A Superadditive Solution

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Abstract

We present a superadditive bargaining solution defined on a class of polytopes in $\mathbb{R}^n$. The solution generalizes the superadditive solution exhibited by MASCHLER and PERLES.

1 Introduction

The Maschler–Perles bargaining solution (Maschler–Perles [4], [5], see also [9] for a textbook presentation) is a mapping defined on 2–dimensional bargaining problems respecting anonymity, Pareto efficiency, and affine transformations of utility. Moreover, this mapping is *superadditive* by which property it is uniquely characterized.

The solution is based on the observation that a *polyhedral* bargaining problem in $\mathbb{R}^2$ is an algebraic sum of "elementary" bargaining problems that are generated by a line segment. By continuity with respect to the Hausdorff metric the solution is then extended to bargaining problems with a smooth Pareto boundary.

Actually it suffices to discuss the solution on the particular class of bargaining problems, which are algebraic sums of line segments generating triangles with equal area. Superadditivity is then a rather immediate matter and continuity suffices fot the extension as previously (see [10]).

By various reasons this solutions has never been very popular compared with, say, the Nash solution. Indeed, Perles [8] proved that a superadditive bargaining solution does not exist for more than 2 players (i.e., bargaining problems in 3 and more dimensions), this is certainly a drawback. Calvo–Gutierrez (see [2]) presented an extension to n–person games, they generalized a procedure to compute the solution but produced no examples. Of course, their approach cannot yield a superadditive solution.

Apart from this, the Maschler–Perles solution requires some techniques in order to be extended to smooth bargaining problems. This procedure and the resulting interpretation of two points travelling along the Pareto curve with a specified velocity, may add to the resistance econimists seem to offer to the concept.

On the other hand one would think that superadditivity is at least as attractive as the IIA axiom characterizing the Nash bargaining solution. Maschler–Perles argue that it supports agreements *ex ante* when the players face lotteries over bargaining problems. In addition, there is the natural interpretation of bargaining *simultaneously* in two environments, meaning that all sums of utility vectors of the two problems are available. Hence, treating the joint bargaining procedures as a single one improves the situation of both players exactly if superadditivity of the solution concept prevails.

Conspicuously, the other outstanding solution emerging in cooperative game theory, the Shapley value, is characterized by additivity (which is interpreted as risk neutrality towards lotteries over games).

Based on this, one should naively expect that generalizing the Shapley value to NTU–games starts out with a superadditive solution. Yet, beginning with Shapley's ([12]) approach and culminating in Aumann's axiomatization [1], many authors have made an effort to justify the Nash solution as the two

person bargaining analogue to the Shapley value from which one is to set out for the genereal NTU approach. Can it be outright rejected that this, at least partially, is also motivated by PERLES counter example?

The Maschler–Perles (superadditive) solution certainly does have its merits and we feel that it is worthwile to attempt a generalization in spite of the many obstacles prevailing. It is the result of this work that a superadditive solution *can* be established on a suitable class of bargaining problems, though certainly not on the full class.

Thus, we produce a class of polyhedral bargaining problems in $\mathbb{R}^n$ which admits of a superaditive bargaining solution. This class is the natural one in the following sense: in two dimensions, the slopes of all simplices (line segments) involved in representing a polyhedral bargaining problen can be *ordered* – in higher dimensions this has to be made a *requirement* defining the appropriate family of bargaining problems. In two dimensions, the "speed of concessions" is determined by the *area of the prisms* generated by line segments. If instead, one regards this as a *density* of a surface measure, then the generalization to arbitrary dimensions is at hand. We can then map the Pareto surface onto a simplex such that the surface measure is translated into a measure absotely continuous with respect to Lebesgue measure. The pre–image of the center of gravity with respect to this measure is the generalized Maschler–Perles solution and it behaves superadditively.

In order to discuss this class a thorough discussion exhibiting the suitable type of convex polyhedra to deal with is inevitable. In two dimensions it is rather obvious that any polyhedral bargaining problem is a sum of triangles. Beginning with three dimensions, this is statement is false. The class of algebraic ("Minkowski") sums of comprehensive simplices (*prisms* as we shall call them) is quite involved and deserves some attention by its own. This leads to a foundation of polyhedra called *cephoids*, a topic treated in [6]. Within that paper we provide the general classification of sums of prisms.

The present paper starts out with a short discussion of cephoids, recalling some facts from [6], but also providing some arguments for the standard class of bargaining problems we have in mind (SECTION 2). In SECTION 3 we introduce the surface measure and in SECTION 4 we introduce the solution concept and prove its superadditivity. SECTION 5 provides some examples and counterexamples. SECTION 6 contains the discussion of a subclass of bargaining problems on which the solution is even unique.

2 Cephoids

A ***bargaining solution*** is given by a ***feasible set*** $U \subseteq \mathbb{R}^n$ of utility vectors (with some regularity conditions) and a ***status quo point*** $\overline{\boldsymbol{x}} \in U$. Players may agree to allot utilities according to some point in U or else will receive their share according to the status quo. A ***bargaining solution*** is a mapping defined on some class of bargaining problems and resulting in a (feasible, Pareto efficient) element of U. The mapping is required to commute with *affine transformation of utility* (simple translation and dilatation of the axis). Therefore, it is common to restrict the discussion to the status quo point $\mathbf{0}$ – in which case it is not mentioned furthermore. Also, one is interested in "individually rational" utilities only, which results in discussing only the nonnegative elements of the feasible set. This will be our viewpoint henceforth.

Thus, we consider just feasible sets $U \subseteq \mathbb{R}^n_+$, which are supposed to be compact, convex and comprehensive (i.e., containing the southwest orthant generated by any of its points). Among these we single out polyhedral sets of a certain type as follows.

Let $\boldsymbol{I} := \{1, \ldots, n\}$ denote the set of players (hence the coordinates of $\mathbb{R}^n$, the "utility space"). Let $\boldsymbol{e}^i$ be the i^{th} unit vector of $\mathbb{R}^n$ $(i \in \boldsymbol{I})$. For any positive vector $\boldsymbol{a} = (a_1, \ldots, a_n) > 0 \in \mathbb{R}^n$ put $\boldsymbol{a}^i := a_i \boldsymbol{e}^i$ $(i \in \boldsymbol{I})$ and associate with $\boldsymbol{a}$ the ***prism*** $\Pi^{\boldsymbol{a}}$ which is given by

$$\Pi^{\boldsymbol{a}} := \operatorname{convH}\left(\left\{\mathbf{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}\right). \tag{2.1}$$

The (outward) face of this prism is the ***simplex*** $\Delta^{\boldsymbol{a}}$ which is given by

$$\Delta^{\boldsymbol{a}} := \operatorname{convH}\left(\left\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n, \right\}\right). \tag{2.2}$$

For any $\boldsymbol{J} \subseteq \boldsymbol{I}$ we obtain the ***subprism*** of $\Pi^{\boldsymbol{a}}$ given by

$$\Pi^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Pi^{\boldsymbol{a}} \mid x_i = 0 \ (i \notin \boldsymbol{J})\}, \tag{2.3}$$

a similar notation is used for the simplex $\Delta^{\boldsymbol{a}}$, we write

$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Delta^{\boldsymbol{a}} \mid x_i = 0 \ (i \notin \boldsymbol{J})\} \tag{2.4}$$

for the subface generated by the coordinates $i \in \boldsymbol{J}$.

The ***Minkowski*** or ***algebraic sum*** of two sets $U, V \subseteq \mathbb{R}^n$ is

$$U + V := \{\overline{\boldsymbol{x}} + \overline{\boldsymbol{y}} \mid \overline{\boldsymbol{x}} \in U, \ \overline{\boldsymbol{y}} \in V\},$$

the general version for summing a finite family of sets is provided ananlogously.

Now we have the following definition (see [6]).

Definition 2.1. *Let* $\boldsymbol{a}^\bullet = (\boldsymbol{a}^{(k)})_{k=1}^K$ *denote a family of positive vectors and let*

$$\Pi = \Pi^{\boldsymbol{a}^\bullet} := \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} \tag{2.5}$$

be the algebraic sum. Then Π *is called a* ***cephoid****.*

The representation of a cephoid by means of a family $\boldsymbol{a}^\bullet$ is in general not unique. In particular, any prism is a sum of "homothetic" copies of itself, e.g., for some $\boldsymbol{a} > \mathbf{0}$

$$\Pi^{\boldsymbol{a}} = \frac{1}{2}\Pi^{\boldsymbol{a}} + \frac{1}{2}\Pi^{\boldsymbol{a}} = \Pi^{\frac{1}{2}\boldsymbol{a}} + \Pi^{\frac{1}{2}\boldsymbol{a}}$$

We provide some examples of cephoids.

Example 2.2. Consider two positive vectors $\boldsymbol{a} = (1, 3, 2)$ and $\boldsymbol{b} = (2, 1, 3)$ and the the two prisms $\Pi^{\boldsymbol{a}}$, $\Pi^{\boldsymbol{b}}$ generated. The cephoid $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ is depicted in Figure 2.1. There are copies of the two generating simplices on the surface of Π, these are given by $\Delta^{\boldsymbol{a}}+\boldsymbol{b}^1$ and $\Delta^{\boldsymbol{b}}+\boldsymbol{a}^2$. The "diamond" is the sum $\Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$ of two lower dimensional faces. It is important that, for the sum of two Pareto efficient points to be Pareto efficient, it is necessary and sufficient that a joint normal exists to both points which, as a consequence, establishes a joint normal to the sum as well.

