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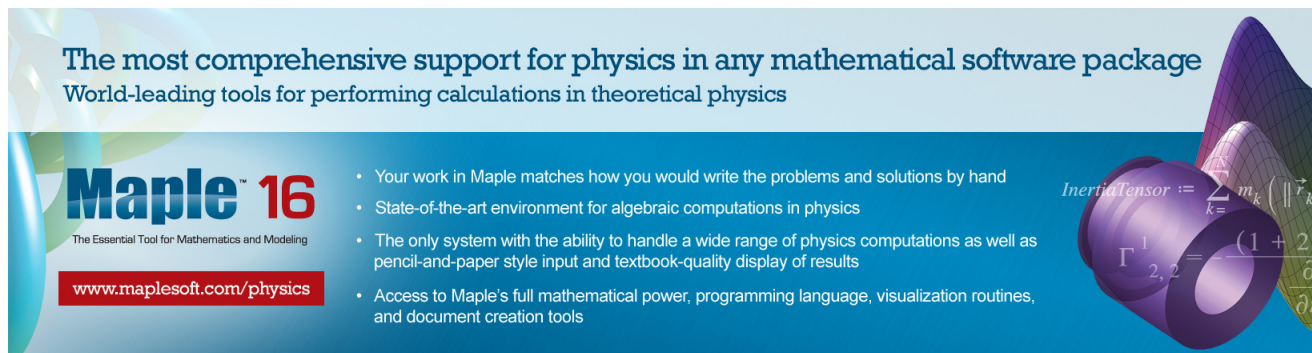
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Spectral and topological properties of a family of generalised Thue-Morse sequences

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The classic middle-thirds Cantor set leads to a singular continuous measure via a distribution function that is known as the Devil's staircase. The support of the Cantor measure is a set of zero Lebesgue measure. Here, we discuss a class of singular continuous measures that emerge in mathematical diffraction theory and lead to somewhat similar distribution functions, yet with significant differences. Various properties of these measures are derived. In particular, these measures have supports of full Lebesgue measure and possess strictly increasing distribution functions. In this sense, they mark the opposite end of what is possible for singular continuous measures. For each member of the family, the underlying dynamical system possesses a topological factor with maximal pure point spectrum, and a close relation to a solenoid, which is the Kronecker factor of the system. The inflation action on the continuous hull is sufficiently explicit to permit the calculation of the corresponding dynamical zeta functions. This is achieved as a corollary of analysing the Anderson-Putnam complex for the determination of the cohomological invariants of the corresponding tiling spaces. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3688337]

Dedicated to Robert V. Moody on the occasion of his 70th birthday.

I. INTRODUCTION

The probably most widely known singular continuous measure emerges as the unique invariant probability measure for the iterated function system[31] of the classic middle-thirds Cantor set. The construction and the distribution function F of the resulting measure are illustrated in Figure 1. Due to its shape, F is known as the Devil's staircase. It is a non-decreasing continuous function that is constant almost everywhere, which corresponds to the fact that the underlying measure gives no weight to single points, but is concentrated on an uncountable set of zero Lebesgue measure (the Cantor set). The Cantor measure is thus both continuous and singular, hence purely singular continuous.

Singular continuous measures occur in a wide range of physical problems, most notably in the theory of non-periodic Schrödinger operators; see Refs. 20 and 23 and references therein for examples. In particular, it is an amazing result that singular continuous spectra are in a certain sense even generic here; compare Refs. 51 and 39. One would also expect the appearance of singular continuous measures in mathematical diffraction theory,[22,30,8] where the Thue-Morse sequence provides one of the few really explicit examples. Recent experimental evidence[54] indicates that this spectral type might indeed be more relevant to diffraction than presumed so far. This case has not yet

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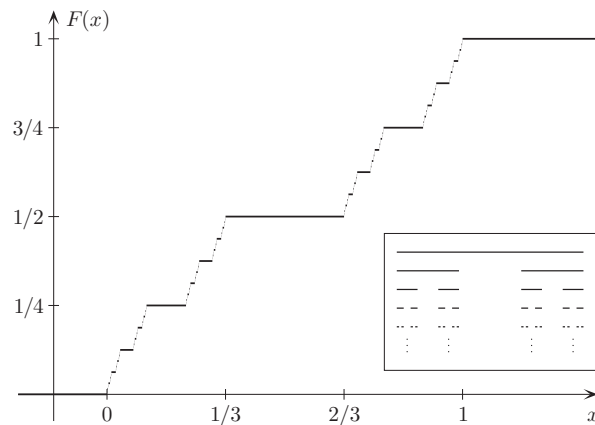
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FIG. 1. The distribution function *F* of the classic middle-thirds Cantor measure. The construction of the Cantor set is sketched in the inset.

received the theoretical attention it deserves, though partial results exist in the dynamical systems literature; compare Refs. 45 and 32.

The Thue-Morse system is an example of a bijective substitution of constant length.[45] This class has a natural generalisation to higher dimensions, and is studied in some detail in Refs. 26 and 27; see also Ref. 55 and references therein for related numerical studies. Bijective substitutions form an important case within the larger class of lattice substitution systems. For the latter, from the point of view of diffraction theory, a big step forward was achieved in Refs. 36–38, where known criteria for pure pointedness in one dimension[24] were generalised to the case of $\mathbb{Z}^d$-action. Moreover, a systematic connection with model sets was established (see Refs. 41 and 42 for detailed expositions and Ref. 35 for a rather complete picture), and there are also explicit algorithms to handle such cases; compare Refs. 28 and 1 and references therein. Nevertheless, relatively little has been done for the case without any coincidence in the sense of Dekking[24] or its generalisation to lattice substitution systems.[36,37] Although it is believed that one should typically expect singular continuous measures for bijective substitutions without coincidence, explicit examples are rare.

As a first step to improve this situation, we investigate a class of generalised Thue-Morse sequences in the spirit of Ref. 34. They are defined by primitive substitution rules and provide a two-parameter family of systems with purely singular continuous diffraction. Below, we formulate a rigorous approach that is constructive and follows the line of ideas that was originally used by Wiener,[53] Mahler,[40] and Kakutani[32] for the treatment of the classic Thue-Morse case. Some of the measures were studied before (mainly by scaling arguments and numerical methods) in the context of dimension theory for correlation measures; compare Refs. 34 and 55 and references therein.

The paper is organised as follows. We begin with a brief summary of the Thue-Morse sequence with its spectral and topological properties, where we also introduce our notation. Sec. III treats the family of generalised Thue-Morse sequences from Ref. 6, where the singular continuous nature of the diffraction spectra is proved and the corresponding distribution functions are derived. Here, we also briefly discuss the connection with a generalisation of the period doubling sequence. The latter has pure point spectrum, and is a topological factor of the generalised Thue-Morse sequence. This factor has maximal pure point spectrum. The diffraction measure of the generalised Thue-Morse system is analysed in detail in Sec. IV, and its Riesz product structure is derived. In Sec. V, we construct the continuous hulls of the generalised Thue-Morse and period doubling sequences as inverse limits of the substitution acting on the Anderson-Putnam cell complex,[3] and employ this construction to compute and relate their Čech cohomologies. The substitution action on the Čech cohomology is then used in Sec. VI to derive the dynamical zeta functions of the corresponding substitution dynamical systems. Finally, we conclude with some further observations and open problems.

II. A SUMMARY OF THE CLASSIC THUE-MORSE SEQUENCE

The classic Thue-Morse (or Prouhet-Thue-Morse, abbreviated as TM below) sequence[2] can be defined via a fixed point of the primitive substitution,

$$\varrho = \varrho_{\mathrm{TM}}: \quad 1 \mapsto 1\bar{1}, \quad \bar{1} \mapsto \bar{1}1 \tag{1}$$

on the binary alphabet $\{1, \bar{1}\}$. The one-sided fixed point starting with 1 reads as

$$v = v_0 v_1 v_2 \ldots = 1\bar{1}\bar{1}1\bar{1}11\bar{1}\ldots, \tag{2}$$

while $\bar{v}$ is the fixed point starting with $\bar{1}$. One can now define a two-sided sequence w by

$$w_i = \begin{cases} v_i, & i \geq 0, \\ v_{-i-1}, & i < 0. \end{cases} \tag{3}$$

It is easy to check that $w = \ldots w_{-2}w_{-1}|w_0 w_{-1}\ldots = \ldots v_1 v_0 | v_0 v_1 \ldots$ defines a 2-cycle under ϱ, and hence a fixed point for ϱ^2, with the central seed $1|1$ being legal. Recall that a finite word is called *legal* when it occurs in the n-fold substitution of a single letter for some $n \in \mathbb{N}$. An iteration of ϱ^2 applied to this seed converges to w in the product topology, which is thus a two-sided fixed point of ϱ^2 in the proper sense.

The sequence w defines a dynamical system (under the action of the group $\mathbb{Z}$) as follows. Its compact space is the (discrete) *hull*, obtained as the closure of the $\mathbb{Z}$-orbit of w,

$$\mathbb{X}^{\mathrm{TM}} = \overline{\{S^i w \mid i \in \mathbb{Z}\}},$$

where S denotes the shift operator (with $(Sw)_i = w_{i+1}$) and where the closure is taken in the local (or product) topology. Here, two sequences are close when they agree on a large segment around the origin (marked by |). Now, $(\mathbb{X}^{\mathrm{TM}}, \mathbb{Z})$ is a strictly ergodic dynamical system (hence uniquely ergodic and minimal.[45,52]) Its unique invariant probability measure is given via the (absolute) frequencies of finite words (or patches) as the measures of the corresponding cylinder sets, where the latter then generate the (Borel) σ-algebra. Its minimality follows from the repetitivity of the fixed point word w, which also implies that $\mathbb{X}^{\mathrm{TM}} = \mathrm{LI}(w)$, where LI stands for the local indistinguishability class. The latter consists of all elements of $\{1, \bar{1}\}^{\mathbb{Z}}$ that are locally indistinguishable from w.

Here, we are interested, for a given $w \in \mathbb{X}^{\mathrm{TM}}$, in the diffraction of the (signed) Dirac comb,

$$\omega = w \delta_{\mathbb{Z}} := \sum_{n \in \mathbb{Z}} w_n \delta_n, \tag{4}$$

where the symbols 1 and $\bar{1}$ are interpreted as weights 1 and -1. This defines a mapping from $\mathbb{X}^{\mathrm{TM}}$ into the signed translation bounded measures on $\mathbb{Z}$ (or on $\mathbb{R}$). Since this mapping is a homeomorphism between $\mathbb{X}^{\mathrm{TM}}$ and its image, we use both points of view in parallel without further mentioning.

Given any $w \in \mathbb{X}^{\mathrm{TM}}$, the *autocorrelation measure* of the corresponding ω exists as a consequence of unique ergodicity. It is defined as the volume-averaged (or Eberlein) convolution,

$$\gamma = \omega \circledast \widetilde{\omega} = \lim_{N \to \infty} \frac{\omega_N * \widetilde{\omega_N}}{2N + 1},$$

where ω_N is the restriction of ω to $[-N, N]$ and $\widetilde{\mu}$ is the "flipped-over" version of the measure μ defined by $\widetilde{\mu}(g) := \overline{\mu(\widetilde{g})}$ for continuous functions g of compact support, with $\widetilde{g}(x) = \overline{g(-x)}$. We use this general formulation to allow for complex weights later on. A short calculation shows that the autocorrelation is of the form

$$\gamma = \sum_{m \in \mathbb{Z}} \eta(m) \, \delta_m$$

with the autocorrelation coefficients,

$$\eta(m) = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} w_n \, \overline{w_{n-m}}. \tag{5}$$

Note that γ applies to *all* sequences of $\mathbb{X}^{TM}$, by an application of the ergodic theorem, as $\eta(m)$ is the orbit average of the continuous function $w \mapsto w_0 \overline{w_{-m}}$ and our dynamical system is uniquely ergodic.

