Remarks on a paper by Skornyakov concerning rings for which every module is a direct sum of left ideals

By

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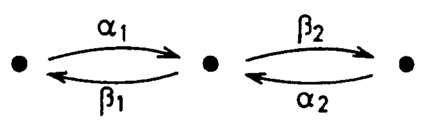
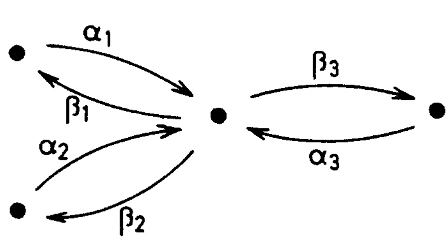
In [5], Skornyakov claims (as his main result) that a ring R is quasi-Frobenius and serial ("generalised uniserial") if and only if every left R-module is a direct sum of left ideals. However, the proof of lemma 4 is incorrect, and there are counter-examples to the theorem itself.

1. Examples. Let k be a field, and R the k-algebra given by the quiver (see [3])



with relations $\alpha_1\beta_2 = \alpha_2\beta_1 = 0$, $\beta_1\alpha_1 = \beta_2\alpha_2$. Then R is quasi-Frobenius, $(\text{rad } R)^3 = 0$. The separated quiver of $R/(\text{rad } R)^2$ is the disjoint union of two quivers of type A_3, namely $\bullet \to \bullet \leftarrow \bullet$ and $\bullet \leftarrow \bullet \to \bullet$, thus any indecomposable module with socle length 2 has multiplicityfree socle (i.e. the composition factors occur with multiplicity ≤ 1 in the socle). Of course, the same is true for the remaining six indecomposable modules. This shows that any indecomposable left module can be embedded into the injective cogenerator ${}_RR$. As a consequence, every left module is the direct sum of left ideals, but R is not serial. — If we consider the k-algebra S given by the quiver



with relations $\alpha_i\beta_j = 0$, for $i \neq j$, and $\beta_1\alpha_1 = \beta_2\alpha_2 = \beta_3\alpha_3$, then we obtain an example of a ring S with two indecomposable left S-modules which are not left ideals, such that, however, every left $M_2(S)$-module is the direct sum of left ideals. (Here, $M_n(R)$ denotes the ring of all $n \times n$-matrices over R).

2. In general. Let R be a quasi-Frobenius ring of finite representation type. Let $S_1, \ldots, S_t$ be the simple left R-modules, $E_1, \ldots, E_t$ their injective hulls. Since ${}_RR$ is injective, ${}_RR = \bigoplus_i E_i^{r_i}$, with $r_i \in \mathbb{N}$, and, since ${}_RR$ is a cogenerator, all $r_i > 0$. If M is a left module with socle $\bigoplus_i S_i^{m_i}$, then M is embeddable into ${}_RR$ if and only if $m_i \leqq r_i$, for all i. Now let $M_1, \ldots, M_q$ be the indecomposable left R-modules. It has been shown in [4] that all M_i are of finite length and that every left R-module is a direct sum of copies of these modules. Let $\bigoplus_i S_i^{m_{ij}}$ be the socle of M_j, and define $s_i = \max_j m_{ij}$. It is now clear that every left R-module is a direct sum of left ideals if and only if $s_i \leqq r_i$ for all i. If we replace R by the matrix ring $M_n(R)$, then the numbers r_i are replaced by nr_i, whereas the numbers s_i are not changed at all. Thus, we see: given a quasi-Frobenius ring R of finite representation type, there exists $n_0 \in \mathbb{N}$ such that for any $n \geqq n_0$, every left $M_n(R)$-module is a direct sum of left ideals.

3. Conversely. Assume every left R-module is a direct sum of left ideals. Then every injective left R-module being embeddable into a free module, has to be projective, thus by a theorem of Faith-Walker [2], R is quasi-Frobenius. Also, the length of the indecomposable left R-modules of finite length is bounded by the length of ${}_RR$, thus by a theorem of Auslander [1], R is of finite representation type.

4. We have seen that the property considered by Skornyakov is not even invariant under Morita equivalence. We note however, that the property is left-right symmetric, and coincides with the property of being a quasi-Frobenius Köthe ring:

Theorem. *Let R be quasi-Frobenius. Then the following properties are equivalent:*

(1) *Every left module is a direct sum of left ideals.*

(2) *Every left module is a direct sum of cyclic modules.*

(1*) *Every right module is a direct sum of right ideals.*

(2*) *Every right module is a direct sum of cyclic modules.*

Proof. (1) $\Rightarrow$ (2). Let M be a left ideal which we may assume to be indecomposable. If M is projective, then M is cyclic. If M is not projective, let $\varepsilon\colon P \to M$ be a minimal projective cover, and K the kernel of ε. Then it is well-known that K is indecomposable and P its injective hull. Since by assumption K is embeddable into ${}_RR$, we see that P is a direct summand of ${}_RR$, thus M is cyclic.

(2) $\Rightarrow$ (1*). Consider the duality $D = \operatorname{Hom}_R(-, {}_RR)$ from left R-modules to right R-modules. Any indecomposable right module of finite length is of the form $D({}_RM)$ for some indecomposable left module ${}_RM$. Now, by assumption, there is a surjective map ${}_RR \to {}_RM$, thus we obtain an injective map $D({}_RM) \to D({}_RR) = R_R$.

References

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