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A GENERALIZATION OF THE AZ IDENTITY

RUDOLF AHLSWEDE and NING CAI

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The identity discovered in [1] can be viewed as a sharpening of the LYM inequality ([3], [4], [5]). It was extended in [2] so that it covers also Bollobás' inequality [6]. Here we present a further generalization and demonstrate that it shares with its predecessors the usefullness for uniqueness proofs in extremal set theory.

1. Introduction

A few years ago Ahlswede and Zhang [1] found the following identity.

Theorem AZ_1. *For every family* $\mathcal{A}\subset 2^\Omega$ *of non-empty subsets of* $\Omega=\{1,2,\dots,n\}$

$$\sum_{X\subset\Omega}\frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}}=1,\quad \textit{where}\quad W_{\mathcal{A}}(X)=\left|\bigcap_{X\supset A\in\mathcal{A}}A\right|.$$

We associate with every $\mathcal{E}\subset 2^\Omega$ the upset $\mathcal{U}(\mathcal{E})=\{U\subset\Omega : U\supset E \text{ for some } E\in\mathcal{E}\}$ and the downset $\mathcal{D}(\mathcal{E})=\{D\subset\Omega : D\subset E \text{ for some } E\in\mathcal{E}\}$.

When $\mathcal{A}$ is an antichain in the poset $(2^\Omega,\supset)$, then the identity becomes

$$\sum_{X\in\mathcal{A}}\frac{1}{\binom{n}{|X|}}+\sum_{X\in\mathcal{U}(\mathcal{A})\setminus\mathcal{A}}\frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}}=1. \tag{1}$$

The LYM inequality is obtained by omission of the second summand, which by definition of $W_{\mathcal{A}}$ can also be written in the form $\sum\limits_{X\notin\mathcal{D}(\mathcal{A})}\frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}}$. We call this the deficiency of the inequality.

More generally, in [2] the Bollobás inequality was lifted to an identity.

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Theorem AZ_2. *For two families* $\mathcal{A}=\{A_1,\dots,A_N\}$ *and* $\mathcal{B}=\{B_1,\dots,B_N\}$ *of subsets of* Ω *with the properties*

(a) $A_i \subset B_i$ *for* $i=1,2,\dots,N$

(b) $A_i \not\subset B_j$ *for* $i \neq j$

$$\sum_{i=1}^{N} \frac{1}{\binom{n-|B_i \setminus A_i|}{|A_i|}} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}} = 1. \tag{2}$$

In [1] it was explained that Theorem AZ_1 gives immediately, what LYM does not, namely the uniqueness part in Sperner's Theorem. In [2] the uniqueness of an optimal configuration of unrelated chains of subsets due to Griggs, Stahl and Trotter [7] was proved with the help of Theorem AZ_2.

Recently, Körner and Simonyi [10] observed the LYM-type inequality:

For $\mathcal{A}=\{A_1,\dots,A_N\}, \mathcal{B}=\{B_1,\dots,B_N\} \subset 2^{\Omega}$ with

$$A_i \cap B_i = \emptyset, A_i \not\subset A_j \cup B_j, B_i \not\subset A_j \cup B_j \quad \text{for} \quad i \neq j$$

$$\sum_{i=1}^{N} \binom{n-|A_i|}{|B_i|}^{-1} + \binom{n-|B_i|}{|A_i|}^{-1} - \binom{n}{|A_i|+|B_i|}^{-1} \leq 1 \tag{3}$$

and they asked (Problem 2) "Is this inequality ever tight?".

This rather modest question was a challenging test of the power of the identities in [1], [2] or, more precisely, of the procedure to produce new identities described in [1].

The outcome is an Ahlswede–Zhang type identity (Theorem 1) which goes considerably beyond Theorem AZ_2. From a special case of this identity we derive *a full characterization* of the cases with equality (Theorem 2) even for a generalized version of (3). In other words we characterize the cases with deficiency zero.

2. The identity

Theorem 1. *Suppose that for a family* $\mathcal{B}=\{B_1,\dots,B_N\}$ *of subsets of* Ω *and a family* $\mathcal{A}^*=\{\mathcal{A}_1,\dots,\mathcal{A}_N\}$ *of subsets of* 2^{Ω}, *where* $\mathcal{A}_i=\{A_i^t : t \in T_i\}$ *for a finite index set* T_i, *we have the properties*

(a) $A_i^t \subset B_i$ *for* $t \in T_i$ *and* $i=1,2,\dots,N$

(b) $A_i^t \not\subset B_j$ *for* $t \in T_i$ *and* $i \neq j$.

