Counterparty credit risk in a multivariate structural model with jumps

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Abstract

We present a multivariate version of a structural default model with jumps and use it in order to quantify the bilateral credit value adjustment and the bilateral debt value adjustment for equity contracts, such as forwards, in a Merton-type default setting. In particular, we explore the impact of changing correlation between names on these adjustments and study the effect of wrong-way and right-way risk.

Keywords: Counterparty risk, Credit Value Adjustment, Debt Value Adjustment, Lévy processes, Normal Inverse Gaussian, Wrong Way Risk. **JEL Classification**: C15, C63, C65, G13

1 Introduction

The aim of this paper is to provide a valuation framework for counterparty credit risk based on a structural default approach which incorporates jumps and dependence between the assets of interest. In this model default is caused by the firm value falling below a prespecified threshold following unforeseeable shocks, which deteriorate its liquidity and ability to meet its liabilities. The presence of dependence between names captures wrong-way risk and right-way risk effects.

In a post-crisis world the correct assessment and management of counterparty credit risk has become a core concern for financial market regulators, playing a substantial contribution in shaping the mutual behavioral pattern of banks and their counterparties. The regulatory landscape has undergone significant changes aimed at reducing the risk of banks failures and increasing financial stability, by strengthening the risk coverage of capital via capital

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requirements related to counterparty risk exposures. Moving away from the Basel I regime introduced in 1988, which assigned capital requirements for banks based on prescribed risk-weights applied to counterparty categories, the Basel II (2005) and, even more, the Basel III (2010) regimes have pointed to the need of an enhanced sensitivity of credit risk measurement. Capital requirements have been linked to more sophisticated measures of counterparty credit risk such as the Credit Valuation Adjustment (CVA), Debit Value Adjustment (DVA) and more recently the potential volatility of the same (VAR of CVA) captured via VAR models in conjunction with stress testing under extreme market scenarios (see Basel, 2010, for example). CVA is the loss due to the default of a counterparty to a specified transaction (possibly involving a third entity). DVA, instead, is to be intended as the additional benefit of one's own default[1].

These changes in regulation have a significant impact on banks behavior and on the pricing of specific types of trades where the underlying exposure profile is significantly large from the banks perspective. Particularly, uncollateralized long dated trades (cross currency swaps, long dated foreign exchange forwards, interest rate swaps with significant carry) became more expensive and banks have either shifted focus on trades that are less credit intensive or tried to mitigate the exposure via mandatory breaks (that reduce the effective duration of the trades) or asking for collateral protection. Among bank counterparties the most affected group has been corporates. Corporates typically engage in derivatives transactions for hedging purposes and traditionally do not have the ability to post high frequency cash collateral against them.

The concept of CVA presents a further angle of relevance for corporates, due to changes in the standards around hedge accounting. Historically, corporates had not been required to measure and record ineffectiveness of the hedges for the CVA and DVA of the derivatives that they transact with their relationship banks. The new IAS/IFRS have called for adjustments in the fair value of derivatives transactions for both CVA and DVA (see IASB, 2011, for example). However, the exact valuation methodology and how to allocate CVA and DVA to individual hedge relationships has not been clearly stated, thus fostering the necessity of a uniform methodology.

This paper aims to provide a solid framework for the assessment of CVA and DVA in presence of right-way risk and wrong-way risk resulting from dependence between credit spreads and underlying transaction, making an academic contribution to a discussion topic that is of extreme relevance in the current financial landscape. Because of the interdependencies between financial assets, the joint evolution of the risk drivers is of particular relevance. In this context, the structural approach to modelling credit risk is appealing as dependence between entities is simple to incorporate; further, it offers an economic rationale behind default as

[1]Concerning the debate on the controversial meaning of DVA, which can be realized only given one's own default, see for example Gregory and German (2012).

this is linked to the fundamentals of a company.

The structural model traces back to Merton (1974), who considered only the possibility of default occurring at the maturity of the contract; first passage time models starting from the seminal contribution of Black and Cox (1976) extend the original framework to incorporate default events at any time during the lifetime of the contract. However, as the driving risk process used is the Brownian motion, all these models suffers of vanishing credit spreads over the short period - a feature not observed in reality. As a consequence, the CVA would be underestimated for short term deals. Improvements aimed at resolving this issue include for example random default barriers and time dependent volatilities (e.g., Brigo and Tarenghi, 2005; Brigo and Morini, 2006), and jumps (see Zhou, 2001; Sepp, 2006; Fiorani et al., 2010, and references therein). In the context of counterparty risk valuation, structural models with jumps have been adopted for example by Lipton and Sepp (2009) and Lipton and Savescu (2012a,b). However, the numerical analysis carried by Lipton and Savescu (2012a,b) is for the case of firm value processes driven by Brownian motions only (see Lipton and Savescu, 2013, as well).

The class of Lévy processes provides a mathematically and computationally tractable tool to incorporate jumps and market shocks in the asset dynamics. Multivariate Lévy processes have been recently explored in the literature: for example Lindskog and McNeil (2003) and Brigo et al. (2007) propose constructions applicable to the class of jump diffusion processes, i.e. processes composed by a Brownian motion and a compound Poisson process, whilst Luciano and Schoutens (2006), Semeraro (2008) and Luciano and Semeraro (2010) focus on the class of time changed Brownian motions, such as the Variance Gamma or the Normal Inverse Gaussian process. Baxter (2007) and Moosbrucker (2006a,b) instead resort to the factor copula approach. These models though present some limitations in that either the proposed construction is class specific, or the range of possible dependencies and the set of attainable values for the correlation coefficient are limited (see Wallmeir and Diethelm, 2012, as well). These restrictions also affect the model put forward by Lipton and Sepp (2009). Ballotta and Bonfiglioli (2014) address these issues via a two factor linear representation of the assets log-returns, obtained as a linear combination of two independent Lévy processes representing respectively the systematic risk factor and the idiosyncratic shock; this model proves to be relatively more general as it can be applied to any Lévy process, and relatively more flexible as it can accommodate the full range of dependencies.

