On the Complexity of Two Dimensional Commuting Local Hamiltonians

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Abstract

The complexity of the commuting local Hamiltonians (CLH) problem still remains a mystery after two decades of research of quantum Hamiltonian complexity; it is only known to be contained in NP for few low parameters. Of particular interest is the tightly related question of understanding whether groundstates of CLHs can be generated by efficient quantum circuits. The two problems touch upon conceptual, physical and computational questions, including the centrality of non-commutation in quantum mechanics, quantum PCP and the area law. It is natural to try to address first the more physical case of CLHs embedded on a 2D lattice, but this problem too remained open apart from some very specific cases [4, 17, 24]. Here we consider a wide class of two dimensional CLH instances; these are k-local CLHs, for any constant k; they are defined on qubits set on the edges of any surface complex, where we require that this surface complex is not too far from being "Euclidean". Each vertex and each face can be associated with an arbitrary term (as long as the terms commute). We show that this class is in NP, and moreover that the groundstates have an efficient quantum circuit that prepares them. This result subsumes that of Schuch [24] which regarded the special case of 4-local Hamiltonians on a grid with qubits, and by that it removes the mysterious feature of Schuch's proof which showed containment in NP without providing a quantum circuit for the groundstate and considerably generalizes it. We believe this work and the tools we develop make a significant step towards showing that 2D CLHs are in NP.

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1 Introduction

1.1 Commuting local Hamiltonians

The Local Hamiltonian (LH) problem is central to the theory of quantum complexity. In 1998 it was proved by Kitaev to be QMA-complete [21], initiating by that the area of quantum Hamiltonian complexity. This result is often considered as the quantum analogue of the celebrated Cook-Levin theorem, which states that the Boolean Satisfiability problem (SAT) is NP-complete [23]. In 2003 Bravyi and Vyalyi [9] raised the question of what is the complexity of the intermediate class in which all terms mutually commute (commuting local Hamiltonians, or CLHs). The question begs an answer not only because the commutation restriction is natural and often made in physics; but this is also a computational probe to the fundamental question: is the uncertainty exhibited by non-commuting operators necessary for quantum systems to exhibit their *full* quantum nature? or, perhaps, it happens to be the (much less expected) case that even commuting quantum systems can express full quantum power.

The CLH problem may seem at first sight to be trivially in NP, since by the commutation condition, there exists a common basis of eigenstates to all terms, where each constraint has a well defined value on each eigenstate; the problem seems like a classical constraint satisfaction problem (CSP). This hope breaks down when realizing that the eigenstates themselves maybe highly complex. While in CSP, a proof for satisfiability is simply a string, i.e. a satisfying assignment, in the quantum case the eigenstates themselves may be highly entangled. Indeed, a beautiful example is Kitaev's toric code [20], whose global entanglement is characterized by topological properties. In the general case, we do not no whether groundstates of CLHs have an efficient classical description at all (that is, a polynomial size classical representation from which the result of any local measurement can be deduced efficiently).

The question of CLHs touches upon some of the most important aspects of quantum many body systems: fundamental, physical and complexity theoretical. For a start, stabilizer codes can be viewed as ground spaces of CLHs; these constitute by far the most common framework for the study of quantum error correcting codes. CLHs are also a very convenient place to start with when studying open problems and toy examples; for example in the study of the quantum PCP conjecture [2, 3, 7] often CLHs are used as a case study (e.g. [5, 13, 18]). Moreover, CLH systems provide the simplest examples for systems obeying the *area law* bounding the entanglement in groundstates of gapped systems[1]. In the one dimensional case, the area law was recently shown in a breakthrough result to provide an efficient classical algorithm for constructing groundstates [22]. In two or higher dimensions such an algorithm cannot be expected, since CLHs become NP hard in 2D. However it is still possible that groundstates satisfying the area law have polynomial size quantum circuits (which may be hard to find). Understanding whether groundstates of 2D CLH systems have efficient descriptions is thus an essential first step towards clarifying how the area law affects the complexity of groundstates.

Despite the importance and fundamental nature of this class, and fourteen years after the problem was posed [9], the complexity of the CLH problem remains a mystery, even in the physically motivated case of 2D. A trivial upper bound to the complexity of the CLH problem is that it belongs to QMA. A simple lower bound exists as well: if we let d denote the dimension of the particles, and let k denote the maximal number of particles that each local

[1] The area law states that the entanglement in the groundstate between two regions grows like the size of the *boundary* between these two regions, rather than their volume.

term acts on, then we may define $CLH(k, d)$ accordingly. Using this notation $CLH(k, d)$ is NP-hard if $k, d \geq 2$. The question becomes then to distinguish between those cases which are within NP, those which are QMA hard, and possibly, the intermediate cases. However, excluding a few special cases of CLH, not much is known.

1.2 Previous results

Bravyi and Vyalyi proved that $CLH(2, d)$, namely the class of instances in which the particle dimensionality d is an arbitrary constant, whereas the interactions only involve two such particles (this is called *two-local CLHs*), is in NP [9]. The proof relies on a decomposition lemma based on the theory of finite dimensional C*-algebra representations [26]. This tool has become essential in all following results about this problem.

Aharonov and Eldar [4] then considered the 3-local case with qubits and qutrits. They showed that $CLH(3, 2) \in NP$ and also that $NE{-}CLH(3, 3) \in NP$ where NE is a geometrical restriction on the interaction called *nearly Euclidean* [4]. An important fact about the proofs for both of these results is that the witness which is sent by the prover is virtually a constant depth quantum circuit which prepares a groundstate for the system, starting from a product state. Hastings called states which can be generated by constant depth quantum circuits "trivial" [18]; the name is justified since indeed, local observables can be computed classically in an efficient way for such states, given the circuit that generates them, because the *light cone* of qubits affecting the output qubits of a local observable is of constant size. Thus, the above mentioned results not only prove containment in NP, but also show that such systems have groundstates with very restricted multi-particle entanglement which is in some sense *local*.

In this regard, Aharonov and Eldar [4] mentioned a tight "threshold" which can be drawn at this point: commuting systems with parameters as above are essentially classical; But, when raising k or d just by 1, i.e when considering $CLH(4, 2)$ or $CLH(3, 4)$, we arrive at a new regime in which the quantum system can exhibit *global* entanglement, namely, the groundstates are no longer trivial (by Hastings' definition). In fact, such systems can exhibit global entanglement even when the system is embedded on a square lattice: Kitaev's toric code [20] is a wonderful example, as it can indeed be shown that groundstates of this code with nearest neighbor interactions cannot be generated by a constant depth quantum circuit [8]. This raises the possibility [4] that general CLH systems with parameters above the "transition point" are too complex for containment in NP, as they allow global entanglement.

There are several examples beyond the transition point which indicate that though global entanglement is possible, it might still be the case that CLH systems remain "classically accessible" even in that regime. First, it is known that despite their global entanglement, toric code states can be constructed in *logarithmic* depth quantum circuits called MERA [1] which moreover, allow local measurements to be simulated classically efficiently. In addition, Schuch proved that CLHs in which all qubits and all 4-local constraints are embedded on a square lattice (generalizing the toric code to general interactions with the same geometry and dimensionality) also belong to NP [24]. Interestingly enough though, Schuch's proof bypasses the question of whether an efficient description of a groundstate exists; instead, the witness which is sent by the prover convinces the verifier that a low energy state exists without describing that state at all. Schuch's result thus leaves open the possibility, suggested in [4], that when crossing the transition point from local to global entanglement mentioned above, groundstates may in general become difficult to describe classically (not including the toric code special case).

