Generalized Brillinger-Like Transforms

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Abstract—We propose novel transforms of stochastic vectors, called the generalized Brillinger transforms (GBT1 and GBT2), which are generalizations of the Brillinger transform (BT). The GBT1 extends the BT to the cases when the covariance matrix and the weighting matrix are singular, and moreover, the weighting matrix is not necessarily symmetric. We show that the GBT1 may computationally be preferable over another related optimal technique, the generic Karhunen–Loève transform (GKLT). The GBT2 generalizes the GBT1 to provide, under the condition we impose, better associated accuracy than that of the GBT1. It is achieved because of the increase in a number of parameters to optimize compared to that in the GBT1.

Index Terms—Brillinger transform (BT), data compression, filtering.

I. INTRODUCTION

Motivation: Recently, the transform of stochastic vectors proposed by Brillinger [1, pp. 368–69, 457] has received considerable attention in signal processing problems (see, e.g., [2]–[7]) as an effective tool for data compression and filtering. The Brillinger transform (BT) F is represented in terms of two matrices, $F = AC$, where C is designated for compression and A for decompression (they are called "compressor" and "decompressor," respectively). We wish to extend the BT to more efficient transforms.

Contribution and novelty: The contribution of this work is threefold. First, in Section II, we extend the BT to a generalized *first-order* BT (GBT1) which is applicable to the cases when the covariance matrix and the weighting matrix are singular, and moreover, the weighting matrix is not necessarily symmetric. The proposed GBT1 is given in terms of pseudoinverse matrices and therefore, it always exists, i.e., it is applicable to cases where the BT is not. This is because BT performs only under the assumption that the covariance matrix and weighting matrix are invertible, and that the weighting matrix is symmetric. These conditions might be restrictive [8] in problems where the BT is applied. We note that the extension of the BT to the GBT1 is not straightforward and requires a new justification based on Facts 1–3 in Section II-A.

Second, although the GBT1 provides the best decompression accuracy, for the same compression ratio, among all transforms of the form $F = AC$ it may happen that the accuracy

is still not satisfactory. In Section III, we propose and justify a new transform called the generalized *second-order* BT (GBT2) which provides, under the condition we impose, better associated accuracy than that of the GBT1.

Third, we show that the GBT1 requires a lower computational load than other related technique, the generic Karhunen–Loève transform (GKLT) [8]–[10]. This issue is elaborated in Section III. Such an analysis provides a better understanding of the two well-known data compression techniques.

A. Truncated SVD

Let the SVD of matrix $M \in \mathbb{R}^{m \times n}$ be given by $M = U_M \Sigma_M V_M^T$, where $U_M = [u_1\ u_2\ \ldots\ u_m] \in \mathbb{R}^{m \times m}, V_M = [v_1\ v_2\ \ldots\ v_n] \in \mathbb{R}^{n \times n}$ are unitary matrices, and $\Sigma_M = \text{diag}(\sigma_1(M), \ldots, \sigma_{\min(m,n)}(M)) \in \mathbb{R}^{m \times n}$ is a generalized diagonal matrix, with the singular values $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq 0$ on the main diagonal. Let

$$P_{M,L} = \sum_{k=1}^{\text{rank}\,(M)} u_k u_k^T \in \mathbb{R}^{m \times m}, \quad P_{M,R} = \sum_{j=1}^{\text{rank}\,(M)} v_j v_j^T \in \mathbb{R}^{n \times n}$$

be the orthogonal projections on the range of M and M^T, respectively, and let

$$[M]_k = \sum_{i=1}^{k} \sigma_i(M) u_i v_i^T \in \mathbb{R}^{m \times n} \qquad (1)$$

for $k = 1, \ldots, \text{rank}\,(M)$, be the truncated SVD of M. For $k > \text{rank}\,(M)$, we define $[M]_k = M\ (= M_{\text{rank}\,(M)})$. For $1 \leq k < \text{rank}\,(M)$, the matrix M_k is uniquely defined if and only if $\sigma_k(M) > \sigma_{k+1}(M)$.