In light of the previous discussion, we adopt the factor construction of Ballotta and Bonfiglioli (2014) to develop a multivariate structural default model for the valuation of bilateral CVA and DVA related to equity contracts such as forwards, via semi-closed analytical formulas which are easily implementable. As by-product we obtain the unilateral case. We analyse in details the case in which the driving process is a Normal Inverse Gaussian (NIG) process (see Barndorff-Nielsen, 1995); this choice is motivated by the fact that the NIG process allows for skewness, excess kurtosis and a fairly rich jump dynamics although parsimonious in terms of number of parameters involved. The focus is on the impact of correlation between entities on the value of CVA and DVA, with particular attention to wrong-way risk and right-way risk; this is explored via sensitivity analysis. The numerical analysis shows that, on the one hand, the proposed model is relatively straightforward to calibrate to observable market data like CDS quotations and option prices; on the other hand, the inclusion of jumps helps the model to improve the fitting of credit spreads over the short period, compared to the classical approach based on purely continuous processes such as the Brownian motion. This in turn is reflected on the different impact of wrong-way and right-way risk on CVA and other relevant metrics of interest between the two distribution assumptions considered in this analysis.

The paper is structured as follows. In Section 2 we introduce the multivariate structural default framework for the valuation of CVA and DVA, for which we obtain a general formula of relative ease of implementation; we further obtain semi-closed analytical formulas in the case in which the driving processes are Brownian motions and NIG processes. We discuss the issue of model calibration in Section 3; in Section 4 we present the results from the sensitivity analysis carried out with respect to the correlation structure as to emphasize the effect of right-way risk and wrong-way risk. Section 5 concludes.

2 A general formula for CVA under a multivariate Lévy structural model

2.1 Lévy processes and multivariate construction via linear transformation

Lévy processes have attracted attention in the financial literature due to the fact that they accommodate distributions with non zero higher moments (skewness and kurtosis) due to the presence of jumps, therefore allowing a more realistic representation of stylized features of market quantities such as assets returns.

A Lévy process $L(t)$ on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$ is a stochastic process with independent and stationary increments whose distribution is infinitely divisible. Further, these processes are fully described by their characteristic function, which in virtue of the celebrated Lévy-Khintchine representation can be written as

$$\phi_L(u;t) = \mathbb{E}\left(e^{iuL(t)}\right) = e^{t\varphi_L(u)}, \quad u \in \mathbb{R},$$

where i is the imaginary unit and $\varphi_X(\cdot)$ represents the so-called characteristic exponent. Examples of Lévy processes commonly used in finance are the Brownian motion and the NIG process. In details, a Brownian motion with diffusion coefficient $\sigma > 0$ has characteristic exponent

$$\varphi_L(u) = -\frac{u^2\sigma^2}{2}, \tag{1}$$

and therefore it is Gaussian distributed with zero mean and variance $\sigma^2 t$; further it is a continuous process. On the other hand, the NIG process is a purely discontinuous process (i.e. a pure jump process) obtained by subordinating a Brownian motion with drift by an independent Inverse Gaussian process. Constructing Lévy processes by subordination has particular economic appeal as, in first place, empirical evidence shows that stock log-returns are Gaussian but only under trade time, rather than standard calendar time (see, e.g. Geman and Ané, 1996). Further, the time-change construction recognizes that stock prices are largely driven by news, and the time between one piece of news and the next is random as is its impact (see Carr et al., 2007, for example). Hence, the NIG process has form

$$L_j(t) = \theta G(t) + \sigma W\left(G(t)\right),$$

for $\theta \in \mathbb{R}$ and $\sigma \in \mathbb{R}^{++}$. $G(t)$ is an Inverse Gaussian process, i.e. a positive increasing Lévy process following an Inverse Gaussian distribution with parameters $\left(t/\sqrt{k}, 1/\sqrt{k}\right)$ (see Barndorff-Nielsen, 1995; Cont and Tankov, 2004, for example), where k is the variance rate of the process $G(t)$. This process models the so-called business time, i.e. the arrival time of market news. $W(t)$ is, instead, the "base" Brownian motion capturing the impact of the arrival of market news on the relevant financial quantities. The resulting characteristic exponent of the NIG process is

$$\varphi_L(u) = \frac{1 - \sqrt{1 - 2iu\theta k + u^2\sigma^2 k}}{k}. \tag{2}$$

The corresponding probability density function is

$$f_t(x) = Ce^{Ax}\frac{K_1(B\sqrt{x^2 + t^2\sigma^2/k})}{\sqrt{x^2 + t^2\sigma^2/k}}, \tag{3}$$

where $K_v(x)$ is the modified Bessel function of the second kind with order v and

$$A = \frac{\theta}{\sigma^2}, \quad B = \frac{\sqrt{\theta^2 + \sigma^2/k}}{\sigma^2}, \quad C = \frac{t}{\pi}e^{t/k}\sqrt{\frac{\theta^2}{k\sigma^2} + \frac{1}{k^2}}.$$

It follows by differentiation of the characteristic exponent that

$$\begin{aligned}
\mathbb{E}\left(L(t)\right) &= \theta t, &(4)\\
\mathbb{V}ar\left(L(t)\right) &= \left(\sigma^2+\theta^2 k\right)t, &(5)\\
s\left(L(t)\right) &= \frac{3\theta k}{\sqrt{\left(\sigma^2+\theta^2 k\right)t}}, &(6)\\
c\left(L(t)\right) &= \frac{3k\left(\sigma^4+6\sigma^2\theta^2 k+5\theta^4 k^2\right)}{\left(\sigma^2+\theta^2 k\right)^2 t}, &(7)
\end{aligned}$$

where $s\left(L(t)\right)$ and $c\left(L(t)\right)$ denote, respectively, the index of skewness and the index of excess kurtosis (see Cont and Tankov, 2004, for example). Hence, the NIG process is fully described by the three parameters (θ,σ,k), which control, respectively, the (sign of the) skewness, the variance, and the excess kurtosis of the process distribution.