Remark 1 (alternative approach): Without the diffraction context, it is possible to directly define the function $\eta \colon \mathbb{Z} \longrightarrow \mathbb{C}$ by (5). It is then a positive definite function on $\mathbb{Z}$, with a representation as the (inverse) Fourier transform of a positive measure μ on the unit circle, by the Herglotz-Bochner theorem.[47] Which formulation one uses is largely a matter of taste. We follow the route via the embedding as a measure on $\mathbb{R}$, so that we get $\widehat{\gamma} = \mu * \delta_{\mathbb{Z}}$ together with its interpretation as the diffraction measure of the Dirac comb ω.   ◇

We can now employ the special structure of our fixed point w to analyse γ. One then finds that $\eta(0) = 1$, $\eta(-m) = \eta(m)$ for all $m \in \mathbb{Z}$, and

$$\eta(m) \,=\, \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n v_{n+m} \tag{6}$$

for all $m \geq 0$. Here, the structure of w and its relation to v was used to derive (6) from (5). Observing that v satisfies $v_{2n} = v_n$ and $v_{2n+1} = \bar{v}_n$ for all $n \geq 0$, one can employ (6) to infer the linear recursion relations,

$$\eta(2m) \,=\, \eta(m) \quad \text{and} \quad \eta(2m+1) \,=\, -\frac{1}{2}\big(\eta(m) + \eta(m+1)\big), \tag{7}$$

which actually hold for all $m \in \mathbb{Z}$. These well-known relations[32] will also follow from our more general results in Sec. III as a special case. One finds $\eta(\pm 1) = -1/3$ from solving the recursion for $m = 0$ and $m = -1$ with $\eta(0) = 1$, while all other values are then recursively determined.

To analyse the *diffraction measure* $\widehat{\gamma}$ of the TM sequence (following Refs. 40 and 32), one can start with its pure point part. Defining $\Sigma(N) = \sum_{n=-N}^{N}\big(\eta(n)\big)^2$, one derives $\Sigma(4N) \leq \frac{3}{2}\Sigma(2N)$ from the recursion (7); see Ref. 5 for the detailed estimate needed. This implies $\frac{1}{N}\Sigma(N) \xrightarrow{N \to \infty} 0$. By Wiener's criterion,[53] this means $\big(\widehat{\gamma}\big)_{pp} = 0$, so that $\widehat{\gamma}$ is a continuous measure (see Wiener's Corollary in Sec. I.7.13 of Ref. 33 or Wiener's Lemma in Sec. 4.16 of Ref. 43 for details).

Defining the (continuous and non-decreasing) distribution function F via $F(x) = \widehat{\gamma}\big([0, x]\big)$, another consequence of (7) is the pair of functional relations,

$$\mathrm{d}F\big(\tfrac{x}{2}\big) \pm \mathrm{d}F\big(\tfrac{x+1}{2}\big) = \begin{Bmatrix} 1 \\ -\cos(\pi x) \end{Bmatrix} \mathrm{d}F(x).$$

Splitting F into its sc and ac parts (which are unique and must both satisfy these relations) now implies backwards that the recursion (7) holds separately for the two sets of autocorrelation coefficients, η_{sc} and η_{ac}, with yet unknown initial conditions at 0. Since this means $\eta_{ac}(1) = -\frac{1}{3}\eta_{ac}(0)$ together with $\eta_{ac}(2m) = \eta_{ac}(m)$ for all $m \in \mathbb{N}$, an application of the Riemann-Lebesgue lemma forces $\eta_{ac}(0) = 0$, and hence $\eta_{ac}(m) = 0$ for all $m \in \mathbb{Z}$, so that also $\big(\widehat{\gamma}\big)_{ac} = 0$; compare Ref. 32. This shows that $\widehat{\gamma}$ is a singular measure. With the previous argument, since $\widehat{\gamma} \neq 0$, we see that it is a purely singular continuous measure. Figure 2 shows an image, where we have used the uniformly converging Volterra-type iteration,

$$F_{n+1}(x) = \frac{1}{2} \int_0^{2x} \big(1 - \cos(\pi y)\big) F_n'(y)\,\mathrm{d}y \qquad \text{with} \qquad F_0(x) = x$$

to calculate F with sufficient precision (note that $F(x+1) = F(x) + 1$, so that displaying F on the interval $[0, 1]$ suffices). In contrast to the Devil's staircase, the TM function is *strictly* increasing, which means that there is no plateau (which would indicate a gap in the support of $\widehat{\gamma}$); see Ref. 5 and references therein for details.

Despite the above result, the TM sequence is closely related to the period doubling sequence, via the (continuous) block map,

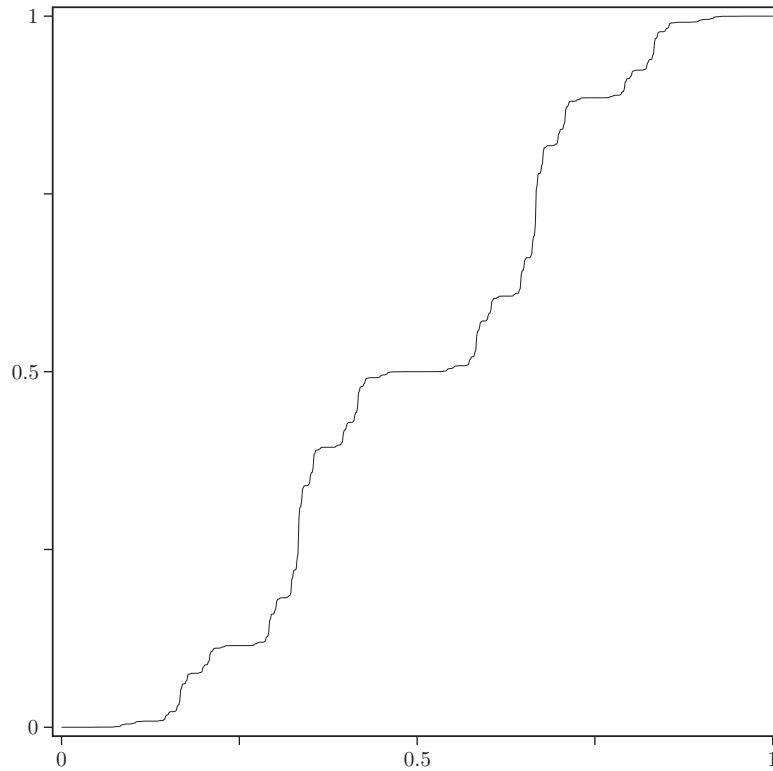$$\phi\colon \quad 1\bar{1}, \bar{1}1 \mapsto a, \quad 11, \bar{1}\bar{1} \mapsto b, \tag{8}$$

FIG. 2. The strictly increasing distribution function of the classic, singular continuous TM measure on [0, 1].

which defines an exact 2-to-1 surjection from the hull $\mathbb{X}^{\mathrm{TM}}$ to $\mathbb{X}^{\mathrm{pd}}$, where the latter is the hull of the period doubling substitution defined by

$$\varrho' = \varrho_{\mathrm{pd}}: \quad a \mapsto ab, \quad b \mapsto aa. \tag{9}$$

Viewed as topological dynamical systems, this means that $(\mathbb{X}^{\mathrm{pd}}, \mathbb{Z})$ is a factor of $(\mathbb{X}^{\mathrm{TM}}, \mathbb{Z})$. Since both are strictly ergodic, this extends to the corresponding measure theoretic dynamical systems, as well as to the suspensions into continuous dynamical systems under the action of $\mathbb{R}$. The dynamical spectrum of the TM system then contains $\mathbb{Z}[\frac{1}{2}]$ as its pure point part,[45] which is the entire dynamical spectrum of the period doubling system. We thus are in the nice situation that a topological factor with maximal pure point spectrum exists which is itself a substitution system.

The period doubling sequence can be described as a regular model set with a 2-adic internal space[16,15] and is thus pure point diffractive. As another consequence, there is an almost everywhere 1-to-1 mapping[50,16] of the continuous hull (see below) onto a (dyadic) solenoid $\mathbb{S} = \mathbb{S}_2$. Here, a solenoid $\mathbb{S}_m$ (with $2 \leq m \in \mathbb{N}$ say) is the inverse limit of the unit circle under the iterated multiplication by m. The dyadic solenoid is obtained for $m = 2$.

The discrete hull $\mathbb{X}^{\mathrm{TM}}$ of the TM sequence has a continuous counterpart (its suspension), which we call $\mathbb{Y}^{\mathrm{TM}}$. Instead of symbolic TM sequences, one considers the corresponding tilings of the real line, with labelled tiles of unit length. Such tilings are not bound to have their vertices at integer positions, and the full translation group $\mathbb{R}$ acts continuously on them. The continuous hull of a TM tiling is then the closure of its $\mathbb{R}$-orbit with respect to the local topology. Here, two tilings are close if, possibly after a small translation, they agree on a large interval around the origin. For the same reasons as in the case of the discrete hull, the corresponding topological dynamical system $(\mathbb{Y}^{\mathrm{TM}}, \mathbb{R})$ is minimal and uniquely ergodic, so that every TM tiling defines the same continuous hull. Similarly, a continuous hull is defined for the period doubling sequence.

According to Ref. 3, the continuous hull of a primitive substitution tiling can be constructed as the inverse limit of an iterated map on a finite CW-complex Γ, called the Anderson-Putnam

(AP) complex. The cells of Γ are the tiles in the tiling, possibly labelled to distinguish different neighbourhoods of a tile within the tiling. Each point in Γ actually represents a cylinder set of tilings, with a specific neighbourhood at the origin. The substitution map on the tiling induces a continuous cellular map of Γ onto itself, whose inverse limit is homeomorphic to the continuous hull of the tiling.

Let Γ_n be the CW-complex of the nth approximation step. It is a nice feature of the corresponding inverse limit space $\mathbb{Y} = \varprojlim \Gamma_n$ that its Čech cohomology $H^k(\mathbb{Y})$ can be computed as the direct (inductive) limit of the cohomologies of the corresponding approximant spaces $H^k(\Gamma_n)$. In our case, we have a single approximant space Γ, and a single map (the substitution map ϱ) acting on it. Consequently, $H^k(\mathbb{Y})$ is the inductive limit of the induced map ϱ^* on $H^k(\Gamma)$. Analogous inverse limit constructions also exist for the hull of the period doubling tiling, $\mathbb{Y}^{\mathrm{pd}}$, and for the dyadic solenoid, $\mathbb{S}$.

As a consequence of the above, there is a 2-to-1 cover $\phi: \mathbb{Y}^{\mathrm{TM}} \to \mathbb{Y}^{\mathrm{pd}}$, and a surjection $\psi: \mathbb{Y}^{\mathrm{pd}} \to \mathbb{S}$ which is 1-to-1 almost everywhere. These maps induce well-defined cellular maps on the associated AP complexes; see also Ref. 49 for a general exposition. We represent these maps by the same symbols, ϕ and ψ. They induce homomorphisms on the cohomologies of the AP complexes, so that we have the following commutative diagram:

$$
\begin{array}{ccccc}
H^k(\Gamma^{\mathrm{sol}}) & \xrightarrow{\psi^*} & H^k(\Gamma^{\mathrm{pd}}) & \xrightarrow{\phi^*} & H^k(\Gamma^{\mathrm{TM}}) \\
\times 2 \downarrow & & \varrho'^* \downarrow & & \downarrow \varrho^* \\
H^k(\Gamma^{\mathrm{sol}}) & \xrightarrow{\psi^*} & H^k(\Gamma^{\mathrm{pd}}) & \xrightarrow{\phi^*} & H^k(\Gamma^{\mathrm{TM}})
\end{array}
. \tag{10}
$$

All these maps are explicitly known. The inductive limits along the columns not only give the cohomologies of the continuous hulls, but also determine the embeddings under the maps ϕ^* and ψ^*. Although $H^1(\mathbb{Y}^{\mathrm{pd}})$ and $H^1(\mathbb{Y}^{\mathrm{TM}})$ are isomorphic, the former embeds (under ϕ^*) in the latter as a subgroup of index 2, which reflects $\mathbb{Y}^{\mathrm{TM}}$ being a two-fold cover of $\mathbb{Y}^{\mathrm{pd}}$. Furthermore, we get $H^1(\mathbb{Y}^{\mathrm{pd}})/\psi^*(H^1(\mathbb{S})) = \mathbb{Z}$. By an application of Proposition 4 of Ref. 18, compare also Example 7 of Ref. 18, this corresponds to the fact that there are exactly two orbits on which the map ψ fails to be 1-to-1. These two orbits are merged into a single orbit under ψ.

The action of the substitution on the cohomology of the AP complex $H^k(\Gamma)$, more precisely the eigenvalues of this action, can be used[3] to calculate the dynamical zeta function of the substitution dynamical system, thus establishing a connection between the action of the substitution on $H^k(\Gamma)$ and the number of periodic orbits of the substitution in the continuous hull. We skip further details at this point because they will appear later as a special case of our two-parameter family, which we discuss next.