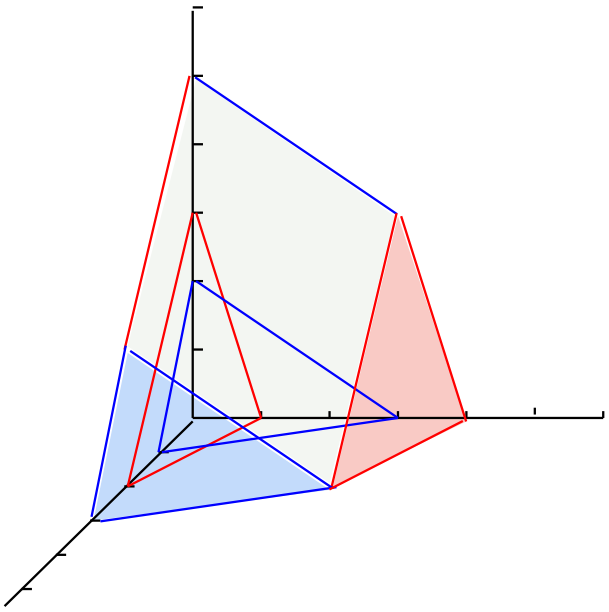

Figure 2.1: Adding two prisms

The representation of Π indicated in Figure 2.1 *is unique* if one requires in addition a minimal sum of summands.

Example 2.3. Now consider Figure 2.2. This is the type of cephoid used in the counter example of Perles [8]. The family $\boldsymbol{a}^\bullet$ is *degenerate* in a well defind sense inasmuch as the two subfaces $\Delta^{\boldsymbol{a}}_{12}$ and $\Delta^{\boldsymbol{b}}_{12}$ are parallel – the normal cone is the same. As a consequence, the surface Δ of Π again consists of two translates of the simplices involved plus a "diamond"– but not uniquely so.

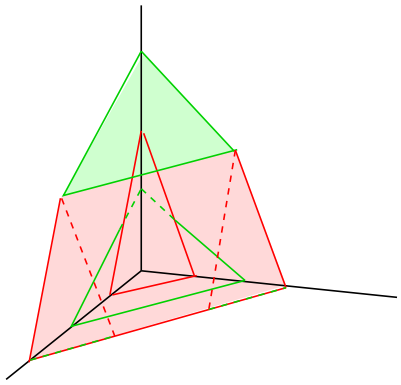

Figure 2.2: The sum of two prisms – parallel subfaces

Example 2.4. In Figure 2.3 the sum of three prisms constitutes a cephoid Π. The generating family of vectors is given by $\boldsymbol{a} = (1, 3, 2)$, $\boldsymbol{b} = (2, 1, 3), \boldsymbol{c} = (3, 2, 1)$. Note that the slopes of two dimensional faces in each x_ix_j-plane are ordered in a cyclic way.

The sum Π shows a translate of each simplex located in the appropriate corner. In addition, there appear three "dimonds", each of them being the sum of two subfaces of the simplices involved. The central vertex is the sum of three vertices of the simplices involved.

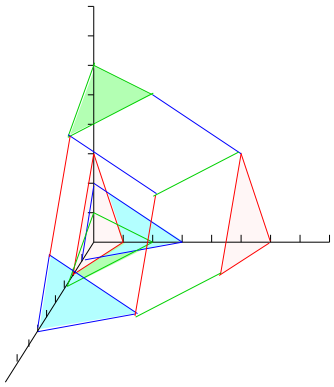

Figure 2.3: The sum of three prisms

Now we add a fourth prism having a joint normal with the central vertex. The result is the cephoid indicated in Figure 2.4. It shows certain symmetries, the translate of the new triangle having replaced the central vertex.

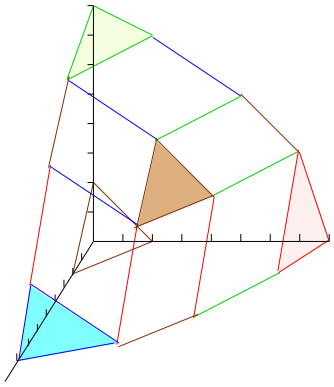

Figure 2.4: Adding a further prism

Continuing this procedure we end up with a general picture of a cephoid in 3 dimensions that looks like Figure 2.5.

We can imagine that this cephoid is the sum of finitely many polyhedra as depicted in Figure 2.3. The construction works under the assumption that there is a joint tangency at each of the central vertices involved. We believe that this is "the" generalization of the two dimensional construction used in the context of establishing the Maschler–Perles solution.

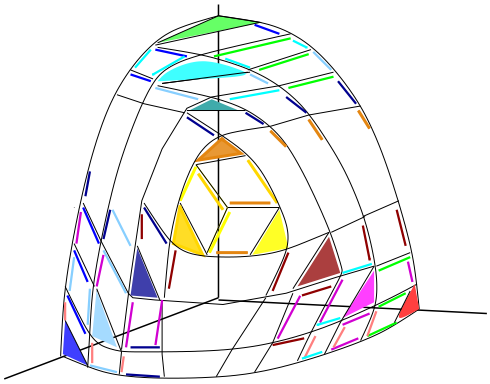

Figure 2.5: The general cephoid in $\mathbb{R}^3$

For dimensions exceeding 3 the picture involves not just simplices and diamonds. In each dimension new types of polyhedra appear on the surface, being the sum of certain subfaces of the prisms involved. The details are found in [6].

A cephoid shows a certain surface structure which is the partially ordered set ("poset") of polyhedra (translates of simplices, diamonds etc.) This structure can be "canonically" represented on a simplex as is explained in [6].

For instance, Figure 2.6 represents the surface structure of the cephoid in Figure 2.1 and Figure 2.7 has a similar surface structure as Figure 2.4. In both cases a multiple of the unit simplex is suitably decomposed in order to reflect the decomposition of the surface of the cephoid under consideration. If we are interested in the structure only, Figure 2.7 is appropriate as we have

chosen all simplices to be of equal area. This is the "canonical representation" used in [6] to classify cephoids.

In the present paper we will employ a "measure preserving representation". E.g., Figure 2.6 contains additional information as the simplices have differing area and the diamond is adapted suitably. This can be thought of as the representation of a modified version of Figure 2.1 such that the prisms involved have differing volumes.

Recall that the two dimensional Maschler–Perles solution involves a bijection of the Pareto curve onto a suitable interval, the length of the representation of each line segment depending on the area of the triangle generated by this line segment. Suitably, the representation in higher dimensions will involve a "surface measure" on each simplex which involves the *volume* of the prism generated. In both cases, the solution is obtained by taking the center of gravity of the representing simplex and constructing its pre–image.

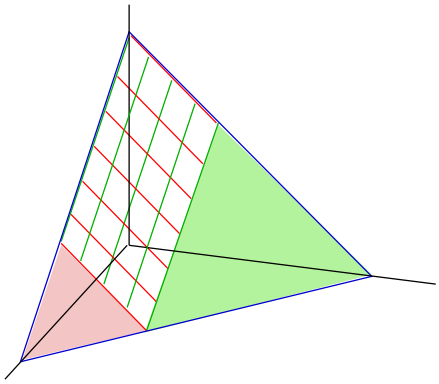

Figure 2.6: Representing the sum of two prisms

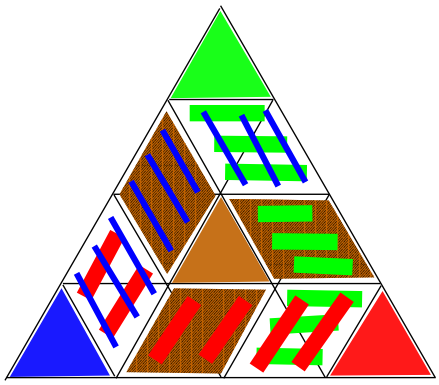

Figure 2.7: Structure of a sum of four prisms

The planar case is in some way "degenerate", yet it serves to represent the surface structure of a cephoid. Uniqueness of the representation requires some type of nondegeneracy or "general position" of the members of a family $\boldsymbol{a}^\bullet$. In two dimensions this means that the tangents or normals of the line segments involved never coincide, thus it could be expressed by a requirement

$$\frac{a_1^{(k)}}{a_2^{(k)}} \neq \frac{a_1^{(l)}}{a_2^{(l)}} \tag{2.6}$$

for any pair $l, k \in \boldsymbol{K}$. In other words, any two subsimplices of dimension 1 are not parallel, therefore the cone of normals is not identical.

In more than two dimensions we have to make sure that the dimension of the cone of joint normals two any family of subsimplices has the minimal dimension. The precise formulation has been given in [6] and is repeated below:

Definition 2.5. *A family $\boldsymbol{a}^\bullet = (\boldsymbol{a}^{(k)})_{k=1}^K$ of positive vectors (as well as the cephoid generated) is said to be* ***nondegenerate*** *if the following conditions hold true:*

1. *For any system of nonempty index sets $\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)} \subseteq \boldsymbol{I}$ with*

$$\bigcup_{k \in \boldsymbol{K}} \boldsymbol{J}^{(k)} \;=\; \boldsymbol{I}$$

the system of linear homogeneous equations in the variables $x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_K$ given by

$$a_i^{(k)} x_i - \lambda_k = 0 \quad (i \in \boldsymbol{J}^{(k)},\ k \in \boldsymbol{K}) \tag{2.7}$$

has a space of solutions $\mathbb{U}$ of dimension

$$\dim \mathbb{U} = n + K - \sum_{k \in \boldsymbol{K}} j_k \tag{2.8}$$

with $j_k = |\boldsymbol{J}^{(k)}|$.