For the construction of Lévy processes in $\mathbb{R}^n$ with dependent components, as discussed in Section 1 we follow Ballotta and Bonfiglioli (2014). Hence, let $\mathbf{X}(t)=(X_1(t),X_2(t),...,X_n(t))^\top$ be a Lévy process in $\mathbb{R}^n$ with dependent components. Further, let us assume that $\mathbf{Y}(t)=(Y_1(t),Y_2(t),...,Y_n(t))^\top$ are Lévy process in $\mathbb{R}^n$ with independent components and $Z(t)$ is a Lévy process in $\mathbb{R}$, independent of $\mathbf{Y}(t)$. Finally, let $a_j\in\mathbb{R}$ for $j=1,...,n$. Then, dependence amongst the risk drivers, $X_j(t)$ for $j=1,...,n$, is modelled via a linear structure so that

$$\mathbf{X}(t)=\mathbf{Y}(t)+\mathbf{a}Z(t), \tag{8}$$

for $\mathbf{a}=(a_1,a_2,...,a_n)$. The stochastic process $Z(t)$ represents the systematic risk component, hence the source of dependence, whilst the processes components of $\mathbf{Y}(t)$ capture the idiosyncratic part of the overall risk. Due to the adopted construction, the description of the multivariate vector of Lévy processes $\mathbf{X}(t)$ only requires information on the univariate Lévy processes $Y_j(t)$, $j=1,\cdots,n$ and $Z(t)$. The joint characteristic function is, in fact

$$\phi_{\mathbf{X}}\left(\mathbf{u};t\right)=\phi_Z\left(\sum_{j=1}^n a_j u_j;t\right)\prod_{j=1}^n\phi_{Y_j}\left(u_j;t\right),\quad \mathbf{u}\in\mathbb{R}^n.$$

Consequently, for each margin process $X_j(t)$, $j=1,...,n$, it follows that the characteristic exponent is

$$\varphi_{Xj}(u_j)=\varphi_{Yj}(u_j)+\varphi_Z(a_j u_j). \tag{9}$$

The resulting coefficient of pairwise linear correlation (bearing in mind the infinitely divisibility property of Lévy processes) is

$$\rho_{jl}^{\mathbf{X}} = \mathbb{C}orr\left(X_j\left(1\right), X_l\left(1\right)\right) = \frac{a_j a_l \mathbb{V}ar\left(Z\left(1\right)\right)}{\sqrt{\mathbb{V}ar\left(X_j\left(1\right)\right)}\sqrt{\mathbb{V}ar\left(X_l\left(1\right)\right)}}. \tag{10}$$

Hence, under the proposed construction $\rho_{jl}^{\mathbf{X}}$ correctly describes the dependence between the components of $\mathbf{X}(t)$. For $a_j, a_l \neq 0$, in fact, $\rho_{jl}^{\mathbf{X}} = 0$ if and only if $Z(t)$ is degenerate, i.e. if the margins are independent; on the other hand, $|\rho_{jl}^{\mathbf{X}}| = 1$ if and only if $\mathbf{Y}(t)$ is degenerate, i.e the components of $\mathbf{X}(t)$ are perfectly (linear) dependent (for the proof of all the statements above, we refer to Ballotta and Bonfiglioli, 2014). Finally, $sign\left(\rho_{jl}^{\mathbf{X}}\right) = sign\left(a_j a_l\right)$.

Multivariate constructions based on equation (8) for the Brownian motion and the NIG process can be obtained by choosing the idiosyncratic components $Y_j(t)$, $j = 1, ..., n$, and the common components $Z(t)$ to be Lévy processes with characteristic exponents as in equations (1) and (2) respectively.

2.2 Counterparty credit risk in the multivariate Lévy structural model

We adopt a structural approach to default and assume that the relevant value processes, under some risk neutral martingale measure[2], is defined as

$$S_j(t) = S_j(0)e^{\left(r - q_j - \varphi_{X_j}(-i)\right)t + X_j(t)}, \quad j = 1, ..., n,$$

where $X_j(t)$ is the j-th component of the multivariate vector of Lévy processes $\mathbf{X}(t)$ given in equation (8), $r > 0$ is the risk-free rate, $q_j > 0$ is a constant cash outflow, n is the number of firms in the market, and $\varphi_{X_j}(-i)$ follows from equation (9).

In particular, we consider the case of three names ($n = 3$), so that $S_1(t)$ and $S_2(t)$ represents the firm value of the (risky) counterparties (i.e. the short and long position respectively) of a derivative contract written on a reference name, denoted as $S_3(t)$, and expiring at T. In this context, q_j for $j = 1, 2$ is the constant cash flow payout ratio, whilst q_3 is the dividend yield paid on security S_3. Further, at this stage we ignore default on the reference name, thus implicitly assuming that its credit quality is stronger than the one of the counterparties. In this paper we focus on the simple framework à la Merton, so that default can occur only at the expiry date of the contract, T, and as soon as the firm value falls below a given level, say K_j, i.e.

$$\ln S_j\left(0\right) + \left(r - q_j - \varphi_{X_j}(-i)\right)T + X_j\left(T\right) \leq \ln K_j.$$

Hence, default on asset j, $j = 1, 2$ is defined as the event $\{Y_j\left(T\right) + a_j Z\left(T\right) \leq l_j\}$, where

[2]We note that in general the given market is incomplete and therefore there are infinitely many risk neutral martingale measures; the availability of market quotes for suitable derivatives instruments, though, allows us to "complete" the market and extract the pricing measure by calibration.

$l_j = h_j - \hat{\mu}_j T$ for $h_j = \ln(K_j/S_j(0))$ and $\hat{\mu}_j = r - q_j - \varphi_{X_j}(-i)$.

In the following, we provide expressions for the counterparty credit risk adjustments metrics from the point of view of Firm 2. In particular, the bilateral CVA is defined as the present value of the expected loss if Firm 2 survives, Firm 1 defaults and the contract written on the reference name has a positive value to Firm 2, i.e.

$$CVA_2 = (1 - R_1)\mathbb{E}\left(1_{(Y_1(T)+a_1Z(T)\leq l_1, Y_2(T)+a_2Z(T)\geq l_2)}\Psi^+\left(S_3(T)\right)\right), \tag{11}$$

where R_1 is the recovery rate on Firm 1 assets, Ψ represents the discounted terminal payoff of the derivative contract on the reference name, and $\Psi^+ = \max(\Psi, 0)$ denotes the positive part of this payoff. We can define the CVA from the point of view of Firm 1 in a similar way. In practice this quantity corresponds to the so-called DVA from the point of view of Firm 2 (with sign changed), that is the expected gain to Firm 2 due to its own default when Firm 1 survives and the contract has a negative value to Firm 2, i.e.

$$DVA_2 = (1 - R_2)\mathbb{E}\left(1_{(Y_1(T)+a_1Z(T)\geq l_1, Y_2(T)+a_2Z(T)\leq l_2)}\Psi^-\left(S_3(T)\right)\right),$$

where $\Psi^- = \max(-\Psi, 0)$ denotes the negative part of the (discounted) contract payoff Ψ, and R_2 is the recovery rate on Firm 2 assets.

The bilateral counterparty value adjustment to Firm 2 is obtained as

$$BVA_2 = CVA_2 - DVA_2.$$

We notice that according to Basel III regulation (Basel, 2010) the DVA does not reduce counterparty exposure, whilst according to International Accounting Standard (IAS39 - see IASB, 2011) the fair value of the contract must recognize computation of both CVA and DVA.