III. A FAMILY OF GENERALISED THUE-MORSE SEQUENCES

The TM sequence is sometimes considered as a rather special and possibly rare example, which is misleading. In fact, there are many systems with similar properties. Let us demonstrate this point by following Refs. 34 and 6, where certain generalisations were introduced. In particular, we consider the generalised Thue-Morse (gTM) sequences defined by the primitive substitutions,

$$
\varrho = \varrho_{k,\ell}: \quad 1 \mapsto 1^k \bar{1}^\ell, \quad \bar{1} \mapsto \bar{1}^k 1^\ell \tag{11}
$$

for arbitrary $k, \ell \in \mathbb{N}$. Here, the one-sided fixed point starting with $v_0 = 1$ satisfies

$$
v_{m(k+\ell)+r} = \begin{cases} v_m, & \text{if } 0 \leq r < k, \\ \bar{v}_m, & \text{if } k \leq r < k+\ell, \end{cases} \tag{12}
$$

for $m \geq 0$, as can easily be verified from the fixed point property. A two-sided gTM sequence w can be constructed as above in Eq. (3). Also analogous is the construction of the discrete hull $\mathbb{X}_{k,\ell}$ as the orbit closure of w under the $\mathbb{Z}$-action of the shift, where $\mathbb{X}_{1,1} = \mathbb{X}^{\mathrm{TM}}$. We will drop the index when it is clear from the context.

Proposition 1: Consider the substitution rule $\varrho = \varrho_{k,\ell}$ of Eq. (11) for arbitrary, but fixed $k, \ell \in \mathbb{N}$. The bi-infinite sequence w that is constructed from the one-sided sequence v of (12) via reflection as in Eq. (3) is a fixed point of ϱ^2, as is $w' = \varrho(w)$. Both sequences are repetitive and locally indistinguishable.

Moreover, w is an infinite palindrome and defines the discrete hull $\mathbb{X} = \overline{\{S^i w \mid i \in \mathbb{Z}\}}$ under the action of the shift S. This hull is reflection symmetric and minimal, and defines a strictly ergodic dynamical system. Similarly, when turning w into a tiling of $\mathbb{R}$ by two intervals of length 1 that are distinguished by colour, the closure $\mathbb{Y}$ of the $\mathbb{R}$-orbit of this tiling in the local topology defines a dynamical system $(\mathbb{Y}, \mathbb{R})$ that is strictly ergodic for the $\mathbb{R}$-action.

Proof: Most claims are standard consequences of the theory of substitution dynamical systems,[45] as applied to ϱ. The fixed point property includes the relation $\varrho^2(w) = w$ together with the legality of the central seed $w_{-1}|w_0$ around the marker. Note that each fixed point of ϱ^n, with arbitrary $n \in \mathbb{N}$, is repetitive and defines the same hull $\mathbb{X}$. The latter is minimal due to repetitivity, and consists precisely of the LI class of w. Since w and w' coincide on the right of the marker, but differ on the left in every position, neither can have a non-trivial period (this would contradict their local indistinguishability). This is an easy instance of the existence of distinct proximal elements.[17] Consequently, $\mathbb{X} = \mathrm{LI}(w)$ cannot contain *any* element with a non-trivial period, so that w and hence ϱ is aperiodic.

The action of the shift is clearly continuous in the product topology. Unique ergodicity follows from the uniform existence of all pattern frequencies (or from linear repetitivity). This means that $(\mathbb{X}, \mathbb{Z})$ defines a topological dynamical system that leads to a strictly ergodic dynamical system $(\mathbb{X}, \mathbb{Z}, \nu)$, where the unique measure ν is defined via the frequencies of patches as the measures of the corresponding cylinder sets. The claim about the extension to the $\mathbb{R}$-action on $\mathbb{Y}$ follows from the suspension of the discrete system, which is easy here because the constant length of the substitution ϱ implies that the canonically attached tiling is the one described. □

Let us mention in passing that the discrete hull $\mathbb{X}$ is a Cantor set, while the local structure of the continuous hull $\mathbb{Y}$ is a product of an interval with a Cantor set; compare Ref. 20 and references therein.

Since each choice of k, ℓ leads to a strictly ergodic dynamical system, we know that all autocorrelation coefficients (as defined by Eq. (5), with $\bar{1} \mathrel{\hat=} -1$) exist. Clearly, we have $\eta(0) = 1$, while several possibilities exist to calculate $\eta(\pm 1) = \frac{k+\ell-3}{k+\ell+1}$.

As before, we turn a gTM sequence $w = (w_n)_{n \in \mathbb{Z}}$ into the Dirac comb,

$$\omega = \sum_{n \in \mathbb{Z}} w_n \, \delta_n, \tag{13}$$

which is a translation bounded measure. Its autocorrelation is of the form

$$\gamma = \sum_{m \in \mathbb{Z}} \eta(m) \, \delta_m \tag{14}$$

with the coefficients $\eta(m)$, which can alternatively be calculated via the one-sided fixed point v as in Eq. (6). Let us now derive a recursion for $\eta(m)$. Since this will be the "golden key" for almost all spectral properties, we provide a detailed proof.

Lemma 1: Consider the gTM sequence defined by the primitive substitution ϱ of (11), for fixed $k, \ell \in \mathbb{N}$. When realised as the Dirac comb of Eq. (13), each element of the corresponding hull $\mathbb{X} = \mathbb{X}_{k,\ell}$ possesses the unique autocorrelation γ of (14), where the autocorrelation coefficients satisfy $\eta(0) = 1$ and the linear recursion,

$$\eta\big((k+\ell)m + r\big) = \frac{1}{k+\ell}\big(\alpha_{k,\ell,r}\,\eta(m) + \alpha_{k,\ell,k+\ell-r}\,\eta(m+1)\big),$$

with $\alpha_{k,\ell,r} = k + \ell - r - 2\min(k, \ell, r, k+\ell-r)$, valid for all $m \in \mathbb{Z}$ and $0 \leq r < k + \ell$. In particular, one has $\eta\big((k+\ell)m\big) = \eta(m)$ for $m \in \mathbb{Z}$.

TABLE I. Conditions on r and t for the splitting of the sum in Eq. (15), and the resulting term for the sum from the recursion relation (12).

	Conditions	Term
	$0 \leq t < k - r$	$v_n v_{n+m}$
	$k - r \leq t < \min(k, k + \ell - r)$	$v_n \bar{v}_{n+m}$
$0 \leq r < k$	$\min(k, k + \ell - r) \leq t < k$	$v_n v_{n+m+1}$
	$k \leq t < \max(k, k + \ell - r)$	$\bar{v}_n \bar{v}_{n+m}$
	$\max(k, k + \ell - r) \leq t < k + \ell$	$\bar{v}_n v_{n+m+1}$
	$0 \leq t < \min(k, k + \ell - r)$	$v_n \bar{v}_{n+m}$
	$\min(k, k + \ell - r) \leq t < k$	$v_n v_{n+m+1}$
$k \leq r < k + \ell$	$k \leq t < \max(k, k + \ell - r)$	$\bar{v}_n \bar{v}_{n+m}$
	$\max(k, k + \ell - r) \leq t < 2k + \ell - r$	$\bar{v}_n v_{n+m+1}$
	$2k + \ell - r \leq t < k + \ell$	$\bar{v}_n \bar{v}_{n+m+1}$

Proof: Existence and uniqueness of γ are a consequence of Proposition 1, via an application of the ergodic theorem. The support of the positive definite measure γ is obviously contained in $\mathbb{Z}$, so that $\gamma = \eta \delta_{\mathbb{Z}}$.

Since $\eta(0) = 1$ is immediate from Eq. (5), it remains to derive the recursion. We now begin with $m \geq 0$ and use formula (6). When $r = 0$, one finds

$$
\eta\big((k+\ell)m\big) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} v_n v_{n+(k+\ell)m} = \lim_{N' \to \infty} \frac{1}{(k+\ell)N'} \sum_{t=0}^{k+\ell-1} \sum_{n=0}^{N'-1} v_{(k+\ell)n+t}\, v_{(k+\ell)(n+m)+t}
$$

$$
\stackrel{(12)}{=} \lim_{N' \to \infty} \frac{1}{(k+\ell)N'} \sum_{t=0}^{k+\ell-1} \sum_{n=0}^{N'-1} \left\{ \begin{array}{l} v_n v_{n+m}, \text{ if } 0 \leq t < k \\ \bar{v}_n \bar{v}_{n+m}, \text{ if } k \leq t < k+\ell \end{array} \right\}
$$

$$
= \lim_{N' \to \infty} \frac{1}{(k+\ell)N'} \sum_{t=0}^{k+\ell-1} \sum_{n=0}^{N'-1} v_n v_{n+m} = \eta(m),
$$

where the penultimate step follows because $\bar{v}_n \bar{v}_{n+m} = v_n v_{n+m}$ due to $v_i \in \{\pm 1\}$.

For general r, one proceeds analogously and finds

$$
\eta\big((k+\ell)m + r\big) = \lim_{N' \to \infty} \frac{1}{(k+\ell)N'} \sum_{t=0}^{k+\ell-1} \sum_{n=0}^{N'-1} v_{(k+\ell)n+t}\, v_{(k+\ell)(n+m)+r+t}\,. \tag{15}
$$

One now has to split the sum over t according to the ten cases of Table I. In each row, the condition for t is formulated in such a way that the difference of the bounds gives the proper multiplicity of the resulting term to the sum, which may be zero in some cases.

Observing $\bar{v}_n = -v_n$, one simply has to add the terms of the form $v_n v_{n+m}$ with their signed multiplicities, which contribute to $\eta(m)$, and those of the form $v_n v_{n+m+1}$, which contribute to $\eta(m+1)$. For instance, when $0 \leq r < k$, one finds the multiplicity of $v_n v_{n+m}$ as

$$
(k-r) - \big(\min(k, k+\ell-r) - (k-r)\big) + \big(\max(k, k+\ell-r) - k\big)
$$

$$
= k + \ell - r - 2\big(\min(k, k+\ell-r) - k + r\big) = k + \ell - r - 2\min(r, \ell)
$$

$$
= k + \ell - r - 2\min(k, \ell, r, k+\ell-r) = \alpha_{k,\ell,r},
$$

where we used $\min(a, b) + \max(a, b) = a + b$ and, in the last line, the inequality $0 \leq r < k$. The required denominator $(k + \ell)$ in the claimed recursion emerges from the splitting as shown above

in Eq. (15). Likewise, the multiplicity for $v_n v_{n+m+1}$ calculates as

$$\big(k - \min(k, k+\ell-r)\big) - \big((k+\ell) - \max(k, k+\ell-r)\big)$$

$$= -\ell + \max(k, k+\ell-r) - \min(k, k+\ell-r)$$

$$= 2k - r - 2\min(k, k+\ell-r) = r - 2\min(r, \ell) = \alpha_{k,\ell,k+\ell-r},$$

where we used that $\min(r, \ell) = \min(k, \ell, r, k + \ell - r)$ holds in this case. The remaining cases (for $k \leq r < k + \ell$) follow from similar calculations; see Table I. This completes the argument for $\eta(m)$ with $m \geq 0$.

Finally, we know that $\eta(-n) = \eta(n)$ holds for all $n \in \mathbb{Z}$. Let $m < 0$ be fixed, so that $m = -|m|$ with $|m| > 0$. If $r = 0$, one simply has

$$\eta\big((k+\ell)m\big) = \eta\big((k+\ell)|m|\big) = \eta(|m|) = \eta(m).$$

When $1 \leq r < k + \ell$, set $m' = |m| - 1$ and $s = k + \ell - r$, so that $m' \geq 0$ and $1 \leq s < k + \ell$. Then, one finds

$$\eta\big((k+\ell)m + r\big) = \eta\big((k+\ell)|m| - r\big) = \eta\big((k+\ell)m' + s\big)$$

$$= \frac{1}{k + \ell}\big(\alpha_{k,\ell,s}\,\eta(m') + \alpha_{k,\ell,k+\ell-s}\,\eta(m'+1)\big) = \frac{1}{k + \ell}\big(\alpha_{k,\ell,r}\,\eta(m) + \alpha_{k,\ell,k+\ell-r}\,\eta(m+1)\big),$$

due to the reflection symmetry of η together with the recursion for positive arguments. $\qquad \square$

Note that the recursion in Lemma 1 uniquely determines all coefficients $\eta(m)$ once $\eta(0)$ is given. Moreover, the recursion is linear, which will have strong consequences later.

Since, for any given $k, \ell \in \mathbb{N}$, every member of the corresponding hull of weighted Dirac combs has the same autocorrelation measure, we speak, from now on, of the autocorrelation measure of the gTM system (for parameters $k, \ell \in \mathbb{N}$). Let us define $\Sigma(N) = \sum_{n=-N}^{N}\big(\eta(n)\big)^2$, where we suppress the parameters k and ℓ. For $k = \ell = 1$, we know a bound on the growth rate of $\Sigma(N)$, namely, $\Sigma(4N) \leq \frac{3}{2}\Sigma(2N)$, from Refs. 32 and 5. For $k + \ell > 2$, we formulate a similar result (with a technically more involved but structurally slightly simpler proof) as follows.

