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Exercise Strategies for American Exotic Options under Ambiguity

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Abstract

We analyze several exotic options of American style in a multiple prior setting and study the optimal exercise strategy from the perspective of an ambiguity averse buyer in a discrete time model of Cox–Ross–Rubinstein style. The multiple prior model relaxes the assumption of a known distribution of the stock price process and takes into account decision maker's inability to completely determine the underlying asset's price dynamics. In order to evaluate the American option the decision maker needs to solve a stopping problem. Unlike the classical approach ambiguity averse decision maker uses a class of measures to evaluate her expected payoffs instead of a unique prior. Given time-consistency of the set of priors an appropriate version of backward induction leads to the solution as in the classical case. Using a duality result the multiple prior stopping problem can be related to the classical stopping problem for a certain probability measure – the worst-case measure. Therefore, the problem can be reduced to identifying the worst-case measure. We obtain the form of the worst-case measure for different classes of exotic options explicitly exploiting the observation that the options can be decomposed in simpler event-driven claims.

JEL subject classification: G12, D81, C61

1 Introduction

The increasing trade volume of exotic options both in the plain form and as component of more sophisticated products motivates the more precise study of these structures. The OTC nature of contracts allows for almost endless variety which comes at the price of tractability and evaluation complexity. The payoff of the option is often conditioned on an event during the lifetime leading to a path dependent structure which is challenging to evaluate.

Most of the literature on this field concentrates on hedging or replication of such structures analyzing the hedging strategy of the seller or deriving the no arbitrage price. This analysis is sufficient in the case of European options as it also captures the problem of the buyer. However, in the case of American options the task of the buyer holding the option in her portfolio differs structurally from the hedging problem of the seller. Unlike the bank/the market the holder of the option is not interested in the risk neutral value of the option but aims to exercise the claim optimally realizing highest possible utility. This valuation in general needs not to be related to the market value of the option as it reflects the personal utility of the holder which depends on investment horizon and objectives and also on the risk attitude of the holder.

Given a stochastic model in discrete time, such as the Cox–Ross–Rubinstein (CRR) model one can easily solve the problem of the buyer using dynamic programming. However, classical binomial tree models impose the assumption of a unique given probability measure driving the stock price process. This assumption might be too strong in several cases as it requires perfect understanding of the market structure and complete agreement on one particular model.

As an example we consider a bank holding an American claim in its trading book. The trading strategy of the bank depends on the underlying model used by the bank. If the model specification is error-prone the bank faces model uncertainty. Being unable to completely specify the model traders rather use multiple prior model instead of choosing one particular model. If the uncertainty cannot be resolved and the accurate model specification is impossible traders prefer more robust strategies as they perform well even if the model is specified slightly incorrect.

Also a risk controlling unit assigning the portfolio value and riskiness uses rather a multiple prior models in order to test for model robustness and to measure model risk. Taking several models into account while performing portfolio distress tests allows to check the sensitivity of the portfolio to model misspecification. Again in a situation of model uncertainty more robust riskiness assignment is desirable as it minimizes model risks.

Similar reasoning can be applied to accounting issues. An investment funds manager making his annual valuation is interested in the value of options in the book that are not settled yet. In case the company applies coherent risk measures as standard risk evaluation tool for future cash flows on the short side, it is plausible to use a multiple prior model evaluating long positions. Finally, a private investor holding American claims in his depot might exhibit ambiguity aversion in the sense of Ellsberg paradox or Knightian uncertainty. Such behavior may arise from lack of expertise or bad quality of information that is available to the decision maker.

Although for different reasons, all the market participants described above face problems that should not be analyzed in a single prior model and need to be formulated as multiple prior problems. In this paper we analyze the problem of the holder of an American claim facing model uncertainty that results in a multiple prior model. We characterize optimal stopping strategies for the buyer that assesses utility to future payoffs in terms of minimal expectation and study how the multiple prior structure affects the stopping behavior.

Multiple prior models have gained much attention in recent studies. Hansen and Sargent (2001) considered the multiple prior models in the context of robust control, Karatzas and Zamfirescu (2003) approached the problem from game theoretical point of view. Delbaen (2002) introduced the notion of coherent risk measures which mathematically corresponds to the approach used in this paper.

The decision theoretical model of multiple priors was introduced by Gilboa and Schmeidler (1989) and further developed to dynamical settings by Epstein and Schneider (2003b). This is the natural extension of the expected utility model when the information is too imprecise. The methods we use in this paper rely heavily on this work.
Epstein and Schneider (2003a) applied the multiple prior model to financial markets and Epstein and Schneider (2003b) addressed the question of learning under uncertainty.

Riedel (2009) considered the general task to optimally stop an adapted payoff process in a multiple prior model and showed that backward induction fails in general. He imposed more structure on the set of priors that ensured the existence of the solution. The cornerstone of the method is the time-consistency of the set of priors which allows the decision maker to change her beliefs about the underlying model as the time evolves. If the set of priors is time-consistent one can proceed as in the classical case (see Snell (1952),Chow, Robbins, and Siegmund (1971) for more detailed analysis) computing the value process of the stopping problem – the multiple prior Snell envelope. It is then optimal to stop as soon as the payoff process reaches the value process. Additionally, the ambiguous optimal stopping problem corresponds to a classical optimal stopping problem for a measure $\hat{P}$ – the so-called worst-case measure (see Riedel (2009), Föllmer and Schied (2004), Karatzas and Kou (1998)).

As an application of the technique Riedel (2009) solves the exercise problem for the buyer of an American put and call in discrete time. A similar problem was analyzed by Nishimura and Ozaki (2007), they considered the optimal investment decision for a firm in continuous time with infinite time horizon under multiple priors which can be related to the perpetual American call. In this paper we follow the lines of Riedel (2009) and analyze several exotic options that have a second source of uncertainty from the perspective of the buyer in a multiple prior setting. We focus on the discrete time version of the problem and develop an ambiguous version of the CRR model. Instead of assuming that the distribution of up- and down- movements of the underlying is known to the buyer we allow the probability of going up on a node to lie in a appropriately modeled set.

This leads to a set of models that agree on the size of up- and down-movement but disagree on the mean return. In this ambiguous binomial tree setting which was first analyzed in Epstein and Schneider (2003a) we aim to apply standard Snell reasoning to evaluate the options. Due to the above mentioned duality result it is enough to calculate the worst-case measure $\hat{P}$ and then to analyze the classical problem under $\hat{P}$. However, the worst-case measure depends highly on the payoff structure of the claim and needs to be calculated for each option separately. If the payoff satisfies certain monotonicity conditions the worst-case measure is easy to derive. The direction/effect of uncertainty is the same for all states of the world and the worst-case measure is then independent on the realization of the stock price process leading to a statical structure that resembles classical models. In the case of more sophisticated payoffs this stationarity of the worst-case measure breaks down and the worst-case measure changes over time depending on the realization of the stock price. This is due to the fact that uncertainty may affect the model in different ways changing the beliefs of decision maker and so the worst-case measure according to the effect that is dominating. This ability to react on information by adjusting the model and to choose the model depending on the payoff is the main structural difference between the classical single measure model and the multiple prior model considered here.

We identify additional sources of uncertainty that lead to the dynamical and path-dependent structure of the worst-case measure. We also analyze the impact of different effects of uncertainty on the overall behavior and the resulting model highlighting differences between the single prior models and the multiple prior model.

In our analysis we decompose the claims in monotone parts as the worst-case measure for monotone problems is well known. We then analyze each claim separately deriving the worst-case measure conditioned on monotonicity. To complete the analysis we paste the measures obtained on subspaces together using time-consistency. This idea is closely linked to the method of pricing derivatives using digital contracts introduced by Ingersoll (2007) and also used by Buchen (2004). However, this literature focuses on European style options and does not cover the dynamical structure analyzed here.

In the case of barrier options the value of the option is conditioned on the event of reaching a trigger. Unlike the plain vanilla option case, the lifetime of an barrier option become uncertain as it depends on the occurrence of the trigger event. This leads to an additional source of uncertainty causing a change in the monotonicity of the value function when the stock price hits the barrier. For example, in the case of an up-and-in put the ambiguity averse decision maker assumes the returns to be low and chooses therefore the measure with the lowest drift before the stock price reaches the barrier. After hitting the barrier she obtains a plain vanilla put option monotone in the underlying and uses therefore the measure with the highest drift. Similar behavior can be observed for other types of barrier options.

The second group of options we focus on are the dual expiry options. Here, the strike of the option is not known at time zero as it is being determined as a function of the underlying's value on a date different from the issue date of the option – the first expiry. Therefore, additional to the uncertainty about the final payoff the decision maker faces uncertainty about the value of the strike before first expiry date.

In the case of shout options the first expiration date, the so-called shout date/freeze date, is determined by the buyer. Here, the investor has to call the bank if she aims to fix the strike. Therefore, the buyer of an shout option faces two stopping problems: First, she has to determine the optimal shouting time in order to set the strike optimally and then the to stop the payoff process optimally. The holder of an shout put gets an put after shouting and thus, anticipates high returns after shouting. Before shouting however he owns a claims whose value is increasing in the price of the underlying which results in low returns anticipated before shouting.

Finally, we analyze options whose payoff function consists of two monotone pieces. Typical examples are straddles and strangles. The buyer of such options presumes a change in the underlying's price but is not sure about the direction of the change. Depending on the value of the underlying the option pays off a call or a put, so as a consequence the actual payoff function becomes uncertain. Here, one can decompose the value of the option in an increasing and a decreasing leg. The buyer of the option changes her beliefs about the returns every time the value switches from decreasing to increasing part of the value function. So, an ambiguity averse buyer of a straddle presumes the stock price to go down in hausse phases and up in baisse phases.

An outline of the paper is as follows. Section 2 introduces the discrete model which is in this form due to Riedel (2009). Section 3 recalls the solution for payoffs monotone in underlying's price introduced in Riedel (2009) and builds the base for the following analysis. Section 4 provides the solution for barrier options options, and Section 5 develops the solution for multiple expiry. Finally, Section 6 discusses U–shaped payoffs

2 Time–Consistent Multiple Priors in discrete Time

We first introduce the basic theoretical setup to evaluate options in multiple prior model. This model has the CRR model as the starting point and was already developed in Riedel (2009) and can be seen as a version of the IID model introduced in Epstein and Schneider (2003a) with a different objective. At the same time the model is the discrete time version of the κ–ignorance model in Epstein and Chen (2002).

Having established the model we discuss the market structure and recall the decision problem of the buyer and the solution method – the multiple prior backward induction introduced by Riedel (2009).

2.1 The stochastic Structure

To set up the model we start with a classical binomial tree. For a fixed maturity date $T \in \mathbb{N}$ we consider a probability space $(\Omega, \mathcal{F}, P_0)$ where $\Omega = \otimes_{t=1}^{T} \mathcal{S}$ with $\mathcal{S} = \{0, 1\}$ is the the set of all sequences with values in $\{0, 1\}$, $\mathcal{F}$ is the σ–field generated by all projections $\varepsilon_t : \Omega \to \mathcal{S}$ and P_0 denotes the uniform on $(\Omega, \mathcal{F})$. By construction, the projections $(\varepsilon_t)_{t=1,\dots,T}$ are independent and identically distributed under P_0 with $P_0(\varepsilon_t = 1) = \frac{1}{2}$ for all $t \le T$. Furthermore, we consider the filtration $(\mathcal{F}_t)_{t=0,\dots,T}$ generated by the projections $(\varepsilon_t)_{t=1,\dots,T}$ where $\mathcal{F}_0$ is the trivial σ–field – $\{\emptyset, \Omega\}$. The event $\varepsilon_t = 1$ represents an up-movement on a tree while the complementary event denotes the down-movement.