2. *For any $\boldsymbol{I}^{(0)} \subseteq \boldsymbol{I}$ the restricted system*

$$\boldsymbol{a}^\bullet\Big|_{\boldsymbol{I}^{(0)}} := \left(\boldsymbol{a}^{(k)}\Big|_{\boldsymbol{I}^0}\right)_{k \in \boldsymbol{K}} \tag{2.9}$$

obtained by restricting the vectors to $\boldsymbol{I}^{(0)}$ satisfies the condition of item *1 in the subspace $\mathbb{R}^{\boldsymbol{I}^{(0)}}$.*

The structure of cephoids generated by nondegenerate families of prisms is exhibited in [6]. The structure of a cephoidal surface is at best represented on (a multiple of) the unit simplex: there is a "canonical" mapping between the two surfaces preserving the partially ordered set of faces. (E.g. Figure 2.7 is the "canonical representation" of Figure 2.4).

Within this paper we replace the "canonical representation" by the "measure preserving representation" in order to imitate the two–dimensional construction of the Maschler–Perles solution. This way the size of the various triangles and diamonds (maximal faces in higher dimensions) depends on the *volume* of the prisms involved.

3 The Surface Measure

We shall use the *volume* in order to define a measure on the *surface* of a cephoid. We start out with a prism. Let $\boldsymbol{a} = (a_1, \ldots, a_n) > 0$ be a positive vector and let $\Pi^{\boldsymbol{a}}$ be the prism associated, the surface is the simplex $\Delta^{\boldsymbol{a}}$.

The volume of $\Pi^{\boldsymbol{a}}$ is

$$V(\Pi^{\boldsymbol{a}}) = \frac{\prod_{i \in \boldsymbol{I}} a_i}{n!}.$$

We shall associate a surface measure of

$$\sqrt[n]{(n!)^{n-1}(V(\Pi^{\boldsymbol{a}}))^{n-1}} \;=:\; \sqrt[n]{v_n(V(\Pi^{\boldsymbol{a}}))^{n-1}} \tag{3.1}$$

to any translate of the surface $\Delta^{\boldsymbol{a}}$. In particular, the simplex $\Delta^{\boldsymbol{e}}$ (the surface of the unit prism $\Pi^{\boldsymbol{e}}$) receives surface measure 1.

Next, let $\boldsymbol{J} = (\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)})$ be a system of index sets that may determine a face

$$\boldsymbol{F} = \Delta^{(1)}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{(K)}_{\boldsymbol{J}^{(K)}} \tag{3.2}$$

of a cephoid. Then the numbers $j_l := |\boldsymbol{J}^{(l)}|$ satisfy

$$(j_1 - 1) + \ldots + (j_K - 1) = n - 1\ , \quad j_1 + \ldots + j_K = n + K - 1\ . \tag{3.3}$$

This is a consequence of the nondegeneracy assumption (see [6]). Consider the Minkowski sum

$$\Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(K)}}\ . \tag{3.4}$$

The (Lebesgue) surface measure of this convex compact polyhedron is a multiple of the surface of the unit simplex, this multiple is denoted by $c_{\boldsymbol{J}}$. Of course the number depends on $j_1, \ldots, j_K$ only and not on the ordering of these indices. Thus we write

$$c_{\boldsymbol{J}} \;=\; c_{j_1,\ldots,j_K} \;:=\; \frac{\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(K)}})}{\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}})}, \tag{3.5}$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure. E.g., for $n = 3$ two triangles will fit into a diamond, hence $c_{13} = 1, c_{22} = 2$. For $n = 4$ three tetrahedra just fill a cylinder and two cylinders fill a cube, hence $c_{114} = c_{141} = c_{411} = 1$, $c_{123} = ... = 3$, and $c_{222} = 6$, etc.

In passing we remark that the coefficient $c_{j_1,\ldots,j_K}$ is the volume of the convex body

$$\begin{aligned} &\mathrm{ConvH}\{0, \boldsymbol{e}^1, \ldots, \boldsymbol{e}^{j_1-1}\} \times \mathrm{ConvH}\{0, \boldsymbol{e}^{j_1}, \ldots, \boldsymbol{e}^{j_1+j_2-1}\} \\ &\times \mathrm{ConvH}\{0, \boldsymbol{e}^{j_1+j_2}, \ldots, \boldsymbol{e}^{j_1+j_2+j_3-1}\} \times \ldots \\ &\ldots \times \mathrm{ConvH}\{0, \boldsymbol{e}^{j_1+j_2\ldots+j_{K-1},}, \ldots, \boldsymbol{e}^{j_1+j_K}\}. \end{aligned} \tag{3.6}$$

This follows from the fact that the subsimplices involved are located in orthogonal subspaces.

Having obtained the above defined "normalizing coefficients" we can now proceed by defining a surface measure on any face of a cephoid.

Definition 3.1. *Let $\boldsymbol{a}^\bullet$ be a positive family of vectors and let $\boldsymbol{F}$ be a maximale face represented via a family of index sets $\boldsymbol{J}$ by*

$$\boldsymbol{F} \;=\; \Delta^{(1)}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{(K)}_{\boldsymbol{J}^{(K)}}. \tag{3.7}$$

*Then the **surface measure** associated with $\boldsymbol{F}$ is given by*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{F}) = c_{\boldsymbol{J}} \sqrt[n]{(v_n)\,[V(\Pi^{(1)})]^{j_1-1} \cdot \ldots \cdot [V(\Pi^{(K)})]^{j_K-1}}. \tag{3.8}$$

Some motivation for this definition can be found in formula (3.6). Moreover, within the following lemma we list some obvious properties of the surface measure. This shows that the surface measure exhibits the "appropriate behaviour".

Lemma 3.2.

1. *For $\boldsymbol{t} = (t_1, \ldots, t_K) > 0$ and $\boldsymbol{t}\boldsymbol{a}^{(\bullet)} = (t_k \boldsymbol{a}^{(k)})_{k \in \boldsymbol{K}}$ let $\boldsymbol{t}\boldsymbol{F}$ denote the face corresponding to a face $\boldsymbol{F}$. Then*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{t}\boldsymbol{F}) \;=\; t_1^{j_1-1} \cdot \ldots \cdot t_K^{j_K-1} \boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{3.9}$$

2. *In particular, for $\boldsymbol{t} = (\varepsilon, \ldots, \varepsilon)$, we obtain from* (3.3)

$$\boldsymbol{\iota}_\Delta(\varepsilon \boldsymbol{F}) \;=\; \varepsilon^{n-1} \boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{3.10}$$

   Equations (3.9) *and* (3.10) *show that $\boldsymbol{\iota}_\Delta(\bullet)$ behaves like the Lebesgue measure of the surface.*

3. *If, for some family $\boldsymbol{a}^\bullet$, we have $\boldsymbol{a}^{(1)} = \ldots = \boldsymbol{a}^{(K)}$, then it follows that a face $\boldsymbol{F}$ represented by* (3.2) *satisfies*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{F}) \;=\; c_{\boldsymbol{J}} \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^{(1)}}) \;. \tag{3.11}$$

4. *More generally, if for some family $\boldsymbol{a}^\bullet$ the volumes satisfy*

$$V(\Pi^{\boldsymbol{a}^{(1)}}) = \ldots = V(\Pi^{\boldsymbol{a}^{(K)}}) \;,$$

   then it follows that a face $\boldsymbol{F}$ represented by (3.2) *satisfies*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{F}) \;=\; c_{\boldsymbol{J}} \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^{(1)}}). \tag{3.12}$$

Proof: The first three items are obtained by obvious computions with volumes and surface areas involving the definition (3.8). The last item is a consequence of the convention established by (3.5). Accordingly, any face has a surface measure which is the appropriate multiple of the surface measure of a simplex generated by the same family of vectors.

q.e.d.

Corollary 3.3. *Let $\boldsymbol{a}^\bullet$ be a family of vectors and let Π, Δ be the cephoid generated and its surface. Let $\boldsymbol{F}$ be a maximal face of Δ represented by $\boldsymbol{J}$ as in* (3.2)*. Then there is a measure $\boldsymbol{\iota}_\Delta$ defined on $\boldsymbol{F}$ which satisfies* (3.8)*, has the properties stated in Lemma 3.2, and is continuous as a function on families $\boldsymbol{a}^\bullet$.*

Proof: For small $\varepsilon_k,\ 0 < \varepsilon_k < 1\ (k \in \boldsymbol{K})$ and $\boldsymbol{x}^{(k)} \in \Delta^{(k)}_{\boldsymbol{J}^{(k)}},\ (k \in \boldsymbol{K})$ let

$$\boldsymbol{F}^{\boldsymbol{\varepsilon}} \ := \ \sum_{k \in \boldsymbol{K}} (1 - \varepsilon_k)\boldsymbol{x}^{(k)} + \varepsilon_k \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \ .$$

Then $\boldsymbol{F}^{\boldsymbol{\varepsilon}}$ is a shrinked copy of $\boldsymbol{F}$ the surface measure $\boldsymbol{\iota}_\Delta$ of which is defined by (3.9). Every face can be decomposed into a union of simplices the number of which is $c_{\boldsymbol{J}}$ (see (3.5)). Decomposing any simplex into copies of sufficiently small $\boldsymbol{F}^{\boldsymbol{\varepsilon}}$ we obtain a σ–additive setfunction $\boldsymbol{\iota}_\Delta$ with the desired properties (the σ–algebra is generated by the relative topology).