Lemma 2: Let $k, \ell \in \mathbb{N}$ with $k + \ell > 2$ be fixed, and let $\eta(n)$ with $n \in \mathbb{Z}$ be the corresponding autocorrelation coefficients from Lemma 1. Then, there is some positive number $q < k + \ell$ such that $\Sigma\big((k+\ell)N\big) \leq q\,\Sigma(N)$ for all $N \in \mathbb{N}$.

Proof: Using the recursions of Lemma 1, one finds

$$\Sigma\big((k+\ell)N\big) = \sum_{n=-(k+\ell)N}^{(k+\ell)N}\big(\eta(n)\big)^2 = \big(\eta((k+\ell)N)\big)^2 + \sum_{r=0}^{k+\ell-1}\sum_{m=-N}^{N-1}\big(\eta((k+\ell)m + r)\big)^2$$

$$= \Sigma(N) + \frac{1}{(k+\ell)^2}\sum_{r=1}^{k+\ell-1}\sum_{m=-N}^{N-1}\big(\alpha_{k,\ell,r}\,\eta(m) + \alpha_{k,\ell,k+\ell-r}\,\eta(m+1)\big)^2$$

$$\leq \frac{\Sigma(N)}{(k+\ell)^2}\bigg((k+\ell)^2 + \sum_{r=1}^{k+\ell-1}\big(\alpha_{k,\ell,r}^2 + \alpha_{k,\ell,k+\ell-r}^2\big)\bigg) + \frac{A}{(k+\ell)^2}$$

with $A = \big(\sum_{m=-N}^{N-1} 2\,|\eta(m)\eta(m+1)|\big)\big(\sum_{r=1}^{k+\ell-1}|\alpha_{k,\ell,r}\,\alpha_{k,\ell,k+\ell-r}|\big)$ being a sum of non-negative terms only. Noting that

$$\sum_{m=-N}^{N-1} 2\,|\eta(m)\eta(m+1)| \leq \sum_{m=-N}^{N-1}\big(\eta(m)\big)^2 + \big(\eta(m+1)\big)^2 \leq 2\,\Sigma(N),$$

one obtains $A \le \Sigma(N) \sum_{r=1}^{k+\ell-1} 2\,|\alpha_{k,\ell,r}\,\alpha_{k,\ell,k+\ell-r}|$. Employing the binomial formula results in

$$\Sigma\big((k+\ell)N\big) \ \le \ \frac{\Sigma(N)}{(k+\ell)^2}\left((k+\ell)^2 + \sum_{r=1}^{k+\ell-1}\big(|\alpha_{k,\ell,r}| + |\alpha_{k,\ell,k+\ell-r}|\big)^2\right).$$

Our claim follows if we show that the term in the large brackets is smaller than $(k+\ell)^3$. For $1 \le r \le k+\ell-1$, we know that $1 \le \min(k,\ell,r,k+\ell-r) \le \min(r,k+\ell-r)$, which implies $|\alpha_{k,\ell,r}| \le k+\ell-r$ and hence

$$|\alpha_{k,\ell,r}| + |\alpha_{k,\ell,k+\ell-r}| \ \le \ k+\ell. \tag{16}$$

Since $k+\ell > 2$ by assumption, the stronger inequality $|\alpha_{k,\ell,1}| + |\alpha_{k,\ell,k+\ell-1}| \le k+\ell-2$ holds for $r = 1$, so that at least one term in the sum is smaller than $k+\ell$. This means that a $q < k+\ell$ exists such that $\Sigma\big((k+\ell)N\big) \le q\,\Sigma(N)$ holds for all $N \ge 1$. □

The recursion derived in Lemma 1 can now be used to show the absence of pure point components (by Wiener's criterion, which will rely on Lemma 2) as well as that of absolutely continuous components (by the Riemann-Lebesgue lemma, which will rely on the special relation $\eta\big((k+\ell)m\big) = \eta(m)$ from Lemma 1), thus establishing that each sequence in this family leads to a signed Dirac comb with purely singular continuous diffraction.

Theorem 1: *Let $k, \ell \in \mathbb{N}$. The diffraction measure of the gTM substitution $\varrho = \varrho_{k,\ell}$ is the Fourier transform $\widehat{\gamma}$ of the autocorrelation measure γ of Lemma 1. It is the diffraction measure of every element of the hull of weighted Dirac combs for ϱ. Moreover, $\widehat{\gamma}$ is purely singular continuous.*

Proof: Since the statement is clear for $k = \ell = 1$ from Sec. II together with Refs. 32 and 5, let $k+\ell > 2$ be fixed. The corresponding autocorrelation is unique by Lemma 1. Since it is positive definite by construction, its Fourier transform exists,[21] and then applies to each element of the hull again. Since the underlying Dirac comb is supported on $\mathbb{Z}$, we know from Theorem 1 of Ref. 4 that $\widehat{\gamma}$ is $\mathbb{Z}$-periodic, hence it can be written as

$$\widehat{\gamma} = \mu * \delta_{\mathbb{Z}} \quad \text{with} \quad \mu = \widehat{\gamma}|_{[0,1)}. \tag{17}$$

Here, μ is a positive measure on the unit interval (which is a representation of the unit circle here), so that the inverse Fourier transform $\check{\mu}$, by the Herglotz-Bochner theorem, is a (continuous) positive definite function on $\mathbb{Z}$ (viewed as the dual group of the unit circle). Since $\gamma = \check{\mu}\,\delta_{\mathbb{Z}}$ by the convolution theorem together with the Poisson summation formula $\widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$, we see that this function is

$$\eta(m) = \int_0^1 e^{2\pi i m x}\,\mathrm{d}\mu(x) = \check{\mu}(m). \tag{18}$$

Let us now use the recursion for η to infer the spectral nature of μ and thus of $\widehat{\gamma}$.

Lemma 2 implies $\frac{1}{N}\Sigma(N) \xrightarrow{N\to\infty} 0$, which means $\big(\widehat{\gamma}\big)_{\mathrm{pp}} = 0$ by Wiener's criterion;[53] see also Refs. 33 and 43. We thus know that $\widehat{\gamma} = \big(\widehat{\gamma}\big)_{\mathrm{sc}} + \big(\widehat{\gamma}\big)_{\mathrm{ac}}$ is a continuous measure, where the right-hand side is the sum of two positive measures that are mutually orthogonal (in the sense that they are concentrated on disjoint sets). Each is the Fourier transform of a positive definite measure with support $\mathbb{Z}$, hence specified by autocorrelation coefficients η_{sc} and η_{ac} which clearly satisfy $\eta_{\mathrm{sc}}(m) + \eta_{\mathrm{ac}}(m) = \eta(m)$ for all $m \in \mathbb{Z}$. The recursion relations for η imply a corresponding set of functional relations for the non-decreasing and continuous distribution function F defined by $F(x) = \widehat{\gamma}\big([0,x]\big)$ for $0 \le x \le 1$. Due to the orthogonality mentioned above, the same relations have to be satisfied by the ac and sc parts separately. This in turn implies that η_{sc} and η_{ac} must both satisfy the recursion relations of Lemma 1, however, with a yet undetermined value of $\eta_{\mathrm{ac}}(0)$, and $\eta_{\mathrm{sc}}(0) = 1 - \eta_{\mathrm{ac}}(0)$.

The recursion of Lemma 1 with $m = 0$ and $r = 1$ gives

$$\eta_{\mathrm{ac}}(1) = \frac{k+\ell-3}{k+\ell+1}\,\eta_{\mathrm{ac}}(0),$$

while $r = 0$ leads to $\eta_{ac}\big((k+\ell)m\big) = \eta_{ac}(m)$ for all $m \in \mathbb{Z}$. Since we have $\lim_{n \to \infty} \eta_{ac}(n) = 0$ from the Riemann-Lebesgue lemma, compare Ref. 33, we must have $\eta_{ac}(m) = 0$ for all $m > 0$. When $k + \ell > 3$, $\eta_{ac}(1) = 0$ forces $\eta_{ac}(0) = 0$, and then $\eta_{ac}(m) = 0$ for all $m \in \mathbb{Z}$ by the recursion, which means $(\widehat{\gamma})_{ac} = 0$. When $k + \ell = 3$, we have $\eta_{ac}(1) = 0$, but the recursion relation for $m = 0$ and $r = 2$ leads to $\eta_{ac}(2) = -\frac{1}{3}\,\eta_{ac}(0)$, hence again to $\eta_{ac}(0) = 0$ with the same conclusion.

As a consequence, $\eta_{sc}(0) = 1$ and $\eta = \eta_{sc}$. We thus have $\widehat{\gamma} = (\widehat{\gamma})_{sc}$ as claimed. □

Remark 2 (diffraction with general weights): If an arbitrary gTM sequence is given, the diffraction of the associated Dirac comb with general (complex) weights $h_{\pm}$ can be calculated as follows. If h is the function defined by $h(1) = h_+$ and $h(\bar{1}) = h_-$, one has

$$\omega_h := \sum_{n \in \mathbb{Z}} h(w_n)\delta_n = \frac{h_+ + h_-}{2}\,\delta_{\mathbb{Z}} + \frac{h_+ - h_-}{2}\,\omega$$

with the ω from Eq. (13). The autocorrelation of ω_h clearly exists and calculates as

$$\gamma_h = \frac{|h_+ + h_-|^2}{4}\,\delta_{\mathbb{Z}} + \frac{|h_+ - h_-|^2}{4}\,\gamma$$

with γ from (14). This follows from $\widetilde{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$ together with $\delta_{\mathbb{Z}} \circledast \omega = \delta_{\mathbb{Z}} \circledast \widetilde{\omega} = 0$, which is a consequence of the fact that 1 and $\bar{1}$ are equally frequent in all gTM sequences. The diffraction is now obtained as

$$\widehat{\gamma_h} = \frac{|h_+ + h_-|^2}{4}\,\delta_{\mathbb{Z}} + \frac{|h_+ - h_-|^2}{4}\,\widehat{\gamma},$$

by an application of the Poisson summation formula $\widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$. Since $\widehat{\gamma}$ is purely singular continuous, this is a diffraction measure with singular spectrum of mixed type. ◇

This diffraction does not display the full dynamical spectrum of the gTM system, which is a well-known phenomenon from the classic TM system.[25] In the latter case, this is "rectified" by the period doubling system as a topological factor. We will return to this question for the gTM systems in Remark 4.

IV. THE DIFFRACTION MEASURE OF THE GTM SYSTEM

Let us consider the diffraction measure in more detail, which we do via a suitable distribution function F for the (continuous) measure $\widehat{\gamma}$. This is done as follows. First, we define $F(x) = \widehat{\gamma}([0,1]) = \mu([0,x])$ for $x \in [0,1]$. The $\mathbb{Z}$-periodicity of $\widehat{\gamma}$ together with $\mu([0,1]) = \eta(0) = 1$ means that F extends to the entire real line via $F(x+1) = 1 + F(x)$. Moreover, since $\widetilde{\gamma} = \gamma$, we know that $\widehat{\gamma}$ is reflection symmetric. With $F(0) = 0$, this implies $F(-x) = -F(x)$ on $\mathbb{R}$, which is our specification of F in this case.

Proposition 2: Let $k, \ell \in \mathbb{N}$ be fixed. The distribution function F of the corresponding diffraction measure is non-decreasing, continuous, and skew-symmetric. Moreover, it satisfies the relation $F(x+1) = 1 + F(x)$ on the real line. Moreover, it possesses the series expansion,

$$F(x) = x + \sum_{m \geq 1} \frac{\eta(m)}{m\pi}\,\sin(2\pi m x),$$

which converges uniformly on $\mathbb{R}$.