Additionally, we define on $(\Omega, \mathcal{F}, P_0)$ a convex set of measures $\mathcal{Q}$ in the following way: We fix an interval $[\underline{p}, \overline{p}] \subset (0, 1)$ for $\underline{p} \le \overline{p}$ and consider all measures whose conditional one step ahead probabilities of going up on a node of the tree remains within the interval $[\underline{p}, \overline{p}]$ for every $t \le T$, i.e.

$$\mathcal{Q} = \left\{ P \in \mathcal{M}_1(\Omega) | P(\varepsilon_t = 1 | \mathcal{F}_{t-1}) \in [\underline{p}, \overline{p}], \forall t \le T \right\} \tag{1}$$

The set $\mathcal{Q}$ is generated by the conditional one-step-ahead correspondence assigning at every node $t \le T$ the probability of going up. correspondence. In particular, $\mathcal{Q}$ contains all product measures defined via $P_p(\varepsilon_{t+1} = 1 | \mathcal{F}_t) = p$ for a fixed $p \in \mathcal{Q}$ and all $t < T$. In the following we denote by $\overline{P} = P_{\overline{p}}$ and by $\underline{P} = P_{\underline{p}}$.

Clearly, the state variables $(\varepsilon_t)_{t=1,\dots,T}$ are independent under all product measures. In general, however, $(\varepsilon_t)_{t=1,\dots,T}$ are correlated. To see this consider the measure P^τ defined via

$$P^\tau(\varepsilon_{t+1} = 1 | \mathcal{F}_t) = \begin{cases} \overline{p} & \text{if } t \le \tau \\ \underline{p} & \text{else} \end{cases}$$

for a stopping time $\tau < T$. As the one-step-ahead probabilities remain in the interval $[\underline{p}, \overline{p}]$ the so defined measure P^τ belongs to $\mathcal{Q}$ for all stopping times $\tau < T$. At the same time the probability of going up on a node depends on the realized path through the value of τ and $(\varepsilon_t)_{t=1,\dots,T}$ are correlated.

The above example reveals an important structural feature of $\mathcal{Q}$: The set of measures is stable under the operation of decomposition in marginal and conditional part. Loosely speaking, it allows the decision maker to change the measure she uses as the time evolves in an appropriate manner. In the example above, the decision maker first uses the measure $\overline{P}$ until an event indicated by the stopping time τ and then changes to $\underline{P}$. Mathematically, this property is equivalent to an appropriate version of the Law of Iterated Expectation and is closely linked to the idea of backward induction. The concept has gained much attention in the recent literature and was also discussed under different notions by Delbaen (2002), Epstein and Schneider (2003a), Föllmer and Schied (2004) and **?**.

The following lemma summarizes crucial properties of the set $\mathcal{Q}$.

Lemma 2.1 *The set of measures defined as in* (1) *satisfies the following properties*

1. *$\mathcal{Q}$ is compact and convex,*
2. *all $P \in \mathcal{Q}$ are equivalent to P_0,*
3. *$\mathcal{Q}$ is time-consistent in the following sense: Let $P, Q \in \mathcal{Q}$, $(p_t)_t, (q_t)_t$ densities of P, Q with respect to P_0. For a fixed stopping time τ define the measure R via*
$$r_t = \begin{cases} p_t & \text{if } t \leq \tau \\ \frac{p_\tau q_t}{q_\tau} & \text{else} \end{cases}$$
then $R \in \mathcal{Q}$.

Due to Lemma 2.1 we can identify the set $\mathcal{Q}$ with the set of the density processes with respect to the measure P_0. In the following we denote by D the density process of $P \in \mathcal{Q}$ with respect to P_0, i.e. $D_t = \frac{dP}{dP_0}|_{\mathcal{F}_t}$ for $P \in \mathcal{Q}$, $t \leq T$. A more detailed analysis of the structure of D can be found in Riedel (2009). The setting is a version of discrete time-consistent multiple priors on a binomial tree discussed in Riedel (2009). Another formulation can be found in Epstein and Schneider (2003b).

2.2 The Market Model

Within the above introduced probabilistic framework we establish the financial market in the spirit of the CRR model. We consider a market consisting of two assets: a riskless bond with a fixed interest rate $r > -1$ and a risky stock with multiplicative increments. For given model parameters $0 < d < 1 + r < u$ and $S_0 > 0$ the stock S evolves according to

$$S_{t+1} = S_t \cdot \begin{cases} u & \text{if } \varepsilon_{t+1} = 1 \\ d & \text{if } \varepsilon_{t+1} = 0 \end{cases} .$$

Without loss of generality, we assume $u \cdot d = 1$. This is a common and appropriate assumption when dealing with exotic options in binomial models, see Cox and Rubinstein (????) for instance.

For every $t \leq T$ the range of possible stock prices is finite and bounded, we denote by

$$E_t = \{S_0 \cdot u^{t-2k} | k \in \mathbb{N}, -t \leq k \leq t\}$$

the set of possible stock prices at time t. Moreover, the filtration generated by the sequence $(S_t)_{t=0,\dots,T}$ coincides with $(\mathcal{F})_{t=0,1,\dots,T}$ and every realized path of S $(s_1, \dots, s_t)$ can be associated with a realization of $(\varepsilon_s)_{s \leq t}$.

As the state variables are not independent under every probability measure $P \in \mathcal{Q}$ in our model the increments of S are correlated in general. The probability of an up-movement depends on the realized path but stays within the boundaries $[\underline{p}, \overline{p}]$ for every $P \in \mathcal{Q}$. As mentioned above the returns are independent and identically distributed under all product measures in $\mathcal{Q}$.

Economically, our model describes a market where the market participants are not perfectly certain about the asset price dynamics. In order to express this uncertainty investors use a class of measures constructed above. This inability to completely determine the underlying probabilistic law may arise from lack or imprecise information or can be part of the distress stress routine as discussed in the introduction. Clearly, the set $\mathcal{Q}$ is the set of possible models the decision maker takes into account. Different choices of $P \in \mathcal{Q}$ correspond to different models. With our specification mean return on stock is uncertain and as one can easily see, $\overline{P}$ corresponds to the highest mean return at every node, while $\underline{P}$ corresponds to the lowest mean return on stock on every node. The specification of $\mathcal{Q}$ is a part of the model and in practice may arise from regulation policies or be imposed by the bank accounting standards, result from statistical consideration or just reflects the degree of ambiguity aversion. The length of the interval $[\underline{p}, \overline{p}]$ determines the range of possible models. As the interval decreases the model converges to the classical binomial tree model and we obtain the classical CRR model as a special case of our model by choosing $\underline{p} = \overline{p}$.

$\mathcal{Q}$ can be interpreted as the set of models the decision maker takes into account. The second difference between the classical binomial tree is the introduction of correlated returns on stock. This allows to incorporate the decision maker's reaction on new arriving information. In our model the investor is allowed to change the model she uses according to the available information. Now the economical implication of time-consistency of $\mathcal{Q}$ becomes clear. Due to this property the multiple prior decision maker is allowed to use the measure $P_1 \in \mathcal{Q}$ until an event indicated by a stopping time τ and then to change his beliefs about the right model using P_2 after τ. The multiple prior decision maker is allowed to adjust the model she uses responding to the state of the market. However, this notion is not the same as classical Bayesian learning as the decision maker has to little information or market knowledge to learn the real distribution. While in the learning process the decision maker updates the model adjusting the set of possible models, here the investor keeps the set of possible models fixed not excluding any of the possible models as the time evolves but choses a particular model at every point of the time reconsidering her choice when new information arrives.

2.3 The Decision Problem

In this setting we consider an investor holding an exotic option in her portfolio. As most of the exotic options are OTC contracts there is usually no functioning market for this derivative or the trading of claims involves high transaction costs. Therefore, in absence of a trading partner the buyer is forced to hold the claim until maturity, so we exclude the possibility of selling the acquired contracts concentrating purely on the exercise decision of the investor. In our analysis we mainly concentrate on institutional investors already holding the derivatives in the portfolio. Therefore, it is plausible to assume risk neutral agents who discount future payoff by the riskless rate.

Remark 2.2 *When having a private investor exposing ambiguity aversion in mind it seems natural also to introduce risk aversion and to discount by individual discount rate δ. As these considerations do not change the structure of the worst-case measure obtained here, we omit this possibility maintaining risk neutrality.*

We consider an American claim $A : \Omega \to \mathbb{R}_+$ written on S and maturing at T that pays off $A(t, (S_s)_{s\leq t})$ when exercised at time t. Note, that we explicitly allow path-dependent structures. The investor holding A in her portfolio aims to maximize her expected payoff choosing an appropriate exercise strategy. As the expectation in our multiple prior setting is not uniquely defined the ambiguity averse decision maker maximizes her minimal expected payoff, i.e.

$$\text{maximize } \inf_{P\in\mathcal{Q}} \mathbb{E}^P A(\tau, (S_s)_{s\leq\tau}) \text{ over all stopping times } \tau \leq T. \qquad (2)$$

The choice of the exercise strategy according to the worst possible model corresponds to conservative value assignment. It treats long book positions in the same way as the coherent risk measures treats short positions[1]. The value of the multiple prior problem stated in (2) $V^{\mathcal{Q}}$ is lower than the value of the problem V^P for every possible model $P \in \mathcal{Q}$. Therefore, this notion minimizes the model risk as the model misspecification within $\mathcal{Q}$ increases the value of the claim.

robustness of the used model and considering the worst possible model.

Remark 2.3

1. *The problem of the long investor stated in* (2) *differs structurally from the task of the seller of the option. The seller of the American claim needs to hedge claim against every strategy of the buyer. To obtain the hedge she solves the optimal stopping problem under the equivalent martingale measure P^*. In the binomial tree the unique equivalent martingale measure P^* is completely determined by parameters r, u and d and does not depend on the mean return. See Hull (2006) for a more detailed analysis. The situation is different for the buyer as she solves the optimal stopping problem under the physical measure taking the mean return into account and being interested in personal utility maximization rather than in risk neutral valuation. Although the buyer and the seller use different techniques assigning value to the options and obtaining different values for the claim there is no contradiction to no arbitrage condition because of the American structure of the claims considered here.*

2. *It is usual to evaluate claims in the book that are not settled yet using mark-to-market approach. The value of the option is then set to be equal to the market price. This makes sense if markets are well functioning or if the investor intends to sell the option on the secondary market rather than hold it until maturity. However, this approach may value the claims wrongly if the market is malfunctioning or there is no market at all as it was seen and still is seen at financial markets these days. Multiple prior value assignment through $V^{\mathcal{Q}}$ is an alternative to the fair value accounting as it provides conservative value assignment by using the worst possible scenario but protects the book value from too pessimistic or overoptimistic views of the market that are due to*

[1]Mathematically, our model is equivalent to a representation of coherent risk measures. See Delbaen (2002) or Riedel (2009) for more detailed analysis.

expectations and do not reflect fundamentals. However, $V^{\mathcal{Q}}$ is not the price for the option it is rather the private value for the investor that may differ from the market view.