B. Special Notation

Let us write (Ω, Σ, μ) for a probability space.[1] We denote by $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_m)^T \in L^2(\Omega, \mathbb{R}^m)$ the signal of interest[2] (a source signal to be estimated) and by $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^T \in L^2(\Omega, \mathbb{R}^n)$ the observed signal where $\mathbf{y}_j, \mathbf{x}_k \in L^2(\Omega, \mathbb{R})$. Let us write $\mathbb{E}[\|\mathbf{x}\|_2^2] = \int_\Omega \|\mathbf{x}(\omega)\|_2^2 d\mu(\omega)$, where $\|\mathbf{x}(\omega)\|_2$ is the Euclidean norm of $\mathbf{x}(\omega) \in \mathbb{R}^m$. We assume that means $E[\mathbf{y}]$ and $E[\mathbf{x}]$ are known. Therefore, without loss of generality, we will assume henceforth that $\mathbf{y}$ and $\mathbf{x}$ have zero means. Then, $E_{\mathbf{yx}} = E[\mathbf{yx}^T] = \{e_{ij}\}_{i,j=1}^{m,n} \in \mathbb{R}^{m \times n}$, where $e_{ij} = \int_\Omega \mathbf{y}_i(\omega) \mathbf{x}_j(\omega) d\mu(\omega)$.

Furthermore, $M^\dagger$ and $M^{1/2}$ denote the Moore–Penrose pseudoinverse matrix and a matrix square root, respectively, for matrix M. For the covariance matrix $E_{\mathbf{xx}}$, we denote $E_{\mathbf{xx}}^{1/2\dagger} := (E_{\mathbf{xx}}^{1/2})^\dagger$. $E_{\mathbf{xx}}^{1/2\dagger}$ is unique since $E_{\mathbf{xx}}$ is positive semidefinite.

[1] $\Omega = \{\omega\}$ is the set of outcomes, Σ a σ-field of measurable subsets of Ω, $\mu : \Sigma[0, 1]$ an associated probability measure on Σ with $\mu(\Omega) = 1$.

[2] $L^2(\Omega, \mathbb{R}^m)$ is the space of square-integrable functions defined on Ω with values in $\mathbb{R}^m$, i.e., such that $\int_\Omega \|\mathbf{x}(\omega)\|_2^2 d\mu(\omega) < \infty$.

C. Review of the BT

Let $k \leq \min\{m,n\}$, $\Gamma \in \mathbb{R}^{m \times m}$ be a *symmetric invertible* weighting matrix, $A \in \mathbb{R}^{m \times k}$ and $C \in \mathbb{R}^{k \times n}$ be matrices of a decompressor and compressor, respectively. The BT of $\mathbf{x}$ is represented by matrices A and C that solve

$$\min_{A,C} \mathbb{E}\left[\|\Gamma(\mathbf{y} - AC\mathbf{x})\|_2^2\right]. \quad (2)$$

Weighting matrix Γ is used, based on *a priori* information, to place a greater importance on some particular entries of the observed data (in this regard, see, e.g., [11]–[13]).[3] Denote $T = \Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{-1} E_{\mathbf{xy}} \Gamma$ and suppose that the SVD of T is given by $T = U_T \Sigma_T U_T^T$. Under the assumption that $E_{\mathbf{xx}}$ is *invertible*, the BT in [1] is given by

$$C = U_{T,k}^T \Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{-1} \quad \text{and} \quad A = \Gamma^{-1} U_{T,k} \quad (3)$$

where $U_{T,k}$ is formed by the first k columns of U_T.

II. GENERALIZED FIRST-ORDER BT (GBT1)

A. Derivation and Justification of GBT1

Here, we consider the generalized BT (GBT1) which is an extension of the BT to the case in which $E_{\mathbf{xx}}$ and Γ are not invertible, and moreover, Γ is not necessarily symmetric.

To this end, denote $T_x = \Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^\dagger E_{\mathbf{xy}} \Gamma^T$, $U_{T_x} \Sigma_{T_x} U_{T_x}^T = T_x$ is the SVD of T_x, $M_L = I + (I - P_{\Gamma,R})S$ and $M_R = I + R(I - P_{E_{\mathbf{xx}}^{1/2},L})$, where S and R are arbitrary matrices. Furthermore, $\|\cdot\|$ denotes the Frobenius matrix norm.