Proof: By construction, F is non-decreasing, and is continuous by Theorem 1. So, $F(x) - x$ defines a 1-periodic continuous function that is skew-symmetric and the difference of two continuous, non-decreasing functions, hence it is of bounded variation. By standard results, see Corollary. 1.4.43 of Ref. 44, it has thus a uniformly converging Fourier series expansion,

$$F(x) - x = \sum_{m=1}^{\infty} b_m\,\sin(2\pi m x).$$

The Fourier coefficient b_m (for $m \in \mathbb{N}$) is

$$b_m = 2 \int_0^1 \sin(2\pi m x) \left(F(x) - x \right) \mathrm{d}x = \frac{1}{m\pi} + 2 \int_0^1 \sin(2\pi m x) \, F(x) \, \mathrm{d}x$$

$$= \frac{1}{m\pi} \int_0^1 \cos(2\pi m x) \, \mathrm{d}F(x) = \frac{1}{m\pi} \int_0^1 e^{2\pi i m x} \, \mathrm{d}F(x) = \frac{\eta(m)}{m\pi}.$$

The first step in the second line follows from integration by parts, while the next is a consequence of the symmetry of $\mathrm{d}F$ together with its periodicity (wherefore the imaginary part of the integral vanishes). Recalling that F (restricted to $[0, 1]$) is the distribution function of the probability measure μ completes the argument. $\qquad\square$

It is interesting that the autocorrelation coefficients occur as Fourier coefficients this way. In preparation of a later result, let us look at this connection more closely. A key observation is that the recursion relations for η from Lemma 1 (which have a unique solution once the initial condition $\eta(0) = 1$ is given, with $|\eta(n)| \leq 1$ for all $n \in \mathbb{Z}$) can also be read as a recursion as follows. Let $\beta \in [-1, 1]^{\mathbb{N}}$ be a sequence and define a mapping Ψ via

$$(\Psi\beta)_{(k+\ell)n+r} := \frac{1}{k+\ell} \begin{cases} \alpha_{k,\ell,r} + \alpha_{k,\ell,k+\ell-r}\,\beta_{n+1}, & \text{if } n = 0 \text{ and } 1 \leq r < k+\ell, \\ (k+\ell)\,\beta_n, & \text{if } n \in \mathbb{N} \text{ and } r = 0, \\ \alpha_{k,\ell,r}\,\beta_n + \alpha_{k,\ell,k+\ell-r}\,\beta_{n+1}, & \text{if } n \in \mathbb{N} \text{ and } 1 \leq r < k+\ell, \end{cases} \qquad (19)$$

which completely defines the sequence $\Psi\beta$. This mapping derives from the recursion for η with positive arguments when $\eta(0) = 1$.

Lemma 3: The mapping Ψ maps $[-1, 1]^{\mathbb{N}}$ into itself, with $\|\Psi\beta\|_\infty \leq \|\beta\|_\infty$. Moreover, for any $\beta^{(0)} \in [-1, 1]^{\mathbb{N}}$, the iteration sequence $\beta^{(N)}$ defined by $\beta^{(N+1)} = \Psi\beta^{(N)}$ for $N \geq 0$ converges pointwise towards $\left(\eta(n) \right)_{n \in \mathbb{N}}$.

Proof: The first claim follows from $|\alpha_{k,\ell,r}| + |\alpha_{k,\ell,k+\ell-r}| \leq k + \ell$, which was used earlier in the proof of Lemma 2, via the triangle inequality. When $r = 0$ or when $k = \ell = r$, one has equality here, so that Ψ is not a contraction on $[-1, 1]^{\mathbb{N}}$ for the supremum norm.

Observe that the iteration for β_1 is closed and reads as

$$\beta_1' = \frac{k+\ell-3}{k+\ell} - \frac{1}{k+\ell}\,\beta_1,$$

which is an affine mapping with Lipschitz constant $\frac{1}{k+\ell}$ and hence a contraction. The iteration for β_1 thus converges exponentially fast (to $\frac{k+\ell-3}{k+\ell+1}$) by Banach's contraction principle.

What happens with the iteration for β_1 determines everything else, because the components β_n with $(k+\ell)^m \leq n < (k+\ell)^{m+1}$ and $m \geq 0$ emerge from β_1 in m steps of the iteration. In particular, the iteration also closes on any finite block with $1 \leq n < (k+\ell)^m$ and fixed $m \in \mathbb{N}$, and shows exponentially fast convergence. Note though that the iteration is only non-expanding as soon as $m > 1$, while Ψ induces an affine mapping with Lipschitz constant,

$$L = \frac{\max\left\{ |\alpha_{k,\ell,k+\ell-r}| \,\big|\, 1 \leq r < k+\ell \right\}}{k+\ell} \leq \frac{k+\ell-1}{k+\ell} < 1,$$

on the components β_n with $1 \leq n < k + \ell$.

Pointwise convergence is now clear, and the limit is the one specified by the original recursion, which proves the claim. $\qquad\square$

The recursion relations for η can also be used to derive a functional equation for the distribution function F. Observe first that

$$
\eta\big((k+\ell)m + r\big) = \int_0^1 e^{2\pi i((k+\ell)m+r)x}\, \mathrm{d}F(x) = \int_0^{k+\ell} e^{2\pi i m x}\, e^{2\pi i \frac{rx}{k+\ell}}\, \mathrm{d}F\big(\tfrac{x}{k+\ell}\big)
$$

$$
= \int_0^1 e^{2\pi i m x}\, e^{2\pi i \frac{rx}{k+\ell}} \sum_{s=0}^{k+\ell-1} e^{2\pi i \frac{rs}{k+\ell}}\, \mathrm{d}F\big(\tfrac{x+s}{k+\ell}\big).
$$

(20)

On the other hand, we know from Lemma 1 that

$$
\eta\big((k+\ell)m + r\big) = \frac{1}{k+\ell}\big(\alpha_{k,\ell,r}\,\eta(m) + \alpha_{k,\ell,k+\ell-r}\,\eta(m+1)\big)
$$

$$
= \int_0^1 e^{2\pi i m x}\, \frac{\alpha_{k,\ell,r} + \alpha_{k,\ell,k+\ell-r}\, e^{2\pi i x}}{k+\ell}\, \mathrm{d}F(x).
$$

(21)

A comparison of (20) and (21) leads to the following result.

Proposition 3: The distribution function F for $k, \ell \in \mathbb{N}$ satisfies the functional equation,

$$
F(x) = \frac{1}{k+\ell}\int_0^{(k+\ell)x} \vartheta\big(\tfrac{y}{k+\ell}\big)\, \mathrm{d}F(y) \quad \text{with} \quad \vartheta(x) = 1 + \tfrac{2}{k+\ell}\sum_{r=1}^{k+\ell-1}\alpha_{k,\ell,r}\, \cos(2\pi r x).
$$

This relation holds for all $x \in \mathbb{R}$, and ϑ is continuous and non-negative.

Proof: Eqs. (20) and (21), which hold for all $m \in \mathbb{Z}$, state the equality of the Fourier coefficients of two 1-periodic Riemann-Stieltjes measures, which must thus be equal (as measures). For all $0 \le r < k + \ell$, we thus have

$$
\sum_{s=0}^{k+\ell-1} e^{2\pi i \frac{rs}{k+\ell}}\, \mathrm{d}F\big(\tfrac{x+s}{k+\ell}\big) = e^{-2\pi i \frac{rx}{k+\ell}}\, \frac{\alpha_{k,\ell,r} + \alpha_{k,\ell,k+\ell-r}\, e^{2\pi i x}}{k+\ell}\, \mathrm{d}F(x).
$$

Fix an integer t with $0 \le t < k + \ell$ and multiply the equation for r on both sides by $\exp\big(-2\pi i \frac{rt}{k+\ell}\big)$. Since $\sum_{r=0}^{k+\ell-1}\exp\big(2\pi i \frac{r(s-t)}{k+\ell}\big) = (k+\ell)\,\delta_{s,t}$, a summation over r followed by a division by $(k + \ell)$ leads to

$$
\mathrm{d}F\big(\tfrac{x+t}{k+\ell}\big) = \frac{1}{k+\ell}\left(1 + \sum_{r=1}^{k+\ell-1}\left(\frac{\alpha_{k,\ell,r}}{k+\ell}\, e^{-2\pi i \frac{r(x+t)}{k+\ell}} + \frac{\alpha_{k,\ell,k+\ell-r}}{k+\ell}\, e^{2\pi i \frac{(k+\ell-r)x-rt}{k+\ell}}\right)\right) \mathrm{d}F(x)
$$

$$
= \frac{1}{k+\ell}\left(1 + \frac{2}{k+\ell}\sum_{r=1}^{k+\ell-1}\alpha_{k,\ell,r}\, \cos\big(2\pi \tfrac{r(x+t)}{k+\ell}\big)\right) \mathrm{d}F(x) = \frac{\vartheta\big(\frac{x+t}{k+\ell}\big)}{k+\ell}\, \mathrm{d}F(x),
$$

which is valid for all $x \in [0, 1)$.

To derive the functional equation, we need to calculate $F(x)$ and hence to integrate the previous relations with an appropriate splitting of the integration region. When $[y]$ and $\{y\}$ denote the integer and the fractional part of y, one finds

$$
F(x) = \int_0^x \mathrm{d}F(y) = \int_0^{\{(k+\ell)x\}} \mathrm{d}F\big(\tfrac{y+[(k+\ell)x]}{k+\ell}\big) + \sum_{t=0}^{[(k+\ell)x]-1}\int_0^1 \mathrm{d}F\big(\tfrac{y+t}{k+\ell}\big),
$$

(22)

which holds for all $x \in [0, 1)$. Observe next that

$$
\int_0^1 \mathrm{d}F\big(\tfrac{y+t}{k+\ell}\big) = \frac{1}{k+\ell}\int_0^1 \vartheta\big(\tfrac{y+t}{k+\ell}\big)\, \mathrm{d}F(y) = \frac{1}{k+\ell}\int_t^{t+1} \vartheta\big(\tfrac{z}{k+\ell}\big)\, \mathrm{d}F(z),
$$

holds for any $0 \le t \le k + \ell$, where we used that $\mathrm{d}F(z - n) = \mathrm{d}F(z)$ for all $n \in \mathbb{Z}$ as a consequence of the relation $F(z + 1) = 1 + F(z)$ for $z \in \mathbb{R}$. Similarly, one obtains

$$\int_0^{\{(k+\ell)x\}} \mathrm{d}F\Big(\tfrac{y+[(k+\ell)x]}{k+\ell}\Big) \;=\; \frac{1}{k+\ell} \int_{[(k+\ell)x]}^{(k+\ell)x} \vartheta\Big(\tfrac{z}{k+\ell}\Big)\, \mathrm{d}F(z)\,.$$

We can now put the pieces in (22) together to obtain the functional equation as claimed, which clearly holds for all $x \in \mathbb{R}$.

The continuity of ϑ is clear. Its non-negativity follows from the functional equation, because we know that F is non-decreasing on $[0, 1]$ (as it is the distribution function of the positive measure μ). If we had $\vartheta(a) < 0$ for some $a \in [0, 1]$, there would be some $\varepsilon > 0$ such that $\vartheta(y) < -\varepsilon$ in a neighbourhood of a, which would produce a contradiction to the monotonicity of F via the functional equation.

Remark 3 (properties of the integration kernel): The non-negative function ϑ of Proposition 3 has various interesting and useful properties. Among them are the normalisation relations,

$$\int_0^1 \vartheta(x)\,\mathrm{d}x \;=\; 1 \quad \text{and} \quad \int_0^{\frac{1}{2}} \vartheta(x)\,\mathrm{d}x \;=\; \frac{1}{2},$$

which follow from $\int_0^1 \cos(2\pi r x)\,\mathrm{d}x = 0$ for $r \ne 0$ together with the 1-periodicity of ϑ and its symmetry (whence we also have $\vartheta(1 - x) = \vartheta(x)$). Another is the bound,

$$\|\vartheta\|_\infty \;\le\; q\,,$$

where q is the number from Lemma 2 (when $k + \ell > 2$) or $q = 2$ (when $k = \ell = 1$). This bound is proved by another use of Eq. (16) and the lines following it. $\diamond$

The functional equation of Proposition 3 provides the basis for the calculation of F by a Volterra-type iteration. To this end, one defines $F_0(x) = x$ (so that $\mathrm{d}F_0(x) = \mathrm{d}x$) together with the recursion

$$F_{n+1}(x) \;=\; \frac{1}{k+\ell} \int_0^{(k+\ell)x} \vartheta\Big(\tfrac{y}{k+\ell}\Big)\, \mathrm{d}F_n(y) \tag{23}$$

for $n \ge 0$. It is clear that each F_n defines an absolutely continuous Riemann-Stieltjes measure, so that one can define densities f_n via $\mathrm{d}F_n(x) = f_n(x)\,\mathrm{d}x$. This gives

$$\int_0^x f_{n+1}(z)\,\mathrm{d}z \;=\; F_{n+1}(x) \;=\; \int_0^x \vartheta(z)\, f_n\big((k+\ell)z\big)\,\mathrm{d}z\,,$$

which results in $f_{n+1}(z) = \vartheta(z)\, f_n\big((k+\ell)z\big)$, and hence in the continuous function,

$$f_n(z) \;=\; \prod_{j=0}^{n-1} \vartheta\big((k+\ell)^j z\big). \tag{24}$$

To put this iteration into perspective, let us introduce the space D of non-decreasing and continuous real-valued functions G on $\mathbb{R}$ that satisfy $G(-x) = -G(x)$ and $G(x + 1) = 1 + G(x)$ for all $x \in \mathbb{R}$. In particular, this implies $G(q) = q$ for all $q \in \frac{1}{2}\mathbb{Z}$. We equip this space with the $\|.\|_\infty$-norm, and thus with the topology induced by uniform convergence, in which the space is closed and complete. Each $G \in D$ defines a positive Riemann-Stieltjes measure on $\mathbb{R}$ that is reflection symmetric and 1-periodic. Also, $G(x) - x$ always defines a continuous, skew-symmetric and 1-periodic function of bounded variation. Our distribution functions F from above are elements of D.