2.4 The Solution Method

If $\mathcal{Q}$ is a singleton the problem stated in 2 can be solved using classical dynamic programing methods. One defines backwards the value process of the problem – the Snell envelope – and stops as soon as the value process reaches the payoff process. This technique fails to hold in the multiple prior setting. See Riedel (2009) for an example. Riedel (2009) extended backward induction to the case of time-consistent multiple priors stating sufficient conditions for the Snell arguments to hold.

Theorem 2.4 (Riedel) *Given a set of measures satisfying conditions stated in Lemma 2.1 and a bounded payoff process X, $X_t = A(t, (S_s)_{s\leq t})$, define the minimax Snell envelope $U^{\mathcal{Q}}$ recursively by*

$$\begin{aligned} U_T^{\mathcal{Q}} &= X_T \\ U_t^{\mathcal{Q}} &= \max\{X_t, \inf_{P\in\mathcal{Q}} \mathbb{E}^P(U_{t+1}^{\mathcal{Q}}|\mathcal{F}_t)\} \text{ for } t < T \end{aligned} \tag{3}$$

Then,

1. *$U^{\mathcal{Q}}$ is the smallest multiple prior $\mathcal{Q}$-supermartingale [2] dominating the payoff process X.*

2. *$U^{\mathcal{Q}}$ is the value process of the multiple prior stopping problem for the payoff process X, i.e.*

$$U_t^{\mathcal{Q}} = \sup_{\tau\geq t} \inf_{P\in\mathcal{Q}} \mathbb{E}^P(X_\tau|\mathcal{F}_t)$$

3. *An optimal stopping rule is then given by*

$$\tau^{\mathcal{Q}} = \inf\{t \geq 0 | U_t^{\mathcal{Q}} = X_t\}.$$

[2] Given a set of measures $\mathcal{Q}$, a multiple prior supermartingale with respect to $\mathcal{Q}$ is an adapted process, say S, satisfying $S_t \geq \operatorname{ess\,inf}_{P\in\mathcal{Q}} \mathbb{E}^P(S_{t+1}|\mathcal{F}_t)$ for $t \in \mathbb{N}$.

Since we are in a finite setting and due to Lemma 2.1 we may replace the inf-operator by the min-operator in the following.

The above result ensures the existence of the solution of problem (2). Moreover, as shown by several authors (for example Föllmer and Schied (2004), Karatzas and Kou (1998), Riedel (2009)) problem (2) is equivalent to a single prior problem for a measure $\hat{P} \in \mathcal{Q}$, i.e. the value function of the multiple prior problem

$$U^{\mathcal{Q}} = U^{\hat{P}} \qquad P_0\text{-a.s.}. \tag{4}$$

The measure $\hat{P}$ is called worst–case measure and can be constructed via backward induction by choosing the worst conditional one-step-ahead probability on every node of the tree and pasting the so obtained densities together at time 0. The worst-case measure is stochastic in general and depends on the payoff process. In our setting one can characterize the worst case measure through the density process D for every payoff process separately. Thus, the worst–case measure is a part of the solution.

Due to equality (4) the optimal stopping strategies $\tau^{\mathcal{Q}}$ of the multiple prior problem and $\tau^{\hat{P}}$ of the problem for the prior $\hat{P}$ coincide. Therefore, the problem can be solved in two steps. In the first step one identifies the worst–case measure $\hat{P}$ and solves the classical problem under $\hat{P}$ in the second step. This technique allows to make use of solutions already obtained in the classical case. For problems where no closed form solution is available the technique reduces numerical complexity by reducing the task to a single model problem where methods are well developed.

Analyzing the exotic options we use this techniques emphasizing the behavioral interpretation of the worst–case measure and highlighting the difference between classical models and the multiple prior approach.

2.5 Options with Monotone Payoffs

In this subsection we provide the solution for claims whose payoffs are monotone in the underlying's price at each time and satisfy the Markov property, i.e. the payoff of the option is not path-dependent. The most famous examples for options of this class are plain vanilla put and call options. The results of this section build the foundation for the analysis of more complicated payoffs in the next sections and were stated in Riedel (2009).

We consider a discounted American claim maturing at T and paying off $X_t = A(t, S_t)$ when exercised at t.

Theorem 2.5 *1. If the payoff function of the claim $A(t,S_t)$ is increasing in S_t for all t, then the multiple prior Snell envelope is $U^{\mathcal{Q}} = U^{\underline{P}}$, and the holder of the claim uses the optimal stopping rule given by $\tau^{\underline{P}} = \inf\{t \geq 0 : A(t,S_t) = U_t^{\underline{P}}\}$.*

2. If $A(t,S_t)$ is decreasing in S_t for all t, the multiple prior Snell envelope is $U^{\mathcal{Q}} = U^{\overline{P}}$, and an optimal stopping rule under ambiguity is given by $\underline{\tau} = \inf\{t \geq 0 : A(t,S_t) = U_t^{\overline{P}}\}$.

The key to this result is the fact that $\underline{P}$ (or $\overline{P}$ resp.) is the worst probability measure in the sense of first–order stochastic dominance and that the payoff is a monotone function of the underlying stock price. Using Theorem 2.5 we can already solve the optimal exercise problem for the call and the put in the multiple prior setting:

Corollary 2.6 (Call) *A risk–neutral buyer of an American call uses an optimal stopping rule for the prior $\underline{P}$. The value of American call at time zero is given via*

$$V_0^{\mathcal{Q}} = U_0^{\underline{P}} = \mathbb{E}^{\underline{P}}\left((S_{\underline{\tau}} - K)^+(1+r)^{-\underline{\tau}}\right),$$

where $\underline{\tau}$ is the optimal stopping time with respect to $\underline{P}$. In particular, if $\underline{p}u + (1-\underline{p})d > 1+r$, the American call is never being exercised.

Corollary 2.7 (Put) *A risk–neutral buyer of an American put uses an optimal stopping rule for the prior $\overline{P}$. The value of American put at time zero is given via*

$$V_0^{\mathcal{Q}} = U_0^{\overline{P}} = \mathbb{E}^{\overline{P}}\left((K - S_{\overline{\tau}})^+(1+r)^{-\overline{\tau}}\right),$$

where $\overline{\tau}$ is the optimal stopping time with respect to $\overline{P}$.

These results help us to find the worst-case measure for more complicated payoffs. Using appropriate decompositions we represent the options as monotone claims. For those monotone claims we can identify the worst-case measure using Theorem 2.5. Pasting the so obtained measures together we construct the desired worst-case measure.

3 Barrier Options

Barrier options are among most traded exotic options and are often used as components of more sophisticated derivatives. The knock-in/knock-out feature of the options leads to a lower premium which has to be paid by the buyer. In return, the buyer is exposed to the risk, for instance in the knock-out case, that the underlying hits the barrier and the option becomes worthless. For knock-in options the buyer faces the risk that the underlying firstly has to hit the barrier level before the option becomes valuable. This singularity of the payoff at the barrier makes the barrier option interesting from the mathematical point of view and challenging to evaluate.

Before stating the results for barrier options we prove a technical theorem which enables us to identify the worst-case measure for different path-dependent payoffs. In the following we need some definitions and simplifications:

For the whole section let us assume that all given barrier levels $H \in \mathbb{R}_+$ lie on the grid of possible asset prices. This makes things significantly easier. In particular, when one aims to specify an explicit formula for the value of the optimal stopping problem. Furthermore, in the case of letting the grid size tending to zero, without the assumption one would have to consider upper- and lower-barriers and interpolate between both in order to obtain reasonable results, see Hull (2006) for a more detailed review.

Now, let $H > S_0,\ t \in [0, T]$. A *first-passage time* τ is defined by

$$\tau : \Omega \longrightarrow [0, T+1], \qquad \tau(\omega) := \inf\{t \geq 0 : S_t(\omega) \geq H\} \wedge T + 1.$$

For $H > S_0$ we call these stopping times depending on the stock price *up-crossing times*. We set $\mathcal{F}_{T+1} := \mathcal{F}_T$ and $\inf \emptyset := \infty$. Similarly, we define *down-crossing times* when $H < S_0$. Clearly, these stopping times depend on the whole path of the price process in general.

In practice, barrier options are said to be weak path-dependent which emphasizes that their payoffs indeed depend on the whole path of the underlying's price, but considering the two- dimensional process consisting of its price and its maximal price, minimal price, respectively, reduces the problem to the usual Markovian case meaning that their payoffs at some time t depend on the two-dimensional process only at that specific time t.

In contrast to plain vanilla options, the payoff process of barrier options is not monotone in S_t for each t but depends on the event of whether the option's underlying has hit the barrier up to time t or not. To express this fact mathematically we use the notion of stochastic intervals. For two stopping times the stochastic interval $[\tau_1, \tau_2[$ is defined by

$$[\tau_1, \tau_2[\ := \left\{(s, \omega) \in [0, T] \times \Omega \,\middle|\, \tau_1(\omega) \leq s < \tau_2(\omega)\right\}.$$

We will often write with slight abuse of notation $\mathbb{1}_{[\tau_1,\tau_2[}(\omega)$ instead of $\mathbb{1}_{[\tau_1,\tau_2[}(t,\omega)$.

By means of stochastic intervals we extend Theorem 2.5 to more general situations. We do not ask for monotonicity all the time but only on stochastic intervals specialized by two stopping times τ_1, τ_2.

In order to identify the worst-case measure we can determine it recursively by using backwards induction for computing the Snell envelope U as shown in Riedel (2009). One observes that the worst-case conditional one-step-ahead probability at time t, denoted by $\hat{P}_t$, is characterized by the equation $U_{t-1} = \min_{P\in\mathcal{Q}} \mathbb{E}^P\left(U_t|\mathcal{F}_{t-1}\right)$. So, in the case of a monotone payoff process X, $\hat{P}_t$ is detected by calculating U_{t-1}, and the monotonicity of U_{t-1} is inherited by the monotonicity of U_t and X_{t-1}, as long as they possess the same.
In the following we will use this study to extend the theory to payoff processes which do not exhibit the same monotonicity at all times but only on different events.
By technical reasons we also need the following stopping times.

Definition 3.1 *Let H be the barrier specifying τ_2. If τ_2 is up-crossing time we define*

$$\sigma_i := \inf\{t \in [\sigma_{i-1}+1, \tau_2[\;\big|\; S_t = H\cdot d\} \wedge T + 1$$

for $1 \leq i \leq T$ with the notation $\sigma_0 := -1$. If τ_2 is down-crossing time we define for $1 \leq i \leq T$

$$\sigma_i := \inf\{t \in [\sigma_{i-1}+1, \tau_2[\;\big|\; S_t = H\cdot u\} \wedge T + 1.$$

These stopping times are needed in order to specify the times/nodes at which there is possibility to reach the second barrier in the next time step with positive probability.

Theorem 3.2 *Let H_1, H_2 be the barrier levels specifying τ_1, and τ_2, respectively. Let the payoff process $X = (X_t)$ be given by*

$$X_t = x(t, S_t, \tau_1, \tau_2) = A(t, S_t)\,\mathbb{1}_{[\tau_1,\tau_2[}(t)$$

where $A(t,\cdot)$ is monotone in S for all t, τ_1, τ_2 are up-crossing times (assume $S_0 < H_1 < H_2$) as defined in **??***, or constant, with $\tau_1 \leq \tau_2$ $P_0 -$ a.e..*
Let $(U_t^{\hat{P}})$ be the Snell envelope of (X_t) under $\hat{P}$.