Due to the factor construction in equation (8), the three events defining the CVA, i.e. default of Firm 1, survival of Firm 2 and positive value of the contract, are mutually independent once we condition on $Z(T)$. Therefore, the CVA formula (11) can be written as

$$\begin{aligned}
& CVA_2 \\
= \; & (1 - R_1)\mathbb{E}\left[\mathbb{E}_Z\left(1_{Y_1(T)\leq l_1 - a_1 z}1_{Y_2(T)\geq l_2 - a_2 Z(T)}\Psi^+\left(S_3(0)e^{\hat{\mu}_3 T + a_3 z + Y_3(T)}\right)\right)\right] \\
= \; & (1 - R_1)\mathbb{E}\left[\mathbb{P}_Z\left(Y_1(T) \leq l_1 - a_1 z\right)\mathbb{P}_Z\left(Y_2(T) \geq l_2 - a_2 Z(T)\right)\right. \\
& \left.\mathbb{E}_Z\left(\Psi^+\left(S_3(0)e^{\hat{\mu}_3 T + a_3 z + Y_3(T)}\right)\right)\right],
\end{aligned} \tag{12}$$

where $\mathbb{E}_Z(\cdot) = \mathbb{E}(\cdot|Z(T))$, $\mathbb{P}_Z(\cdot) = \mathbb{P}(\cdot|Z(T))$ denote respectively the conditional expectation and the conditional probability with respect to $Z(T)$. By similar argument, the DVA to Firm 2 can be written as

$$
\begin{aligned}
& DVA_2 \\
= \; & (1-R_2)\mathbb{E}\left(\mathbb{P}_Z\left(Y_1\left(T\right) \geq l_1 - a_1 z\right)\mathbb{P}_Z\left(Y_2\left(T\right) \leq l_2 - a_2 Z\left(T\right)\right)\right. \\
& \left.\mathbb{E}_Z\left(\Psi^-\left(S_3(0)e^{\hat{\mu}_3 T + a_3 z + Y_3(T)}\right)\right)\right). \qquad (13)
\end{aligned}
$$

The case of a single risky counterparty, originating the unilateral CVA, in which Firm 2 will survive for sure, can be easily dealt with by letting $l_2 \to -\infty$. In this case, the unilateral CVA[3] reads as

$$
\begin{aligned}
& CVA_2 \\
& = (1-R_1)\mathbb{E}\left(\mathbb{P}_Z\left(Y_1\left(T\right) \leq l_1 - a_1 z\right)\mathbb{E}_Z\left(\Psi^+\left(S_3(0)e^{\hat{\mu}_3 T + a_3 z + Y_3(T)}\right)\right)\right). \qquad (14)
\end{aligned}
$$

The explicit computation of the above expressions requires concrete assumptions on the nature of the underlying contract and the distribution of the risk drivers. Concerning the nature of the contract, here we consider the case of a long forward contract on S_3. In particular, the $T-$value of a forward contract expiring in U, $U > T$, with forward price K_3 can be written as

$$
\begin{aligned}
S_3(T)e^{-q_3(U-T)} - K_3 e^{-r(U-T)} &= S_3(0)e^{-q_3(U-T)}e^{\left(r - q_3 - \varphi_{X_3}(-i)\right)T + X_3(T)} - K_3 e^{-r(U-T)} \\
&= \alpha(Z)\left(S_3(0)e^{\left(r - q_3 - \varphi_{Y_3}(-i)\right)T + Y_3(T)} - K(Z)\right),
\end{aligned}
$$

where the second equality follows from the linear structure of the risk driver and

$$
\begin{aligned}
\alpha(Z) &= e^{-q_3(U-T) - \varphi_Z(-a_3 i)T + a_3 Z(T)}, \\
K(Z) &= K_3 e^{-r(U-T)}/\alpha(Z).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \mathbb{E}_Z\left(\Psi^+\left(S_3(0)e^{\hat{\mu}_3 T + a_3 z + Y_3(T)}\right)\right) = \alpha(Z)\mathbb{E}_Z\left(\left(S_3(0)e^{\left(r - q_3 - \varphi_{Y_3}(-i)\right)T + Y_3(T)} - K(Z)\right)^+\right) \\
& = \alpha(Z)\left(e^{-q_3 T}S_3(0)\mathbb{E}_Z\left(e^{-\varphi_{Y_3}(-i)T + Y_3(T)}1_A\right) - K(Z)e^{-rT}\mathbb{P}_Z\left(A\right)\right) \qquad (15)
\end{aligned}
$$

with $A = \{Y_3(T) \geq \ln K(Z) - \left(r - q_3 - \varphi_{Y_3}(-i)\right)T\}$. Let $\mathbb{P}^*$ be a probability measure

[3] Similar calculations leads to the unilateral DVA by letting $l_1 \to -\infty$ in equation (13).

equivalent to the chosen risk neutral one, and defined by the density process

$$\eta(t) = \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\varphi_{Y_3}(-i)t + Y_3(t)} \tag{16}$$

(see Geman et al., 1995, for example); then, the Bayes formula implies that the conditional expected exposure reported in equation (15) returns

$$S_3(0)e^{-q_3 U - \varphi_Z(-a_3 i)T + a_3 Z(T)}\mathbb{P}^*_Z(A) - K_3 e^{-rU}\mathbb{P}_Z(A). \tag{17}$$

We note the similarity between the conditional expected exposure and the price of a vanilla European call option.

We note at this stage that the pricing equations (12) and (17) also hold in the case of any multivariate semimartingale respecting the construction (8).

Workable formulae for the CVA and the DVA of the given forward position can be obtained once a specific assumption is made about the type of Lévy process adopted. In this work we consider in first place the case of a Brownian motion, which is regarded as a benchmark. However, due to its very restrictive nature as continuous process, which cannot accommodate the non-zero credit spread observed for very short maturities, we also consider the case of a NIG process. As discussed in Section 2.1, a NIG process is a pure jump process and therefore is flexible enough to accommodate non-zero skewness and excess kurtosis in the firm dynamics. This allows us to capture non-zero credit spreads as well as the simultaneous jump to default for Firm 1 and the positive exposure on the underlying contract for Firm 2. In addition, in the case of Lévy processes the pricing of plain vanilla options is nowadays well established via either numerical integration of the density function, if available in closed form, or numerical inversion of the characteristic function, which is always available in closed form, using for example the Carr-Madan technique (Carr and Madan, 1999) or similar approaches, such as the Fourier cosine expansion (Fang and Oosterlee, 2008). This makes the (numerical) computation of the CVA relatively straightforward, as discussed in the next sections.