Theorem 1: The GBT1 is represented by matrices $A_{\text{GB}} \in \mathbb{R}^{m \times k}$ and $C_{\text{GB}} \in \mathbb{R}^{k \times n}$ such that

$$A_{\text{GB}} = M_L \Gamma^\dagger U_{T_x,k} \quad \text{and} \quad C_{\text{GB}} = U_{T_x,k}^T \Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^\dagger M_R. \quad (4)$$

The minimal norm GBT1 is given by $A_{\text{GB}} = \Gamma^\dagger U_{T_x,k}$ and $C_{\text{GB}} = U_{T_x,k}^T \Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^\dagger$. For the fixed compression ratio $c = \frac{k}{\min\{m,n\}}$, the associated error is given by

$$\begin{aligned} \varepsilon_{\text{GBT1}} &= \min_{A,C} \mathbb{E}\left[\|\Gamma(\mathbf{y} - AC\mathbf{x})\|_2^2\right] \\ &= \|(\Gamma E_{\mathbf{yy}} \Gamma^T)^{1/2}\|^2 - \alpha_1 \end{aligned} \quad (5)$$

where $\alpha_1 = \sum_{j=1}^{r_x} \sigma_j(T_x)$, $r_x = k$ if $k \leq \text{rank}(T_x)$ and $r_x = \text{rank}(T_x)$ if $k > \text{rank}(T_x)$.

Remark 1: If $E_{\mathbf{xx}}^\dagger = E_{\mathbf{xx}}^{-1}$, and matrix Γ is symmetric and invertible then (5) represents the error associated with the BT.

The proof of Theorem 1 is based on the following facts.

Fact 1: Let $U_M \Sigma_M V_M^T = M$ be the SVD of $M \in \mathbb{R}^{m \times n}$, $\text{rank}(M) = r$, $k < r$, $U_{M,k} = [u_1 \ u_2 \ \dots \ u_k]$. Then,

$$[M]_k = U_{M,k} U_{M,k}^T M. \quad (6)$$

For $k \geq r$, $[M]_k = [M]_r = M$.

Proof: For convenience, we denote $U_k = U_{M,k}$, $\Sigma_k = \text{diag}(\sigma_1, \dots, \sigma_k)$, $\sigma_j = \sigma_j(M)$, $j = 1, \dots, k$, and $V_k = [v_1, \dots, v_k]$. Furthermore, let $\mathbb{O}_{k,r}$ be $k \times r$ zero matrix and let

$$U_{r-k} = [u_{k+1}, \dots, u_r], \quad V_{r-k} = [v_{k+1}, \dots, v_r]$$

$$\Sigma_{r-k} := \text{diag}(\sigma_{k+1}, \dots, \sigma_r) \quad \text{and} \quad \mathbb{O}_{k,r-k} \in \mathbb{R}^{k \times (r-k)}.$$

[3]It is customary to choose small weights where the errors associated with particular entries of observed data are expected to be large, and vice versa.

Then, for $I_k \in \mathbb{R}^{k \times k}$, (1) implies

$$\begin{aligned} [M]_k &= U_k [I_k \ \mathbb{O}_{k,r-k}] \begin{bmatrix} \Sigma_k & \mathbb{O}_{k,r-k} \\ \mathbb{O}_{k,r-k}^T & \Sigma_{r-k} \end{bmatrix} V_r^T \\ &= U_k U_k^T [U_k \ U_{r-k}] \Sigma_r V_r^T = U_k U_k^T M. \end{aligned}$$ ∎

We write $\mathcal{N}(M)$ for the null space of matrix M. *Fact 2:* Let $P \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{n \times m}$. Then,

$$\mathcal{N}(P^T) \subseteq \mathcal{N}(Q^T) \Rightarrow PP^\dagger Q = Q. \quad (7)$$

Proof: By Lemma 23 in [8, p. 167]

$$\mathcal{N}(P^T) \subseteq \mathcal{N}(Q^T) \Rightarrow Q^T (P^\dagger)^T P^T = Q^T.$$

Then, (7) follows. ∎

Fact 3: The following equality holds:

$$\Gamma \Gamma^\dagger \left[\Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}\right]_k = \left[\Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}\right]_k. \quad (8)$$

Proof: We have $\mathcal{N}(\Gamma^T) \subseteq \mathcal{N}\left(E_{\mathbf{xx}}^{1/2\dagger} E_{\mathbf{yx}}^T \Gamma^T\right)$ and $\mathcal{N}\left(E_{\mathbf{xx}}^{1/2\dagger} E_{\mathbf{yx}}^T \Gamma^T\right) \subseteq \mathcal{N}\left([E_{\mathbf{xx}}^{1/2\dagger} E_{\mathbf{yx}}^T \Gamma^T]_k\right)$. Thus, $\mathcal{N}(\Gamma^T) \subseteq \mathcal{N}\left([E_{\mathbf{xx}}^{1/2\dagger} E_{\mathbf{yx}}^T \Gamma^T]_k\right)$. Then, (7) implies (8). ∎