1. *If $A(t,\cdot)$ is decreasing in S for all t, the multiple prior Snell envelope is $U = U^{\hat{P}}$ and the worst-case measure $\hat{P}$ is induced by the density*

$$\hat{D}_t := 2^t \prod_{u\le t\wedge\tau_1} \big(\varepsilon_u \underline{p} + (1-\varepsilon_u)\big)(1-\underline{p}) \prod_{u\in\,]\tau_1,\,t\wedge T]} \big(\varepsilon_u \overline{p} + (1-\varepsilon_u)(1-\overline{p})\big)$$

for all $t \le T$. An optimal stopping rule under ambiguity is given by $\hat{\tau} = \inf\Big\{t \in [\tau_1, \sigma_1] \;\big|\; X_t = U_t^{\hat{P}}\Big\} \wedge T$.

2. *If $A(t,\cdot)$ is increasing in S for all t, the multiple prior Snell envelope is $U = U^{\hat{P}}$ and the worst-case measure $\hat{P}$ is induced by the density*

$$\hat{D}_t := 2^t \prod_{u\le t\wedge\tau_2:\ u\neq\sigma_i+1} \big(\varepsilon_u \underline{p} + (1-\varepsilon_u)(1-\underline{p})\big) \prod_{u\le t:\ u=\sigma_i+1} \big(\varepsilon_u \overline{p} + (1-\varepsilon_u)(1-\overline{p})\big) \prod_{u\in\,]\tau_2,\,t\wedge T]} \big(\varepsilon_u \overline{p} + (1-\varepsilon_u)(1-\overline{p})\big)$$

for all $t \le T$ and all occurring $1 \le i \le T$. An optimal stopping rule under ambiguity is given by $\hat{\tau} = \inf\Big\{t \in [\tau_1, \sigma_1] \;\big|\; X_t = U_t^{\hat{P}}\Big\} \wedge T$.

The proof will be given in appendix. It relies heavily on the theory about the multiple prior Snell envelope constructed by backwards induction which in turn needs time-consistency of the set of priors as a crucial tool.

Remark 3.3 *Note that if $\tau_2 \le T$ there exists always $i \le T$ such that $\tau_2 = \sigma_i + 1$. Note also that the worst-case measure is not unique. After time τ_2 the conditional one-step-ahead probabilities only have to satisfy $\hat{P}(\varepsilon = 1 \mid \mathcal{F}_{t-1}) \in [\underline{p}, \overline{p}]$ for all $t > \tau_2$ since the claim's payoff is always 0 after τ_2. Furthermore, exercising always happens at time σ_1 at the latest. Afterwards the payoff from exercising is always 0. Therefore, the density of the worst-case measure is only relevant for the decision maker up to step σ_1 since afterwards she will not possess the option anymore.*

A similar result as above also holds for down-crossing times. The only difference is the monotonic behavior of X and U which changes for down-crossing times. As a consequence, the densities of the worst-case measures change. So, we will state the theorem without giving the proof since it would be almost a copy of the one of Theorem 3.2.

Theorem 3.4 *Take the same assumptions as in Theorem 3.2 except the one for τ_1 and τ_2 being now either down-crossing times or constant again. Thus, assume $S_0 > H_1 > H_2$.*

1. *If $A(t,\cdot)$ is decreasing in S for all t, the multiple prior Snell envelope is $U = U^{\hat{P}}$ and the worst-case measure $\hat{P}$ is given by the density*

$$\hat{D}_t := 2^t \prod_{u\leq t\wedge\tau_2:\ u\neq\sigma_i+1} \big(\varepsilon_u\overline{p} + (1-\varepsilon_u)(1-\overline{p})\big) \prod_{u\leq t:\ u=\sigma_i+1} \big(\varepsilon_u\underline{p} + (1-\varepsilon_u)(1-\underline{p})\big)$$
$$\prod_{u\in\,]\tau_2,\,t\wedge T]} \big(\varepsilon_u\underline{p} + (1-\varepsilon_u)(1-\underline{p})\big)$$

   for all $t \leq T$ and all occurring $1 \leq i \leq T$. An optimal stopping rule under ambiguity is given by
   $\hat{\tau} = \inf\Big\{t \in [\tau_1, \sigma_1] \;\Big|\; X_t = U_t^{\hat{P}}\Big\} \wedge T.$

2. *If $A(t,\cdot)$ is increasing in S for all t, the multiple prior Snell envelope is $U = U^{\hat{P}}$ and the worst-case measure $\hat{P}$ is given by the density*

$$\hat{D}_t := 2^t \prod_{u\leq t\wedge\tau_1} \big(\varepsilon_u\overline{p} + (1-\varepsilon_u)(1-\overline{p})\big) \prod_{u\in\,]\tau_1,\,t\wedge T]} \big(\varepsilon_u\underline{p} + (1-\varepsilon_u)(1-\underline{p})\big)$$

   for all $t \leq T$. An optimal stopping rule under ambiguity is given by
   $\hat{\tau} = \inf\Big\{t \in [\tau_1, \sigma_1] \;\Big|\; X_t = U_t^{\hat{P}}\Big\} \wedge T.$

Remark 3.5 *An extension of the theorems to cases when τ_1 is up-crossing time and τ_2 down-crossing time, or vice versa, is also possible. One can also skip the condition $\tau_1 \leq \tau_2$. This is just an assumption made to avoid too many cases that have to be distinguished when proving the theorem and stating the density. In special cases it is also possible to extend the theorem to payoff processes which are finite sums of summands of the shape as in the theorems. Later, we are illustrating this for an up-and-out ladder option.*

The theorems above allow to analyze options which do not have the same monotonicity all the time but conditioned on certain events. Here, these technically demanding results lead quickly to the solutions. Generally, the next subsections are applications of the above stated theorem.

3.1 Simple Barrier Options

In this part we will apply the preceding theory to single barrier options. The payoff of a single barrier option depends on the underlying's price hitting a barrier during the lifetime of the contract. In-options become valuable when the asset price hits a predetermined barrier H. If this does not happen over the lifetime of the contract, the option remains worthless. In contrast, out-options become worthless when the stock price reaches a barrier.

While exercising American put and call can be reduced easily to the single prior case by using monotonicity and stochastic dominance, see Corollaries 2.6 and 2.7, the picture is quite more involved in the case of American up-and-in put and down-and-in call barrier options.

First, we consider the American up-and-in put with strike price K and barrier H. We assume $H > K$ and, to avoid the trivial case, $H > S_0$. Let $T > 0$ be the contract's maturity. Denote by

$$\tau_H = \inf\{t \geq 0 | S_t \geq H\} \wedge T + 1$$

the lined-in time. At time τ_H and afterwards, the barrier option coincides with a plain vanilla American put initiated at τ_H, expiring at T and strike K.

Firstly the holder of such an option faces uncertainty about the event whether the option is being knocked in. After knock-in she faces the same uncertainty as holding a plain vanilla put. Both uncertainties work in contrary directions. At the beginning her ambiguity aversion leads to the assumption of having lowest drift in the option's underlying's price, and after knock-in, she assumes highest drift which results in highest marginal probabilities for up-movements of the option's underlying.

This is stated in the next corollary and proven by means of Theorem 3.2.

Corollary 3.6 (Up-and-in put) *For an American barrier up-and-in put option with data as specified above the ambiguity averse agent uses the following prior $\hat{P}$ specified by the density*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_H} \big(\varepsilon_u \underline{p} + (1-\varepsilon_u)(1-\underline{p})\big) \prod_{u \in \,]\tau_H,\, t \wedge T]} \big(\varepsilon_u \overline{p} + (1-\varepsilon_u)(1-\overline{p})\big) \quad \textit{for } t \leq T.$$

Hence, the value of the option at time t from the perspective of the ambiguity averse buyer is given by

$$U_t^{\mathcal{Q}} = U_t^{\hat{P}} = \mathbb{E}^{\hat{P}}[X_{\hat{\tau}} \mid \mathcal{F}_t], \tag{5}$$

where $\hat{\tau}$ is an optimal stopping time given by
$\hat{\tau} = \inf\left\{ t \in [\tau_H, T] \;\middle|\; X_t = U_t^{\hat{P}} \right\}$.

PROOF: We apply Theorem 3.2 part 1. Set $\tau_1 := \tau_H$ and $\tau_2 := T+1$. The discounted payoff process is given by $X_t = (K - S_t)^+/(1+r)^t \mathbb{1}_{[\tau_H, T+1[}$ for all $t \in [0,T]$. Since $A(t, S_t) := (K - S_t)^+/(1+r)^t$ is monotone decreasing in S_t for each t, Theorem 3.2 part 1 applies. □

From the density of the worst-case measure we see that the pessimistic buyer thus presumes a change of drift at knock-in. Before the option becomes valuable, she uses the lowest mean return in her computations, and afterwards, she uses the measure where the underlying asset price has maximal mean return. This corresponds to the lowest conditional one-step-ahead probabilities for up-movements before knock-in, and to the highest ones afterwards, respectively. Furthermore, one sees that the worst-case measure $\hat{P}$ is the pasting of $\overline{P}$ after $\underline{P}$ at τ_H and therefore, it exhibits a non-stationary structure.

Let us rewrite equation (5) to get an explicit formula for the value process $(U_t^{\mathcal{Q}})$. By the last corollary $U_t^{\mathcal{Q}} = U_t^{\hat{P}}$ for all $t \leq T$. Hence, for $t \leq \tau_H$ we obtain by Bayes' rule (see Karatzas and Shreve (1991), Lemma 5.3) and law of iterated expectation

$$\begin{aligned} U_t^{\mathcal{Q}} &= \mathbb{E}^{\hat{P}}[X_{\hat{\tau}} | \mathcal{F}_t] = \mathbb{E}^{P_0}\left(X_{\hat{\tau}} \frac{\hat{D}_{\hat{\tau}}}{\hat{D}_t} \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}^{P_0}\left[\mathbb{E}^{P_0}\left(X_{\hat{\tau}} \frac{\hat{D}_{\hat{\tau}}}{\hat{D}_{\tau_H}} \middle| \mathcal{F}_{\tau_H} \right) \frac{\hat{D}_{\tau_H}}{\hat{D}_t} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\underline{P}}\left[\mathbb{E}^{\overline{P}}\left[X_{\hat{\tau}} \middle| \mathcal{F}_{\tau_H} \right] \middle| \mathcal{F}_t \right], \end{aligned} \tag{6}$$

where $X = (X_t)$ denotes the discounted payoff process.
If $t > \tau_H$

$$U_t^{\hat{P}} = \mathbb{E}^{\hat{P}}[X_{\hat{\tau}} | \mathcal{F}_t] = \mathbb{E}^{\overline{P}}\left[(K - S_{\hat{\tau}})^+/(1+r)^{\hat{\tau}} \,\middle|\, \mathcal{F}_t \right]$$

which equals the value of a plain vanilla American put in the ambiguity-averse framework discounted to time 0.