Hastings provided two other results proving upper bounds on the complexity of the CLH problem in certain cases. In [18] he considered k-local CLHs whose interaction graphs are 1-*localizable*; roughly speaking, these are instances whose interaction graphs can be mapped to graphs continuously, such that the preimage of every point is of bounded diameter. This extends the result of [9] that two local Hamiltonians are in NP, to slightly more general constructions which are in some sense, two-local in every local region. In another result of Hastings [17], he considered CLHs on a planar lattice, and proved that the problem is in NP under certain restrictive conditions on the C*-algebraic decomposition (essentially, that when dividing the lattice to stripes, the transformation which disentangles adjacent stripes, a'la Bravyi and Vyalyi [9], is local). Hastings also provided parts of a proof that 2D CLH is in NP, and suggested that the proof will be completed elsewhere, however this was not done.

We note that an interesting clue pointing in fact in the other direction, namely suggesting that the CLH problem could be harder than NP, was given recently by Gosset, Mehta and Vidick [14]; they show that a certain problem regarding the connectivity of the ground space of CLHs is as hard as that of general LHs. It is suggested in [14] that this is probably true even for CLHs in 2D, though this remains to be worked out.

We are left with the mystery: possibly the above "classical" examples are just special cases, and in the general case above the low parameters threshold, global entanglement prevents an efficient description of the groundstates of CLHs; or maybe, the "classicality" of the entanglement in the toric code groundstates as well as in the other examples mentioned [17,24] is generic for all CLHs, and thus the problem lies in NP.

1.3 Results

We consider a wide subclass of CLH in 2D. Specifically, we consider $CLH(k, 2)$ instances (i.e with qubits) where the qubits are arranged on the edges of a polygonal complex $\mathcal{K}$ whose underlying topological space is a surface. We refer to those as *2D complexes*[2]. The local terms live on the vertices of $\mathcal{K}$ (these are called stars), and on its faces (plaquettes), where each of these terms acts on the edges attached to the vertex or the face, respectively. In Section 2, this class is formally defined and denoted by $2D-CLH^*(k, 2)$. We shall emphasize that the Hamiltonian terms need not be of the form of products of σ_x or σ_z Paulis as in Kitaev's surface codes, but can be general operators on the relevant qubits (as long as they commute). Moreover, the locality parameter k, which in this case equals the maximal degree of vertices and faces of $\mathcal{K}$ (a degree of a face is the number of its edges), is an arbitrary constant as well.

An example of a polygonal complex, where each vertex and each face has a degree of at most 5. One may define on this complex a $2D - CLH^*(5, d)$ instance by assigning to each star and plaquette a Hamiltonian acting on the attached edges, where those Hamiltonians mutually commute.

We note that there is no restriction whatsoever on the topology of the complex $\mathcal{K}$; it can be of any genus, and may or may not include a boundary. We impose one condition on $\mathcal{K}$, which is a metric-geometric condition that we call quasi-Euclidity (though of similar flavor, it shouldn't be confused with the nearly-Euclidean condition of [4]). This condition ensures that the surface induced by the complex admits a triangulation in which the triangles may be slim (as in hyperbolic geometry) and may be fat (as in elliptic geometry) but only up to some constant. This makes the complex in some sense Euclidean up to a constant distortion, and prevents "wild" situations. Any physically natural 2D setting should be covered by this.

[2] despite some friction with ordinary *simplicial 2-complexes* as in e.g [19] which do not necessarily define topologically a surface

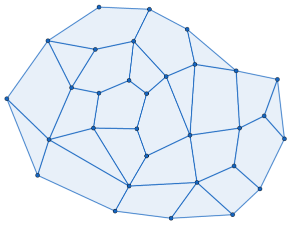


Figure 1.1 Polygonal complex.

Our main two results are:

▶ **Theorem 1.** *The $2D - CLH^*(k,2)$ problem on quasi-Euclidean complexes is in NP.*

▶ **Theorem 2.** *For any instance of $2D - CLH^*(k,2)$ defined on a quasi-Euclidean complex, there exists a polynomial depth quantum circuit which prepares a groundstate.*

Importantly, these results replace the mysterious feature of Schuch's result [24] providing a proof for containment in NP without an efficient groundstate description, by one in which the groundstate can be efficiently classically described; this seems to strengthen the common feeling that containment in NP should go hand in hand with efficient description for the groundstate. Moreover, our results hold for a wide class of cases, which includes not only the 4-local case in a square lattice of Schuch [24], but CLHs with arbitrary locality k, that are defined on any quasi-Euclidian 2D complex. We remark that our definition of $2D - CLH^*(k,2)$ unfortunately does not capture the most general k-local quantum systems of qubits embedded on a surface (see Section 2).

1.4 Proof overview

Our starting point is a folklore quantum algorithm for preparing the groundstates of the toric code. Recall that the toric code Hamiltonian [20] acts on qubits set on the edges of an $n \times n$ grid with boundary conditions which make it topologically a torus. The Hamiltonian has two types of constraints, one for each vertex (star) denoted s, and one for each face (plaquette) denoted p:

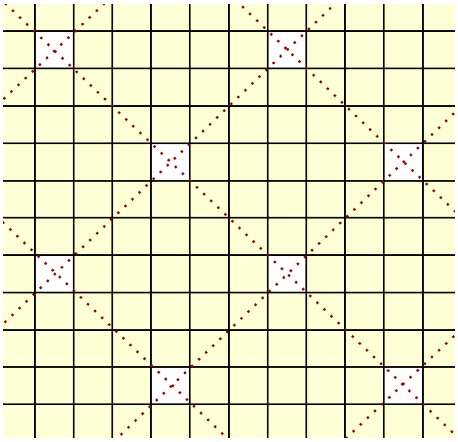
$$A_s = \bigotimes_{e \in s} \sigma_z^e, \qquad B_p = \bigotimes_{e \in p} \sigma_x^e, \qquad\qquad H = -\sum_s A_s - \sum_p B_p \tag{1.1}$$

The groundstates of this Hamiltonian form a code space, and exhibit global-entanglement.

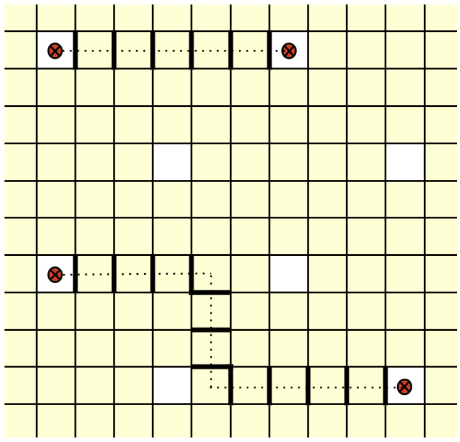
Consider creating "holes" in the torus, by removing a small fraction of the plaquettes, in a regular manner. Figure 1.2 (A) shows how by removing enough plaquettes we are left with a *punctured Hamiltonian* $\tilde{H}$, which involves *two local* interactions between *super-particles* comprised each of constantly many qubits. By [9] there is a constant depth quantum circuit which prepares a groundstate (denote it $|\psi\rangle$) for $\tilde{H}$.

This doesn't seem at first as real progress, since $|\psi\rangle$ is a trivial state, whereas groundstates of the original Hamiltonian are globally entangled. The key idea is that now we can *correct for the plaquettes we have removed*, using the known idea of applying string operators connecting pairs of "holes".