Let $k, \ell \in \mathbb{N}$ be fixed. Define a mapping Φ by $G \mapsto \Phi G$, where

$$\Phi G(x) \;=\; \frac{1}{k+\ell} \int_0^{(k+\ell)x} \vartheta\Big(\tfrac{y}{k+\ell}\Big)\, \mathrm{d}G(y)\,. \tag{25}$$

Clearly, our previous iteration (23) can now be written as $F_{n+1} = \Phi F_n$ with the initial condition $F_0(x) = x$, where $F_0 \in D$.

Lemma 4: The operator Φ maps D into itself. Moreover, for arbitrary $F^{(0)} \in D$, the iteration sequence defined by $F^{(N+1)} := \Phi F^{(N)}$ for $N \geq 0$ converges uniformly to the continuous distribution function F of Proposition 2.

Proof: Let $G \in D$. It is clear that ΦG is again continuous (since ϑ and G are continuous) and non-decreasing (since ϑ is non-negative by Proposition 3). The skew-symmetry follows from $dG(-y) = -dG(y)$ by a simple calculation. Finally, one has

$$\Phi G(x+1) = \frac{1}{k+\ell} \int_0^{(k+\ell)(x+1)} \vartheta\left(\tfrac{y}{k+\ell}\right) dG(y) = \Phi G(x) + I,$$

where the remaining integral I, using ϑ from Proposition 3 and the periodicity of dG, is

$$I = \frac{1}{k+\ell} \int_{(k+\ell)x}^{(k+\ell)x+(k+\ell)} \vartheta\left(\tfrac{y}{k+\ell}\right) dG(y) = \frac{1}{k+\ell} \int_0^{k+\ell} \vartheta\left(\tfrac{y}{k+\ell}\right) dG(y)$$

$$= \frac{1}{k+\ell} \sum_{t=0}^{k+\ell-1} \int_t^{t+1} \vartheta\left(\tfrac{y}{k+\ell}\right) dG(y) = \frac{1}{k+\ell} \sum_{t=0}^{k+\ell-1} \int_0^1 \vartheta\left(\tfrac{y+t}{k+\ell}\right) dG(y)$$

$$= 1 + \frac{2}{(k+\ell)^2} \sum_{r=1}^{k+\ell-1} \alpha_{k,\ell,r} \int_0^1 \sum_{t=0}^{k+\ell-r} \cos\left(2\pi \tfrac{ry+rt}{k+\ell}\right) dG(y) = 1.$$

The last step follows because the sum under the integral vanishes as a consequence of the relation $\sum_{t=0}^{m-1} \exp\left(2\pi i \, \tfrac{rt}{m}\right) = 0$ for $1 \leq r < m$.

To establish the convergence, let us first show pointwise convergence $F^{(N)}(x) \xrightarrow{N\to\infty} F(x)$ for all $x \in \mathbb{R}$. Observe that $F^{(N)}(x) - x$ defines a 1-periodic, continuous function on $\mathbb{R}$ for each $N \in \mathbb{N}_0$, and so does $F(x) - x$. Their restrictions to $[0, 1]$ define regular (signed) measures on the unit circle, $\nu^{(N)}$ and ν say, with Fourier-Stieltjes coefficients $a_n^{(N)}$ and a_n, where $n \in \mathbb{Z}$. The latter coefficients follow from Proposition 2, and the former from the observation that every $G \in D$ has a uniformly converging Fourier series of the form

$$G(x) = x + \sum_{n=1}^\infty \frac{\beta_n}{n\pi} \sin(2\pi n x),$$

where $|\beta_n| \leq 1$ by an application of Theorem II.4.12 of Ref. 56. Here, we have $\beta^{(N+1)} = \Psi \beta^{(N)}$ with the mapping Ψ of Lemma 3, with initial condition $\beta^{(0)}$ and convergence $\beta_n^{(N)} \xrightarrow{N\to\infty} \eta(n)$ for each $n \in \mathbb{N}$. Consequently, $a_n^{(N)} \xrightarrow{N\to\infty} a_n$ for all $n \in \mathbb{Z}$, which means that $\nu^{(N)}(p) \xrightarrow{N\to\infty} \nu(p)$ for any trigonometric polynomial p with period 1, and hence (by an application of the Stone-Weierstrass theorem[46]) for all continuous functions on $[0, 1]$. This proves weak convergence $\nu^{(N)} \xrightarrow{N\to\infty} \nu$ on the unit circle. Since all measures are absolutely continuous, with continuous Radon-Nikodym densities, this implies pointwise convergence $F^{(N)}(x) - x \xrightarrow{N\to\infty} F(x) - x$ for all $x \in [0, 1]$, and hence (by periodicity) the pointwise convergence on $\mathbb{R}$ claimed above.

Uniform convergence on $\mathbb{R}$ now follows from that on $[0, 1]$, which can be shown via the "stepping-stone" argument from the proof of Theorem. 30.13 of Ref. 19. For completeness, we spell out the details. Given $\varepsilon > 0$, there are numbers $m \in \mathbb{N}$ and $0 = x_0 < x_1 < \cdots < x_m = 1$ with

$$|F(x_i) - F(x_{i-1})| = F(x_i) - F(x_{i-1}) < \varepsilon$$

for $1 \leq i \leq m$. Also, for all sufficiently large $N \in \mathbb{N}$, one has
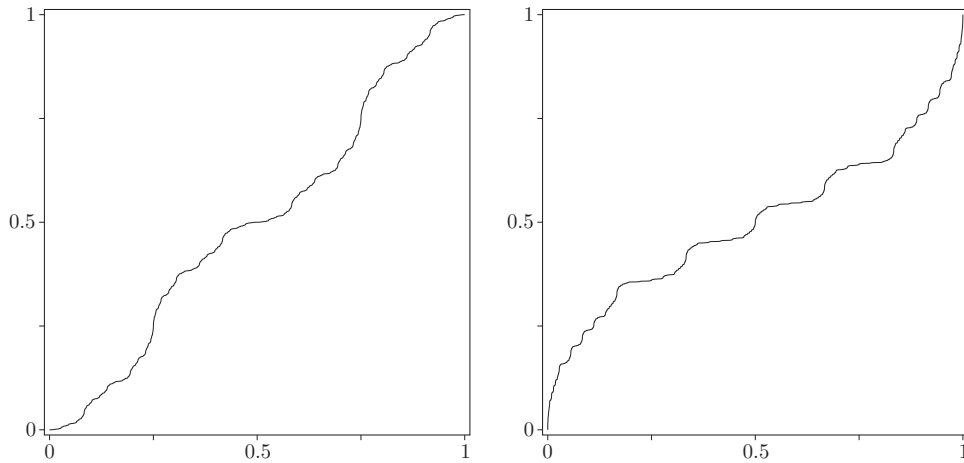
$$|F^{(N)}(x_i) - F(x_i)| < \varepsilon$$

FIG. 3. The continuous and strictly increasing distribution functions of the generalised Thue-Morse measures on [0, 1] for parameters $(k, \ell) = (2, 1)$ (left) and $(5, 1)$ (right).

for all $0 \leq i \leq m$. Since $F^{(N)}(1) = F(1) = 1$, consider an arbitrary $x \in [0, 1)$, so that $x_{i-1} \leq x < x_i$ for precisely one $i \in \{1, 2, \ldots, m\}$. Using monotonicity, this implies the inequalities

$$F(x_{i-1}) \; \leq \; F(x) \; \leq \; F(x_i) \; < \; F(x_{i-1}) + \varepsilon$$

and

$$F(x_{i-1}) - \varepsilon \; < \; F^{(N)}(x_{i-1}) \; \leq \; F^{(N)}(x) \; \leq \; F^{(N)}(x_i) \; < \; F(x_i) + \varepsilon \; < \; F(x_{i-1}) + 2\varepsilon.$$

Together, they give $|F^{(N)}(x) - F(x)| < 2\varepsilon$, which holds for all $x \in [0, 1]$, and then uniformly for all $x \in \mathbb{R}$, as $F^{(N)} - F$ is 1-periodic for all $N \in \mathbb{N}_0$.

Due to our convergence results, Eq. (24) means that the measure $\widehat{\gamma}$ has a (vaguely convergent) representation as the infinite *Riesz product* $\prod_{n \geq 0} \vartheta\big((k + \ell)^n x\big)$. The entire analysis is thus completely analogous to that of the original TM sequence and shows that the latter is a typical example in an infinite family. Two further cases are illustrated in Figure 3.

Remark 4 (pure point factors): The block map (8) applies to any member of the gTM family, and always gives a 2-to-1 cover of the hull $\mathbb{X}^{\mathrm{pd}}_{k,\ell}$ of the *generalised period doubling* (gpd) substitution,

$$\varrho' = \varrho'_{k,\ell}: \quad a \mapsto b^{k-1}ab^{\ell-1}b, \quad b \mapsto b^{k-1}ab^{\ell-1}a. \tag{26}$$

Since we always have a coincidence (at the k-th position) in the sense of Dekking,[24] they all define systems with pure point spectrum (which can be described as model sets in the spirit of Refs. 16 and 15) – another analogy to the classic case $k = \ell = 1$. Also, for given $k, \ell \in \mathbb{N}$, the dynamical system $(\mathbb{X}^{\mathrm{pd}}_{k,\ell}, \mathbb{Z})$ is a topological factor of $(\mathbb{X}^{\mathrm{TM}}_{k,\ell}, \mathbb{Z})$.

Moreover, the dynamical spectrum of the gTM system contains $\mathbb{Z}[\frac{1}{k+\ell}]$ as its pure point part, which happens to be the entire spectrum of the gpd system. The latter fact can be derived from the support of the gpd diffraction measure, via the general correspondence between the dynamical and the diffraction spectrum for pure point systems.[37,12] The detailed calculations can be done in analogy to the treatment of the period doubling system in Refs. 15 and 7, after an explicit formulation of the one-sided fixed point of the gpd substitution, which results in the Fourier module $\mathbb{Z}[\frac{1}{k+\ell}]$. Consequently, the gpd system is a topological factor of the gTM system with maximal pure point spectrum.

The gpd system can be described as a model set (with suitable p-adic type internal space). From Ref. 13, we know that there exists an almost everywhere 1-to-1 "torus" parameterisation via a solenoid $\mathbb{S}_{k+\ell}$. Here, the inflation acts as multiplication by $k + \ell$. In fact, the solenoid provides the maximum equicontinuous (or Kronecker) factor of the gTM system. $\diamond$

V. TOPOLOGICAL INVARIANTS

The formulation via Dirac combs embeds the symbolic sequences into the class of translation bounded measures on $\mathbb{R}$, as described in a general setting in Ref. 12. It is thus natural to (also) consider the continuous counterpart of the discrete hull $\mathbb{X}$ in the form

$$\mathbb{Y} = \overline{\{\delta_t * \omega \mid t \in \mathbb{R}\}},$$

where ω is an arbitrary element of $\mathbb{X}$ (which is always minimal in our situation) and the closure is now taken in the vague topology. Here, $\mathbb{Y}$ is compact, and $(\mathbb{Y}, \mathbb{R})$ is a topological dynamical system. Note that the continuous tiling hull mentioned earlier is topologically conjugate, wherefore we use the same symbol for both versions. The discrete hull $\mathbb{X}$, in this formulation, is homeomorphic with the compact set

$$\mathbb{Y}_0 = \{\nu \in \mathbb{Y} \mid \nu(\{0\}) \neq 0\} \subset \mathbb{Y},$$

while $(\mathbb{Y}, \mathbb{R})$ is the suspension of the system $(\mathbb{Y}_0, \mathbb{Z})$.

We are now going to construct the continuous hulls of the generalised Thue-Morse and period doubling sequences as inverse limits of the inflation map acting on an AP-complex,[3] and use this construction to compute their Čech cohomology. At first sight, there are infinitely many cases to be considered. We note, however, that the AP complex Γ depends only on the atlas of all r-patterns occurring in the tiling, for some bounded r. In the original AP construction,[3] Γ consists of 3-patterns; tiles with one collar tile on the left and one on the right. How these collared tiles are glued together in the complex is then completely determined by the set of 4-patterns occurring in the tiling. Therefore, there are only finitely many different AP complexes Γ to be considered.