Proof of Theorem 1: Let us denote $\mathbf{w} = \Gamma \mathbf{y}$. Recall that $\mathbb{E}\left[\|\mathbf{w}\|_2^2\right] = \int_\Omega \text{tr}\{\mathbf{w}(\omega)\mathbf{w}(\omega)^T\}d\mu(\omega) = \text{tr}(E_{\mathbf{ww}})$ (since $\|M\|^2 = \text{tr}(MM^T)$; see, e.g., [8, p. 167]). Then, for $F_1 = AC$

$$\begin{aligned} &\mathbb{E}\left[\|\Gamma(\mathbf{y} - AC\mathbf{x})\|_2^2\right] \\ &= \text{tr}\{E_{\mathbf{ww}} - E_{\mathbf{wx}} F_1^T \Gamma^T - \Gamma F_1 E_{\mathbf{xw}} + \Gamma F_1 E_{\mathbf{xx}} F_1^T \Gamma^T\} \\ &= \left\|E_{\mathbf{ww}}^{1/2}\right\|^2 - \left\|E_{\mathbf{wx}} E_{\mathbf{xx}}^{1/2\dagger}\right\|^2 \\ &\quad + \left\|E_{\mathbf{wx}} E_{\mathbf{xx}}^{1/2\dagger} - \Gamma F_1 E_{\mathbf{xx}}^{1/2}\right\|^2. \end{aligned} \quad (9)$$

The latter is true because $E_{\mathbf{wx}} E_{\mathbf{xx}} E_{\mathbf{xx}}^\dagger = E_{\mathbf{wx}}$ (see [8, p. 168]) and $E_{\mathbf{xx}}^\dagger E_{\mathbf{xx}}^{1/2} = E_{\mathbf{xx}}^{1/2\dagger}$. Denote by $\mathbb{R}(m,n,k)$ the set of all $m \times n$ matrices of rank at most k. It has been shown in [14] and [15] that the solution to problem

$$\min_{F_1 \in \mathbb{R}(m,n,k)} \left\|E_{\mathbf{wx}} E_{\mathbf{xx}}^{1/2\dagger} - \Gamma F_1 E_{\mathbf{xx}}^{1/2}\right\|^2 \quad (10)$$

is given by

$$F_1 = M_L \Gamma^\dagger \left[\Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}\right]_k E_{\mathbf{xx}}^{1/2\dagger} M_R \quad (11)$$

(see Remark 2 below), where by Fact 1

$$\left[\Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}\right]_k = U_{T,k} U_{T,k}^T \Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}. \quad (12)$$

If we denote $F_1 = F_{\text{GB1}} = A_{\text{GB}} C_{\text{GB}}$ then (11) and (12) imply (4). Furthermore, if in (9), F_1 is replaced with F_{GB1} then

$$\begin{aligned} &\Gamma F_1 E_{\mathbf{xx}}^{1/2} \\ &= \Gamma M_L \Gamma^\dagger \left[\Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}\right]_k E_{\mathbf{xx}}^{1/2\dagger} M_R E_{\mathbf{xx}}^{1/2} \end{aligned} \quad (13)$$

where $\Gamma M_L \Gamma^\dagger = \Gamma \Gamma^\dagger$ and $E_{\mathbf{xx}}^{1/2\dagger} M_R E_{\mathbf{xx}}^{1/2} = E_{\mathbf{xx}}^{1/2\dagger} E_{\mathbf{xx}}^{1/2}$. Therefore, (13) implies

$$\Gamma T E_{\mathbf{xx}}^{1/2} = \Gamma \Gamma^\dagger \left[\Gamma E_{\mathbf{yx}} E_{\mathbf{xx}}^{1/2\dagger}\right]_k E_{\mathbf{xx}}^{1/2\dagger} E_{\mathbf{xx}}^{1/2}.$$

Here, on the basis of Lemma 42 in [8, p. 311]

$$\left[\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\right]_k E_{\mathbf{xx}}^{1/2\dagger}E_{\mathbf{xx}}^{1/2} = \left[\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\right]_k$$

and (8) is also true. Thus,

$$\varGamma M_L \varGamma^\dagger \left[\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\right]_k E_{\mathbf{xx}}^{1/2\dagger} M_R E_{\mathbf{xx}}^{1/2}$$
$$= \left[\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\right]_k. \tag{14}$$