Formula (6) also demonstrates exactly what is interpreted above by analyzing the density of the worst-case measure. For $t = 0 < \tau_H$, for instance, equation (6) becomes

$$
\begin{aligned}
U_0^{\hat{P}} &= \mathbb{E}^{\underline{P}}\left[\mathbb{E}^{\overline{P}}\left[X_{\hat{\tau}} \,|\, \mathcal{F}_{\tau_H}\right] \,\Big|\, \mathcal{F}_0\right] \\
&= \mathbb{E}^{\underline{P}}\big[\mathbb{E}^{\overline{P}}[(K - S_{\hat{\tau}})^+/(1+r)^{\hat{\tau}-\tau_H} \big| \mathcal{F}_{\tau_H}]/(1+r)^{\tau_H}\big] \\
&= \mathbb{E}^{\underline{P}}\left(\sum_{i=0}^{T} \mathbb{E}^{\overline{P}}_{\{\tau_H=i\}}[(K - S_{\hat{\tau}})^+ \mathbb{1}_{\{\tau_H=i\}}/(1+r)^{\hat{\tau}-\tau_H}]/(1+r)^{\tau_H}\right) \\
&= \sum_{i=0}^{T} \mathbb{E}^{\overline{P}}_{\{\tau_H=i\}}[(K - S_{\hat{\tau}})^+/(1+r)^{\hat{\tau}-i}]/(1+r)^i \underline{P}(\tau_H = i) \\
&= \sum_{i=0}^{T} \mathbb{E}^{\overline{P}}_{\{\tau_H=i\}}[(K - S_{\hat{\tau}})^+/(1+r)^{\hat{\tau}-i}]/(1+r)^i \frac{J_H}{i}\binom{i}{\frac{i+J_H}{2}} \underline{p}^{\frac{i+J_H}{2}} (1-\underline{p})^{\frac{i-J_H}{2}},
\end{aligned}
\tag{7}
$$

where J_H is the positive integer such that $H = S_0 u^{J_H}$. For a derivation of the formula used in the last line see Feller (1968). The expectation in the last line denotes the value under ambiguity of an American plain vanilla put starting at time i with initial price of the underlying $S_i = H$.
For finite maturity T there does not exist a constant early exercise boundary. Thus, it is not possible to find a closed-form binomial expression for the American put and the American up-and-in put, see also Reimer and Sandmann (1995).

Using Theorem 3.4 part 2 we obtain the analogous result for an American down-and-in call option with barrier $H < S_0$. In the following we omit the proves, or treat them only briefly, respectively, since they are also applications of the last theorems. For an American down-and-in call option the discounted payoff process X is given by $X_t = (S_t - K)^+/(1+r)^t \,\mathbb{1}_{\{\tau_H \geq t\}}$ for all $t \leq T$, where $\tau_H := \inf\{t \geq 0 | S_t \leq H\} \wedge T + 1$ and $\tau_2 := T + 1$. Using Theorem (3.4) part 2 for a barrier level $H < S_0$ we derive

Corollary 3.7 (Down-and-in call) *The ambiguity averse agent uses the following prior $\hat{P}$ given by the density*

$$\hat{D}_t := 2^t \prod_{u \leq t \wedge \tau_H} \big(\varepsilon_u \overline{p} + (1-\varepsilon_u)(1-\overline{p})\big) \prod_{u \in]\tau_H,\, t \wedge T]} \big(\varepsilon_u \underline{p} + (1-\varepsilon_u)(1-\underline{p})\big) \quad \textit{for } t \leq T.$$

Similar to an up-and-in barrier put option a down-and-in barrier call equals a plain vanilla call option when the underlying hits the barrier level H. As in (6) we can derive an analogous formula for the value process (U_t^Q) of the down-and-in call option. For $t \leq \tau_H$ we obtain

$$U_t^Q = \mathbb{E}^{\overline{P}} \left[\mathbb{E}^{\underline{P}} \left[X_{\hat{\tau}} \middle| \mathcal{F}_{\tau_H} \right] \middle| \mathcal{F}_t \right], \tag{8}$$

where $\hat{\tau}$ is an optimal stopping time for this considered problem. Assuming $\underline{p}u + (1 - \underline{p})d > 1 + r$ leads to the fact that $\hat{\tau} = T$, see Corollary 2.6. In this case one can derive a binomial closed-form solution.

Another example is an up-and-out call option. The option is *knocked out* when a prespecified barrier level is reached. This means the claim becomes worthless. Using Theorem 3.2 part 2 we obtain by setting $\tau_1 := 0$, $\tau_2 := \tau_H$ and the here reasonable assumptions $H > S_0$

Corollary 3.8 (Up-and-out call) *The ambiguity averse agent uses for an up-and-out call option specified by the discounted payoff $X_t := (S_t - K)^+/(1 + r)^t \, \mathbb{1}_{[0,\tau_H[}$ for all $t \leq T$ the following prior $\hat{P}$ given by the density*

$$\hat{D}_t := 2^t \prod_{u \leq \tau_H \wedge t:\ u \neq \sigma_i + 1} \left(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p}) \right) \prod_{u \leq t:\ u = \sigma_i + 1} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right) \prod_{u \in]\tau_H, t \wedge T]} \left(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p}) \right)$$

for all $t \leq T$ and all occurring $1 \leq i \leq T$. In particular, a sufficient condition for early exercise of the American up-and-out call at time t is specified when the price at time t is larger or equal to $\frac{H \cdot d}{(1+r)^{T-t}} + K \left(1 - \frac{1}{(1+r)^{T-t}} \right)$.

Proof: The agent uses the stated prior density due to Theorem (3.2) part 2. The early exercise payoff at each time is bounded from above by $H \cdot d - K$. Therefore, early exercise at time t is optimal if

$$\begin{aligned} &(S_t - K)(1 + r)^{T-t} \geq H \cdot d - K \\ \iff &S_t \geq \frac{H \cdot d}{(1 + r)^{T-t}} + K \left(1 - \frac{1}{(1 + r)^{T-t}} \right). \end{aligned}$$

See also Reimer and Sandmann (1995). □

Note that the early exercise condition is always satisfied for $t = \sigma_1$. Thus, early exercise occurs at time σ_1 at the latest. Hence, the decision maker always exercises the option when there is knock-out danger meaning that the option's underlying might hit the barrier in the next time period and become worthless. As a consequence, the decision maker does not directly experience changes of the conditional one-step-ahead probabilities at all occurring times $\sigma_i + 1$ since she exercises the option at or even before time σ_1.

Remark 3.9 *Assuming additionally in Corollary 3.8 that the inequality $\underline{p}u + (1-\underline{p})d > 1+r$ is satisfied, the American up-and-out call is exactly exercised at time σ_1. This can be derived by the following reasoning: The value of the American up-and-out call under ambiguity being still alive at a fixed time t with $S_t \leq H \cdot d^2$ is larger or equal to*

$$\begin{aligned}
&\frac{1}{1+r}\,\mathbb{E}^{\hat{P}}\left((S_{t+1}-K)^+ \mid S_t \leq H \cdot d^2\right) = \frac{1}{1+r}\,\mathbb{E}^{\underline{P}}\left((S_{t+1}-K)^+ \mid S_t \leq H \cdot d^2\right) \\
&\quad = \frac{1}{1+r}\left((S_t \cdot u - K)^+ \underline{p} + (S_t \cdot d - K)^+(1-\underline{p})\right) \\
&\quad \geq \max\left\{\frac{1}{1+r}\left((S_t \cdot u - K)\underline{p} + (S_t \cdot d - K)(1-\underline{p})\right), 0\right\} \\
&\quad \geq \max\left\{\left(S_t - \frac{K}{1+r}\right), 0\right\} \geq (S_t - K)^+ \quad \textit{for all } S_t \leq H \cdot d^2.
\end{aligned}$$

The first inequality follows by assumption $\underline{p}u + (1-\underline{p})d > 1+r$. This shows that the sufficient condition for early exercise is not satisfied for all $S_t \leq H \cdot d^2$. Thus, in this case early exercise is only optimal at time σ_1 when the price equals $H \cdot d$.

Without the early exercise feature, the ambiguity averse buyer of the option would face all changes of the conditional one-step-ahead probabilities that might occur during the option's lifetime. This is being illustrated in the next subsection when considering a so-called ladder option.

Down-and-out put options behave analogously to up-and-out call options. Besides, there are four further types of barrier options exhibited with one single barrier level. But due to their structure, their payoffs possess the same monotonicity in S at each time and the worst-case measure for these types can be identified by using results from the previous section. As a consequence, in these cases the worst-case measure does not feature path-depending varying conditional one-step-ahead probabilities induced by ambiguity as we saw before.

3.2 Multiple Barrier Options

The above reasoning can also be applied to options endowed with more than one barrier. As mentioned in Remark 3.5 one can use the theorems to attain the worst-case measure for options with both a knock-in and a knock-out barrier level, or for out-options, additionally exhibited with a further barrier level which replaces the preceding one after some prespecified time progress.

This shall be demonstrated in the following. We examine ladder options and focus on the special case of an up-and-out ladder call option expiring at time T with two barrier levels H_1 and H_2. We assume $S_0 < H_1 < H_2$. This claim resembles a single up-and-out barrier call option with the additional feature that after some prespecified date $t_1 \in (0, T)$ the knock-out barrier changes from H_1 to the higher level H_2, meaning that the first barrier H_1 is only valid up to time t_1. Afterwards, the second barrier H_2 is valid and determines the knock-out event.

As we will see, this difference related to single up-and-out call options might have impact on the buyer's early exercise strategy which in turn might affect the significance of varying conditional one-step-ahead probabilities when the underlying's price is close to the first barrier before time t_1. The reason is quite obvious. Now, in the case of the underlying's price is close to H_1 before time t_1, the buyer might find it not optimal to exercise the claim. For instance, if the second barrier H_2 is very high, or, the first barrier H_1 is close to the strike price K, the benefit from exercising before time t_1 will be quite low compared to possible future exercise payoffs after time t_1. This situation is being expressed by the required inequality below, see (9).

Formally, the discounted payoff at time t of such a ladder call option with strike price $K < H_1$ and maturity T is given by

$$\begin{aligned} X_t &= \begin{cases} (S_t - K)^+/(1+r)^t, & \text{if } t \le t_1 \text{ and } t < \tau_{H_1} \\ (S_t - K)^+/(1+r)^t, & \text{if } t > t_1, t < \tau_{H_2} \text{ and } t < \tau_{H_1} \\ 0, & \text{else} \end{cases} \\ &= (S_t - K)^+/(1+r)^t \, \mathbb{1}_{[0, \tau_{H_1} \wedge t_1[} + (S_t - K)^+/(1+r)^t \, \mathbb{1}_{[t_1, \tau_{H_2} \wedge \tau_{H_1}[} \end{aligned}$$

whereas $\tau_{H_1} := \inf\{t \in [0, t_1] \big| S_t = H_1\} \wedge T + 1$ and
$\tau_{H_2} := \inf\{t \in]t_1, T] \big| S_t = H_2\} \wedge T + 1$. Here, $[t_1, t_1[$ is defined as the empty set. In order to represent the density of the worst-case measure we need as

in Definition 3.1 the following stopping times:

$$\sigma_i := \inf\{t \in [\sigma_{i-1} + 1, \tau_{H_1} \wedge t_1 - 1[\; \big| \; S_t = H_1 \cdot d\} \wedge T + 1$$

for all $1 \leq i < t_1$ with the notation $\sigma_0 := -1$ and

$$\gamma := \inf\{t \in [t_1, \tau_{H_2}[\; \big| \; S_t = H_2 \cdot d\} \wedge T + 1.$$

Furthermore, for $t \leq T$ let $\Omega_{(t)} := \bigotimes_{i=1}^{t}\{0, 1\}$ denote the set of all paths in Ω up to time t.