On each face, the measure $\boldsymbol{\iota}_\Delta$ is actually a multiple of the Lebesgue measure $\boldsymbol{\lambda}$ and in view of *item 3* of Lemma 3.2, $\boldsymbol{\iota}_\Delta$ behaves compatibly when evaluated on different faces.

q.e.d.

Example 3.4. Consider the sum of two prisms. Let $\boldsymbol{a}, \boldsymbol{b} > 0$ and consider the polyhedron $\Pi^{\boldsymbol{ab}} \ := \ \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$. Figure 3.1 shows the situation. The two vectors are in general position. The surface consists of the translates $\boldsymbol{b}^1 + \Pi^{\boldsymbol{a}}$

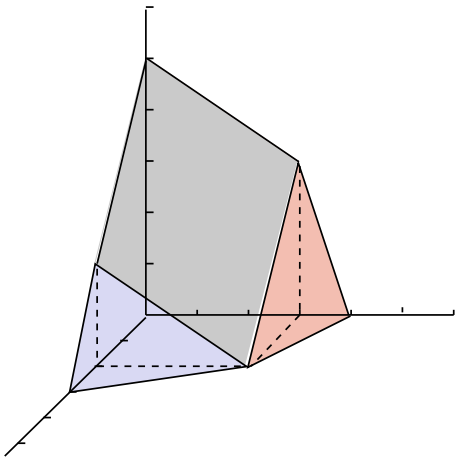

Figure 3.1: The sum of two prisms

and $\boldsymbol{a}^2 + \Pi^{\boldsymbol{b}}$ and the "diamond"

$$\Lambda^{\boldsymbol{ab}} \;:=\; \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13} \tag{3.13}$$

which is the sum of the two subsimplices indicated.

Now we decompose the surface of $\Pi^{\boldsymbol{ab}}$ into certain triangles as indicated in Figure 3.2. The surface $\Lambda^{\boldsymbol{ab}}$ is decomposed into 4 equal diamonds. The $\boldsymbol{\iota}_\Delta$–measure of these is clearly $\frac{1}{4}$ of the $\boldsymbol{\iota}_\Delta$–mesure of $\Lambda^{\boldsymbol{ab}}$. Each diamond in turn has the area which is twice the area of a simplex.

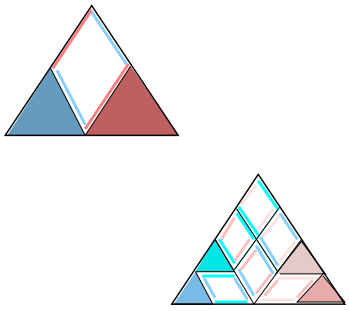

Figure 3.2: Scheme of the decomposition of the surface of $\Pi^{\boldsymbol{ab}}$

Figure 3.2 provides sketch of the decomposition of the surface of $\Pi^{\boldsymbol{ab}}$. This sketch refers to a sum of homothetic multiples of the unit simplex, but the situation is structurally the same as the one for a decomposition of the surface in Figure 3.1.

Definition 3.5. *We call the measure $\boldsymbol{\iota}_\Delta$ the **surface measure**.*

Remark 3.6.

1. *Let $\boldsymbol{e} := (1, \ldots, 1)$ The measure $\boldsymbol{\iota}_\Delta$ on $\Delta^{\boldsymbol{e}}$ is the Lebesgue measure $\boldsymbol{\lambda}$ normalized to $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{e}}) = 1$.*
2. *A sum of homothetic prism is a multiple of one of those prisms. While the surface structure is not unique, the surface measure is seen to be a multiple of Lebesgue measure – independently on a homothetic decomposition and the surface structure. The measure $\boldsymbol{\iota}_\Delta$ behaves consistently with any surface structure.*

The similarity between the structure of the surface of $\Pi^{\boldsymbol{ab}}$ and the homothetic sum represented in Figure 3.1 will now be formalized. We create a mapping which carries the surface of a cephoid onto the one of a suitable multiple of the unit simplex such that the surface measure is transformed into Lebesgue measure.

Consider a family $\boldsymbol{a}^\bullet$. For every $k = 1, \ldots, K$ let $\alpha_k := \boldsymbol{\iota}_\Delta(\Delta^{(k)})$. Define

$$\widehat{\boldsymbol{a}}^{(k)} \;:=\; \sqrt[n-1]{\alpha_k}\,\boldsymbol{e} \quad (k \in \boldsymbol{K}) \tag{3.14}$$

and put

$$\widehat{\Pi}^{(k)} := \Pi^{(\widehat{\boldsymbol{a}}^{(k)})} := \Pi^{\sqrt[n-1]{\alpha_k}\boldsymbol{e}} \quad , \quad \widehat{\Delta}^{(k)} := \Delta^{(\widehat{\boldsymbol{a}}^{(k)})} := \Delta^{\sqrt[n-1]{\alpha_k}\boldsymbol{e}} \tag{3.15}$$

such that

$$\boldsymbol{\iota}_{\Delta}(\widehat{\Delta}^{(k)}) = \alpha_k \boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{e}}) = \alpha_k \tag{3.16}$$

holds true for $k \in \boldsymbol{K}$. Also, let

$$\widehat{\alpha} \;:=\; \sum_{k=1}^{K} \sqrt[n-1]{\alpha_k} \quad , \quad \widehat{\Delta} \;:=\; \Delta^{\widehat{\alpha}\boldsymbol{e}} \quad , \quad \widehat{\Pi} \;:=\; \widehat{\Pi}^{\widehat{\alpha}\boldsymbol{e}}. \tag{3.17}$$

We shall arrange the surface structure of $\widehat{\Pi}$ in a way such that the surface structure of Π is preserved. This is achieved by mapping the extremals of the faces of Δ bijectively onto certain corresponding vectors of $\widehat{\Delta}$ such that the surface measure is transported into the Lebesgue measure.

To this end, consider a family $\boldsymbol{a}^{\bullet}$ in general position. By nondegeneracy every vertex is a unique sum of vertices of the $\Delta^{\boldsymbol{a}^{(k)}}$ involved. More precisely, for every vertex $\boldsymbol{u}$ of Δ, there is a unique mapping $\mathbf{i}_{\bullet}$ such that $\boldsymbol{u}$ can be written via

$$\begin{aligned} \mathbf{i}_{\bullet} \;:\quad & \boldsymbol{K} \to \boldsymbol{I} \\ \boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}_{\bullet}} \;:=\; & \sum_{k \in \boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \;. \end{aligned} \tag{3.18}$$

Now we have

Definition 3.7. *1. Let $\boldsymbol{u}$ be a vertex on Δ and let $\mathbf{i}_{\bullet}$ be the corresponding mapping as described by* (3.18)*. Then*

$$\widehat{\boldsymbol{u}} \;:=\; \kappa(\boldsymbol{u}) \;:=\; \sum_{k \in \boldsymbol{K}} \widehat{\boldsymbol{a}}^{(k)\mathbf{i}_k} \tag{3.19}$$

is the ***measure preserving representation*** *of $\boldsymbol{u}$ on $\widehat{\Delta}$.*

2. Let $\boldsymbol{F}$ be a face of Δ and let $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^L$ be its extremal points. Then the convex hull of the images, i.e.,

$$\kappa(\boldsymbol{F}) \;:=\; \widehat{\boldsymbol{F}} \;:=\; \mathrm{ConvH}\{\kappa(\boldsymbol{u}^1), \ldots, \kappa(\boldsymbol{u}^L)\}, \tag{3.20}$$

is the ***measure preserving representation*** *of $\boldsymbol{F}$ on $\widehat{\Delta}$.*

3. Let $\mathcal{V}$ be the poset of faces of Δ and let

$$\widehat{\mathcal{V}} \;:=\; \kappa(\mathcal{V}) \;:=\; \{\kappa(\boldsymbol{F}) \,\big|\, \boldsymbol{F} \in \mathcal{V}\} \tag{3.21}$$

be the collection of images of faces under the mapping κ. Then $\widehat{\mathcal{V}}$ is the ***measure preserving representation*** *of $\mathcal{V}$ on $\widehat{\Delta}$.*

Theorem 3.8. *$\widehat{\mathcal{V}}$ is a poset which is isomorphic to $\mathcal{V}$. Hence $(\Delta, \mathcal{V})$ and $(\widehat{\Delta}, \widehat{\mathcal{V}})$ are combinatorically equivalent.*

This is a standard procedure in convex geometry (see [3]). The mapping κ is bijective between the vertices of Δ and the appropriate subset of grid vectors as described in equations (3.18) and (3.19). The minimum of two faces (whenever it exists) is obtained by taking the intersection of the corresponding two sets of extremal points. Similarly, if the maximum of two faces exists, then it is obtained via the union of the sets of extremal points.