For obvious reasons, we want to extend the factor map $\phi \colon \mathbb{Y}_{k,\ell}^{\mathrm{TM}} \to \mathbb{Y}_{k,\ell}^{\mathrm{pd}}$ between the continuous hulls to the cell complexes approximating them, compare (10). We therefore choose the cell complex $\Gamma_{k,\ell}^{\mathrm{TM}}$ to consist of tiles with one extra layer of collar on the right, compared to the cell complex $\Gamma_{k,\ell}^{\mathrm{pd}}$, so that we obtain, for each pair (k, ℓ), a well-defined factor map $\phi \colon \Gamma_{k,\ell}^{\mathrm{TM}} \to \Gamma_{k,\ell}^{\mathrm{pd}}$, which we denote by the same symbol.

While the extra collar seems to complicate things, we can compensate this by a simplification compared to the original AP construction. In Ref. 29, it was shown that a computation using a complex with one-sided collars already yields the correct cohomology groups, even though the inverse limit is not necessarily homeomorphic to the continuous hull of the tiling. The minimal setup therefore consists of AP complexes $\Gamma_{k,\ell}^{\mathrm{TM}}$ with tiles having two-sided collars, and complexes $\Gamma_{k,\ell}^{\mathrm{pd}}$ having tiles with left collars only. All collars have thickness one. A straightforward analysis shows that only three cases have to be distinguished: $k = \ell = 1$ (the classical case), either $k = 1$ or $\ell = 1$ (but not both), and $k, \ell \geq 2$. The complexes $\Gamma_{k,\ell}^{\mathrm{TM}}$ are all sub-complexes of the complex shown in Figure 4. If $k = \ell = 1$, the two loops on the left and right have to be omitted, because the corresponding patterns 111 and $\bar{1}\bar{1}\bar{1}$ do not occur in the classic TM sequence. Likewise, in the case $k, \ell \geq 2$, the lens in the centre has to be omitted, whereas the full complex has to be used in all remaining cases.

For the gpd sequences, an approximant complex that includes all different cases as sub-complexes can be constructed in a similar way. This complex is shown in Figure 5. As we have labelled the edges in Figure 4 with both the 3-patterns in the gTM sequence and the corresponding
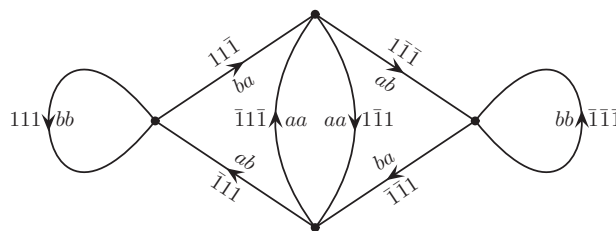

FIG. 4. The universal complex for the gTM system. Edges are labelled both with the corresponding collared tile of the gTM tiling and its image in the gpd tiling.
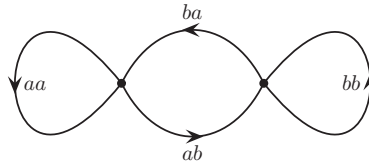
FIG. 5. The universal complex for the gpd system. Its edges correspond to tiles with one-sided (left) collar.

image under ϕ in the gpd sequence, it is easy to see how the factor map ϕ maps the complex of Figure 4 to that of Figure 5: the loops on the left and right of Figure 4 are both mapped onto the right loop of Figure 5, the central lens in Figure 4 is wrapped twice around the left loop in Figure 5, and the rhombus in Figure 4 is wrapped twice around the central loop in Figure 5. Therefore, $\Gamma_{k,\ell}^{\mathrm{pd}}$ consists of the left and the central loop of Figure 5 in the case $k = \ell = 1$ (the classical case), the central and the right loop of Figure 5 in the case $k, \ell \geq 2$, and of all three loops in the remaining cases. It is obvious that the map ϕ is uniformly 2-to-1 also at the level of the approximant complexes.

The cohomology of the continuous hulls $\mathbb{Y}_{k,\ell}^{\mathrm{TM}}$ and $\mathbb{Y}_{k,\ell}^{\mathrm{pd}}$ is now given by the direct limit of the induced maps $\rho_{k,\ell}^*$ and $\rho_{k,\ell}'^*$ on the cohomology of the AP complexes, $H^*(\Gamma_{k,\ell}^{\mathrm{TM}})$ and $H^*(\Gamma_{k,\ell}^{\mathrm{pd}})$. Although there are only three different cell complexes in each case, there is an infinite family of maps $\rho_{k,\ell}^*$, respectively $\rho_{k,\ell}'^*$, but these can be parameterised by k and ℓ.

We first define a basis of the homology of the full complexes of Figures 4 and 5, subsets of which will be used for all k and ℓ, and pass later to the corresponding dual basis for the cohomology. The homology of the full complex $\Gamma_{k,\ell}^{\mathrm{TM}}$ is generated by the basis

$$c_1^{\mathrm{TM}} = 1\bar{1}1, \quad c_2^{\mathrm{TM}} = \bar{1}1\bar{1}, \quad c_3^{\mathrm{TM}} = 1\bar{1}\bar{1}1, \quad c_4^{\mathrm{TM}} = 1, \quad c_5^{\mathrm{TM}} = \bar{1},$$

where the words on the right have to be thought of as being repeated indefinitely. Likewise, the homology of $\Gamma_{k,\ell}^{\mathrm{pd}}$ is generated by the cycles,

$$c_1^{\mathrm{pd}} = a, \quad c_2^{\mathrm{pd}} = ab, \quad c_3^{\mathrm{pd}} = b.$$

Clearly, we have

$$\phi_*(c_1^{\mathrm{TM}}) = c_1^{\mathrm{pd}} + c_2^{\mathrm{pd}}, \quad \phi_*(c_2^{\mathrm{TM}}) = c_1^{\mathrm{pd}} + c_2^{\mathrm{pd}}, \quad \phi_*(c_3^{\mathrm{TM}}) = 2c_2^{\mathrm{pd}},$$
$$\phi_*(c_4^{\mathrm{TM}}) = c_3^{\mathrm{pd}}, \quad \phi_*(c_5^{\mathrm{TM}}) = c_3^{\mathrm{pd}}.$$

Here, the lower asterisk in ϕ_* denotes the induced action on homology, whereas ϕ^* denotes the action on cohomology.

In order to go into more detail, we have to distinguish the three different cases. We begin with the classical case, $k = \ell = 1$. Here, the relevant basis elements are c_1^{TM}, c_2^{TM}, and c_3^{TM} for TM, and c_1^{pd} and c_2^{pd} for pd. On these, the substitution acts as

$$\rho_*(c_1^{\mathrm{TM}}) = c_1^{\mathrm{TM}} + c_2^{\mathrm{TM}}, \quad \rho_*(c_2^{\mathrm{TM}}) = c_1^{\mathrm{TM}} + c_2^{\mathrm{TM}}, \quad \rho_*(c_3^{\mathrm{TM}}) = 2(c_1^{\mathrm{TM}} + c_2^{\mathrm{TM}}) - c_3^{\mathrm{TM}},$$
$$\rho_*'(c_1^{\mathrm{pd}}) = c_2^{\mathrm{pd}}, \quad \rho_*'(c_2^{\mathrm{pd}}) = 2c_1^{\mathrm{pd}} + c_2^{\mathrm{pd}},$$

where ρ_* and ρ_*' again denote the induced action on homology. If we express these maps as matrices A^{TM} and A^{pd} acting from the left on column vectors, with respect to the basis above, and likewise define a matrix P for the action of ϕ_*, we obtain

$$A^{\mathrm{pd}} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}, \quad A^{\mathrm{TM}} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}. \tag{27}$$

The corresponding action on the dual basis of cohomology is simply given by the transposed matrices, or by the same matrices considered as acting from the right on row vectors. We adopt the latter viewpoint here. It is easy to verify that the matrices (27) satisfy $A^{\mathrm{pd}}P = PA^{\mathrm{TM}}$. In other words,

the substitution action commutes with that of ϕ. A^{pd} and A^{TM} have left eigenvectors and eigenvalues

$$\begin{pmatrix} 1 & 2 & | & 2 \\ 1 & -1 & | & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -1 \\ 1 & -1 & 0 & | & 0 \end{pmatrix}. \tag{28}$$

Next, we look at the case $k, \ell \geq 2$, which is still relatively simple. Here, we have to work with the basis elements c_3^{TM}, c_4^{TM}, and c_5^{TM} for gTM, and c_2^{pd} and c_3^{pd} for gpd. The action of the substitution is then given by

$$\begin{aligned}
\rho_*(c_3^{\text{TM}}) &= 3c_3^{\text{TM}} + (2(k+\ell) - 6)(c_4^{\text{TM}} + c_5^{\text{TM}}), \\
\rho_*(c_4^{\text{TM}}) &= c_3^{\text{TM}} + (k-2)c_4^{\text{TM}} + (\ell-2)c_5^{\text{TM}}, \\
\rho_*(c_5^{\text{TM}}) &= c_3^{\text{TM}} + (\ell-2)c_4^{\text{TM}} + (k-2)c_5^{\text{TM}}, \\
\rho'_*(c_2^{\text{pd}}) &= 3c_2^{\text{pd}} + 2(k+\ell-3)c_3^{\text{pd}}, \\
\rho'_*(c_3^{\text{pd}}) &= 2c_2^{\text{pd}} + (k+\ell-4)c_3^{\text{pd}}.
\end{aligned}$$

The corresponding matrices $A_{k,\ell}^{\text{pd}}$, $A_{k,\ell}^{\text{TM}}$, and $P_{k,\ell}$ read as

$$A_{k,\ell}^{\text{pd}} = \begin{pmatrix} 3 & 2 \\ 2(k+\ell-3) & k+\ell-4 \end{pmatrix}, \; A_{k,\ell}^{\text{TM}} = \begin{pmatrix} 3 & 1 & 1 \\ 2(k+\ell)-6 & k-2 & \ell-2 \\ 2(k+\ell)-6 & \ell-2 & k-2 \end{pmatrix}, \; P_{k,\ell} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and satisfy $A_{k,\ell}^{\text{pd}} P_{k,\ell} = P_{k,\ell} A_{k,\ell}^{\text{TM}}$. The matrices $A_{k,\ell}^{\text{pd}}$ and $A_{k,\ell}^{\text{TM}}$ have left eigenvectors and eigenvalues,

$$\begin{pmatrix} 2 & 1 & | & k+\ell \\ k+\ell-3 & -2 & | & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 1 & 1 & | & k+\ell \\ k+\ell-3 & -1 & -1 & | & -1 \\ 0 & 1 & -1 & | & k-\ell \end{pmatrix}. \tag{29}$$

Finally, the case where $\min(k, \ell) = 1$, but $k + \ell > 2$, requires the full AP complex. For notational ease, we consider the case $k = 1$, $\ell \geq 2$. The case $\ell = 1$, $k \geq 2$ is completely analogous. The substitution then acts as follows on our basis:

$$\begin{aligned}
\rho_*(c_1^{\text{TM}}) &= c_2^{\text{TM}} + c_3^{\text{TM}} + (\ell-1)c_4^{\text{TM}} + (2\ell-3)c_5^{\text{TM}}, \\
\rho_*(c_2^{\text{TM}}) &= c_1^{\text{TM}} + c_3^{\text{TM}} + (\ell-1)c_5^{\text{TM}} + (2\ell-3)c_4^{\text{TM}}, \\
\rho_*(c_3^{\text{TM}}) &= c_1^{\text{TM}} + c_2^{\text{TM}} + c_3^{\text{TM}} + (2\ell-3)(c_4^{\text{TM}} + c_5^{\text{TM}}), \\
\rho_*(c_4^{\text{TM}}) &= c_2^{\text{TM}} + (\ell-2)c_5^{\text{TM}}, \\
\rho_*(c_5^{\text{TM}}) &= c_1^{\text{TM}} + (\ell-2)c_4^{\text{TM}}, \\
\rho'_*(c_1^{\text{pd}}) &= c_2^{\text{pd}} + (\ell-1)c_3^{\text{pd}}, \\
\rho'_*(c_2^{\text{pd}}) &= c_1^{\text{pd}} + 2c_2^{\text{pd}} + (2\ell-3)c_3^{\text{pd}}, \\
\rho'_*(c_3^{\text{pd}}) &= c_1^{\text{pd}} + c_2^{\text{pd}} + (\ell-2)c_3^{\text{pd}}.
\end{aligned}$$

The corresponding matrices are

$$
A_{k,\ell}^{\mathrm{pd}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ \ell-1 & 2\ell-3 & \ell-2 \end{pmatrix}, \quad
A_{k,\ell}^{\mathrm{TM}} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \ell-1 & 2\ell-3 & 2\ell-3 & 0 & \ell-2 \\ 2\ell-3 & \ell-1 & 2\ell-3 & \ell-2 & 0 \end{pmatrix},
$$

$$
P_{k,\ell} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.
$$

In this case, $A_{k,\ell}^{\mathrm{TM}}$ and $A_{k,\ell}^{\mathrm{pg}}$ have left eigenvectors and eigenvalues

$$
\begin{pmatrix} 1 & 2 & 1 & | & \ell+1 \\ \ell & \ell-2 & -2 & | & -1 \\ 1 & 1-\ell & 1 & | & 0 \end{pmatrix} \quad \text{and} \quad
\begin{pmatrix} 3 & 3 & 4 & 1 & 1 & | & \ell+1 \\ 1-\ell & 1-\ell & 2-\ell & 1 & 1 & | & -1 \\ 1 & -1 & 0 & 1 & -1 & | & 1-\ell \\ 2-\ell & 2-\ell & 2-2\ell & 1 & 1 & | & 0 \\ 2-\ell & \ell-2 & 0 & 1 & -1 & | & 0 \end{pmatrix}. \tag{30}
$$

As the cohomology of the continuous hull is given by the direct limit of the action of the substitution on the cohomology of the AP approximant complexes, it is determined by the non-zero eigenvalues of the matrices $A_{k,\ell}^{\mathrm{TM}}$ and $A_{k,\ell}^{\mathrm{pd}}$. For all k and ℓ, these eigenvalues are $k+\ell$ and -1 for the gpd case, and $k+\ell$, -1, and $k-\ell$ in the gTM case, where the last eigenvalue is relevant only if $k\neq\ell$. Since $H^0(\Gamma) = \mathbb{Z}$ if Γ is connected, and the substitution action on $H^0(\Gamma)$ is trivial, with eigenvalue 1, we can summarise our cohomology results as follows.