To do this, we first *measure* in the state $|\psi\rangle$ each of the plaquette terms which were removed. Due to the commutation relations, the resulting state is *still* a groundstate of $\tilde{H}$ but now it is also an eigenstate of the toric code, with a known eigenvalue for each of the



(a) Punctured Hamiltonian



(b) Logical Operators

Figure 1.2 (A) The white squares are the holes. The dotted lines induce a partition of the set of qubits (edges) to squares (tilted in 45 degrees), which are the super-particles, each containing a constant number of qubits. Every local term (star or plaquette) of the punctured Hamiltonian acts on qubits which belong to at most 2 super-particles. (B): A hole with a spot inside indicates an excitation (i.e. a violation). The dotted lines are string logical operators (copaths) which annihilate particles in pairs. The edges in bold denote the qubits on which the logical operator acts.

terms. Viewing the toric code as a subcode of the punctured code (the groundspace of the punctured Hamiltonian $\tilde{H}$), what we now need is a set of *logical operators* in the punctured code, that act within it and can transform our state into a toric code groundstate.

To this end, we recall the notion of *string* operators which are Pauli operators acting on the paths (strings) connecting a pair of holes [20]. Such an operator changes the values of the measurements corresponding to the constraints in both holes, while keeping all the other values intact. Notice that this process always works on *pairs* of holes. The dependency relations between the local terms ($\prod_s A_s = \prod_p B_p = \mathbf{1}$) [20] imply that for any eigenstate of the toric code there is an *even* number of plaquette (and also star) terms which are in their excited states. Since all plaquettes in the punctured Hamiltonian are satisfied (i.e., not excited), it follows that there is an even number of excited plaquettes out of those which we removed, and thus such a pairing exists.

Note that we could have actually removed *all* plaquettes, resulting in a punctured Hamiltonian $\tilde{H}$ consisting only of A_s terms; Starting with the state $|0^n\rangle$, which is a groundstate of $\tilde{H}$, we could then proceed as in the above algorithm, to derive a groundstate of the toric code (without any help of the prover). We will make use of both approaches in this paper; the "regular holes" approach is the one we will generalize (conceptually) to more general instances, while the second more specific approach is used as a subroutine in our final algorithm, for technical reasons. We will thus present and prove it formally in Section 4.

1.4.1 Physical interpretation

The toric code has a physical interpretation which will be very useful for us [20]. The value of the edges in the σ_x and σ_z basis are interpreted as a $\mathbb{Z}_2$ vector potential or electric field, respectively. When a constraint is violated, we interpret this as if an elementary excitation, or a particle, is created. The star constraints can be viewed as requiring that the electric flux from the vertex (namely the values of the qubits in the computational basis) is zero, i.e., that this vertex will have no electrical charge. If a vertex constraint is violated, we say that there is an "electric charge" at that vertex. Likewise, the plaquette constraints require that the magnetic flux which passes through the face is zero (mod 2). If a plaquette constraint is violated we say that there is a "magnetric vortex" in this plaquette [20]. The toric code consists of the states in which neither electrical charges, nor magnetic vortices appear. The punctured system however allows particles to be created at the sites which we have removed. After measuring these terms, we know exactly where these particles are. It is left to annihilate them. Having a closed surface with no boundary, such as the torus, the total charge on it, as well as the total magnetic flux passing through it, must be zero (as Gauss and Stoke's laws imply, respectively). This means that there must be an even number of electrical charges, and an even number of magnetic vortexes, which can then be annihilated in pairs, by what is called "string operators" connecting pairs of charges or pairs of vortexes (see [20]). In the above algorithm for the toric code we only needed to annihilate magnetic vortices (plaquettes).

1.4.2 From toric code to general $2D - CLH^*(k, 2)$

It is far from clear how the methods above concerning the toric code can be applied to general 2D CLH systems; after all, surface codes seem to be an extremely restricted type of 2D CLHs (where the local terms must take the form of tensor products of either σ_x or σ_z Pauli operators), whereas we are concerned with arbitrary commuting local terms. Theorem 13 in Section 5 provides our first main step in the proof: we show that all $2D - CLH^*(k, 2)$ instances are "equivalent to the toric code permitting boundaries". This in particular means that if all terms, stars and plaquettes, act non-trivially on all of their attached edges, (plus $\mathcal{K}$ is closed, i.e topologically has no boundary), then the instance is, up to a minor modification, equal to the toric code. In the general case, terms may act trivially on some of their qubits (edges); we will call such edges boundary/coboundary edges. Theorem 13 says that $2D - CLH^*(k, 2)$ instance are virtually the toric code, except for those essentially 1D behaving boundary areas (and thus the term "permitting boundaries"). The proof of this structure theorem relies heavily on the C*-algebraic techniques mentioned earlier. We emphasize that Theorem 13 holds only after some transformation of the instance to one with no "classical qubits" whose value is simply a classical bit which can be provided by the prover (see subsection 3.3).

1.4.3 Constructing the Punctured Hamiltonian

The above equivalence theorem raises the idea of using a similar algorithm as for the toric code groundstates, and somehow handling the special boundary/coboundary qubits. However, we encounter two challenges. First, we do not have sufficient control on operators near the boundary/coboundary. If we carelessly tear out holes in their vicinity, we might not know how to repair them- the correcting process of the toric code heavily relies on the specific commutation and anti-commutation relations between a string operator and the Hamiltonian terms (equation 1.1). We handle this difficulty by tearing out holes only in the interior regions (that is regions without boundary/coboundary qubits) where we do have resemblance to the toric code. It turns out that there is no need to tear holes close to boundary/coboundary qubits as in some sense these special qubits are already punctured: by definition such qubits are not surrounded by Hamiltonians acting on them non-trivially.

The second challenge is that we do no longer have the dependencies $\prod_s A_s = \prod_p B_p = \mathbf{1}$ that ensured earlier an even number of excitations of any given type, and so the idea of fixing holes in pairs is irrelevant. In the physical interpretation, the latter means that the

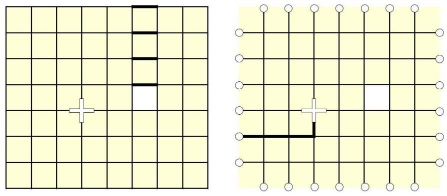


Figure 1.3 Logical operators.

total charge on the manifold can be different than 0 since now flux can escape through the boundary. In section 6 we show that the curse of boundaries is in fact a blessing, since now we can also dump excitations to the boundary/coboundary with string operators, similarly to logical operators in surface codes [10] (figure 1.4).

The latter idea, which can be viewed as the main conceptual idea in the paper, introduces a new challenge - we have two types of special qubits. Boundary qubits give rise to copath string logical operators whereas coboundary qubits give rise to path string logical operators. We cannot expect that puncturing only plaquette terms out of the surface will allow us to fix them later on. Figure 1.3 shows simple examples of systems in which only one type of term (star/plaquette) have access to the boundary/coboundary via copath/path. In short, plaquettes play nicely with boundary edges whereas stars play nicely with coboundary edges.

The white plaquette and the white plus indicate holes. In a complex with boundary but no coboundary only plaquette holes can be connected via a copath to utilize a logical operator, whereas in a complex with coboundary but no boundary only star holes can be connected via paths to utilize a logical operator.