The canonical representation is the suitable projection of the outer surface Δ of a cephoid Π on an $n-1$–dimensional subset. E.g., the poset of faces of Figures 2.7 and 2.3 are combinatorically equivalent. Also, we can visualize the surface of 4–dimensional cephoids on a suitable multiple of the unit simplex of $\mathbb{R}^3$ (a tetrahedron), which will serve to discuss several important examples in SECTION 5.

Remark 3.9. *In a well defined sense, the mapping κ constitutes a piecewise linear isomorphism between Δ and $\widehat{\Delta}$.*

Remark 3.10. *Let $\boldsymbol{K} = \bigcup_{k=1}^{n} \boldsymbol{K}_n$ be a decomposition of $\boldsymbol{K}$ and consider the vector*

$$\widehat{\boldsymbol{x}} = (\sum_{k\in\boldsymbol{K}_1} \widehat{\boldsymbol{a}}^{(k)1}, \ldots, \sum_{k\in\boldsymbol{K}_n} \widehat{\boldsymbol{a}}^{(k)n}). \tag{3.22}$$

Then $\boldsymbol{x} := \kappa^{-1}(\widehat{\boldsymbol{x}})$ is an extremal point of Δ. The mapping $\boldsymbol{i}_\bullet$ defines uniquely a decomposition of $\boldsymbol{K}$ via

$$\boldsymbol{K}_i := \{k \in \boldsymbol{K} \mid \boldsymbol{i}_k = i\}$$

whenever we assume nondegeneracy of the family $\boldsymbol{a}^\bullet$. Yet, in $\widehat{\Delta}$ the representation of $\widehat{\boldsymbol{x}}$ by means of vertices as in (3.22) will in general not be unique.

It is obviously possible to establish a measure preserving representation for an arbitrary cephoid not "in general position". Figure 2.2 shows, that we cannot expect uniqueness. We will discuss the resulting problem (discontinuity) in Example 5.2.

Remark 3.11. *However, a measure preserving representation is fruitfully extended to families $\boldsymbol{a}^\bullet$ with the following property:*

There is a decomposition of $\boldsymbol{K}$, say $\boldsymbol{K} = \bigcup_{k=1}^{r} \boldsymbol{L}_\rho$ such that the members of each family $(\boldsymbol{a}^{(k)})_{k\in\boldsymbol{L}_\rho}$ are homothetic a and a family $(\boldsymbol{a}^{(\rho)})_{\rho=1\ldots r}$ of representatives of each $\boldsymbol{L}_\rho$ is nondegenerate in the sense of Definition 2.5. In other words, the familiy is nondegenerate up to some homothetic copies.

*Indeed, in this case points on the surfaces Δ and $\widehat{\Delta}$ can be identified consistently despite the fact that there are non–unique representations of some extremals given by vectors of the type exhibited in (3.22). A family $\boldsymbol{a}^\bullet$ satisfying this slightly relaxed condition will be called **weakly nondegenerate**.*

4 A Bargaining Solution

Within this section we attempt to describe a generalization of the MASCHLER–PERLES solution ([4]). A counterexample provided by PERLES([8]) shows that we cannot expect a superadditive solution on the full class of bargaining solutions. Yet, it will be possible to construct subclasses on which superadditive solutions exist and are even uniquely defined.

We begin with the definition generalizing the two–dimensional version.

Definition 4.1. *Let* Π *be a cephoid and let* $\widehat{\Delta} = \Delta^{\widehat{\alpha}\boldsymbol{e}}$ *carry its measure preserving representation. Let*

$$\boldsymbol{\mu}(\Pi^{\widehat{\alpha}\boldsymbol{e}}) \;:=\; \frac{\widehat{\alpha}}{n}\boldsymbol{e}$$

denote the barycenter of $\Delta^{\widehat{\alpha}\boldsymbol{e}}$*. Then we define*

$$\boldsymbol{\mu}(\Pi) \;:=\; \widehat{\boldsymbol{\kappa}}^{-1}(\widehat{\alpha}\boldsymbol{e}) \tag{4.1}$$

to be the ***solution*** *of* Π*.*

For the following development it is convenient to introduce two assumptions concerning the family of vectors $(a^{(k)})_{k=1}^{K}$ characterizing a cephoid $\Pi = \sum_{k=1}^{K} \Pi^{a^{(k)}}$.

First of all we assume that all prisms involved have *equal volume*. This assumption is not as severe as it may seem on first sight. For, a prism generated by a rational vector can be replaced by a homothetic sum of small multiples of itself. This way, any family $\boldsymbol{a}^\bullet$ with volumes being multiples of the same small number qualifies. Of course, we loose nondegeneracy by this procedure – but *weak* nondegeneracy is preserved.

As the final definition of the Maschler–Perles solution in the two dimensional case involves continuity with respect to the Hausdorff metric, the above assumption does not seem to be too strong. Within the framework of the 2–dimensional solution theory it can be assumed without loss of generality.

Our second requirement will be that the total number K of prisms involved is a multiple of the dimension n. This can be achieved in a similar way by replacing each prism by a sum of n homothetic $\frac{1}{n}$–copies of itself.

Thus, we come up with the following definition.

Definition 4.2. *A family* $\boldsymbol{a}^\bullet$ *of positive vectors as well as the cephoid* Π *generated are called* ***standard*** *if the following conditions are satisfied.*

1. $\boldsymbol{a}^\bullet$ *is weakly nondegenerate (see Remark 3.11).*
2. *The* K *prisms involved have equal volume.*
3. n *is a divisor of* K*.*

Lemma 4.3. *Let*

$$\Pi \;=\; \sum_{k=1}^{K} \Pi^{a(k)}$$

be a standard cephoid. Then $\boldsymbol{\mu}(\Pi)$ *has in each coordinate* $\frac{K}{n}$ *summands, i.e., there is a decomposition* $\boldsymbol{K} = \boldsymbol{K}_1 \cup \ldots \cup \boldsymbol{K}_n$ *with* $|\boldsymbol{K}_1| = \ldots = |\boldsymbol{K}_n| = \frac{K}{n}$ *such that*

$$\boldsymbol{\mu}(\Pi) = \left(\sum_{k \in \boldsymbol{K}_1} a_1^{(k)} \;,\; \ldots, \sum_{k \in \boldsymbol{K}_n} a_n^{(k)} \right) \tag{4.2}$$

holds true.

Proof: Consider the measure preserving representation on $\widehat{\Delta} = \Delta^{\widehat{\alpha}\boldsymbol{e}}$ (see Definition 3.7). As the volumes of all prisms involved are equal, there is a positive number α_0 satisfying $\sqrt[n-1]{\alpha_k} = \alpha_0 \;\; (k \in \boldsymbol{K})$.

Thus

$$\widehat{\boldsymbol{a}}^{(k)} = \alpha_0 \boldsymbol{e} \;\; (k \in \boldsymbol{K}) \tag{4.3}$$

holds true.

Therefore the barycenter of $\widehat{\Delta}$ is given by

$$\begin{aligned} \boldsymbol{\mu}(\widehat{\Pi}) = \quad & \frac{\boldsymbol{e}}{n}\,\widehat{\alpha} \quad &= \; & \frac{\boldsymbol{e}}{n} \sum_{k \in \boldsymbol{K}} \sqrt[n-1]{\alpha_k} \\ = \quad & \frac{\boldsymbol{e}}{n} K \alpha_0 \quad &= \; & (K_0\alpha_0, \ldots, K_0\alpha_0), \end{aligned} \tag{4.4}$$

where $K_0 \;:=\; \frac{K}{n}$ is an integer.

Now consider the pre–image $\boldsymbol{\mu}(\Pi) = \kappa^{-1}(\boldsymbol{\mu}(\widehat{\Pi}))$. In view of Definition 3.7 (see also formula (3.18)), there is mapping $\boldsymbol{i}_\bullet : \boldsymbol{K} \to \boldsymbol{I}$ such that

$$\boldsymbol{\mu}(\Pi) \;=\; \sum_{k \in \boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \;\;,\;\; \boldsymbol{\mu}(\widehat{\Pi}) \;=\; \sum_{k \in \boldsymbol{K}} \widehat{\boldsymbol{a}}^{(k)\mathbf{i}_k} \;=\; (K_0, \ldots, K_0)\alpha_0, \tag{4.5}$$

holds true. In view of 4.3 the sets $\boldsymbol{K}_i \;:=\; \{k \,|\, \boldsymbol{i}_k = i\}$ necessarily satisfy

$$|\boldsymbol{K}_1| \;=\; \ldots \;=\; |\boldsymbol{K}_n| \;=\; K_0 \;. \tag{4.6}$$

q.e.d.

Definition 4.4. *Let* $a^\bullet$ *denote a standard family of positive vectors.*

$$\Pi \;=\; \sum_{k=1}^{K} \Pi^{a(k)}$$

We call $\boldsymbol{a}^\bullet$ *as well as* Π ***well ordered*** *if there is a decomposition of* $\boldsymbol{K}$*, say*

$$\boldsymbol{K} = \bigcup_{i \in \boldsymbol{I}} \overline{\boldsymbol{K}}_i \tag{4.7}$$

such that

1. $\left|\overline{\boldsymbol{K}}_i\right| = \left|\overline{\boldsymbol{K}}_j\right|$ $(i,j \in \boldsymbol{I})$,
2. $a_i^{(k)} \geq a_j^{(l)}$ $(k \in \overline{\boldsymbol{K}}_i, l \notin \overline{\boldsymbol{K}}_i)$

holds true.