2.3 The Gaussian specification

Using the same notation introduced in Section 2.1, we assume that $Y_j(T)$, $j = 1, 2, 3$ and $Z(T)$ are independent Brownian motions with diffusion coefficients $\gamma_1, \gamma_2, \gamma_3, \gamma_Z$ respectively. Consequently, $X_j(T)$, $j = 1, 2, 3$ is a Brownian motion with diffusion parameters $\sigma_1, \sigma_2, \sigma_3$ such that $\sigma_j^2 = \gamma_j^2 + a_j^2\gamma_Z^2$, $j = 1, 2, 3$. It follows that the exponential compensator is

$$\varphi_{X_j}(-i) = \frac{\sigma_j^2}{2} = \frac{\gamma_j^2}{2} + \frac{a_j^2\gamma_Z^2}{2} = \varphi_{Y_j}(-i) + \varphi_Z(-a_j i), \quad j = 1, 2, 3.$$

Under this assumption, it follows from equations (12) and (17) that the CVA to Firm 2 for a long position in a forward contract on the reference name is

$$CVA_2 = (1 - R_1) \int_{\mathbb{R}} \Phi\left(\frac{l_1 - a_1 z}{\gamma_1 \sqrt{T}}\right) \Phi\left(-\frac{l_2 - a_2 z}{\gamma_2 \sqrt{T}}\right) g(z)\, f_T(z)\, dz, \tag{18}$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard Normal random variable, $f_T(z)$ is the density function at T of the (rescaled) Brownian motion controlling the common risk component, and

$$\begin{aligned} g(Z(T)) &= S_3(0) e^{-q_3 U - \frac{a_3^2 \gamma_Z^2}{2} T + a_3 Z(T)} \Phi\left(\frac{\ln \frac{S_3(0) e^{(r-q_3)U}}{K_3} - \frac{\sigma_3^2}{2} T + a_3 Z(T)}{\gamma_3 \sqrt{T}} + \gamma_3 \sqrt{T}\right) \\ &\quad - K_3 e^{-rU} \Phi\left(\frac{\ln \frac{S_3(0) e^{(r-q_3)U}}{K_3} - \frac{\sigma_3^2}{2} T + a_3 Z(T)}{\gamma_3 \sqrt{T}}\right). \end{aligned}$$

The computation of the integral in equation (18) can be easily performed via a numerical integration method, such as Gaussian quadrature (see Press et al., 2007, for example).

2.4 The NIG specification

Using the multivariate construction as given in equation (8), we assume that $Y_j(T)$ and $Z(T)$ follow independent NIG processes. In particular, $Y_j(T)$ is obtained by subordinating a Brownian motion with drift $\beta_j \in \mathbb{R}$ and volatility $\gamma_j > 0$ by an unbiased IG subordinator with variance rate $\nu_j > 0$, whilst $Z(T)$ is obtained by subordinating a Brownian motion with drift $\beta_Z \in \mathbb{R}$ and volatility $\gamma_Z > 0$ by an unbiased IG subordinator with variance rate ν_Z. $X_j(T)$ is then the sum of two independent NIG processes. The exponential compensator is

$$\begin{aligned} \varphi_{X_j}(-i) &= \varphi_{Y_j}(-i) + \varphi_Z(-a_j i) \\ &= \frac{1 - \sqrt{1 - 2\beta_j \nu_j - \gamma_j^2 \nu_j}}{\nu_j} + \frac{1 - \sqrt{1 - 2a_j \beta_Z \nu_Z - a_j^2 \gamma_Z^2 \nu_Z}}{\nu_Z}, \quad j = 1, 2, 3. \end{aligned}$$

By similar argument as reported in the Appendix (see equations A.1-A.3), the change of measure defined by equation (16) implies that, under $\mathbb{P}^*$, $Y_3(T)$ is a NIG process obtained by subordinating an arithmetic Brownian motion with drift $\beta_3 + \gamma_3^2$ and diffusion parameter γ_3 by an Inverse Gaussian process with parameters $(T/\sqrt{\nu_3}, \sqrt{(1 - 2\beta_3 \nu_3 - \gamma_3^3 \nu_3)/\nu_3})$. Therefore, the CVA to Firm 2 of a long position in the forward contract on the reference entity (equations

12 and 17) is

$$CVA_2$$
$$= (1-R_1)\int_{\mathbb{R}} \bar{F}\left(d_1, -\frac{\beta_1}{\gamma_1}, \frac{T}{\sqrt{\nu_1}}, \frac{1}{\sqrt{\nu_1}}\right) \bar{F}\left(d_2, \frac{\beta_2}{\gamma_2}, \frac{T}{\sqrt{\nu_2}}, \frac{1}{\sqrt{\nu_2}}\right) g(z)\, f_T(z)\, dz, \quad (19)$$

with

$$\begin{aligned} g(Z(T)) &= S_3(0)e^{-q_3 U - \frac{1-\sqrt{1-2a_3\beta_Z\nu_Z - a_3^2\gamma_Z^2\nu_Z}}{\nu_Z}T + a_3 Z(T)}\bar{F}\left(d_3, \frac{\beta_3+\gamma_3^2}{\gamma_3}, \frac{T}{\sqrt{\nu_3}}, \sqrt{\frac{1-s}{\nu_3}}\right) \\ &\quad -K_3 e^{-rU}\bar{F}\left(d_3, \frac{\beta_3}{\gamma_3}, \frac{T}{\sqrt{\nu_3}}, \frac{1}{\sqrt{\nu_3}}\right), \end{aligned}$$

and

$$\begin{aligned} \bar{F}(\xi, \chi, a, b) &\doteq \frac{a}{\sqrt{2\pi}}e^{ab}\int_0^\infty \Phi\left(\frac{\xi}{\sqrt{z}} + \chi\sqrt{z}\right) z^{-3/2} e^{-\frac{1}{2}\left(\frac{a^2}{z}+b^2 z\right)} dz, \quad (20) \\ d_1 &= \frac{l_1 - a_1 Z(T)}{\gamma_1}, \qquad d_2 = -\frac{l_2 - a_2 Z(T)}{\gamma_2}, \\ d_3 &= \frac{\ln\frac{S_3(0)e^{(r-q_3)U}}{K_3} - \varphi_{X_3}(-i)T + a_3 Z(T)}{\gamma_3}, \qquad s = \nu_3\left(2\beta_3 + \gamma_3^2\right). \end{aligned}$$

The integral in equation (19) can be computed via numerical quadrature. In particular, the COS method of Fang and Oosterlee (2008) is used to numerically compute the probability and the cumulative density functions of the NIG process. In order to benchmark this numerical solution, we have also used both the analytical expression of the probability and the cumulative density functions, see equations (3) and (20) respectively, and Monte Carlo simulation.