Then with $\mathcal{A}=\bigcup_{i=1}^{N} \mathcal{A}_i$

$$\sum_{i=1}^{N} \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n-|B_i - \bigcup_{t \in S} A_i^t|}{|\bigcup_{t \in S} A_i^t|}^{-1} + \sum_{X \notin \mathcal{D}(\mathcal{B})} \frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}} = 1. \tag{4}$$

The specialisation $|T_i|=1$ for $i=1,\dots,N$ gives Theorem AZ_2. The proof goes again by counting chains. A key tool in [2] was

Lemma 1. *For two sets $A, B \subset \Omega$ with $A \subset B$ exactly* $\frac{n!}{\binom{n-|B\setminus A|}{|A|}}$ *maximal chains in* $(2^\Omega, \subset)$ *meet* $\{X : A \subset X \subset B\}$. ∎

Using the principle of inclusion–exclusion this generalizes to

Lemma 2. *For $B \subset \Omega$ and $\mathscr{C} \subset 2^\Omega$ with $C \subset B$ for all $C \in \mathscr{C}$ exactly*

$$n! \sum_{k=1}^{|\mathscr{C}|} (-1)^{k-1} \sum_{\mathscr{C}' \subset \mathscr{C}, |\mathscr{C}'|=k} \binom{n - |B \setminus \cup_{C \in \mathscr{C}'} C|}{|\cup_{C \in \mathscr{C}'} C|}^{-1}$$

maximal chains in $(2^\Omega, \subset)$ *meet* $\{X : C \subset X \subset B \text{ for some } C \in \mathscr{C}\}$. ∎

Proof of Theorem 1. The number of maximal chains leaving $\mathcal{U}(\mathscr{A})$ at U is

$$(n - |U|)! W_{\mathscr{A}}(U)(|U| - 1)!$$

Since the sets $\mathscr{X}_i = \{X : A_i^t \subset X \subset B_i \text{ for some } t \in T_i\}$ $(i = 1, 2, \ldots, N)$ are disjoint we have

$$\sum_{i=1}^{N} \sum_{X \in \mathscr{X}_i} (n - |X|)! W_{\mathscr{A}}(X)(|X| - 1)! + \sum_{X \in \mathcal{U}(\mathscr{A}) - \cup \mathscr{X}_i} (n - |X|)! W_{\mathscr{A}}(X)(|X| - 1)! = n!$$

By the definition of $W_{\mathscr{A}}$ the last summand can be written in the form

$$\sum_{X \notin \mathscr{D}(\mathscr{B})} \frac{W_{\mathscr{A}}(X) n!}{|X| \binom{n}{|X|}} \quad \text{and by Lemma 2}$$

$$\sum_{X \in \mathscr{X}_i} (n - |X|)! W_{\mathscr{A}}(X)(|X| - 1)! = n! \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n - |B_i \setminus \cup A_i^t|}{|\cup_{t \in S} A_i^t|}^{-1}. \quad ∎$$

3. On zero deficiency

We characterize here a case of zero deficiency, that is, the property

$$\sum_{X \notin \mathscr{D}(\mathscr{B})} \frac{W_{\mathscr{A}}(X)}{|X| \binom{n}{|X|}} = 0. \tag{5}$$

Theorem 2. *Under the assumptions of Theorem 1 and the additional conditions*

(c) $A_i^t \cap A_i^{t'} = \emptyset$ *for all i and $t, t' \in T_i$ with $t \neq t'$*

(d) $|T_i| \geq 2$ *and* $N \geq 2$

we have, that the identity

$$\sum_{i=1}^{N}\sum_{k=1}^{|T_i|}(-1)^{k-1}\sum_{S\subset T_i,|S|=k}\binom{n-|B_i-\cup_{t\in S}A_i^t|}{|\cup_{t\in S}A_i^t|}^{-1}=1 \tag{6}$$

holds exactly if

(i) $|A_i^t|=1$ *for all* $t\in T_i$ *and all* i.

(ii) $B_1\setminus\cup_{t\in T_1}A_1^t=B_2\setminus\cup_{t\in T_2}A_2^t=\ldots=B_N\setminus\cup_{t\in T_N}A_N^t=B$, *say.*

(iii) $\sum_{i=1}^{N}|T_i|=n-|B|$.

In words, the B_i *have a common part* B *and each* B_i *has a rest of singletons* A_i^t. *The* B_i*'s exhaust* Ω.

In the proof we use a well-known identity, which follows by iterative application of Pascal's identity.