Example 4.5. For $n = 3$, Concider a ***cyclic*** case in which the coordinates of the vectors are ordered as follows:

$$\begin{array}{ccccccccc} a_1^{(1)} & \geq & \dots & \geq & a_1^{(K)} & & & & \\ a_2^{(\frac{K}{3}+1)} & \geq & \dots & \geq & a_2^{(K)} & \geq & a_2^{(1)} & \geq \dots \geq & a_2^{(\frac{K}{3})} \\ a_2^{(\frac{2K}{3}+1)} & \geq & \dots & \geq & a_3^{(K)} & \geq & a_3^{(1)} & \geq \dots \geq & a_3^{(\frac{2K}{3})} \end{array} \tag{4.8}$$

Note that Figure 2.3 represents a cyclic polyhedron (assuming that the volume of the prisms is equal). However, the orientation in the setup suggested by this figure is clockwise, i.e., mathematically negative. The orientation above is mathematically positive. Yet, both versions are well ordered.

A similar observation can be made regarding the polyhedron in Figure 2.4. This can be seen as cyclic (negatively oriented): the central simplex can be obtained as a sum of three homothetic prisma by writing it as a sum of three of its thirds. The other prisms must be assumed to have equal volume or to be decomposable in a suitable way.

Lemma 4.6. *If $\boldsymbol{a}^\bullet$ is well ordered, then*

$$\boldsymbol{\mu}(\Pi) = \left(\sum_{k \in \overline{\boldsymbol{K}}_1} a_1^k \ , \ \dots \ , \ \sum_{k \in \overline{\boldsymbol{K}}_n} a_n^k \right), \tag{4.9}$$

that is, $\boldsymbol{\mu}(\Pi)$ collects the $\frac{K}{n}$ largest vectors with respect to each coordinate.

Proof: By Lemma 4.3 we know that the solution satisfies

$$\boldsymbol{\mu}(\Pi) = \left(\sum_{k \in \boldsymbol{K}_1} a_1^{(k)} \ , \ \dots, \sum_{k \in \boldsymbol{K}_3} a_n^{(k)} \right) \tag{4.10}$$

holds true with $|\boldsymbol{K}_1| = \dots = |\boldsymbol{K}_n| = \frac{K}{n}$. Now suppose that some $k_1 \in \overline{\boldsymbol{K}}_1$ is not contained in K_1, i.e., the summand $a_1^{(k_1)}$ does not appear in the first sum in (4.10). Then it is contained in some other set $\boldsymbol{K}_i$, assume for simplicity that this is $\boldsymbol{K}_2$. Necessarily, there is $k_2 \in \overline{\boldsymbol{K}}_2$ that is not contained in $\boldsymbol{K}_2$. Again assume for simplicity, that $k_2 \in \boldsymbol{K}_3$ holds true and find k_3 such that $k_3 \in \overline{\boldsymbol{K}}_3$, $k_3 \notin \boldsymbol{K}_3$. Proceeding this way, we must close the circle after finitely many steps, again let us assume that this is after n steps. Thus we have found $k_n \in \overline{\boldsymbol{K}}_n$, $k_n \notin \boldsymbol{K}_n$ such that $k_n \in \boldsymbol{K}_1$ is the case. Now we exchange the indices cyclically, i.e., consider the vector

$$\overline{\boldsymbol{x}} := \left(\sum_{k \in (\boldsymbol{K}_1 \setminus \{k_n\}) \cup \{k_1\}} a_1^{(k)} \ , \ \sum_{k \in (\boldsymbol{K}_2 \setminus \{k_1\}) \cup \{k_2\}} a_2^{(k)} \ , \ \dots \ , \ \sum_{k \in (\boldsymbol{K}_n \setminus \{k_{n-1}\}) \cup \{k_n\}} a_n^{(k)} \right)$$

Because of Definition 4.4 , we have increased the coordinates in each position. But the vector $\overline{\boldsymbol{x}}$ is a sum of vertices of the simplices involved, hence it cannot Pareto dominate the vector $\boldsymbol{\mu}(\Pi)$ which is located on Δ.

q.e.d.

We are now in the position to prove a version of superadditivity.

Theorem 4.7. *The mapping $\boldsymbol{\mu}$ behaves superadditively along decompositions of a well ordered polyhedron. That is, if Π is a well ordered polyhedron and $\Pi = \Upsilon + \Psi$, then (Υ, Ψ are cephoids and)*

$$\boldsymbol{\mu}(\Pi) \geq \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi). \tag{4.11}$$

Proof:

1$^{\text{st}}$STEP :

First of all, consider the case that both Υ and Ψ are sums of those prisms that generate Π. That is, assume

$$\Upsilon = \sum_{k \in \boldsymbol{I}} \Pi^{\boldsymbol{a}^{(k)}} \quad , \quad \Psi = \sum_{k \in \boldsymbol{J}} \Pi^{\boldsymbol{a}^{(k)}}$$

with suitable disjoint index sets $\boldsymbol{I}, \boldsymbol{J}$ satisfying $\boldsymbol{I} \cup \boldsymbol{J} = \boldsymbol{K}$. In each family the prisms have equal volume. Possibly n is not a divisor of $|\boldsymbol{I}|$ or $|\boldsymbol{J}|$. If so, we replace each prism by a sum of n homothetic $\frac{1}{n}$–copies of itself. This does not change the order property of Π and preserves weak nondegeneracy. Hence we can at once assume w.l.o.g that Υ and Ψ are standard.

According to Lemma 4.3 we know that

$$\begin{aligned} \boldsymbol{\mu}(\Upsilon) &= \left(\sum_{k \in \boldsymbol{I}_1} a_1^{(k)} \ , \ \ldots \ , \ \sum_{k \in \boldsymbol{I}_n} a_n^{(k)} \right), \\ \boldsymbol{\mu}(\Psi) &= \left(\sum_{k \in \boldsymbol{J}_1} a_1^{(k)} \ , \ \ldots \ , \ \sum_{k \in \boldsymbol{J}_n} a_n^{(k)} \right), \end{aligned} \tag{4.12}$$

with

$$\begin{aligned} |\boldsymbol{I}_1| = \ldots = |\boldsymbol{I}_n|, \\ |\boldsymbol{J}_1| = \ldots = |\boldsymbol{J}_n|. \end{aligned}$$

Obviously, we have

$$|\boldsymbol{I}_1 + \boldsymbol{J}_1| = \ldots = |\boldsymbol{I}_n + \boldsymbol{J}_n|,$$

and as the sum of all n terms is K, each of them has to be $\frac{K}{n}$. Now consider the first cordinate of the solutions. We obtain

$$\boldsymbol{\mu}_1(\Upsilon) + \boldsymbol{\mu}_1(\Psi) = \sum_{k \in \boldsymbol{I}_1 \cup \boldsymbol{J}_1} a_1^{(k)} \leq \sum_{k \in \boldsymbol{K}_1} a_1^{(k)} = \boldsymbol{\mu}_1(\Pi), \tag{4.13}$$

as the 1^{st} coordinate of the $\boldsymbol{a}^{(k)}$ is maximal in $\boldsymbol{K}_1$.

2ndSTEP : Next, assume that there are n–adic numbers $t_k = \frac{r_k}{2^{Tn}}$ $(k \in \boldsymbol{K})$ such that

$$\Upsilon = \sum_{k \in \boldsymbol{K}} \Pi^{t_k \boldsymbol{a}^{(k)}} \quad , \quad \Psi = \sum_{k \in \boldsymbol{K}} \Pi^{(1-t_k)\boldsymbol{a}^{(k)}} \tag{4.14}$$

holds true. It is no loss of generality to assume that all these numbers have a common basis 2^{Tn}. Therefore, as in our introductory remark, we can decompose every prism in each of the families into small homothetic multiples of each other untill all prisms involved have equal volume and each prism $\Pi^{t_k \boldsymbol{a}^{(k)}}$ is a sum of such prisms with equal volume. We may then apply the result of the first step in order to prove superadditivity in the above sense.

3rdSTEP :

Now suppose that the decomposition is arbitrary. By a well known criterion (see PALLASCHKE–URBANSKI ([7]), Theorem 8.3.3 or SCHNEIDER([11]), Theorem 3.2.8) the two polyhedra Υ and Ψ have to satisfy equation 4.14 possibly with non n–adic real numbers t_k, $0 \leq t_k \leq 1$ $(k \in \boldsymbol{K})$. But the $n--$adic numbers are dense and it is not hard to see that, whenever Υ and Ψ are approximated using decompositions of Π that are n–adic in the sense of the $2^{nd}STEP$, then the solution behaves continuously. Superadditivity follows, therefore, from the result of the $2^{nd}STEP$.

q.e.d.

5 Examples

We present some examples that demonstrate the merits and demerits of the solution concept.