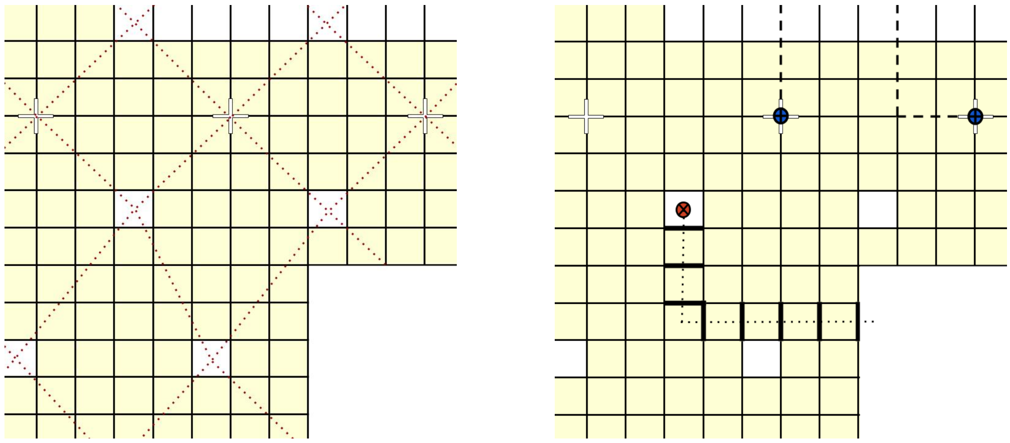
A major technical effort in the paper is proving Lemma 15 which roughly states that for any adjacent plaquette and star, at least one of them has access to the boundary/coboundary (unless they are both already touching the boundary/coboundary), hence a hole in one of them will be fixable

With this in mind, we construct the punctured Hamiltonian as follows: we start by considering the set $\mathcal{W}$ of "fixable" terms. These are terms which are not in the boundary of the system (and thus are in the form of a toric code term) and in addition have access to the boundary or coboundary via a copath or path depending on whether it is a plaquette or star term respectively (see Definition 14 and Figure 6.1). By Lemma 15 the fixable holes are very “dense”. We shall not hesitate to remove all of those terms since, by how the elements of the set $\mathcal{W}$ were chosen, we can correct their values later on.

We call the Hamiltonian obtained by removing all of the terms in $\mathcal{W}$ the *punctured Hamiltonian* $\tilde{H}$.

1.4.4 2-locality of the punctured Hamiltonian

Lemma 15 guarantees that at any large enough constant size area, either there are boundary qubits (recall these are qubits which are acted trivially by at least one of its surrounding terms) which may serve as a hole, or else there must be a fixable term in that area, i.e a member of $\mathcal{W}$, which was removed. In the case of the grid it is now very simple to generate a 2-local structure among constant size super-particles: just consider a coarse grained grid of 5×5, and use Lemma 15 to conclude that there must be some hole inside each 5×5 square. However we are allowing much more general geometries than the grid; it is here and only here, that we make use of the quasi-Euclidity condition. This is what allows us to follow a similar process, and to tear holes in some regular manner. Technically, we need



(a) Punctured Hamiltonian. **(b)** Logical Operators.

Figure 1.4 (A) Even when boundaries/coboundaries exist, one can tear out holes to obtain a 2-local instance w.r.t superparticles of constant size. (B) After measuring each hole, it remains to correct it if needed by connecting it to the boundary/coboundary via a string operator depending on the hole type (i.e plaquette/star).

to apply Moore's bound [6, 15] to bound the number of edges (qubits) which belong to any super-particle resulting from the process; together with some other combinatorial arguments the proof goes through.

Now that the punctured Hamiltonian is 2-local, we again are guaranteed that a groundstate can be generated by a constant depth quantum circuit [9]. This is the only place where the prover is needed. Note that this groundstate is in general not the groundstate of the original Hamiltonian, yet, the fact that we have torn out only terms of $\mathcal{W}$, namely the fixable terms, implies that we can apply the approach of measuring them and correcting them with string operators to the boundary/coboundary of the system (Figure 1.4 (B)).

1.5 Organization of the paper

In Section 2 we formalize the problem. Section 3 gives some background: "the induced algebra", "classical qubits", and notations. Section 4 provides the efficient algorithm for generating toric code states which we use as a subroutine. Section 5 contains Theorem 13, stating that $2D - CLH^*(k, 2)$ instances are "equivalent to the toric code permitting boundaries". Based on this, in Section 6 we prove lemma 15 which shows that many fixable terms (those with "access to the boundary") exist, and define the punctured Hamiltonian, in which all these terms are removed. In Section 7 we show that the punctured Hamiltonian is indeed 2-local with respect to super-particles of constant size. Section 8 combines all these results to prove Theorems 1,2. In Section 9 we discuss the results, their implications, and state open questions.

This version does not include all proofs in their complete form. Those can be found in the more techinical version of this paper [6].

2 Formulation of the problem

2.1 Definitions

▶ **Definition 1** (CLH instance). An instance of $CLH(k, d)$ consists of a set of Hamiltonian terms (Hermitian matrices) acting on n qudits (particles of dimension d), where each term acts non-trivially on at most k of the n qudits. The norm of each term is bounded by 1, and the terms mutually commute.

To be precise, we note that as usual, the Hermitian matrices are given with entries represented by poly(n) bits.

We consider the cases where the CLH instance is defined on a 2D complex. The type of complexes we allow (see definition bellow) is a generalization of a simplicial 2-complex; while in simplicial complexes the 2-cells must be 2-simplexes (triangles), we allow the 2-cells to be any simple polygon. Topologically speaking, we may define a simple polygon to be any set homeomorphic to the closed disk $D = \{\mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}|| \leq 1\}$ with some choice of a finite amount (at least three) of points on its boundary to be called the vertices of the polygon. The arcs on the boundary which connect two adjacent vertices are called the sides of the polygon. Such complexes are often called *polygonal complexes* [15].

▶ **Definition 2** (polygonal complex). A polygonal complex $\mathcal{K}$ is a collection of points (called 0-cells or vertices), line segments (1-cells, or edges), and simple polygons (2-cells, or faces) glued to each other such that:

1. Any side of a 2-cell in $\mathcal{K}$ is a 1-cell in $\mathcal{K}$. Every endpoint of a 1-cell in $\mathcal{K}$ is a 0-cell in $\mathcal{K}$.
2. The intersection of any two distinct 2-cells of $\mathcal{K}$ is either empty or else it is a single 1-cell (along with its endpoints). The intersection of any two distinct 1-cells of $\mathcal{K}$ is either empty or else it is a single 0-cell.

If all polygons have exactly three vertices then $\mathcal{K}$ is called a simplicial 2-complex. The 1-skeleton of $\mathcal{K}$ is by definition the graph obtained by removing all 2-cells from $\mathcal{K}$. Finally, $\mathcal{K}$ is called two dimensional (2D) if the topological space which it defines $\mathcal{S} = \bigcup \mathcal{K}$ is a surface.

By surface we mean the topological definition of a surface[3] allowing boundaries [25]; that is a topological space such that each point in the interior has a neighborhood homeomorphic to $\mathbb{R}^2$ whereas each point in the boundary has a neighborhood which is homeomorphic to the the upper plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. We shall remark that if $\mathcal{K}$ is finite (which will be the only case we consider) then $\mathcal{S}$ is compact. If in addition $\mathcal{S}$ has no boundary (in the ordinary topological sense) then we say that $\mathcal{S}$ (and thus also $\mathcal{K}$) is closed.

Note that 2D polygonal complexes have the property that every 1-cell is the face of at most two 2-cells (one if that 1-cell is in the boundary, and two if it is in the interior). That is because if 3 or more 2-cells are attached at that 1-cell then the neighborhoods of points in the interior of that 1-cell are neither homeomorphic to $\mathbb{R}^2$ nor to the upper plane.

The 1-skeleton of $\mathcal{K}$ admit the natural graph metric in which the distance between any two vertices is the length of the minimal path between them, where the length of every edge is 1.

▶ **Definition 3** (triangulation). A *triangulation* of a topological space X is a finite simplicial 2-complex $\mathcal{T}$ together with a homeomorphism $f : \mathcal{T} \to X$. The 2-cells of $\mathcal{T}$ are called the *triangles* of the triangulation.

[3] In many texts (e.g [19]) second countability and Hausdorff are required in the definition as well. In our case however, we are only considering finite polygonal complexes which always satisfy these two conditions.