Lemma 3. $\sum_{k=1}^{m}(-1)^{k-1}\binom{M}{m-k}=\binom{M-1}{m-1}$. ∎

Proof of Theorem 2. From (i) and (ii) we derive in terms of $\beta=|B|$

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{|T_i|}(-1)^{k-1}\sum_{S\subset T_i,|S|=k}\binom{n-|B_i\setminus\cup_{t\in S}A_i^t|}{|\cup_{t\in S}A_i^t|}^{-1}\\
&=\sum_{i=1}^{N}\sum_{k=1}^{|T_i|}(-1)^{k-1}\binom{|T_i|}{k}\binom{n-(\beta+|T_i|)+k}{k}^{-1}\\
&=\sum_{i=1}^{N}\sum_{k=1}^{|T_i|}(-1)^{k-1}\frac{|T_i|!(n-(\beta+|T_i|))!}{(|T_i|-k)!(n-(\beta+|T_i|)+k)!}\\
&=\sum_{i=1}^{N}\frac{|T_i|!(n-(\beta+|T_i|))!}{(n-\beta)!}\sum_{k=1}^{|T_i|}(-1)^{k-1}\binom{n-\beta}{|T_i|-k}\\
&=\sum_{i=1}^{N}\frac{|T_i|!(n-(\beta+|T_i|))!}{(n-\beta)!}\frac{(n-\beta-1)!}{(|T_i|-1)!(n-(\beta+|T_i|))!},
\end{aligned}$$

by Lemma 3, and now by (iii)

$$=\sum_{i=1}^{N}\frac{|T_i|}{n-\beta}=1. \tag{7}$$

We assume now that (6) holds and derive (i), (ii), and (iii). By Theorem 1 we have deficiency zero, that is,

$$W_{\mathcal{A}}(X)=0\quad\text{for all}\quad X\notin\mathcal{D}(\mathcal{B}). \tag{8}$$

For the quantity

$$m = \min\{|A_i^t| : 1 \le i \le N, t \in T_i\} \tag{9}$$

we show first that it equals 1, then we establish (i) and (ii), and finally (iii).

Step 1. W.l.o.g. we can assume $|A_1^1| = m$. For any $y \in \Omega \setminus B_1$ consider $A_1^1 \cup \{y\}$. Thus clearly $A_1^1 \cup \{y\} \not\subset B_1$ and by condition (b) also $A_1^1 \cup \{y\} \not\subset B_j$ for $j \neq 1$. Therefore $A_1^1 \cup \{y\} \notin \mathscr{D}(\mathscr{B})$ and by (8) $W_{\mathscr{A}}(A_1^1 \cup \{y\}) = 0$. By the minimality of A_1^1 in $\mathscr{A}$ and the definition of $W_{\mathscr{A}}$ every m-subset of $A_1^1 \cup \{y\}$ must be in $\mathscr{A}$. In particular for any $a \in A_1^1$ the set $(A_1^1 \setminus \{a\}) \cup \{y\}$ is in $\mathscr{A}$. Since it is not in $\mathscr{A}_1$ it must be in some $\mathscr{A}_j$ with $j \neq 1$.

W.l.o.g. we can assume it to be A_2^1. Furthermore, since $A_1^1 \neq A_1^2$ we can require the a choosen above to be from $A_1^1 \setminus A_1^2$. Also, since by (b) $A_1^2 \not\subset B_2$ there is $z \in A_1^2 \setminus B_2, z \neq a$. As previously we conclude that $A_2^1 \cup \{z\} \notin \mathscr{D}(\mathscr{B})$ and that the m-set

$$(A_2^1 \cup \{z\}) \setminus \{y\} = (A_1^1 \setminus \{a\}) \cup \{z\} \in \mathscr{A}.$$

However, we also have $(A_1^1 - \{a\}) \cup \{z\} \in \mathscr{A}_1$ and by (c) $A_1^1 \cap ((A_1^1 - \{a\}) \cup \{z\}) = \emptyset$. This implies $A_1^1 = \{a\}$ and $m = 1$.

Step 2. After relabelling we can assume now $A_1^1 = \{1\}$ and $B_1 = \{1, 2, \dots, \ell\}$. By the arguments in Step 1 we get $\{1, k\} \notin \mathscr{D}(\mathscr{B})$ and $\{1, k\} \supset \{k\} \in \mathscr{A}$ whenever $k > \ell$. By (b)for all $t \in T_i$ and $i \ge 2$ A_i^t has an element, say e, with $e > \ell$. However, since $\{e\} \in \mathscr{A}$ by (a), (b) and (c) actually A_i^t must equal $\{e\}$. We thus know that A_i^t is a singleton for all $i \ge 2$ and $t \in T_i$. Now we can let any $i \ge 2$ take the role of 1 in the previous argument and get that all A_1^t are also singletons. We have proved (i).