Corollary 3.10 (Ladder call option) *Let all data be given as above, in particular, let us suppose the strict inequality of Corollary 3.8. Additionally, suppose that the value function satisfies for all $\omega_{(t)} \in \Omega_{(t)}$ with $S_t(\omega_{(t)}) = H_1 d$ the following inequality*

$$X_t(\omega_{(t)}) < (1 - \overline{p})U_{t+1}(\omega_{(t)}, 0) \tag{9}$$

for all $t < \tau_{H_1} \wedge t_1$. Then the ambiguity averse buyer of this ladder option uses the following prior $\hat{P}$ specified by the density

$$\hat{D}_T := 2^T \prod_{u \leq \tau_{H_2} \wedge T:\ u \neq \sigma_i + 1 \ and \ u \neq \gamma + 1} \big(\varepsilon_u \underline{p} + (1 - \varepsilon_u)(1 - \underline{p})\big)$$
$$\prod_{u \leq T:\ u = \sigma_i + 1 \ or \ u = \gamma + 1} \big(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p})\big)$$
$$\prod_{u \in]\tau_{H_2}, T]} \big(\varepsilon_u \overline{p} + (1 - \varepsilon_u)(1 - \overline{p})\big)$$

for all occurring $1 \leq i < t_1$. In particular, the agent uses the canonical optimal stopping rule $\hat{\tau}$ which is less or equal to $\gamma \wedge T$.

Proof: Examination of the proof of Theorem 3.2 reveals that we can also apply the second part of the theorem to this special situation since the time interval $[0, T]$ is divided into two disjoint intervals and $A(t, S_t) := (S_t - K)^+/(1 + r)^t$, which is increasing in S_t for all $t \leq T$, remains the same on both intervals. Thus, applying the theorem on both subintervals yields the density for the ambiguity averse agent. The canonical optimal stopping rule is also specified by the theorem. The requested inequality (9) guarantees that the decision maker does not exercise the valuable option at node $H_1 \cdot d$ when being located before time t_1. From the beginning of time t_1 the same arguments as in the case of an usual up-and-out call option (see Corollary 3.8) lead to an optimal stopping time. So, this ladder option is held up to time γ at most. □

This example exhibits many modifications of conditional one-step-ahead probabilities. On the one hand, whenever it is possible that the option might knock-out in the next time period, the conditional one-step-ahead probability switches and thus puts high weight on an up-movement of the underlying's price. On the other hand, whenever the option does not knock out in that specified period, the conditional probability switches back to the marginal probability weighting before since an option's knock-out is not possible in the next forthcoming time period. Unlike an American up-and-out call, the decision maker experiences all these varying marginal probabilities before time t_1 since now early exercise is not optimal due to the required assumptions.

4 Multiple Expiry Options

In this section we analyze exotic options that are characterized by several expiry dates. At every expiry the owner of the option has the right to modify the contract conditions reseting the strike or the maturity in a predefined way. New conditions of the contract depend on the underlying's value at expiry dates and are not known to the buyer at time zero. Therefore, additionally to the uncertainty about future underlying's value the decision maker faces uncertainty about future contract conditions while evaluating the option. The expiry dates can be predefined points in time (forward start options) or random dates chosen by the buyer or seller of the contract (shout options).

Such options can be seen as a sequence of claims where every claim expires at a predefined date and pays off a new born claim expiring at the next expiry date. In the case of European claims the expiry dates are deterministic corresponding to forward start options. In the case of shout options we face American claims leading to stochastic expiry dates. In general, multiple expiry options can be entitled with any number of expiry dates, here, we consider dual expiry options where contract conditions change exactly once. Kwok and Wu (2004) analyze shout options with infinite number of shout possibilities and establishes a relation to lookback options.

4.1 Shout Options

Shout options are contracts that give the buyer the right to reset the strike at a date chosen by the buyer. The event of reseting the contract features is called shouting and gives the structure its name. The reset right allows the investor to benefit from market movements by choosing a favorable strike. At the same time she can lock in already realized profits ensuring against an unfavorable stock movement.
Shout options are often used by professional investors as a cheaper alternative to lookback options. Whereas the buyer of the lookback option has the right to sell the stock at the maximal price the owner of the shout option has to call her bank and to freeze the price at which she can sell/buy at any time σ before maturity. The structure becomes active and the buyer has to have enough understanding of the market in order to set the strike as close as possible to the peak. Mathematically, the buyer faces an optimal stopping problem, aiming to set the strike optimally.

In the following we analyze shout puts focusing on a more special case later on. The same analysis can be performed for call options.

At time zero the buyer of a shout put receives a plain vanilla put option with strike K_0 and maturity T with additional right to modify the strike of the contract once at any time prior to maturity by calling her bank and fixing the strike in a predefined way. At the time of shouting the buyer locks in the realized profits by receiving a cash payment $(K_0 - S_\sigma)$, additionally, she receives a new option of European style with strike $K_1 = f(S, \sigma)$, where f is a $\mathcal{F}_\sigma$ measurable function of the whole path $S = (S_1, \ldots, S_T)$. At maturity the buyer receives the positive part of the difference between the strike K_1 and the final stock price, i.e. $(K_1 - S_T)^+$. The contract is then specified by the initial strike K_0, the function f determining the new strike K_1 and the maturity date T. This structure allows the investor to lock in realized profits protecting himself against downside risk by receiving the cash payment and at the same time to participate on future upside with the new born option.

To simplify the analysis we consider a particular shout option – the so called single shout floor that allows for closed form solutions even in finite time. The initial strike of the single shout floor K_0 is equal to zero and the strike K_1 is given by $K_1 = f(S_\sigma) = S_\sigma$. The buyer shouts once at $\sigma \leq T$ fixing the strike at S_σ. At the expiry date she receives a payoff that corresponds to the payoff profile of an European put i.e. $(S_\sigma - S_T)^+$. Thus, the buyer of this shout option has to solve the following problem

$$\text{Maximize } \min_{P \in \mathcal{Q}} \mathbb{E}^P((S_\sigma - S_T)^+/(1+r)^T) \text{ over all stopping times } \sigma \leq T \tag{10}$$

Note, that unlike the American put, the exercise date is fixed but the birth date has to be determined optimally by the buyer. Determining the optimal starting time/shouting time constitutes the optimal stopping problem for the single shout option. The task is to optimally start the payoff process rather then stop it which can be seen as purchasing a new issued European option with a fixed maturity. We will maintain this parallel during our analysis.

However, we cannot apply our standard theory of backward induction to the problem stated in (10) because the payoff $(S_\sigma - S_T)^+/(1+r)^T$ obtained from stopping at any stopping time $\sigma \leq T$ depends on the value of the stock at maturity and is for this reason not adapted to the filtration $(\mathcal{F})_{t=1,\dots,T}$. To overcome this difficulty we condition the payoff on the available information and consider the following payoff process

$$X_t = \min_{P \in \mathcal{Q}} \mathbb{E}^P((S_t - S_T)^+/(1+r)^T | \mathcal{F}_t). \tag{11}$$

For every $t \leq T$ we can interpret X_t as discounted multiple prior value of the shout floor if shouted at t. At the same time it corresponds to the value of an at-the-money European put issued at t and maturing at T evaluated under multiple priors.[3]

Using the appropriate version of the law of iterated expectations one can easily see that for all stopping times $\sigma \leq T$ we have

$$\min_{P \in \mathcal{Q}} \mathbb{E}^P((S_\sigma - S_T)^+/(1+r)^T) = \min_{P \in \mathcal{Q}} \mathbb{E}^P(X_\sigma) \tag{12}$$

Therefore, we can reformulate the problem stated in (10) equivalently in the following way

$$\text{Maximize } \min_{P \in \mathcal{Q}} \mathbb{E}^P((X_\sigma) \text{ over all stopping times } \sigma < T \tag{13}$$

where the payoff process X is defined via (11). Thus, the optimal stopping time found for (13) is also optimal for the problem (10) and the values of

[3] Strictly speaking, the value of the European put issued at t and maturing at T differs from the expression (11) by a discount term

the two problems coincide. Again, we can interpret the problem as optimal investment in a put with a fixed investment horizon.

We solve the problem in two steps: first we compute X_t – the explicit value of the shout option freezed at t for all $t \leq T$ and derive the worst-case measure after shouting. In the second step, we identify the worst-case measure before shouting reducing the problem to the single prior case.

To compute X_t for a fixed $t \leq T$ we note that the uncertainty about the strike is resolved at the time of shouting. The strike becomes a constant and as a consequence the claim becomes a plain vanilla European put. As the payoff of the put is decreasing in S_t for all $t \leq T$ by Theorem (2.5) we conclude that the worst-case measure is given by $\overline{P}$ and we have

$$\begin{aligned} X_t =& \min_{P \in \mathcal{Q}} \mathbb{E}^P \left((S_t - S_T)^+ / (1+r)^T | \mathcal{F}_t \right) \\ =& \mathbb{E}^{\overline{P}}((S_t - S_T))^+ / (1+r)^T | \mathcal{F}_t). \end{aligned}$$

Additionally, under $\overline{P}$ the increments of the underlying between t and $T - \Delta(S_t, S_T)$ are independent for all $t \leq T$ which leads to

$$\begin{aligned} X_t =& S_t \cdot \mathbb{E}^{\overline{P}}((1 - \Delta(S_t, S_T))^+ / (1+r)^T | \mathcal{F}_t) \\ =:& S_t \cdot g(t, T, \overline{P}) \end{aligned} \tag{14}$$

where

$$g(t, T, \overline{P}) = (1+r)^{-T} \sum_{k=0}^{k^*(t)} \left(\binom{n}{k} \right) \overline{p}^k \cdot (1 - \overline{p})^{n-k} \left(1 - d^{n-2k} \right)$$

and $k^*(t) =: \max \left\{ k : k < \frac{T-t}{2} \right\}$

The above equation provides the value of the embedded option contained in the shout contract maturing at T at the time of shouting. At the same time it corresponds to the value of the at-the-money European put issued at $t \leq T$ and maturing at T.

The buyer of a shout option uses $\overline{P}$ to evaluate the option after shouting. Moreover, the value of a freezed shout floor is homogeneous of degree one in the current stock price S_t. As the expiry date T and the measure $\overline{P}$ remain fixed we omit them in the following and write $g(t, T, \overline{P})$ as $g(t)$.

To complete the analysis it remains to determine the worst-case measure before shouting and to solve the optimal stopping problem for X under the worst-case measure.

As $g(t) > 0$ for all $t \leq T$ X_t as a function of t and S_t is increasing in S_t for all $t \leq T$. Again using Theorem (2.5) we conclude that the worst-case measure of problem (13) is given by $\underline{P}$.