3 Model Calibration

The parameters fitting is performed in two steps; firstly, we obtain the parameters of the margin processes $X_j(T)$ by direct calibration to market data. Secondly, we recover the parameters of the processes $Y_j(T)$, $j = 1, \cdots, 3$, $Z(T)$ and the loading coefficients from the correlation matrix.

In the calibration procedure, we at first estimate the parameters of the margins using information from CDS and option markets and thereafter, resorting to the historically estimated correlation structure and to suitable convolution restriction aimed at preserving the distribution of the margins, we calibrate the parameters of the common factor and the idiosyncratic processes. This approach is reasonable from the point of view of practitioners as the product we are aiming to price is illiquid, and therefore market quotes necessary to calibrate the full correlation matrix are not available.

Full details are given in the following of this section.

3.1 Calibration to market data of the margin process

In the first step we calibrate the margin processes $X_j(T)$ to market data for each asset. This is achieved by fitting the term structure of credit spreads extracted from CDS quotations for the two counterparties (S_1 and S_2), and by fitting call and put option prices on the reference entity (S_3). The procedure is described in details as follows.

By a standard bootstrap procedure, see for example O'Kane and Turnbull (2003), we extract default probabilities from market quotations of CDS. Out of the term structure of default probabilities, we compute the term structure of credit spread[4], defined as

$$CS(0,T) = -\frac{1}{T}\ln(1 - PD\,(0,T) + (1-R)\,PD\,(0,T)). \tag{21}$$

Thereafter, we solve the following non linear least square fit problem

$$\min_{h,\lambda}\sum_{i=1}^{n}\left(CS^{mkt}\,(0,T_i) - CS^{model}\,(0,T_i;h,\lambda)\right)^2,$$

with respect to the unknown log-leverage $h = \ln(K/S(0))$ and the parameters of the margin process (i.e. $\lambda \equiv (\theta_j, \sigma_j, k_j)$ for $j = 1,2$ if we adopt the NIG model and $\lambda = \sigma_j$ if we opt for the Gaussian one). In the above minimization problem, $CS^{mkt}(0,T)$ denotes the credit spreads computed according to fromula (21) using bootstrapped market default probability for maturity T, whilst $CS^{model}\,(0,T;h,\lambda)$ denotes the one computed according to either the NIG or the Gaussian model. In particular, the model credit spreads can be computed using the following expressions for the marginal default probability in the Gaussian case

$$PD^{gauss}\,(0,T;h,\sigma) = \Phi\left(\frac{h - \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

and in the NIG setting

$$PD^{nig}\,(0,T;h,\sigma) = \bar{F}\left(\frac{h - (r - q - \varphi_X(-i))\,T}{\sigma}, -\frac{\theta}{\sigma}, \frac{T}{\sqrt{k}}, \frac{1}{\sqrt{k}}\right)$$

[4] Here, the credit spread refers to the additional return paid by a defaultable zero-coupon with respect to a default free one.

with

$$\varphi_X(-i) = \frac{1}{k}\left(1 - \sqrt{1 - 2\theta k - \sigma^2 k}\right),$$

and $\bar{F}(\cdot)$ as given by equation (20).

The calibration to the option price surface is performed by solving the following minimization problem

$$\min_{\lambda} \sum_{i=1}^{M} \sum_{\iota=1}^{N} \left(O^{mkt}\left(K_j, T_\iota\right) - O^{model}\left(K_i, T_\iota; \lambda\right)\right)^2,$$

where $\lambda \equiv (\theta_3, \sigma_3, k_3)$ in the case of the NIG model and $\lambda = \sigma_3$ in the case of the Gaussian one; $O^{mkt}(K,T)$ are the market prices of options with strike K and time to maturity T, with the convention that we use only out-of-the-money call and put options. Similarly, $O^{model}(K,T;\lambda)$ denote the corresponding option model prices, which are calculated using the Black-Scholes formula in the Gaussian case and according to expression (A.4) reported in the Appendix in the NIG case.

For both calibration procedures, the initial parameter set has been randomized 100 times around sensible starting values; the output from the best 5 calibrations has been averaged and used in our numerical experiment.

The market data used refer to quotation on June 26, 2014. In particular, we identify as counterparties a corporate firm, ENI, and a financial firm, Deutsche Bank (DB). They enter a forward contract on Brent Crude Oil. The fitting to market credit spreads is given in Table 1, where we report the ENI and DB market credit spreads (columns 2 and 5) and the corresponding fitted credit spreads, using the Gaussian (column 3 for ENI and column 6 for DB) and the NIG (column 4 for ENI and column 7 for DB) model specifications. Option quotes refer to Brent Financial (European) Options quoted on the Chicago Mercantile Exchange, expiring on August 11, 2014[5]. Market prices and fitting performance of the Gaussian and NIG specifications are given in Table 2.

The parameters resulting from the different calibrations are given in Table 3 in the Gaussian case and in Table 4 in the NIG one; the corresponding quality of the calibration is reported in Table 1 and 2 for convenience of reading. This is measured by the RMSE between market and model credit spread and option prices. In particular, we note the better fit allowed by the NIG model as compared to the Gaussian one, especially in generating more realistic and accurate levels of short-term credit spreads, and therefore showing the importance of including jumps in the relevant dynamics of interest. The introduction of jumps in the underlying dynamics also helps to significantly reduce the calibration error for option

[5]The underlying futures contract is expiring on August 14, 2014. The Futures price is 113.76 USD per barrel.

prices.

3.2 Calibration of the idiosyncratic and common factors

In the second step, given the set of parameters of the margin process $X_j(T)$ identified as described above and reported in Tables 3 and 4, we recover the parameters of the common factor and of the idiosyncratic components exploiting the model assumption $X = Y + aZ$. The actual fitting method depends on chosen model.