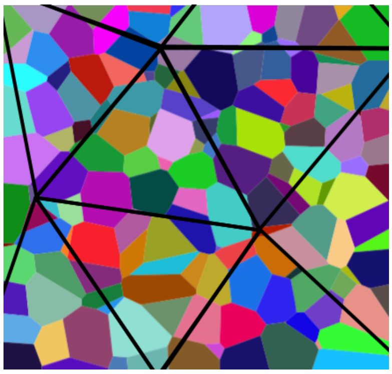


Figure 2.1 Quasi-Euclidean polygonal complex.

The following definition is inspired by metric geometry in which hyperbolic spaces are roughly defined to be metric spaces which have only r-slim triangles - triangles which do not contain any ball of radius r; whereas elliptic metric spaces are such which have a bound on the *diameter* of triangles [11].

▶ **Definition 4** (quasi-euclidean 2D complex)**.** Let $\mathcal{K}$ be a 2D polygonal complex with underlying surface S. A triangulation of $\mathcal{S}$ is said to be $(r, R)-$quasi-Euclidean for some $0 < r < R$ if each of its triangles contains a ball of radius r in $\mathcal{K}$ (w.r.t metric defined above) and the subgraph in it is of diameter at most R. The degree of a triangulation is by definition the maximal degree of its 1-skeleton. In the case where $\mathcal{S}$ admits such a triangulation we say that $\mathcal{K}$ is (r, R)-quasi-Euclidean.

We emphasize that there is no demand from the triangulation to be in any sort in accordance with the complex structure of $\mathcal{K}$ (e.g vertices of $\mathcal{T}$ do not need to be located on vertices of $\mathcal{K}$).

A triangulation $\mathcal{T}$ (dark lines) of the surface $\mathcal{S}$ on which the complex $\mathcal{K}$ lies. $\mathcal{T}$ is (r, R)-quasi-Euclidean with $r = 2$,$R = 12$ since each triangle contains a ball of radius 2 but its diameter is less than 12. The makes $\mathcal{K}$ a (r, R)-quasi-Euclidean complex. Having each triangle contain a ball of radius $r \geq 2k$ (here $k = 7$) ensures that there exists a polygon which is contained in the triangle, as well as all other polygons touching it. The fact that the diameter of each triangle is at most R implies that the number of edges in each triangle is bounded by a number dependent only on R and k, by Moore's bound [15].

▶ **Definition 5** ($2D - CLH^*(k, d)$ instance)**.** Consider instances x of $CLH(k, d)$ for which:

1. There exists a two dimensional polygonal complex $\mathcal{K}$.
2. There exists a 1-1 mapping between qudits of x and edges of $\mathcal{K}$.
3. There exists a 1-1 mapping between local terms of x and the set of vertices and faces of $\mathcal{K}$.
4. If h corresponds to a vertex v then the set of qudits $\{q_1, ..., q_r\}$ which h acts on corresponds to the set of edges $\{e_1, ..., e_r\}$ attached to v.
5. If h corresponds to a face f then the set of qudits $\{q_1, ..., q_r\}$ which h acts on corresponds to the set of edges $\{e_1, ..., e_r\}$ which are in the boundary of f.

We consider the restriction of this class to quasi-Euclidean complexes - those which admit a (r, R)-quasi-Euclidean triangulation of degree D, for some arbitrary constants $D > 0$ and $R > r > 2k$. We call such $2D - CLH^*(k, d)$ instances *quasi-Euclidean*.

The quasi-Euclidean condition doesn't limit the topology in any way. Specifically, for any compact surface $\mathcal{S}$ there exists a quasi-Euclidean polygonal complex $\mathcal{K}$ such that $\mathcal{S}$ is its underlying surface (i.e $\mathcal{S} = \bigcup \mathcal{K}$) [25]. This condition is needed only in Section 7. Hence in the following we ignore it and treat general $2D - CLH^*(k, 2)$ instances; only in Section 7 we will mention this condition again.

Another possible way to define a CLH on a 2D polygonal complex is to place the qudits on the *vertices* rather than the edges, and then local terms are associated with faces alone. We denote the class of such instances by $2D - CLH(k, d)$ (i.e without the star symbol). The latter definition captures the notion of a 2D system in a more general way: every $2D - CLH^*(k, 2)$ instance can be converted to a $2D - CLH(k, 2)$ whereas the converse is true only when the instance has no vertices of degree 3. In addition, if our results can be generalized to $2D - CLH^*(k, d)$ for arbitrary d, this will in fact imply that they also hold for $2D - CLH(k, d)$, under a mild condition similar to quasi-Euclidity (see [6]).

To each of those classes corresponds the local Hamiltonian problem of deciding, given $a < b$ with $b - a < \frac{1}{poly(n)}$, whether the ground energy of the system (i.e the sum of all local terms) is bellow a or above b, provided the promise that one of these cases hold. We use the same notation to denote both the class of such instances (as in Theorem 2) and the corresponding decision problem (as in Theorem 1).

3 Notation and Background

3.1 Notations

Throughout this paper we use $\mathcal{H}$ to denote Hilbert spaces, q to denote qubits, and accordingly $\mathcal{H}_q$ to denote the Hilbert space associated with the qubit q. $\mathcal{K}$ denotes the complex on which the $2D-CLH^*(k, 2)$ is defined whereas $\mathcal{S}$ denotes its underlying surface. We use s to denote stars, p to denote plaquettes and let $|s|$ and $|p|$ denote the degree of a star or a plaquette, i.e the number of edges which belong to s or to p. A_s denotes the local term which corresponds to s and B_p denotes the local term which corresponds to p. h denotes a local term in general. We say that two stars (plaquettes) are adjacent if they share an edge, and say that a star and plaquette are adjacent if they share two edges (which is the only way a star and a plaquette can intersect). When more geometrical aspects are discussed we will consider vertices instead of stars denoted by v, edges instead of qubits denoted by e and faces instead of plaquettes denoted by f. We let H denote the sum of all local terms $H = \sum_s A_s + \sum_p B_p$ where s and p range over the stars and plaquettes of the instance. When we construct a punctured Hamiltonian, i.e a Hamiltonian obtained by removing some terms from the original one, we will always denote it by $\tilde{H}$.

3.2 The induced algebra

▶ **Definition 6** (induced algebra). Let h be an operator on a tensor product Hilbert space $\mathcal{H}_{q_1} \otimes \mathcal{H}_{q_2}$ and let $h = \sum_{i=1}^{m} h_{q_1}^i \otimes h_{q_2}^i$ be a Schmidt decomposition[4] of h. The induced algebra of h on $\mathcal{H}_{q_1}$ is denote by $\mathcal{A}_{\mathcal{H}_{q_1}}^h$ or in short $\mathcal{A}_{q_1}^h$ and is defined to be the C*-algebra generated by $\{I\} \cup \{h_{q_1}^i\}_{i=1}^m$ (I denotes the identity operator).

[4] That is to say: $h_{q_1}^i \in \mathcal{L}(\mathcal{H}_{q_1})$, $h_{q_2}^i \in \mathcal{L}(\mathcal{H}_{q_2})$ for each i and the that sets $\left\{h_{q_1}^i\right\}_{i=1}^m$, $\left\{h_{q_2}^i\right\}_{i=1}^m$ are orthogonal with respect to the Hilbert-Schmidt inner product i.e $tr({h_{q_l}^i}^\dagger \cdot h_{q_l}^j) = 0$ for any $i \neq j$ and $l = 1, 2$.

3.3 Classical qubits

The equivalence to the toric code which we are aiming for can be shown only after performing a certain reduction of removing "classical qubits". Classical qubits are classical in the sense that they do not participate in the entanglement of the system and consequently, the prover may hand us its correct value as a classical bit.