Theorem 2: *The Čech cohomology of the continuous hull of the gTM sequences is given by* $H^0(\mathbb{Y}_{k,\ell}^{\mathrm{TM}}) = \mathbb{Z}$, *and*

$$
H^1(\mathbb{Y}_{k,\ell}^{\mathrm{TM}}) = \begin{cases} \mathbb{Z}[\frac{1}{k+\ell}] \oplus \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{|k-\ell|}], & \text{if } |k-\ell| \geq 2, \\ \mathbb{Z}[\frac{1}{k+\ell}] \oplus \mathbb{Z}^2, & \text{if } |k-\ell| = 1, \\ \mathbb{Z}[\frac{1}{k+\ell}] \oplus \mathbb{Z}, & \text{if } k = \ell. \end{cases}
$$

The Čech cohomology of the continuous hull of the generalised period doubling sequences is given by

$$
H^0(\mathbb{Y}_{k,\ell}^{\mathrm{pd}}) = \mathbb{Z}, \quad H^1(\mathbb{Y}_{k,\ell}^{\mathrm{pd}}) = \mathbb{Z}[\tfrac{1}{k+\ell}] \oplus \mathbb{Z},
$$

valid for any pair $k, \ell \in \mathbb{N}$. $\qquad\square$

Since the cohomology of the hulls is given by the direct limit of the columns of the diagram

$$
\begin{array}{ccccc}
H^k(\Gamma_{k,\ell}^{\mathrm{sol}}) & \xrightarrow{\psi^*} & H^k(\Gamma_{k,\ell}^{\mathrm{pd}}) & \xrightarrow{\phi^*} & H^k(\Gamma_{k,\ell}^{\mathrm{TM}}) \\
\times(k+\ell)\downarrow & & \varrho'^*\downarrow & & \downarrow\varrho^* \\
H^k(\Gamma_{k,\ell}^{\mathrm{sol}}) & \xrightarrow{\psi^*} & H^k(\Gamma_{k,\ell}^{\mathrm{pd}}) & \xrightarrow{\phi^*} & H^k(\Gamma_{k,\ell}^{\mathrm{TM}})
\end{array} \tag{31}
$$

(compare (10)), we now have access also to the homomorphisms in the sequence

$$
H^*(\mathbb{S}_{k+\ell}) \xrightarrow{\psi^*} H^*(\mathbb{Y}_{k,\ell}^{\mathrm{pd}}) \xrightarrow{\phi^*} H^*(\mathbb{Y}_{k,\ell}^{\mathrm{TM}}). \tag{32}
$$

For all three spaces, $H^0 = \mathbb{Z}$, and the maps ϕ^* and ψ^* acting on them are isomorphisms. Further, it is easy to see that ψ^* embeds $H^1(\mathbb{S}_{k+\ell}) = \mathbb{Z}[\frac{1}{k+\ell}]$ non-divisibly in $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{pd}})$; it is simply mapped isomorphically onto the summand $\mathbb{Z}[\frac{1}{k+\ell}]$ in $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{pd}})$. The same cannot be said of the second map, however. For all pairs k, ℓ, the matrix P maps the eigenvector of $A_{k,\ell}^{\mathrm{pd}}$ with eigenvalue -1 to

twice the corresponding eigenvector of $A_{k,\ell}^{\mathrm{TM}}$. As a result, the quotient $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{TM}})/\rho^*(H^1(\mathbb{Y}_{k,\ell}^{\mathrm{pd}}))$ has a torsion component $\mathbb{Z}_2$; the summand $\mathbb{Z}$ of $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{pd}})$ coming from the eigenvalue -1 is mapped to $2\mathbb{Z}$ in $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{TM}})$. This result was already known for the classic period doubling and Thue-Morse sequences,[18] and extends to the generalised sequences. Therefore, in the case $k = \ell$, where the cohomologies of gTM and gpd are isomorphic as groups, one should rather consider $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{pd}})$ as subgroup of index 2 of $H^1(\mathbb{Y}_{k,\ell}^{\mathrm{TM}})$.

VI. DYNAMICAL ZETA FUNCTIONS

The continuous hull permits to employ the Anderson-Putnam method[3] for the calculation of the dynamical zeta function of the *inflation* action on $\mathbb{Y}$. The dynamical zeta function[48] of a substitution can be viewed as a generating function for the number of fixed points $a(n)$ under n-fold substitution via

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} z^n\right). \tag{33}$$

Knowing the fixed point counts $a(n)$, one can calculate the number $c(n)$ of cycles of length n from the formula

$$c(n) = \frac{1}{n} \sum_{d|n} \mu\left(\tfrac{n}{d}\right) a(d),$$

where μ is the Möbius function from elementary number theory (and should not be confused with the measure μ that appeared above).

Anderson and Putnam[3] showed how the dynamical zeta function of a substitution tiling can be expressed by the action of the substitution on the cohomology of the AP complex Γ. In the one-dimensional case, the zeta function is given by

$$\zeta(z) = \frac{\det(\mathbb{1} - zA^0)}{\det(\mathbb{1} - zA^1)},$$

where A^k is the matrix of the substitution action on $H^k(\Gamma)$. In our case, A^0 is a 1×1 unit matrix, and A^1 is diagonalisable, so that we can rewrite the zeta function as

$$\zeta(z) = \frac{1 - z}{\prod_i (1 - z\lambda_i)},$$

where λ_i are the eigenvalues of A^1. For the gTM and gpd tilings, these eigenvalues have been derived above, so that we arrive (after a simple calculation) at the following theorem.

Theorem 3: *Let $k, \ell \in \mathbb{N}$. The generalised Thue-Morse sequence defined by the inflation $\varrho_{k,\ell}$ of (11) possesses the dynamical zeta function,*

$$\zeta_{k,\ell}^{\mathrm{TM}}(z) = \frac{1 - z}{(1 + z)(1 - (k + \ell)z)(1 - (k - \ell)z)},$$

while the induced generalised period doubling sequence, as defined by $\varrho'_{k,\ell}$ of (26), leads to

$$\zeta_{k,\ell}^{\mathrm{pd}}(z) = \frac{1 - z}{(1 + z)(1 - (k + \ell)z)}.$$

In particular, when $k = \ell$, one has $\zeta^{\mathrm{TM}} = \zeta^{\mathrm{pd}}$.   □

For our two systems, the corresponding fixed point counts, for $n \in \mathbb{N}$, can now be calculated from Eq. (33) to be

$$a_{k,\ell}^{\mathrm{pd}}(n) = (k + \ell)^n - \left(1 - (-1)^n\right) \quad \text{and} \quad a_{k,\ell}^{\mathrm{TM}}(n) = a_{k,\ell}^{\mathrm{pd}}(n) + (k - \ell)^n. \tag{34}$$

It is an interesting exercise to relate the orbits of the two systems according to the action of the mapping ϕ in agreement with these counts.

There is an interesting connection between the inflation on $\mathbb{Y}_{k,\ell}^{\mathrm{TM}}$ or $\mathbb{Y}_{k,\ell}^{\mathrm{pd}}$ and the multiplication by $(k + \ell)$ on the (matching) solenoid $\mathbb{S}_{k+\ell}$, which emerges from the torus parameterisation.[10,50,13] This solenoid is a set on which the multiplication is invertible. It is constructed via a suitable inverse limit structure (under iterated multiplication by $(k + \ell)$), starting from the 1-torus (or unit circle), represented by the unit interval $[0, 1)$ with arithmetic taken mod 1. Together with the above, we obtain the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{Y}_{k,\ell}^{\mathrm{TM}} & \xrightarrow{\phi} & \mathbb{Y}_{k,\ell}^{\mathrm{pd}} & \xrightarrow{\psi} & \mathbb{S}_{k+\ell} \\
\varrho \downarrow & & \varrho' \downarrow & & \downarrow \times(k+\ell) \\
\mathbb{Y}_{k,\ell}^{\mathrm{TM}} & \xrightarrow{\phi} & \mathbb{Y}_{k,\ell}^{\mathrm{pd}} & \xrightarrow{\psi} & \mathbb{S}_{k+\ell}
\end{array}
\tag{35}
$$

where ψ denotes the torus parameterisation for the generalised period doubling sequence, in the spirit of Refs. 50, 13, and 16. The mapping ψ is 1-to-1 almost everywhere. Like for the classic period doubling sequence, it fails to be 1-to-1 on exactly two translation orbits, which are mapped to a single orbit.

Counting *finite* periodic orbits under the multiplication action on the solenoid, however, means that the inverse limit is not needed, so that the corresponding dynamical (or Artin-Mazur) zeta function coincides with that of the toral endomorphism that is represented by multiplication by $m = k + \ell \geq 2$. This, in turn, is given by

$$
\zeta_m^{\mathrm{sol}}(z) = \frac{1 - z}{1 - mz}
$$

by an application of Theorem 1 of Ref. 11. The number of fixed points is given by $a_m^{\mathrm{sol}}(n) = m^n - 1$, where $m = k + \ell$ as before for a comparison with (34).

VII. FURTHER DIRECTIONS

A natural question concerns the robustness of the singular continuous spectrum under simultaneous permutations of positions in $\varrho(1)$ and $\varrho(\bar{1})$, as considered in Ref. 55. For $k = \ell = 1$, the two possible rules are the Thue-Morse rule $(1\bar{1}, \bar{1}1)$ and its partner $(\bar{1}1, 1\bar{1})$, written in obvious shorthand notation. Since the squares of these two rules are equal, they define the same hull, and hence the same autocorrelation.

There are three possible rules for $k = 2$ and $\ell = 1$, namely, $(11\bar{1}, \bar{1}\bar{1}1)$, $(1\bar{1}1, \bar{1}1\bar{1})$, and $(\bar{1}11, 1\bar{1}\bar{1})$. The first is our $\varrho_{2,1}$ from above, while the third result in a fixed point that is the mirror image, and hence possesses the same autocorrelation. The middle rule, however, has the periodic fixed point $\ldots 1\bar{1}1\bar{1}1\bar{1} \ldots$ and hence autocorrelation $\gamma = (\delta_0 - \delta_1) * \delta_{2\mathbb{Z}}$. Its Fourier transform reads as

$$
\widehat{\gamma} = \delta_{\mathbb{Z}+\frac{1}{2}},
$$

which is a periodic pure point measure. By general arguments, one can see that the diffraction measure of the balanced situation (with 1 and $\bar{1}$ being equally frequent) must be of pure type, and cannot be a mixture. On the basis of the results from Refs. 26 and 27, it is then clear that, given a permutation, the diffraction is either pure point or purely singular continuous. It remains an interesting question to decide this explicitly for general k and ℓ, and a general permutation.

Moreover, it is clear that similar structures exist in higher dimensions. Indeed, starting from the treatment of bijective lattice substitution systems in Refs. 26 and 27, it is possible to demonstrate the singular continuous nature for bijective substitutions with trivial height lattice and a binary alphabet, and to calculate it explicitly in terms of Riesz products. Details will be explained in a forthcoming paper.[9]

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