Then, for $F_1 = F_{\text{GB1}} = A_{\text{GB}}C_{\text{GB}}$ and $A_{\text{GB}}, C_{\text{GB}}$ given by (4), the last term in (9) is written as

$$\left\|\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger} - \varGamma F_{\text{GB1}}E_{\mathbf{xx}}^{1/2}\right\|^2$$

$$= \left\|\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger} - \left[\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\right]_k\right\|^2 = \sum_{j=k+1}^{\text{rank}\,(T_x)} \sigma_j(T_x)$$

since $\|\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\|^2 = \sum_{j=1}^{\text{rank}\,(T_x)} \sigma_j(\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger})$. Therefore, the error representation in (5) follows. Interestingly, (5) does not depend on S and R, i.e., any their choice (e.g., $S = \mathbb{O}_{m,m}, R = \mathbb{O}_{n,n}$) results in the same associated error. ∎

Remark 2: In [14] and [15], the proof of (11) is based on $\|\varLambda - \varGamma F_1 G\| = \|\widetilde{\varLambda} - Z\|$, where $\varLambda = E_{\mathbf{wx}}E_{\mathbf{xx}}^{1/2\dagger}$, $G = E_{\mathbf{xx}}^{1/2}$, $\widetilde{\varLambda} = U_\varGamma^T \varLambda V_G$, $Z = \Sigma_\varGamma \widehat{F_1} \Sigma_G \widehat{F_1} = V_\varGamma^T F_1 U_G$.

Example 1: Let $E_{\mathbf{yx}}, E_{\mathbf{xx}}$ be formed by 6×3 sample matrices Y and $X = Y + 0.1\Theta$, where Y has uniformly distributed entries and Θ has normally distributed entries with mean 0 and variance 1. The entries are chosen randomly. Then, $E_{\mathbf{yx}}$ and $E_{\mathbf{xx}}$ are singular and the BT is not applicable. The GBT1 is applicable and, for $k = 2$ and diagonal $\varGamma$ with random entries within $(0, 1)$, the associated error is 0.02.

B. Numerical Load. Comparison With the GKLT

In a number of applied problems, dimensions m, n of associated covariance matrices are large. For instance, in the DNA array analysis [16], [17], $m = O(10^4)$. In this case, the associated numerical load needed to compute the covariance matrices increases significantly. Therefore, a method which requires a lower associated numerical load is, of course, more preferable. In addition to the advantages of the GBT1 mentioned in Section I, another advantage is that the computational load for the GBT1 is less than that for the GKLT [8]. The GKLT is the optimal technique related to GBT1 and is represented here by (11). The computational schemes for the GBT1 and GKLT are different. In particular, the GBT1 requires to compute $U_{T_x,k}$ from the eigendecomposition of the $m \times m$ symmetric matrix T_x, whereas the GKLT requires to compute the whole SVD of the nonsymmetric $m \times n$ matrix $\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}$.

Table I provides estimates of the number of flops required to compute $U_{T_x,k}$, for the GBT1, and the SVD of $\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}$, for the GKLT, by the Golub–Reinsch SVD method and the R-bidiagonalization method (R-SVD), for $m = n$, as it given in [18, p. 254].

The GBT1 computational load (L_{GBT1}) consists of computation of matrices $E_{\mathbf{xx}}^\dagger$ and $U_{T_x,k}$, and matrix products $E_{\mathbf{yx}}E_{\mathbf{xx}}^\dagger$, T_x, and $A_{\text{GB}}C_{\text{GB}}$. The GKTL computational load (L_{GKLT}) consists of computation of matrix $E_{\mathbf{xx}}^{1/2\dagger}$, matrix

TABLE I
ESTIMATES OF FLOP NUMBERS

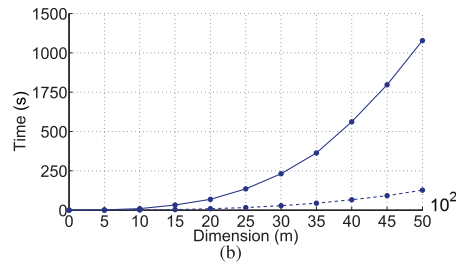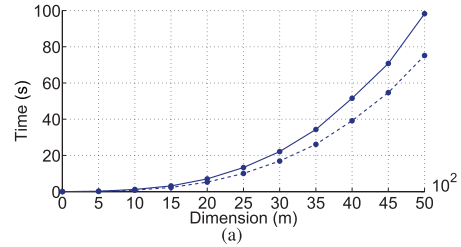| Required	Flops	
	Golub–Relisch SVD	R-SVD
U_{T_x}	$12m^3$	$13m^3$
SVD of $\varGamma E_{\mathbf{yx}}(E_{\mathbf{xx}}^{1/2})^\dagger$	$21m^3$	$26m^3$