Example 5.1. Recall Figure 2.3 (Example 2.4) which is generated by the family of positive vectors $\boldsymbol{a} = (1,3,2)$, $\boldsymbol{b} = (2,1,3)$, $\boldsymbol{c} = (3,2,1)$. We consider some variants of this example given by

$$P = \Pi^{l_a \boldsymbol{a}} + \Pi^{l_b \boldsymbol{b}} + \Pi^{l_c \boldsymbol{c}}$$

with positive constants l_a, l_b, l_c.

To begin with, let $\boldsymbol{a}^\bullet$ be given via

$$\begin{aligned} \boldsymbol{a}^{(1)} = \ldots = \boldsymbol{a}^{(4)} &= \boldsymbol{a} = (1,3,2)\ ; \\ \boldsymbol{a}^{(5)} = \boldsymbol{a}^{(6)} = \boldsymbol{a}\ ;\ \boldsymbol{a}^{(7)} = \boldsymbol{a}^{(8)} &= \boldsymbol{b} = (2,1,3)\ ; \\ \boldsymbol{a}^{(9)} = \ldots = \boldsymbol{a}^{(12)} &= \boldsymbol{c} = (3,2,1)\ . \end{aligned} \tag{5.1}$$

Now put

$$\overline{\boldsymbol{K}}_1 := \{9,\ldots,12\},\ \overline{\boldsymbol{K}}_2 := \{1,\ldots,4\},\ \overline{\boldsymbol{K}}_3 := \{5,\ldots,8\}. \tag{5.2}$$

Then $\boldsymbol{a}^\bullet$ is well ordered in the sense of Definition 4.4 as

$$\begin{aligned} c_1 &\geq a_1, b_1 \\ a_2 &\geq b_2, c_2 \\ b_3 &\geq a_3 \geq c_3 \end{aligned}$$

holds true. Therefore we can apply Theorem 4.7 which shows that $\boldsymbol{\mu}$ behaves superadditively along any decomposition of

$$\Pi = \sum_{k=1}^{K} \Pi^{a^{(k)}} = \Pi^{6\boldsymbol{a}} + \Pi^{2\boldsymbol{b}} + \Pi^{4\boldsymbol{c}}\ .$$

Indeed, the proof of Theorem 4.7 can immediately specified; we observe that $\boldsymbol{\mu}$ collects the largest quantities in each coordinate. As it turns out, we obtain

$$\begin{aligned} \boldsymbol{\mu} &= 4\boldsymbol{c}^1 + 4\boldsymbol{a}^2 + 2\boldsymbol{b}^3 + 2\boldsymbol{a}^3 \\ &= (4\times 3, 4\times 3, 2\times 3 + 2\times 2) \\ &= (12,12,10)\ . \end{aligned} \tag{5.3}$$

Obviously, the procedure works for any triple (l_a, l_b, l_c) satisfying

$$l_b \leq l_c \leq l_a\ ,\quad l_a + l_b = 2l_c. \tag{5.4}$$

Moreover, we may exchange the roles of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in a cyclic order. Then we obtain similar statements whenever

$$\begin{gathered} l_a \leq l_b \leq l_c\ ,\quad l_a + l_c = 2l_b \\ \text{or} \\ l_c \leq l_a \leq l_b\ ,\quad l_c + l_b = 2l_a\ . \end{gathered} \tag{5.5}$$

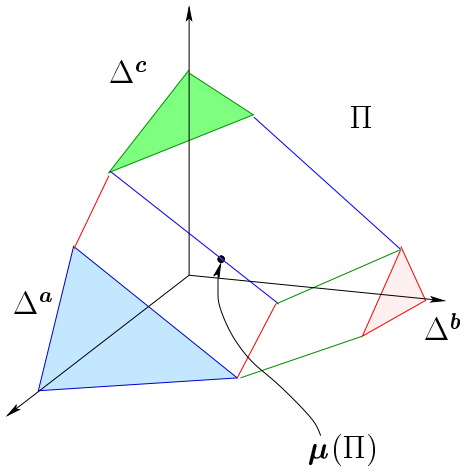

Figure 5.1: The solution for a variant of Figure 5.1

The smallest term is permitted to be 0, e.g., $l_c = 0,\ l_b = 2l_a$ is feasible. This yields the sum of two prisms similar to Figure 2.1, but the translate of $\Pi^{2\boldsymbol{b}}$ has twice the area of the one of $\Pi^{\boldsymbol{a}}$.

In order to clarify the situation, Figure 5.1 depicts the case $(l_a, l_b, l_c) = (3, 1, 2)$ which is structurally the same as the one treated above with $(l_a, l_b, l_c) = (6, 2, 4)$. We observe that

$$\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{b}}) = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{c}}) = \sqrt[3]{6^2} \ := \ \alpha$$

and it is convenient to compute quantities in terms of α. Hence, the simplex for the measure preserving representation is $6\alpha\Delta^{\boldsymbol{e}}$ which is the union of 36 simplices of area α (see Figure 5.2).

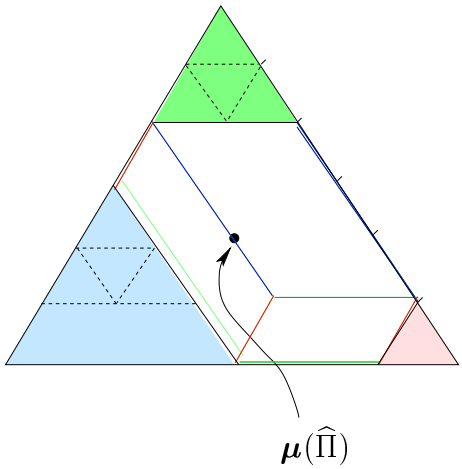

Figure 5.2: The representation of Figure 5.1

The translate of $\Delta^{\boldsymbol{b}}$ is represented by a simplex with area of one unit α, $\Delta^{\boldsymbol{c}}$ is reflected by a simplex with 4 units, and $\Delta^{\boldsymbol{a}}$ receives 9 units. It is seen that the barycenter $\boldsymbol{\mu}(\widehat{\Pi}) = \alpha(2, 2, 2)$ corresponds to

$$\begin{aligned} \boldsymbol{\mu}(\Pi) = &\ 2\boldsymbol{c}^1 + 2\boldsymbol{a}^2 + \boldsymbol{b}^3 + \boldsymbol{a}^3 \\ = &\ (6,6,5)\ . \end{aligned} \tag{5.6}$$

The Nash solution for this example is $\boldsymbol{\nu}(\Pi) = (6, \frac{27}{4}, \frac{9}{2})$. This point (maximizing the coordinate product) is located on the edge connecting $\boldsymbol{\mu}(\Pi) = (6,6,5)$ and the vertex $\boldsymbol{\kappa} = (6,9,3)$, more precisely,

$$\boldsymbol{\nu} = \frac{\boldsymbol{\kappa}}{4} + \frac{3\boldsymbol{\mu}}{4}.$$

Thus, the superadditive solution gives slightly more to player 3 and slightly less to player 2 compared to the Nash solution and, in addition, treats players 1 and 2 equally.

Example 5.2. Now let us assume that (l_a, l_b, l_c) does *not* obey the conditions (5.4) or (5.5), then $\boldsymbol{\mu}$ may not behave superadditively. E.g., if we choose

$$\Pi = \Delta^{2\boldsymbol{a}} + \Delta^{\boldsymbol{b}} \ = \ \Delta^{\boldsymbol{a}} + \Delta^{\boldsymbol{a}} + \Delta^{\boldsymbol{b}},$$

then

$$\boldsymbol{\mu}(\Pi) = \boldsymbol{a}^2 + \boldsymbol{a}^3 + \boldsymbol{b}^1 \ = \ (2,3,2)$$

– there is no way to "collect the largest values in each coordinate". Thus, it turns out that

$$\boldsymbol{\mu}(\Delta^{2\boldsymbol{a}}) + \boldsymbol{\mu}(\Delta^{\boldsymbol{b}}) = \left(\frac{4}{3}, \frac{7}{3}, \frac{7}{3}\right)$$

dominates $\boldsymbol{\mu}(\Pi)$ with respect to the third coordinate. Hence, $\boldsymbol{\mu}$ is not superadditive "on Π". Apparently, some version of "cyclic orientation" admits of a superadditive solution for a certain type of cephoid – this is indeed the meaning of Definition 4.4. Now, there are obviously cases which do not admit any version of "orientation" at all. The foremost candidate is provided by Figure 2.2. Studying this version (and the above examples) sheds light on the problem of nonexistence exhibited by PERLES' counterexample.

Indeed, let $\beta := 5$ and consider the cephoid generated by the family of vectors $\boldsymbol{a} = (\beta, \beta, \frac{1}{\beta^2}), \boldsymbol{b} = (\beta, \beta, \beta)$. Then $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ is indicated in Figure 5.3.

Due to the fact that $\Delta^{\boldsymbol{a}}_{12}$ and $\Delta^{\boldsymbol{b}}_{12}$ have the same normal cone the representation of the trapezoid $\Xi := \Delta^{\boldsymbol{b}} + \Delta^{\boldsymbol{a}}_{12}$ is not unique. E.g. $\Xi = (\boldsymbol{a}^1 + \Delta^{\boldsymbol{b}}) \cup (\Delta^{\boldsymbol{a}}_{12} + \Delta^{\boldsymbol{b}}_{23})$ as well as $\Xi = (\boldsymbol{a}^2 + \Delta^{\boldsymbol{a}}) \cup (\Delta^{\boldsymbol{a}}_{12} + \Delta^{\boldsymbol{b}}_{13})$ holds true. Hence there is no "canonical" or "measure preserving" representation of the surface of Π. There are, however, representations.

