mathrm{sRep}(Z_2^f)$ is the symmetric fusion category (SFC) formed by the representations of the fermion-number-parity symmetry Z_2^f, where the nontrivial representation is assigned Fermi statistics, and BFC is a unitary braided fusion category.

For quantum systems with symmetry, we have the following result: all $(2+1)$D gapped bosonic phases with a finite unitary on-site symmetry G_H are classified by (Barkeshli *et al.*, 2014; Lan, Kong, and Wen, 2016b)

$$[G_\Psi \subset G_H; \mathrm{Rep}(G_\Psi) \subset \mathrm{BFC} \subset \mathrm{MTC}; c], \tag{24}$$

where $\mathrm{Rep}(G_\Psi)$ is the SFC formed by the representations of G_Ψ, where all representations are assigned Bose statistics, and MTC is a minimal modular extension of the BFC. This classification includes symmetry-breaking orders, SPT orders, topological orders, and symmetry-enriched topological (SET) orders described by the projective symmetry group (Wen, 2002). SET orders of time-reversal or reflection symmetry are classified by Barkeshli *et al.* (2016). More discussions on SET orders can be found in Hung and Wan (2013), Hung and Wen (2013), Lu and Vishwanath (2013), Mesaros and Ran (2013), Xu (2013), and Chang *et al.* (2015).

We have a similar result for fermion systems: all $(2+1)$D gapped fermionic phases with unitary finite on-site symmetry G_H^f are classified by (Lan, Kong, and Wen, 2016b)

$$[G_\Psi^f \subset G_H^f; \mathrm{sRep}(G_\Psi^f) \subset \mathrm{BFC} \subset \mathrm{MTC}; c], \tag{25}$$

where $\mathrm{sRep}(G_\Psi^f)$ is the SFC formed by the representations of G_Ψ^f, where some representations are assigned Fermi statistics. However, we are still struggling to obtain a systematic theory of topological order in $(3+1)$D, 28 years after the introduction of the concept.

Those results imply that the long-range entanglement in $(2+1)$D is described by an unfamiliar mathematics—tensor category theory. This is the mathematics for the quantum topology, and it is the quantum topology (instead of classical topology) that forms the mathematical foundation of topological order (i.e., long-range entanglement). This explains the title of this paper "quantum-topological phases of matter," which really means "highly entangled phases of matter."

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