Fig. 1. Time versus matrix dimension m used to execute the GBT1 (dashed line) and GKLT (solid line), for (a) $s = 2m$ and (b) $s = m/2$.

product $\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}$, the SVD of $\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}$, and matrix product $\varGamma^\dagger \left[\varGamma E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}\right]_k E_{\mathbf{xx}}^{1/2\dagger}$. Matrix product of $m \times n$ and $n \times p$ requires approximately $2mnp$ flops, for large n. On this basis, for $m = n$, $\varGamma = I$, $S = \mathbb{O}_{m,m}, R = \mathbb{O}_{n,n}$, and U_{T_x} and the SVD of $E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}$ evaluated by the R-SVD

$$L_{\text{GBT1}} = 35m^3 + 2m^2k \quad \text{and} \quad L_{\text{GKLT}} = 52m^3.$$

For U_{T_x} and the SVD of $E_{\mathbf{yx}}E_{\mathbf{xx}}^{1/2\dagger}$ evaluated by the Golub–Relisch SVD

$$L_{\text{GBT1}} = 34m^3 + 2m^2k \quad \text{and} \quad L_{\text{GKLT}} = 47m^3.$$

Of course, L_{GBT1} and L_{GKLT} are evaluated approximately. A difference between L_{GBT1} and L_{GKLT} increases if the covariance matrices are singular. Example (2) illustrates it.

Example 2: We simulate $\mathbf{y}$ and $\mathbf{x}$ by matrices $Y \in \mathbb{R}^{m \times s}$ and $X \in \mathbb{R}^{m \times s}$ as in Example (1) but with different m, s. The diagrams in Fig. 1(a) and (b) represent time versus matrix dimension m used to execute the GBT1 and GKLT, for $c = m/4$. For $s = m/2$, the covariance matrices are singular. Fig. 1 illustrates the GBT1 advantage considered above, i.e., that the GBT1 is faster than the GKLT.

III. GENERALIZED SECOND-ORDER BT (GBT2)

Now, we consider the generalized second-order BT (GBT2) in the form $F(\mathbf{x}) = DC_1\mathbf{x} + DC_2\mathbf{v}$ that contains three matrices to optimize, D, C_1, and C_2, i.e., one matrix more than the GBT1. Here, $\mathbf{v}$ is an "auxiliary" signal used to further optimize the transform. We show that, because of the increase

in a number of parameters to optimize, the GBT2 may provide better accuracy than that of the GBT1.

A. Determination of D, C_1, and C_2

Optimal $D \in \mathbb{R}^{m \times k}$, $C_1 \in \mathbb{R}^{k \times n}$, and $C_2 \in \mathbb{R}^{k \times n}$ solve

$$\min_{D, C_1, C_2} \mathbb{E}\left[\|\Gamma(\mathbf{y} - D[C_1\mathbf{x} + C_2\mathbf{v}])\|_2^2 \right]. \tag{15}$$

To find D, C_1, C_2, we denote $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix}$, $N_L = I + (I - P_{\Gamma,R})S$, and $N_R = I + K(I - P_{E_{\mathbf{zz}}^{1/2},L})$, where S and K are arbitrary matrices. Matrix T_z and its SVD are defined similar to T_x and the SVD of T_x, respectively, in Section II. Here, $\mathbf{v}$ is still arbitrary.

Theorem 2: Optimal $D = D^{(1)}$, and $C_1 = C_1^{(1)}$ and $C_2 = C_2^{(1)}$ are given by

$$D^{(1)} = N_L \Gamma^\dagger U_{T_z,k}, [C_1^{(1)}, C_2^{(1)}] = U_{T_z,k}^T \Gamma E_{\mathbf{yz}} E_{\mathbf{zz}}^\dagger N_R. \tag{16}$$

For the fixed compression ratio c, the associated error is

$$\varepsilon_{\text{GBT2}}(\mathbf{v}) = \min_{D, C_1, C_2} \mathbb{E}\left[\|\Gamma(\mathbf{y} - D[C_1\mathbf{x} + C_2\mathbf{v}])\|_2^2 \right]$$

$$= \|(\Gamma E_{\mathbf{yy}})^{1/2}\|^2 - \alpha_2(\mathbf{v}) \tag{17}$$

where $\alpha_2(\mathbf{v}) = \sum_{j=1}^{r_z} \sigma_j(T_z)$, $r_z = k$ if $k \leq \text{rank}(T_z)$ and $r_z = \text{rank}(T_z)$ if $k > \text{rank}(T_z)$.