Remark 4.1 *It might be surprising at the first sight that the value of the put contained in the shout contract at the time of shouting increases if the strike increases. The reason for this observation contradicting the usual intuition is the fact that the strike is not a constant at the moment of issuance of the option. The value of the claim at the time of shouting is increasing with respect to the difference between strike and the current stock price. Economically, a higher S_t at the time of shouting increases the strike of the new born option enlarging the in-the- money region of the option.*

As a result of the above discussion on the monotonicity of the claim we obtain the following:

Corollary 4.2 (Shout put) *A risk-neutral buyer of an single shout floor option uses the optimal stopping rule for the prior $\hat{P}$ given by the density*

$$\hat{D}_t = 2^t \prod_{v=1}^{\sigma \wedge t} \big(\underline{p} \cdot \varepsilon_v + (1-\underline{p}) \cdot (1-\varepsilon_v)\big) \prod_{v=\sigma+1}^{t} \big(\overline{p} \cdot \varepsilon_v + (1-\overline{p}) \cdot (1-\varepsilon_v)\big). \tag{15}$$

Summing up, we conclude that the value of the shout floor is given by

$$U_t^{\mathcal{Q}} = \begin{cases} \mathbb{E}^{\underline{P}}\left(\mathbb{E}^{\overline{P}}\left((S_\sigma - S_T)^+/(1+r)^T | \mathcal{F}_\sigma\right)\right), & \text{if } t < \sigma \\ \mathbb{E}^{\overline{P}}\left((S_\sigma - S_T)^+/(1+r)^T | \mathcal{F}_t\right) & \text{else} \end{cases}$$

The decision maker changes her beliefs about mean returns at the first expiry date. Before shouting and freezing the strike she presumes low returns of the stock that keeps the in-the-money region of the option small and the embedded put option less worth; after shouting she receives a put option and therefore changes her belief — being pessimistic, she presumes now that the risky asset will have high returns. This change of beliefs causes the difference in the values of the classical result and the multiple prior result.

5 Quasi-convex payoffs

In the last section we consider options that consist of two monotone parts. Typical examples are options having U–shaped payoff including straddles, strangles or short option strategies. Investors buying such options are speculating on change in the underlying's value without specifying the direction of the change. Depending on the actual price of the underlying falling or rising stock increases the profit of the investor. This fact leading to different monotonicity type with respect to underlying causes different beliefs at different stock prices. One can think of this effect as of getting different payoff function conditioned on the stock price. Thus, stock price uncertainty induces uncertainty about the payoff function. To illustrate this idea consider a straddle: by exercising the straddle above the strike the buyer gets a payment of $(S_t - K)$ which corresponds to a call. Otherwise, she gets the payoff of a put – $(K - S_t)$. Thus, depending on the current stock price the payoff function changes, at time zero the actual payoff function is not known to the buyer. This uncertainty about the payoff function cannot be resolved over time in general.

Remark 5.1 *Mathematically, payoffs described above correspond to quasi-convex/quasi-concave payoff functions. Note, that we still deal with functions defined on natural numbers and have to be careful when using the term quasi-convex. Strictly speaking, the notion we use corresponds to discrete convexity studied intensively in the context of indivisible goods (see for example Murota (1998) for a general introduction). In one dimensional setting discrete convexity reduces to the following: A set $E \subset \mathbb{N}$ is convex if all points in E are contained in the convex hull of E. The definition of quasi-convex is then straight forward.*

In our analysis we concentrate on piecewise linear, U–shaped payoffs paying off $f(t, S_t)$ when exercised at $t \leq T$ where f has the following form:

$$f(t, S_t) = c_1 \cdot (K_1 - S_t)^+ + c_2 \cdot (S_t - K_2)^+$$

for $c_1,\ c_2 \in \mathbb{R}$, $K_1 \leq K_2$. However, our results apply to more general functions as quadratic or ladder functions. We show that the Snell envelope $U_t^{\mathcal{Q}}$ at time $t \leq T$ is a quasi-convex function in S_t if the claim is Markovian.

Lemma 5.2 *If the discounted payoff function $A(t, S_t)$ is quasi-convex in its second variable for every $t \leq T$, then the Snell envelope $U_t^{\mathcal{Q}}$ is given by a quasi-convex function $v(t, x)$, i.e. given $S_t = x_t$*

$$U_t^{\mathcal{Q}} = v(t, x_t) = \sup_{\tau \geq t} \min_{P \in \mathcal{Q}} \mathbb{E}^P(A(\tau, S_\tau)|S_t = x_t)$$

Proof: We have to show that for every $t \leq T$ the value function $v(t, \cdot)$ depends only on the value of the stock at time t and that the quasi-convexity of the payoff function carries over to the value function. We do it via backward induction.

Before applying backward induction we note that in one-dimensional case a function $g : E \to \mathbb{N}$ is quasi-convex if and only if there exists a $\hat{x} \in E$ such that $g(x) \geq g(\hat{x})$ holds for all $x \in E$. If $\hat{x}$ belongs to the boundary of E the function g is monotone. If $\hat{x}$ belongs to the interior of E g consists of two monotone parts and reaches its minimum at $\hat{x}$. In any case, quasi-convexity reduces to the existence of a unique minimum in one dimensional case.

For $t = T$ we clearly have for all possible values of $S_T = x_T$

$$U_T^{\mathcal{Q}} = A(T, x_T)$$

where $A(T, \cdot)$ is a quasi-convex function.

For $t+1 < T$ we assume that for any value of $S_{t+1} = x_{t+1} \in E_{t+1}$ the value function $v(t + 1, \cdot)$ is quasi-convex function depending only on the current value of the stock. Because of quasi-convexity there exists a unique minimum m_{t+1} and a unique

$$\hat{x}_{t+1} = \inf\{x_{t+1} \in E_{t+1} | v(t + 1, x_{t+1}) = m_{t+1}\}.$$

The function $v(t + 1, \cdot)$ is decreasing on the set $\{x_{t+1} \leq \hat{x}_{t+1}\}$ and increasing on the set $\{x_{t+1} \geq \hat{x}_{t+1}\}$.

In $t < T$ we then have for any value $S_t = x_t$

$$\begin{aligned} U_t^{\mathcal{Q}} &= \max\{A(t, S_t), \min_{P \in \mathcal{Q}} \mathbb{E}^P \left(U_{t+1}^{\mathcal{Q}} | \mathcal{F}_t\right)\} \\ &= \max\{A(t, x_t), \min_{P \in \mathcal{Q}} \mathbb{E}^P \left(U_{t+1}^{\mathcal{Q}} | S_t = x_t\right)\} \\ &= \max\left\{A(t, x_t), \hat{p}_{t+1} v(t + 1, x_t \cdot u) + (1 - \hat{p}_{t+1}) v(t + 1, x_t \cdot d)\right\} \\ &= v(t, x_t) \end{aligned} \tag{16}$$

where $\hat{p}_{t+1} \in [\underline{p}, \overline{p}]$ is the marginal of the worst-case measure $\hat{P}$ at time t. Since $v(t+1, \cdot)$ is independent of the realized past, the minimizer $\hat{p}_{t+1}$ depends only on the value of x_t. This proves that the value function at time t $v(t, \cdot)$ depends only on current value of the underlying.

To prove quasi-convexity we analyze the structure of the continuation value in equation (16)

$$u(t, x_t) := \hat{p}_{t+1} v(t+1, x_t \cdot u) + (1 - \hat{p}_{t+1}) v(t+1, x_t \cdot d)$$

for different values of $S_t = x_t$.

On the set

$$E_t^d = \{x_t \in E_t | x_t \leq \hat{x}_{t+1} \cdot d\} \tag{17}$$

$x_t \cdot d < x_t \cdot u < \hat{x}_{t+1}$ and therefore using the induction hypothesis we can conclude that the function $u(t+1, \cdot)$ is decreasing as a convex combination of two increasing functions. Similarly, for all

$$E_t^i = \{x_t \in E_t | x_t \geq \hat{x}_{t+1} \cdot u\} \tag{18}$$

we have $\hat{x}_{t+1} < x_t \cdot d < x_t \cdot u$ and the function increases on the above set with the same argument.

Because of the binomial tree structure of the state space and the fact that

$$E_{t+1} = \{E_t \cdot u^k | k \in \{-1; 1\}\}$$

equations (17) and (18) partition the set of possible values of S_t and E_t can be written as

$$E_t = \{x_t \in E_t | x_t \leq \hat{x}_{t+1} \cdot d\} \cup \{x_t \in E_t | x_t \geq \hat{x}_{t+1} \cdot u\}$$

Because of monotonicity of $u(t, \cdot)$ on E_t^d and E_t^i the minimum of $u(t, \cdot)$ is unique. This shows that the function $u(t, \cdot)$ is quasi-convex.

To complete the proof we recall that $A(t, x_t)$ is quasi-convex by assumption. Thus, the function defined by equation (16) is a quasi-convex function as maximum of two quasi-convex functions. Clearly, the value function at time t depends only on the current stock price and given $S_t = x_t$ we can write $U^{\mathcal{Q}}$ as a function $v(t, x_t)$. □

The quasi-convexity of the value function implies that for every $t \leq T$ we can separate the space E_t on which the value of the claim is monotone allowing to determine the worst-case measure. The decomposition point is the minimizer of the value function $\hat{x}_t$ which is constructed in the proof of Lemma 5.2.

Having analyzed the shape of the value function we now can compute the worst-case measure with the following argument. If asset prices are low, the value function is decreasing. Therefore, with the same argument as for simple American options, one can show that $\overline{P}$ is the worst-case measure here. In the other region on the contrary, $\underline{P}$ is the worst-case measure. At a predefined level $\hat{x}_t$ the investor changes his beliefs and so the mean return on stock under the measure. We then have the following

Lemma 5.3 (Straddle) *The buyer of a straddle uses the optimal stopping rule for the measure $\hat{P}$ with density*

$$\hat{D}_t = 2^t \prod_{v \le t, S_v \in E_v^i} \big(\underline{p} \cdot \varepsilon_v + (1-\underline{p}) \cdot (1-\varepsilon_v)\big) \prod_{v \le t, S_v \in E_v^d} \big(\overline{p} \cdot \varepsilon_v + (1-\overline{p}) \cdot (1-\varepsilon_v)\big).$$

PROOF: We consider the value function on the continuation region where for a given $S_t = x_t$ we have $U_t^{\mathcal{Q}} = v(t, x_t)$

$$v(t, x_t) = \min_{p_{t+1} \in [\underline{p}, \overline{p}]} (p_{t+1} v(t+1, x_t \cdot u) + (1 - p_{t+1}) v(t+1, S_t \cdot d)$$

As $v(t, \cdot)$ is decreasing on E_t^d, the worst-case measure on this set is given by $\overline{P}$. With the same argument the worst-case measure $\hat{P}$ is $\underline{P}$ on E_t^i, i.e.

$$\hat{P}[\varepsilon_{t+1} = 1 | \mathcal{F}_t] = \begin{cases} \underline{p} & \text{on } \{x_t \ge \hat{x}_{t+1} \cdot u\} \\ \overline{p} & \text{on } \{x_t \le \hat{x}_{t+1} \cdot d\} \end{cases} . \tag{19}$$

where $\hat{x}_{t+1}$ is the minimizer of $v(t+1, \cdot)$. Using the definition of $\underline{p}$ and $\overline{p}$ and pasting the densities together one obtains the result. □

Under $\hat{P}$ the process (S_t) becomes mean-reverting in an appropriate sense pushing S_t down if it is high and up if it is low. This corresponds to the intuition: the ambiguity averse decision maker anticipates low mean returns in hausse phases and high mean returns when the stock value is low. Unlike previous cases the uncertainty about the payoff function here cannot be resolved before T in general. The change of the measure occurs every time the stock price crosses the critical value $\hat{x}_t$ forcing the decision maker to change her beliefs about mean returns.