Also we have arrived at the following configuration: $B_i \supset A_i = \cup_{t \in T_i} A_i^t$ and $B_i \cap A_j = \emptyset$ for $i \neq j$. We claim now that $B_i = A_i \cup C$, where $C = \Omega \setminus \cup_{i=1}^N A_i$. To see this, suppose that $c \in C$ and $c \notin B_i$. Then for any $a \in A_i$ $\{a, c\} \notin \mathscr{D}(\mathscr{B})$ and thus $W_{\mathscr{A}}(\{a, c\}) = 0$. This, however, contradicts $W_{\mathscr{A}}(\{a, c\}) = |\{a\}| = 1$.

We have established (ii) with $B = C$. (6), together with the equations leading to (7), give now also (iii). ∎

Finally we present a consequence of Theorem 2, which in particular gives a positive answer to the question of Körner and Simonyi mentioned in the Introduction.

Corollary. *If we are given for $t = 1, 2$ and $i = 1, 2, \dots, N$ sets $A_i^t \subset \Omega$ with $A_i^1 \cap A_i^2 = \emptyset$ and $A_i^t \not\subset A_j^1 \cup A_j^2$ for $t = 1, 2$ and $i \neq j$ then*

$$\sum_{i=1}^N \binom{n - |A_i^1|}{|A_i^1|}^{-1} + \binom{n - |A_i^2|}{|A_i^2|}^{-1} - \binom{n}{|A_i^1| + |A_i^2|}^{-1} = 1 \tag{10}$$

exactly if

(i') $|A_i^t| = 1$ *for* $t = 1, 2$ *and* $i = 1, 2, \dots, N$

(ii') *n is even and* $N = \frac{n}{2}$.

There is a direct proof of this Corollary which is shorter than the one via Theorem 2.

Proof. With the choice $\mathcal{B}_i = \bigcup_{t \in T_i} A_i^t$ formula (6) takes the form

$$\sum_{i=1}^{N} \sum_{k=1}^{|T_i|} (-1)^{k-1} \sum_{S \subset T_i, |S|=k} \binom{n - \sum_{t \notin S} |A_i^t|}{\sum_{t \in S} |A_i^t|}^{-1} = 1$$

and if $T_i = \{1,2\}$ for all i this becomes (10). (i) specializes to (i'), (ii) is true by definition of B_i with $B = \emptyset$. (iii) specializes to $N \cdot 2 = n$ and thus (ii'). ∎

4. On general cloud antichains

A family $\mathcal{A}^* = \{\mathcal{A}_1, \dots, \mathcal{A}_N\}$ of subsets of 2^Ω is a cloud-antichain, if

(11) $$A_i \not\subset A_j \text{ for } A_i \in \mathcal{A}_i, A_j \in \mathcal{A}_j \text{ with } i \neq j.$$

They have been analyzed in [2] for $N = 2$. In particular, in the case $|\mathcal{A}_i| = M$ for $i = 1, \dots, N$ we are interested in the maximal length $N(n, M)$ of these antichains.

Clearly, for $\mathcal{A} = \bigcup_{i=1}^N \mathcal{A}_i$

(12) $$W_{\mathcal{A}}(X) = W_{\mathcal{A}_i}(X) \text{ for } X \in \mathcal{A}_i$$

and therefore by Theorem AZ_1

(13) $$\sum_{i=1}^{N} \sum_{X \in \mathcal{A}_i} \frac{W_{\mathcal{A}_i}(X)}{|X|\binom{n}{|X|}} + \sum_{X \in \mathcal{U}(\mathcal{A}) \setminus \mathcal{A}} \frac{W_{\mathcal{A}}(X)}{|X|\binom{n}{|X|}} = 1.$$

Notice that $n! \sum_{X \in \mathcal{B}} \frac{W_{\mathcal{B}}(X)}{|X|\binom{n}{|X|}}$ counts the number, say $\sigma(\mathcal{B})$, of saturated chains meeting a member of $\mathcal{B}$.

We can derive from (13) a bound on $N(n, M)$, if we have a bound or even exact result for the following seemingly basic quantity:

(14) $$s(M, n) = \min\{\sigma(\mathcal{B}) : \mathcal{B} \subset 2^\Omega, |\mathcal{B}| = M\}.$$

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Rudolf Ahlswede, Ning Cai

Universität Bielefeld,
Fakultät für Mathematik,
D-4800 Bielefeld 1, Germany
hollmann@math.10.mathematic.
uni-bielefeld.de