▶ **Definition 7** (trivial qubit)**.** A qubit (or qudit) is called trivial, if no local term acts on it non-trivially.

▶ **Definition 8** (classical qubit)**.** A qubit (or qudit) is called classical if its Hilbert space can be decomposed into a direct sum of 1-dimensional subspaces which are invariant under all local terms in the Hamiltonian H.

When we say that a Hamiltonian h acts trivially on a certain qubit we simply mean that it can be written as $h = I \otimes h'$ where I is the identity operator on that qubit, and h' acts only on other qubits.

Note that due to the low dimension of qubits, once such a non-trivial direct sum decomposition exists then the subspaces must be one dimensional and so the qubit is classical. Note also that every trivial qubit is in particular classical - any direct sum decomposition will do. The following claim says that whenever there is a classical qubit q, the instance can be reduced to a new instance in which it is a trivial qubit.

▶ **Claim 9** (removing classical qubits)**.** *To derive theorems 1,2 it is sufficient to prove it under the restriction of $2D - CLH^*(k, 2)$ to instances with the condition that every classical qubit is trivial.*

This claim is the key idea in the proof that the 2-local commuting Hamiltonian problem lies in NP [9]; In fact, one can easily construct a formal proof for claim 9 using the same arguments as in [9] (see [6]).

Thus, we shall assume from now on that all classical qubits were turned to be trivial qubits.

4 Generating a toric code state

The toric code is a special case of a $2D - CLH^*(k, 2)$ instance. We shall not restrict to the particular setting of a grid on a torus, so by saying toric code we refer to any $2D-CLH^*(k, 2)$ instance defined on a closed complex $\mathcal{K}$ (i.e it topologically has no boundary) with the usual star and plaquette local terms (equation 1.1).

Starting with the state $|0\rangle^{\otimes n}$, we measure *all* plaquettes and record the measurement results by $\bar{\lambda} = (\lambda_p)_p$ (where $\lambda_p = \pm 1$). As a result, the system collapses to a state corresponding to the measured values: $|\psi_{\bar{\lambda}}\rangle$. Note that $|\psi_{\bar{\lambda}}\rangle$ is a toric code state (i.e a groundstate of the Hamiltonian given in equation 1.1) precisely when $\lambda_p = 1$ for each plaquette p.

Whenever we have two plaquettes p_1,p_2 with $\lambda_{p_1} = \lambda_{p_2} = -1$ we can connect them by a copath γ^*, apply $L^* = \bigotimes_{e \in \gamma^*} Z_e$, and obtain a new state $|\psi_{\bar{\lambda}'}\rangle$ where λ and λ' are the same except for the value on the plaquettes p_1,p_2 (see [6] for a more elaborate explanation of logical and string operators). In other words, a pair of plaquette terms which are in their excited state can always be relaxed. After matching pairs of excitations, and annihilating them by applying string operators between them, we obtain a toric code state. It is thus left to show that such a matching always exists:

▶ **Claim 10** (even amount of excitations). *The number of plaquettes p for which $\lambda_p = -1$ is even.*

Proof. Since $\mathcal{K}$ is closed so $\prod_p B_p = \mathbf{1}$ (and also $\prod_s A_s = \mathbf{1}$). Therefore:

$$|\psi_{\bar{\lambda}}\rangle = \mathbf{1}\,|\psi_{\bar{\lambda}}\rangle = \prod_p B_p\,|\psi_{\bar{\lambda}}\rangle = \prod_p \lambda_p\,|\psi_{\bar{\lambda}}\rangle = (\prod_p \lambda_p)\,|\psi_{\bar{\lambda}}\rangle$$

It follows that $\prod_p \lambda_p = 1$. ◀

This is summarized in the following algorithm:

Algorithm - constructing a toric code state (folklore):

1. Start with the tensor product state $|0\rangle^{\otimes n}$.
2. For each star p measure B_p and record the measured value λ_p.
3. As long as $-1 \in \{\lambda_p\}_p$ choose two stars p_1, p_2 with $\lambda_{p_1} = \lambda_{p_2} = -1$, find a copath γ^* connecting them (with some linear time path-finding classical algorithm) and apply Z along that copath, that is the operator $L = \bigotimes_{q\in\gamma} Z_q$. Then change the values of $\lambda_{p_1}, \lambda_{p_2}$ from -1 to 1.

It is not hard to be convinced that a similar approach works also for a variation of the toric code where each term is as in the toric code but with some scalar factor (see [6]). This remark is relevant since the equivalence to the toric code (which we formulate in the following section) allows such factors.

5 Equivalence to the toric code

We now formulate the notion of equivalence between general $2D - CLH^*(k, d)$ instances and the toric code.

▶ **Definition 11** (boundary/coboundary qubit). A qubit is said to be in the *boundary* of the system if it is acted non-trivially by at most one plaquette; it said to be in the *coboundary* of the system if it is acted non-trivially by at most one star. Other qubits are said to be in the *interior*. A local term which acts only on interior qubits is said to be in the *interior of the system*.

Qubits that live on edges which are topologically on the boundary of the manifold are of course in the boundary of the system; however qubits which are (topologically) in the interior of the manifold can also be in the boundary/coboundary of the *system* if a Hamiltonian term *acts trivially* on them. When this happens, these qubits serve, in spirit, as “holes”. We will later exploit this fact in order to tear out holes only in the *interior* of the system to obtain the 2-local punctured Hamiltonian and a constant depth circuit that generates groundstate for it.

Following [9], we will make use of the notion of induced algebras (Definition 6) of any term in the Hamiltonian, on any set of qubits it acts on. The induced algebra from a star (plaquette) term s (p) on qubits $q_1, ...q_r$ is denoted $\mathcal{A}^s_{q_1,...,q_r}$ ($\mathcal{A}^p_{q_1,...,q_r}$). In addition, given an operator h we denote by $\langle h\rangle$ the algebra generated by this operator. We can now state the definition of equivalence to the toric code:

▶ **Definition 12** (equivalence to the toric code permitting boundaries)**.** An instance of $2D - CLH^*(k,2)$ is said to be *equivalent to the toric code* if its underlying surface $\mathcal{S}$ is closed (it topologically doesn't have boundary) and there exists a choice of basis for each qubit such that $A_s \in \left\langle Z^{\otimes|s|} \right\rangle \backslash \mathbb{C} \cdot I$, $B_p \in \left\langle X^{\otimes|p|} \right\rangle \backslash \mathbb{C} \cdot I$ for any s, p.

An instance is said to be *equivalent to the toric code permitting boundaries* if there exists a choice of basis for each qubit such that:

1. $\mathcal{A}^s_{q_1,\ldots,q_r} = \langle Z^{\otimes r} \rangle$ for any star s, for $(q_1, \ldots, q_r)$ a copath of qubits of s which are not in the coboundary, with no two consecutive qubits in the boundary.
2. $\mathcal{A}^p_{q'_1,\ldots,q'_r} = \langle X^{\otimes r} \rangle$ for any plaquette p, for $(q'_1, \ldots, q'_r)$ a path of qubits of p which are not in the boundary, with no two consecutive qubits in the coboundary.

▶ **Theorem 13** (equivalence to the toric code permitting boundaries)**.** *Every $2D - CLH^*(k,2)$ instance (after removing all classical qubits as described in subsection 3.3) is equivalent to the toric code permitting boundaries. In particular, if it has no qubits which are in the boundary or in the coboundary then it is equivalent to the toric code.*

The first step in the proof is to classify the possible induced algebras of a Hamiltonian on a single qubit in the interior and show that these algebras are always generated by a *single* Pauli operator (i.e., an operator which is equal to a Pauli matrix up to a change of basis). This can be done quite easily using the ordinary C*-algebraic techniques as in [9, 24]. The main technical part is to establish severe restrictions on the induced algebras on *pairs* of qubits (which are in the interior, roughly), and essentially showing that they must be similar to those of the toric code. This analysis involves a close and fairly technical study of the implication of the commutation relations between the Hamiltonians on the algebras that they induce.