Proof: The proof is similar to the proof of Theorem 1 subject to the appropriate changes in notation. ∎

B. Determination of $\mathbf{v}$

1) First Method: The practical approach is to replace the cost in (15) with the sample version, $J(D, C_1, C_2, Y, V) := \|\Gamma(Y - D[C_1X + C_2V])\|^2 = \|G - HV\|^2$, where $Y \in \mathbb{R}^{m \times s}$, $X \in \mathbb{R}^{n \times s}$ are samples of $\mathbf{y}$, $\mathbf{x}$, respectively, $V \in \mathbb{R}^{n \times s}$, $G = \Gamma(Y - DC_1X)$ and $H = \Gamma DC_2$. In this setting, we determine optimal V iteratively as follows.

First, for an arbitrary V, determine optimal $D^{(1)}, C_1^{(1)}, C_2^{(1)}$ such that $J(D^{(1)}, C_1^{(1)}, C_2^{(1)}, Y, V) = \min_{D,C_1,C_2} J(D, C_1, C_2, Y, V)$. Then, $D^{(1)}, C_1^{(1)}, C_2^{(1)}$ are given as in (16) where $E_{\mathbf{yz}}$ is replaced with $s^{-1}YZ^T$ where $Z = [X^T, V^T]^T$.

Write $Y^{(1)} = \Gamma^\dagger \Gamma D^{(1)}[C_1^{(1)}X + C_2^{(1)}V]$. Second, find $D^{(2)}, C_1^{(2)}, C_2^{(2)}$ such that $J(D^{(2)}, C_1^{(2)}, C_2^{(2)}, Y^{(1)}, V) = \min_{D,C_1,C_2} J(D, C_1, C_2, Y^{(1)}, V)$. Find the minimal-norm $V^{(1)}$ such that $J(D^{(2)}, C_1^{(2)}, C_2^{(2)}, Y^{(1)}, V^{(1)}) = \min_V J(D^{(2)}, C_1^{(2)}, C_2^{(2)}, Y^{(1)}, V)$. Then,

$$V^{(1)} = H^{(2)\dagger} \Gamma(Y^{(1)} - D^{(2)}C_1^{(2)}X), \tag{18}$$

where $H^{(2)} = \Gamma D^{(2)}C_2^{(2)}$. Then the procedure is repeated with $V^{(1)}$ instead of V until the tolerance for the estimate of Y is achieved.

Convergence will be elaborated and reported elsewhere in the near future.

2) Second Method: In particular, $\mathbf{v}$ can be chosen as $\mathbf{v} = \mathbf{x}^2$, where $\mathbf{x}^2$ is given by $\mathbf{x}^2(\omega) = [\mathbf{x}_1^2(\omega), \dots, \mathbf{x}_n^2(\omega)]^T$ and $\mathbf{x}_j^2(\omega) = [\mathbf{x}_j(\omega)]^2$. Although this choice of $\mathbf{v}$ is simple, it is not optimal.

Furthermore, the error representations in (5) and (17) imply $\varepsilon_{\text{GBT1}} - \varepsilon_{\text{GBT2}}(\mathbf{v}) = \alpha_2(\mathbf{v}) - \alpha_1$. Therefore, if $\alpha_1 < \alpha_2(\mathbf{v})$
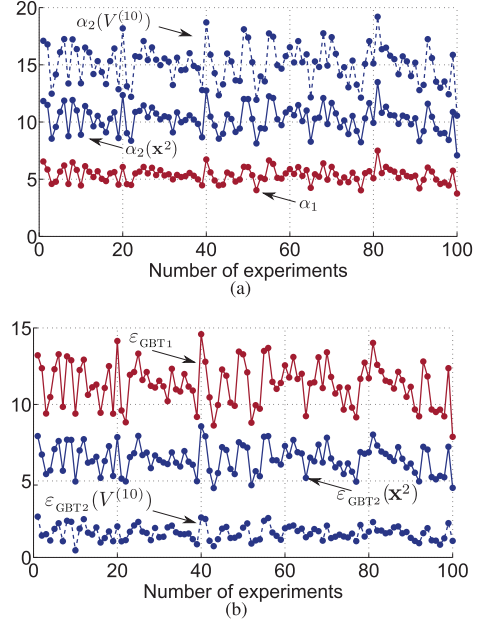

Fig. 2. Example 4: Illustrations to conditions $\alpha_1 < \alpha_2(\mathbf{x}^2)$, $\alpha_1 < \alpha_2(V^{(10)})$ (a) and MSEs (b).

then, for the same compression ratio c, the error associated with the GBT2 is less than that of the GBT1, i.e., $\varepsilon_{\text{GBT2}}(\mathbf{v}) < \varepsilon_{\text{GBT1}}$. The condition $\alpha_1 < \alpha_2(\mathbf{v})$ can be used in testing experiments as those in Examples (3) and (4) below.