We note that, as correlation is not directly observable due to lack of liquidity of suitable derivative instruments, in the following we use the sample correlation based on a sample size of 2 years log-returns to estimate the correlation between the log-returns of the firm values (ENI and DB) and contract's underlying variable (Brent Crude Oil spot price). The estimated correlation matrix is given in Table 5.

In the following, we discuss in details the procedure for recovering the parameters of the idiosyncratic and common processes for the case of Gaussian and NIG models; we note though that this procedure can be easily adapted to other Lévy specifications, like Variance Gamma or CGMY processes.

3.2.1 Gaussian specification

If we assume that Y and Z are independent Gaussian processes, X is Gaussian as well and we have simply to decompose its variance in two components, one originated by the idiosyncratic factor, and the other one by the common factor. The estimated correlation matrix is used to calibrate the free parameters, i.e. the loading factors a_j, $j = 1, \cdots, 3$. For simplicity, but without lack of generality, we assume $\gamma_Z = 1$; consequently, once we have determined the sensitivity coefficients a_j according to the given correlation matrix, the variances of the idiosyncratic components are obtained by

$$\gamma_j^2 = \sigma_j^2 - a_j^2, j = 1, 2, 3,$$

where σ_j^2 are the estimated variances of the margin processes given in Table 3. The resulting parameters are summarized in Table 5, columns 6 and 7.

3.2.2 NIG specification

If we assume that the idiosyncratic and the common factor are both NIG, in general X is not NIG[6]. In practice, we choose the parameters of the common component such that the distribution of the sum $Y + aZ$ does not deviate from a NIG one, in a sense to be specified

[6]Indeed, we recall that the NIG distribution is closed under convolution if and only if the distribution of Y and Z share the same parameters, a quite restrictive assumption in our framework.

below, with the constraint of preserving the observed correlation structure, as captured by the loading coefficients a_j, $j = 1, \cdots, 3$, and the cumulants of the margin process, by controlling for the parameters of the idiosyncratic process. This leaves us with 3 parameters, those referring to the common component Z. We can fix the free parameters so that the integrated distance between the characteristic function of the margin, i.e. $\phi_{X_j}(u)$, and the characteristic function due to the factor structure, i.e. $\phi_{Y_j}(u)\phi_Z(a_j u)$, is as small as possible for every value of the Fourier variable u. In practice, we fix the parameters of the common component so that the following expression

$$\sum_{i=1}^{3} \int \left|\phi_{X_i}(u) - \phi_{Y_j}(u)\phi_Z(a_j u)\right|^2 du,$$

is minimized. This criterion refers to the econometric literature where model parameters are estimated by fitting the theoretical characteristic function to the empirical one, see for example Feuerverger and Mureika (1977). From a computational point of view, the minimization of the above objective function is very fast.

The coefficients $a_j, j = 1, \cdots, 3$ are chosen so that the estimated correlation between the different names is recovered, whilst the parameters of the idyosincratic components are chosen by imposing equality between the first four cumulants of X_j and of $Y_j + a_j Z$. If we let c_k^s, $k = 1, ..., 4, s = X, Y, Z$ to be the cumulant of order k of the random quantity s, the factor structure and the independence of Y and Z imply

$$c_k^{X_i} = c_k^{Y_i} + a_i^k c_k^Z.$$

In practice, it is not possible to guarantee that the above equality holds whatever the order of the cumulant. So we restrict the parameters of the idiosyncratic components in such a way that the above equality is satisfied only for $k = 1, ...4$. Therefore, the implied cumulants of Y_j given the factor structure are

$$\delta_k^{Y_j} := c_k^{X_j} - a_j^k c_k^Z,$$

and we impose the equality with theoretical cumulants, i.e. $\delta_k^{Y_j} = c_k^{Y_j}, k = 1, \cdots, 4$. This allows us to recover the parameters of each component Y via

$$\beta = \frac{3\delta_2^2\delta_3}{-4\delta_3^2 + 3\delta_2\delta_4},$$
$$\gamma^2 = \frac{\delta_2(-5\delta_3^2 + 3\delta_2\delta_4)}{-4\delta_3^2 + 3\delta_2\delta_4},$$
$$\nu = \frac{-4\delta_3^2 + 3\delta_2\delta_4}{9\delta_2^3}$$
$$\mu = \delta_1 - \theta,$$

where we have omitted the dependence on the component index j. We recall that fitting the cumulants amounts to fit the characteristic function and its derivatives up to the maximum cumulant order considered but only at the origin. If the distribution to be recovered is fully determined by its moments, this is plausible choice. A different motivation for using this procedure is given in Eriksson et al. (2009). The resulting parameters are given in Table 5, columns 8 to 11. The resulting cumulative distribution and probability density functions of the linear combination $Y + aZ$ are plotted against the ones of X in Figure 1, where we also report the resulting error of $Y + aZ$ in reproducing X. Results are shown for the case of Brent Crude Oil; similar results are also obtained for the two counterparties and are available from the Authors upon request.

4 Numerical Results

The aim of this section is to use the models developed in Section 2, and calibrated to market data in Section 3, to study the counterparty risk adjustments originating from a forward contract entered by DB and ENI, written on Brent Crude Oil. Particular attention is paid to the impact of right-way risk and wrong-way risk (captured by different values of the correlation between the counterparty and the underlying asset) on the credit adjustments of interest.

4.1 CVA and DVA: bilateral vs unilateral adjustments

The three parties in our example are DB as the forward short position (S_1), ENI as the forward long position (S_2), and Brent Crude Oil as the reference name of the forward contract (S_3). The forward price is fixed at its no-arbitrage value.

Using the calibrated parameters reported in Tables 3, 4 and 5, we obtain the corresponding unilateral and bilateral CVA and DVA as discussed in Section 2. Results are reported in Table 6, the second column reporting the relevant figures obtained under the Gaussian model using the closed-form solutions developed in Section 2.3; the remaining columns, instead, reporting the estimates obtained under the NIG model. As discussed in Section 2.4, in this case CVA and DVA can only be approximated by choosing a suitable numerical approximation; for this experiment, we have adopted the COS method of Fang and Oosterlee (2008) which is benchmarked against Monte Carlo. The corresponding Monte Carlo estimates together with the 95% confidence interval are reported in the final three column. We note that the 95% confidence interval generated by the Monte Carlo simulation always contains the price produced by the COS algorithm (the Monte Carlo algorithm uses 10^7 iterations, whilst the COS method is set with $N = 2^{10}$ terms in the series expansion and a truncation range governed by $L = 10$ - see Section 5.1 Fang and Oosterlee, 2008).