An immediate implication of Theorem 13 is that we now know how to generate a groundstate for any $2D - CLH^*(k,2)$ instance which has no qubits in the boundary or coboundary of the system, since such instances are equivalent to the toric code.

6 Construction of punctured Hamiltonian

We are now ready to show how we can generate a groundstate of an arbitrary quasi-Euclidean $2D - CLH^*(k,2)$ instance, even when there are qubits in the boundary/coboundary.

▶ **Definition 14** (access to the boundary/coboundary)**.** A star s is said to have *access to the coboundary* if there exists a path γ starting from s which ends at a coboundary edge such that $L = \bigotimes_{q\in\gamma} X_q$ anti-commutes with A_s and commutes with any other local term. Similarly, a plaquette p is said to have *access to the boundary* if there exists a copath γ^* starting from p which ends at a boundary edge such that $L^* = \bigotimes_{q\in\gamma^*} Z_q$ anti-commutes with B_p and commutes with any other local term.

Access to the boundary or coboundary means that either L^* or L serve as an appropriate logical operator for the corresponding plaquette or star respectively in the sense that it flips its value while keeping the value of all other constraints in tact.

▶ **Lemma 15** (**Main lemma:** access to the boundary/coboundary)**.** *Let s, p be adjacent star and plaquette which are in the interior of the system. Then either s has access to the coboundary, or p has access to the boundary.*

The proof of Lemma 15 relies on the a further study of the induced algebras near the boundary/coboundary of the system. The idea is to start with an edge shared by s and p and start drawing a ribbon from it which is briefly a juxtaposition of a path and an adjacent

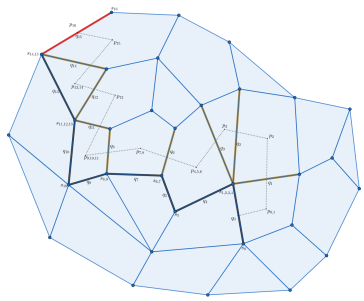


Figure 6.1 A ribbon to the boundary

copath (see Figure 6.1). We do this until we encounter a boundary/coboundary edge. At areas far from the boundary, we are in a regime which look like the toric code and thus the desired commutation and anti-commutation relations hold. Near the boundary/coboundary there are enough restrictions on the induced algebras to conclude that either the path or the copath within the ribbon can serve as the support for a logical operator which can correct p or s respectively (see [6]).

The labeled edges in bold form a ribbon. The stars and plaquettes which are shared between two adjacent edges in the ribbon are labeled as well. Ribbons include in them both a path which can easily be seen in the figure and a copath which is drawn as a dotted line. The last qubit of the ribbon is in the boundary and so we say that the first qubit has access to the boundary via a copath. See [6] for a full explanation.

Construction of punctured Hamiltonian: Let $\mathcal{W}$ denote the set consisting of all stars and plaquettes in the interior of the system which have access to the coboundary or to the boundary, respectively. This set can be thought of as the set of "fixable" terms. Let $\tilde{H}$ be the *punctured Hamiltonian:* the local Hamiltonian obtained by replacing all terms which are in $\mathcal{W}$ by the identity operator.

7 2-locality of the punctured Hamiltonian

We now show with the help of Lemma 15 that the punctured Hamiltonian $\tilde{H}$ has so many holes that it is 2-local.

The division to superparticles is based on the quasi-Euclidean condition (this is the only place we use this condition). Recall that by definition, the quasi-Euclidity condition (Definition 4) provides us with a triangulation $\mathcal{T}$ of $\mathcal{S}$ of degree $D = \mathcal{O}(1)$ such that each triangle contains a ball of radius $2k$ and is of diameter $R = \mathcal{O}(1)$ (with the ordinary graph metric with edge length 1).

We now construct a graph which will help us divide the qubits to superparticles. The vertices of this graph will be associated with terms in the Hamiltonian. A local term can be associated with a point in the surface in a natural way: each star is naturally realized as the vertex which is associated with it, and each plaquette p is associated with some arbitrarily chosen point in its interior to be called "the center of the plaquette". This allows us to precisely speak of a local term as a point on the surface.

▶ **Claim 16** (punctured triangles)**.** *For each triangle $T \in \mathcal{T}$ there exists a term h of $\tilde{H}$ such that all of the edges attached to it (i.e the edges associated with the qubits which h acts on), are fully contained in T and moreover, h acts trivially on at least one of its qubits.*

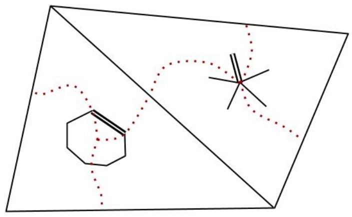


Figure 7.1 Choosing a holes in each triangle and separating to regions.

The idea of the proof is very simple: T contains a ball of radius $2k$ and thus must contain a pair of adjacent star and plaquette which are in the interior. By Lemma 15 at least one of them belongs to $\mathcal{W}$ so let h be that term.

Choose such a term h for every triangle $T \in \mathcal{T}$ and call it "the center of the triangle T". Such a term acts trivially on some edge e (when considered as a term in $\tilde{H}$; if h was removed, then this term is in fact the identity). In addition, for each 1-cell of $\mathcal{T}$, that is a side of a triangle $T \in \mathcal{T}$, choose some point in its interior to be called "the center of the 1-cell". Then connect each triangle center with 3 paths to the centers of the sides of T. Those paths should be non intersecting, contained in the interior of T (except at the end of the paths) and in addition must satisfy one more condition: clearly, those three non-intersecting paths divide T into 3 regions; the paths should be drawn such that e belongs to one region and all the other edges of h belong to the two other regions (that way h will act non-trivially on at most 2 regions). To be sure that such paths can always be drawn, it suffices to show it for an equilateral triangle - this can of course be done. Then the general case is obtained as a homeomorphism of the triangle (see figure 7.1).

According to Claim 16, every triangle includes a local term h which acts trivially on (at least) one of its edges e (this edge is marked as a double edge). Whether a star term or a plaquette term, we can connect it to the three triangle sides with three paths (dotted curves) such that e belongs to one region, and the other edges belong to the two other regions.

This construction gives rise to a graph G which highly resembles $\mathcal{T}^*$ the dual of $\mathcal{T}$. The vertices of G consist of the chosen triangles center of $\mathcal{T}$, as well as the centers of 1-cells of triangles in $\mathcal{T}$ which are on the boundary of $\mathcal{S}$. Between any two vertices of G corresponding to the centers of two triangle T_1, T_2 which share a side (i.e $T_1 \cap T_2$ is a 1-cell of $\mathcal{T}$) let there be an edge; in addition, for every triangle which has a side on the boundary of the surface, let there be an edge between the triangle center and the boundary. The edges of G are drawn on $\mathcal{S}$ as the paths constructed in the previous paragraph.

Consequently, vertices of $\mathcal{T}$ are in one-to-one correspondence with faces of G. Those faces induce a partition $\mathcal{P}$ of the set of qubits $\mathcal{Q}$ according to the face of G which they belong to (if an edge of $\mathcal{K}$ touches more then one face of $\mathcal{T}^*$ then join it to one of those faces arbitrarily) [12]. We accordingly have: $\mathcal{H} = \bigotimes_{q\in\mathcal{Q}} \mathcal{H}_q = \bigotimes_{P\in\mathcal{P}} \mathcal{H}_P$ with $\mathcal{H}_P := \bigotimes_{q\in P} \mathcal{H}_q$. We refer to each cluster $P \in \mathcal{P}$ and to its Hilbert space $\mathcal{H}_P$ as a super-particle.