Indeed, take the simplex $36\Delta^{\boldsymbol{e}}$ (Figure 5.4). As $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) = 1$ and $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{b}}) = 25$ we can assign a measure of 1 unit to the image of (the translate of)$\Delta^{\boldsymbol{a}}$ and 25 units to the image (a translate) of $\Delta^{\boldsymbol{b}}$ which leaves 10 units for the image of a diamond. Two possibilities are presented in Figure 5.4. Taking the

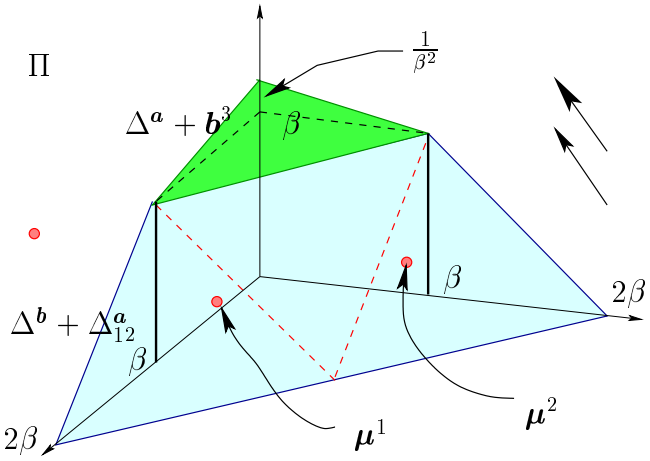

Figure 5.3: The solution for a variant of Figure 2.3

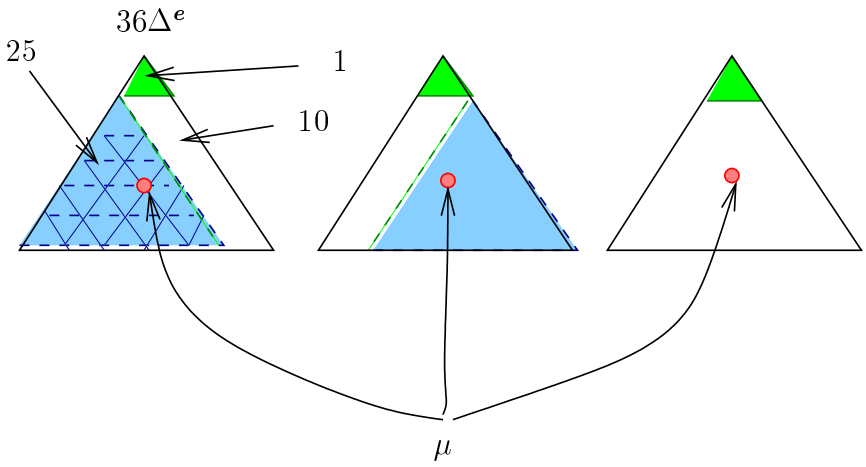

Figure 5.4: Representing the cephoid of Figure 5.3

barycenter in both situations and transporting it backwards yields two points $\boldsymbol{\mu}^1$ and $\boldsymbol{\mu}^2$. Thus, there is no "canonical" procedure for the definition of a solution.

Moerover, it would seem that the appropriate way of mapping the surface of Π onto $36\Delta^e$ is indicated by the third sketch in Figure 5.4 as this way we would preserve the poset of surface polyhedra.

As far as a point valued solution is concerned, there seems no way to extend it to this kind of bargaining problem. In fact, the counterexample of PERLES [8] exhibits just this difficulty. Perles proved, that on the family of polyhedra he considered, any superadditive solution has to be continuous. It is rather obvious that his proof can be extended to all cephoids. The present example shows that continuity of our solution cannot prevail if degenerate cephoids are admitted. While this is not another non–existence proof, it sheds light on Perles' procedure.

6 Remarks on Uniqueness

In two dimensions the uniqueness of the superadditive solution is based on two fundamental facts. First of all, the sum of two prisms (triangles in this case) with equal volume can be transformed into a a symmetric parallelogram by means of an "linear transformation of utility" (a dilatation of the axes).

Secondly, let there be given a cephoid which is a sum of triangles with equal surface measure (i.e., induced by a well ordered family). Then we can split off successively the outermost triangles on the "right" and on the "left" side and take the sum which is a parallelogram with the above property. The central corner of admits of a joint normal with all the "inner" triangles and corners. Thus, we can decompose the cephoid into a sum of parallelograms with joint normal at the central corners. Any superadditive solution behaves additively on this decomposition, hence it is uniquly defined.

The details can be seen in the presentation of Maschler–Perles [4], in [9], or more recently in [10].

In three and more dimensions the procedure can be repeated. However, we cannot expect that all well ordered cephoids can be decomposed into a sum of symmetric parallelepipeds with central corners admitting of a joint normal.

For example, consider Figure 2.3. The family of positive vectors generating Π is given by $\boldsymbol{a} = (1,3,2)$, $\boldsymbol{b} = (2,1,3)$, and $\boldsymbol{c} = (3,2,1)$, obviously Π is invariant under all permutations of the axes and hence every bargaining solution chooses the unique corner point $(3,3,3)$. Whenever we apply a linear transformation of utility on Π, this property does not change.

Other than on two dimensions, however, this is *not* the general situation of a well ordered family of three prisms up to linear transformation of utility.

E.g., suppose we have three positive vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in $\mathbb{R}^3$ such that

$$\begin{array}{ccccc} a_1 & < & a_3 & < & a_2 \\ b_2 & < & b_1 & < & b_3 \\ c_3 & < & c_2 & < & c_1 \end{array} \tag{6.1}$$

is the case and the volume of the correponding prisms is equal, i.e.,

$$a_1a_2a_3 \;=\; b_1b_2b_3 \;=\; c_1c_2c_3 = 6V(\Delta^{\boldsymbol{a}}) \;=\; 6V(\Delta^{\boldsymbol{b}}) \;=\; 6V(\Delta^{\boldsymbol{c}}) \tag{6.2}$$

holds true. Then there is a linear transformation of utility transporting the sum into a symmetric cephoid if and only if

$$a_1c_2b_3 \;=\; b_1a_2c_3 \;=\; c_1b_2a_3 = 6V(\Delta^{\boldsymbol{a}}) \tag{6.3}$$

is true. This means, that

$$(a_1, c_2, b_3)\ ,\ (b_1, a_2, c_3)\ ,\ (c_1, b_2, a_3)$$

constitute a well ordered family as well (inspection shows that the orientation changes).

In order to generalize this concept, we call a family of vectors $\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(n)}$ (and the cephoid resulting) ***symmetric up to l.t.u*** if there is a linear transformation ("of utility")

$$L \ : \ \mathbb{R}^n \to \mathbb{R}^n \ , \ L(\boldsymbol{x}) \ = \ (\alpha_1 x_1, \ldots, a_n x_n) \ \ (\boldsymbol{x} \in \mathbb{R}^n)$$

(with positive $\alpha_1, \ldots, \alpha_n$)) such that

$$\Pi^\star = \sum_{l=1}^{n} \Pi^{(L(\boldsymbol{a}^{(l)}))}$$

is symmetric.

Note that a cephoid which is symmetric up to l.t.u admits of a central vertex, say $\overset{\circ}{\boldsymbol{a}}$ such that $L(\overset{\circ}{\boldsymbol{a}})$ is located on the diagonal of $\mathbb{R}^n$. It follows at once from symmetry and Pareto efficiency that $\boldsymbol{\mu}(\Pi) = \overset{\circ}{\boldsymbol{a}}$ holds true.

Definition 6.1. *A well ordered family* $\boldsymbol{a}^\bullet \ = \ \left(\boldsymbol{a}^{(k)}\right)_{k \in \boldsymbol{K}}$ *is called* ***sufficiently symmetric*** *if*

1. *There exists a further decomposition (apart from the one mentioned in Definition 4.4)*

   (6.4) $$\boldsymbol{K} \ = \ \boldsymbol{L}_1 + \ldots + \boldsymbol{L}_{\frac{K}{n}}$$

   such that $|\boldsymbol{L}_\rho| = n$ *and every family* $\left(\boldsymbol{a}^{(k)}\right)_{k \in \boldsymbol{L}_\rho}$ *is symmetric up to l.t.u.*

2. *The central vertices* $\overset{\circ}{\boldsymbol{a}}^{(\rho)}$ *of the symmetric cephoids*

   $$\Pi^{[\rho]} = \sum_{k \in \boldsymbol{L}_\rho} \boldsymbol{a}^{(k)}$$

   admit of a joint normal.

For example, the cephoids of Figures 2.3, 2.4, and 2.5 are sufficiently symmetric.

Naturally, we have the following Theorem.

Theorem 6.2. *1. For n= 2, the sufficiently symmetric cephoids are (Hausdorff–)dense within the compact, convex, comprehensive sets.*

2. For every sufficiently symmetric cephoid there is a unique superadditive solution, this solution is $\boldsymbol{\mu}$.

The proof is obvious.

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