6 Conclusion

This paper studies the optimal exercise strategies of the buyer of various American options in a framework that allows for model uncertainty in discrete time. The imprecise information about the correct probability measure driving the stock price process in the market generates different models with varying conditional one-step-ahead probabilities used by the buyer. The buyer then is allowed to change the measure, and so the model she uses and to assign the value to the claim according to the worst possible model. While the solution for plain vanilla options is straightforward in the model the situation differs if the payoff of the option becomes more sophisticated. The effect of uncertainty differs over time leading to a dynamical structure of the worst-case measure. This paper analyzes different effects of uncertainty highlighting the structural difference between the standard models used in Finance and the multiple prior models: the buyer of the option adapts her beliefs to the state of the world and the overall effect of model uncertainty. A natural next step is to extend the theory to continuous market models and to analyze exotic options in that framework.

A Proof of Theorem 3.2

PROOF: We give proof for decreasing A in S_t for all $t \leq T$. The second case works analogously. For notational simplicity we write $\omega_{(t)}$ for an element in $\bigotimes_{i=1}^{t}\{0,1\} \subseteq \Omega$. Furthermore, for a stopping time τ we introduce for each $t \leq T$ the restriction τ^t of τ to pathes in Ω running up to time t:

$$\begin{aligned} \tau^t : \bigotimes_{i=1}^{t} \{0,1\} &\longrightarrow [0,t] \cup \{T+1\} \\ \omega_{(t)} &\longmapsto \tau^t(\omega_{(t)}) = \begin{cases} \tau(\omega_{(t)}), & \text{if } \tau(\omega_{(t)}) \leq t \\ T+1, & \text{if else} \end{cases} . \end{aligned}$$

The restricted stopping are being used in order to be mathematically more exact.

We start the proof with

Lemma A.1 *Let $(U_t^{\mathcal{Q}})$ be the multiple prior Snell envelope of X as defined in Theorem 3.2. Assume that $U_t^{\mathcal{Q}}$ is given by the function $u(t, S_t, \tau_1^t, \tau_2^t)$ for* *all $t \leq T$. Then for all $t \in [0, T-1]$ and all $k \in [1, T-t]$*

$$u(t, S, t, T+1) \geq u(t+k, S, t+k, T+1).$$

PROOF: The inequality follows directly by the inequality

$$u(t, S, t, T+1) \geq u(t+k, S, t, T+1) = u(t+k, S, t+k, T+1).$$

The inequality always holds for claims of American style whose undiscounted payoff does only depend on the underlying's price S at each time. For the special choice of τ_1^t and τ_2^t it therefore also holds for the considered claims of the theorem. The equality holds since the claim is already knocked-in. $\square$

Using theory about multiple prior Snell envelope, see Riedel (2009), we show by backwards induction in t that $U_t^{\mathcal{Q}} = u(t, S_t, \tau_1^t, \tau_2^t)$ for all t such that u has the following properties:

(i) for $t < \tau_1^t$: $u\left(t, \cdot, \tau_1^t(\cdot), \tau_2^t(\cdot)\right) \nearrow$ in $S \leq \bar{S}^1$,
where $\bar{S}_t^1$ is determined by $\tau_1^t(\bar{S}^1) = t$

(ii) for $t \in [\tau_1^t, \tau_2^t[$: $u\left(t, \cdot, \tau_1^t(\cdot), \tau_2^t(\cdot)\right) \searrow$ in S

(iii) for $t \geq \tau_2^t$: $u\left(t, \cdot, \tau_1^t(\cdot), \tau_2^t(\cdot)\right) = 0$ for all S.

First, note that u is well-defined due to the definition of the payoff process X. (u complies with the definition of a function since X_t which only depends on S_t, τ_1^t, and τ_2^t, does for each $t \leq T$.) For $t = T$ we have

$$\begin{aligned}
U_T^{\mathcal{Q}}(\cdot) = X_T(\cdot) = \mathbb{1}_{[\tau_1^T, \tau_2^T[}(T, \cdot)\, A(T, S_T(\cdot)) \\
&= \begin{cases} 0, & \text{if } \tau_1^T = T+1 \text{ or } \tau_2^T \leq T \\ A(T, S_T), & \text{if } \tau_2^T = T+1 \text{ and } T \geq \tau_1^T \end{cases} \\
&= \begin{cases} 0 = u(T, S_T, \tau_1^T, \tau_2^T)\ \forall S_T, & \text{if } \tau_1^T = T+1 \text{ or } \tau_2^T = T \\ A(T, S_T) = u(T, S_T, \tau_1^T, T+1)\ \forall S_T, & \text{if } \tau_1^T \leq T < \tau_2^T \end{cases} .
\end{aligned}$$

So, $U_T^{\mathcal{Q}}$ satisfies the representation and the properties by the assumptions on $X_T, A(T, \cdot)$, respectively.

In the induction step for $t < T$ we handle the different cases separately. First, assume $t \in [\tau_1^t, \tau_2^t[$, say $\tau_1^t(\omega_{(t)}) =: k \leq t$: Then

$$U_t^{\mathcal{Q}}(\omega_{(t)}) = \max\left\{X_t(\omega_{(t)}), \min_{P\in\mathcal{Q}}[U_{t+1}|\mathcal{F}_t(\omega_{(t)})]\right\}$$
$$\stackrel{\text{(IH)}}{=} \max\left\{X_t(\omega_{(t)}), \min_{p_{t+1}\in[\underline{p},\overline{p}]}\right.$$
$$\left.\left\{p_{t+1}u(t+1, S_tu, k, \tau_2^{t+1}(\omega_{(t)},1)) + (1-p_{t+1})u(t+1, S_td, k, \tau_2^{t+1}(\omega_{(t)},0))\right\}\right\}.$$

By induction hypothesis and due to $\tau_2^{t+1}(\omega_{(t)},0) \geq \tau_2^{t+1}(\omega_{(t)},1)$, properties (ii) and (iii) for $t+1$ imply
$u(t+1, S_td, k, \tau_2^{t+1}(\omega_{(t)},0)) \geq u(t+1, S_tu, k, \tau_2^{t+1}(\omega_{(t)},1))$. Therefore,

$$U_t^{\mathcal{Q}}(\omega_{(t)}) = \max\left\{X_t(\omega_{(t)}), \overline{p}u(t+1, S_tu, k, \tau_2^{t+1}(S_tu)) + (1-\overline{p})u(t+1, S_td, k, T+1)\right\}$$
$$= \hat{U}_t(\omega_{(t)}).$$

Hence, in this case $U_t^{\mathcal{Q}}$ is a function $u(t, S_t, \tau_1^t, \tau_2^t)$ which is decreasing in S since $A(t,\cdot)$ is decreasing in S by assumption, and $u(t+1,\cdot,k,\tau_2^t(\cdot))$ is monotone decreasing in S by induction hypothesis (property (ii), (iii), respectively).
Second, if $t \geq \tau_2^t(\omega_{(t)}) =: l < T$, and $\tau_1^t(\omega_{(t)}) =: k < l$:

$$U_t^{\mathcal{Q}}(\omega_{(t)}) = \max\left\{X_t(\omega_{(t)}), \min_{p_{t+1}\in[\underline{p},\overline{p}]}\right.$$
$$\left.(p_{t+1}u(t+1, S_tu, k, l) + (1-p_{t+1})u(t+1, S_td, k, l))\right\}$$
$$= 0,$$

since $X_t(\omega_{(t)}) = 0$ by assumption and $u(t+1,\cdot,k,l) = 0$ by induction hypothesis (property (iii)).
Third, assume the case $t < \tau_1^t(\omega_{(t)}) = T+1$:
Then $X_t = 0$ and therefore we get in the <u>first case</u> when $\tau_1^{t+1}(\omega_{(t)},1) = T+1$

$$U_t^{\mathcal{Q}}(\omega_{(t)}) = \min_{p_{t+1}\in[\underline{p},\overline{p}]}\left\{p_{t+1}u(t+1, S_tu, \tau_1^{t+1}(\omega_{(t)},1), T+1)\right.$$
$$\left.+(1-p_{t+1})u(t+1, S_td, \tau_1^{t+1}(\omega_{(t)},0), T+1)\right\}$$
$$= \underline{p}u(t+1, S_tu, T+1, T+1) + (1-\underline{p})u(t+1, S_td, T+1, T+1)$$

by induction hypothesis (property (i)). Hence, $p_{t+1} = \underline{p}$ and $u(t, \cdot, T{+}1, T{+}1)$ is increasing in S.
In the <u>second case</u> when $\tau_1^{t+1}(\omega_{(t)}, 1) = t + 1$:

$$\begin{aligned} U_t^{\mathcal{Q}}(\omega_{(t)}) &= \min_{p_{t+1}\in[\underline{p},\overline{p}]} \{p_{t+1}u(t+1, S_t u, t+1, T+1) \\ &\quad + (1-p_{t+1})u(t+1, S_t d, T+1, T+1)\} \\ &= \underline{p}u(t+1, S_t u, t+1, T+1) + (1-\underline{p})u(t+1, S_t d, T+1, T+1) \\ &= u(t, S_t, T+1, T+1) \end{aligned}$$

again by induction hypothesis (property (i) since $S_t \cdot u = \bar{S}^1$) and we obtain $p_{t+1} = p_{\tau_1^{t+1}} = \underline{p}$. In order to show the monotonicity note that by induction hypothesis (property (i)) the last expression is greater or equal to $\underline{p}u(t+1, S_t, T+1, T+1) + (1-\underline{p})u(t+1, S_t dd, T+1, T+1)$ which again is equal to $u(t, S_t d, T+1, T+1)$ (see the first case).
Thus, for showing property (i) we just have to prove that $u(t, \bar{S}^1, t, T+1) \geq u(t, \bar{S}^1 d, T+1, T+1)$. Using property (i) of induction hypothesis we obtain

$$\begin{aligned} u(t, \bar{S}^1 d, T+1, T+1) &= \underline{p}u(t+1, \bar{S}^1, t+1, T+1) \\ &\quad + (1-\underline{p})u(t+1, \bar{S}^1 \cdot d^2, T+1, T+1) \\ &\leq \underline{p}u(t+1, \bar{S}^1, t+1, T+1) \\ &\quad + (1-\underline{p})u(t+1, \bar{S}^1, t+1, T+1) \\ &= u(t+1, \bar{S}^1, t+1, T+1) \\ &\leq u(t, \bar{S}^1, t, T+1). \end{aligned}$$

The last inequality is due to Lemma A.1. This completes the proof and $(U_t^{\mathcal{Q}})$ satisfies the same recursion as $(\hat{U}_t)$. Thus, $(U_t^{\mathcal{Q}}) = (\hat{U}_t)$ follows and the worst-case measure $\hat{P}$ is specified by the density $\hat{D}_T$ as claimed.

An optimal stopping time is given by $\hat{\tau}$. This follows by general theory, see Riedel (2009). The time boundary σ of the optimal stopping rule is due to the claim's knock-out feature. □

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