C. Models of Compressor and Decompressor by GBT2

Compressed signal is represented by $\mathbf{u} = C_1\mathbf{x} + C_2\mathbf{v}$. Compressor and decompressor are represented by C_1 and C_2, and D, respectively.

Example 3: Let $\boldsymbol{x} = \boldsymbol{y} + \sigma^2 \boldsymbol{\xi}$ where $\boldsymbol{y} \in L^2(\Omega, \mathbb{R}^{100})$ is a uniformly distributed random vector, $\sigma \in \mathbb{R}$, $\mathbf{v} = \mathbf{x}^2$ and $\boldsymbol{\xi} \in L^2(\Omega, \mathbb{R}^{100})$ is white noise. Vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{\xi}$ are assumed to be independent. Therefore, $E_{\mathbf{yy}} = \{\eta_{ij}\}_{i,j=1}^{100}$, $\eta_{ij} = \begin{cases} 1/3, & i = j, \\ 1/4, & i \neq j, \end{cases}$ $E_{\mathbf{yx}} = E_{\mathbf{yy}}$, $E_{\boldsymbol{\xi}\boldsymbol{\xi}} = \sigma^2 I$,

$E_{\mathbf{xx}} = E_{\mathbf{yy}} + E_{\xi\xi}$, $E_{\mathbf{yx}^2} = \{\lambda_{ij}\}_{i,j=1}^{100}$, $\lambda_{ij} = \begin{cases} 1/4, & i = j \\ 1/6, & i \neq j, \end{cases}$

$E_{\mathbf{yz}} = [E_{\mathbf{yy}} E_{\mathbf{yx}^2}]$, where $\mathbf{z} = [\mathbf{x}^T (\mathbf{v})^T]^T$, $E_{\mathbf{x}^2\mathbf{x}^2} = \{\mu_{ij}\}_{i,j=1}^{100}$, $\mu_{ij} = \begin{cases} 1/5, & i = j \\ 1/9, & i \neq j, \end{cases}$ $E_{\xi^2\xi^2} = 2\sigma^4 I$, and $E_{\mathbf{zz}} = \begin{bmatrix} E_{\mathbf{xx}} & E_{\mathbf{yy}} \\ E_{\mathbf{yy}} & E_{\mathbf{x}^2\mathbf{x}^2} + E_{\xi^2\xi^2} \end{bmatrix}$. Let, e.g., $k = 50$ and $\sigma = 0.4$. Then, $\alpha_1 = 27.7$, $\alpha_2 = 30.0$, and $\varepsilon_{\text{GBT1}} = 7.01$, $\varepsilon_{\text{GBT2}} = 5.8$.

Example 4: Let $\boldsymbol{y}$ be a uniformly distributed random vector, $\mathbf{x} = s_1\boldsymbol{y} + s_2\boldsymbol{\delta}$, $s_1, s_2 \in [0.1, 10]$ and $\boldsymbol{\delta}$ is a Gaussian random vector with mean zero and variance one. The covariance matrices are formed from samples $Y \in \mathbb{R}^{50 \times 150}$ and $X \in \mathbb{R}^{50 \times 150}$. In each test, Y, X, and diagonal Γ with entries within $(0, 1)$ are chosen randomly. In Fig. 2(a), for $s_1 = 0.5$ and $s_2 = 8$, conditions $\alpha_1 < \alpha_2(\mathbf{x}^2)$, $\alpha_1 < \alpha_2(V^{(10)})$ are illustrated where values of α_1 and α_2 are represented versus experiment numbers. In Fig. 2(b), for the same s_1 and s_2, values of the errors are given. In all 100 experiments, the GBT2 with optimized V given by $V^{(10)}$ provides better associated accuracy. In fact, Fig. 2 represents typical diagrams of the experiments we made for different values of $s_1, s_2 \in [0.1, 10]$.

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