The dark lines are the quasi-Euclidean triangulation. The dotted curves are the edges of the graph G realized as the chosen paths in $\mathcal{S}$. The faces of G induce a partition of the qubits into superparticles. The fatness of the triangles and the bounded degree of the triangulation implies that the superparticles' size is constant.

So far we have only used the "slimness" of a triangle condition in the definition of quasi-Euclidean condition. Here is where we need the bound on the fatness of triangles and the upper bound on its degree.

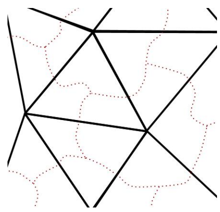


Figure 7.2 The graph embedding of G in $\mathcal{S}$.

▶ **Claim 17** (constant sized super-particles). *Each super-particle includes at most $D \cdot k^{R+2}$ qubits (in particular $\mathcal{O}(1)$).*

The proof is a straightforward consequence of Moore's bound [6, 15] which provides a bound on the number of vertices and edges which are contained in some ball in a graph of bounded degree. Since each vertex of $\mathcal{K}$ is of degree at most k, and the diameter of each triangle is at most R we immediately obtain a bound on the number of edges in each triangle and thus in each super-particle (since the number of triangles it intersects is at most D - the maximal degree of $\mathcal{T}$).

▶ **Claim 18** (punctured Hamiltonian is 2-local). *Each local term of $\tilde{H}$ acts on at most two super-particles.*

The proof follows by the observation that the only plaquettes/stars which act on 3 super-particles are the ones near the vertices of G which were removed (see Figure 7.2)

8 Completing the algorithm and the proofs for Theorems 1 & 2.

We now proof Theorems 1 & 2. By Claims 17, 18, it is possible to prepare a ground space $|\tilde{\psi}\rangle$ of $\tilde{H}$, using a constant depth quantum circuit. Given such a groundstate, we measure every $h \in \mathcal{W}$ one by one. Actually it will be simpler to measure $I - 2 \cdot \pi_h$ instead where π_h is the orthogonal projector onto the ground space of h. Record that result of the measurement by λ_h. Accordingly, having $\lambda_h = 1$ indicates that $|\tilde{\psi}\rangle$ is already a groundstate of h whereas $\lambda_h = -1$ indicates an excitation at that spot. The state we had $|\tilde{\psi}\rangle$ collapses by these measurements to a new state $|\psi\rangle$ which is an eigenstate of every $h \in \mathcal{W}$, while still being in the ground space of $\tilde{H}$. Recall that the set of terms we measured (the set $\mathcal{W}$) all have access to the boundary (Definition 14). Thus their value can be changed via string logical operators while not effecting the value of any other term. This is summarized by the following algorithm:

Algorithm (constructing a groundstate for an arbitrary quasi-Euclidean $2D - CLH^*(k, 2)$ instance):

1. If the instance has no boundary or coboundary qubits, then it is equivalent to the toric code, so apply algorithm 4 and terminate.
2. Else, generate a groundstate of $\tilde{H}$ with a constant depth quantum circuit.
3. For each term $h \in \mathcal{W}$ which was removed, measure $I - 2 \cdot \pi_h$, and record the measurement value as $\lambda_h = \pm 1$. (π_h is the orthogonal projector onto the groundspace of h).
4. Fix every $h \in \mathcal{W}$ for which $\lambda_h = -1$: if h is a star term s, find a path γ from s to the coboundary and apply $L = \bigotimes_{q \in \gamma} X_q$. If h is a plaquette term p, find a copath γ^* from p to the boundary and apply $L^* = \bigotimes_{q \in \gamma^*} Z_q$.

This proves Theorem 2. Theorem 1 follows as well: if the instance has no boundary/-coboundary qubits (and this can of course be checked efficiently by the verifier) then the system is equivalent to the toric code, so it's ground energy can be computed easily (see [6] for the case where the local terms are as in the toric code only upto a factor of a scalar). Otherwise, the problem of computing the ground energy of H reduces to computing the ground energy of $\tilde{H}$, since the verifier knows that any groundstate of $\tilde{H}$ can be corrected to a (possibly other) groundstate of $\tilde{H}$ such that all terms in $\mathcal{W}$ are satisfied (i.e the energy with respect to the terms in $\mathcal{W}$ is minimal). It is thus left to note that $\tilde{H}$ is a 2-local CLH, and this problem is in NP by [9].

9 Discussion

An interesting property of the algorithm is that all of the *quantum* operations are summed up to have only *constant* depth. Indeed, the algorithm consists of three steps: a constant depth quantum circuit that generates a groundstate for the punctured Hamiltonian, a non-constant depth computation of path finding which can be carried out in a *classical* manner, and finally a constant depth quantum circuit of logical operators (tensor product of Pauli operators).

This observation regards the complexity of the algorithm, but it is interesting also conceptually. While the quantum circuit presented here is of polynomial depth, it is enough for the verifier to obtain only a constant depth circuit description, and verify that it is indeed a groundstate of the punctured Hamiltonian, in order to be know the ground energy of the whole system (since the verifier knows that these holes can always be fixed). This means that while the time it takes to generate a groundstate for the system is concentrated on creating global entanglement, all the hardness and potential frustration of the groundstate comes into play only at the level of local entanglement of the groundstate of the 2-local punctured system.

Moreover, our results shed new light on the possible threshold phenomenon suggested in [4]. Recall that this threshold (described above in subsection 1.2) regards the fact that up until $k = 3, d = 3$, and also for $k = 2$ and arbitrary d, $CLH(k, d)$ always have trivial groundstate, which in turn implies that those problems are in NP. The threshold refers to the fact one cannot expect the exact same phenomenon for higher parameters since then there are systems with topological quantum order which are known to have no trivial groundstates. It is thus interesting that our proof extends this trivial state phenomenon even beyond this transition point into the regime of potentially global entanglements, in the sense that even here the prover hands us a description of a trivial state - a ground state of $\tilde{H}$ (even though it cannot in general be a groundstate of the actual instance). This raises the question of whether such a property holds for more general CLHs.

Can these results be extended to all 2D systems? A generalization from qubits to qudits of dimension larger than 2 would imply this, under the quasi-Euclidity assumption. Thus, the main open problem is to generalize our results to higher dimensional particles. We note that in any case one can still tear holes in a regular manner (using e.g the quasi-Euclidity assumption) to obtain a punctured Hamiltonian which is 2-local with respect to superparticles, and thus has a trivial groundstate. The problem is that we do not know how to fix those holes later on: our characterization of $2D - CLH^*(k, 2)$ instances (i.e Theorem 13) and of fixable terms (namely, the creation of logical operators in Lemma 15) strongly uses the fact that the particles are 2 dimensional. It is open whether further generalization could be derived using more general characterizations of commuting local Hamiltonians, perhaps over general finite groups (e.g the quantum double model [20]).

We mention that if indeed the results can be generalized to qudits, it might also be possible to generalize to 3D manifolds or more, perhaps in an inductive manner.

A more technical question is whether the quasi-Euclidity condition can be relaxed. Quasi-Euclidity seems closely related to the notion of 1-localizablity introduced in Hastings' paper [18] already mentioned (In fact, the quasi-Euclidity condition we use can be replaced by the technical assumption used in [18] regarding the girth of the complex; we could then deduce the existence of a groundstate for $\tilde{H}$ from 1-localizabilty instead of 2-locality). This raises the question of whether manifolds which are very non-Euclidean and which have low girths, can exhibit much more complex multi-particle entanglement (we mention in this context [16]).

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