For the case under consideration, regardless of the chosen model, the CVA to Firm 2 is always smaller than the corresponding DVA, particularly in the unilateral case. These value differences reflect the interdependency between the three names involved in the transaction: the joint probability that Firm 1 defaults, Firm 2 survives and the call option defining the CVA expires in the money is 0.27%, whilst the joint probability that Firm 1 survives, Firm 2 defaults and the put option defining the DVA expires in the money is 0.45%; in the unilateral case, instead, these probabilities are 0.28% in the case of the joint default of Firm 1 and non-zero call payoff against 0.60% for the joint default of Firm 2 and non-zero put payoff.

Further, we observe a reduction in the value of the CVA (DVA) as we include in the calculation one's own default, i.e. as we move from the unilateral case in which only the counterparty's default is included, to the bilateral case, in which the default of both parties is taken into account. As it can be seen from Table 1, Firm 1 (DB) has worst credit quality and therefore higher default probabilities when compared to Firm 2 (ENI). Consequently, the unilateral and bilateral CVAs are at similar level, whilst unilateral and bilateral DVAs can differ up to 30% of the adjustment value. The figures in this Table are also relevant for regulatory and accounting purposes. Indeed, we recall that Basel III sets a capital charge depending on the unilateral CVA of the contract and no compensation can be considered due to the investor's own default. The picture is though quite different when we consider accounting rules. In this case, according to IAS/IFRS, balance sheets must explicitly acknowledge the bilateral counterparty value adjustment (BVA) of the contract. In the example illustrated by Table 6, if we believe in a NIG world, the long (short) side must post a regulatory capital according to the unilateral exposure, i.e. 4.2039 (14.0070). For balance sheet purposes however, the relevant figure is 4.1031-9.8202 (the opposite value for the short side)[7]. This sets up a clear dilemma for a bank which has to decide whether they should enter into a derivative contract with a counterparty. Indeed, the decision can be substantially different if regulatory or accounting rules are taken into account.

Finally, Table 6 shows the substantial price difference between bilateral and unilateral

[7]Similar considerations would hold in a Gaussian world.

CVAs and DVAs obtained under the Gaussian model and the NIG model: this range between 75% and 90% of the value of the adjustment. These prices discrepancies are to be traced back to the ability of the NIG model accommodate for non-zero (negative) skewness and excess kurtosis in the distribution of the log-returns, as compared to the Gaussian model, as shown in Tables 3 and 4 as well, and consequently to offer a more realistic portray of the actual market credit spreads and default probabilities, as discussed in Section 3. These features then result in higher probabilities of default, credit spreads and, ultimately, higher values of the CVA (DVA) when compared to the ones obtained under the Gaussian model in correspondence of different maturities of the forward contract. This is clearly illustrated in Figure 2 for the case of the CVA. The price difference is particularly evident over the very short time horizon, as expected.

A possible way to improve the performance of the Gaussian model would be to find the value of the volatility parameters, σ_j, $j = 1, 2, 3$, such that the (marginal) default probabilities of S_1 and S_2 and the price of the call option on S_3 are matched under the two model specifications. In an independent world, this would imply the equality between the CVAs obtained under the NIG model and the Gaussian one. In presence of correlation though, the price difference would be still quite significant, as shown in Figure 3. This is due to the fast decay of the tail dependence typical of the Gaussian distribution (see Embrechts et al., 2001, for example), which leads to underestimating the impact of the risk of default and "contagion" between the entities involved, and ultimately the impact of right-way/wrong-way risk.

4.2 Right-way risk and wrong-way risk

In this section, the study of right-way risk and wrong-way risk is carried out in form of a sensitivity analysis of CVA and DVA versus the correlation coefficient between the counterparty and the underlying asset of the forward contract, ρ_{13}. This is achieved by perturbing ρ_{13} about its estimated value, re-fitting the model parameters according to the new correlation matrices as described in Section 3, and re-computing the values of CVA and DVA keeping all other parameters unchanged. As in the previous section we have shown its ability to offer a more realistic description of market features, in the following we work only with the NIG model.

In Figures 4-5, we show the CVA for different levels of the default barriers (top panel) and different levels of correlation between the forward seller and the forward underlying asset (bottom panels). Consistent with intuition, the value of the CVA to the forward buyer increases with the default probability of the seller (i.e. when the default threshold K_1 increases) and decreases when its own probability of default increases (i.e. when the default threshold K_2 increases). In more details, Figure 4 - bottom left and right panels - shows the CVA in the case in which the three parties are positively correlated. In this situation, when the correlation between the seller and the underlying asset increases, for a fixed percentage deterioration in the seller firm value, the underlying asset value decreases more and so does the CVA to the buyer as the call option moves out-of-the-money. In other words, the higher the default probability of Firm 1, the lower the expected exposure for Firm 2 in case Firm 1 defaults (right-way risk), and consequently the lower the corresponding value of the CVA. Further, the joint probability of the seller to default and the call option on the reference name to expire in the money decreases when ρ_{13} increases, as illustrated in the top two panels in Figure 6. Analogous consideration hold when there is negative correlation between the three names. The CVA to the forward buyer, in fact, will have a higher value when the forward seller and the underlying asset are highly "anti-correlated", as shown in Figure 5 (bottom right and left panels). In this case, the higher Firm 1 default probability, i.e. the worst its credit quality, the higher Firm 2 expected exposure in case Firm 1 defaults, as the probability that the call option of the buyer moves in-the-money increases (wrong-way risk).

A rise in the value of the correlation between the seller and the reference name, ρ_{13}, also causes the joint probability of the seller to survive and the put option on the reference name to expire in the money to decrease, as shown in the bottom two panels in Figure 6. Therefore the DVA to the forward buyer is lower for higher positive levels of ρ_{13}, and higher for 'more negative' levels of ρ_{13}. This is depicted in Figures 7-8, which also show that the DVA decreases when the seller default probability increases (i.e. K_1 increases), whilst it increases with the buyer default probability (i.e. in correspondence of higher levels of K_2).

The corresponding bilateral counterparty value adjustment (BVA) is presented in Figures 9-10. As expected, the BVA to the buyer is negative when it is more likely that the buyer will default (higher values of K_2 - bottom left panel). The BVA becomes positive though when the probability of default of the seller increases (top and bottom right panel).

5 Conclusions

In this paper, we have developed a multivariate structural default framework with jumps to quantify counterparty credit risk, bilateral/unilateral credit value adjustment and bilateral/unilateral debt value adjustment for equity derivatives on a given reference name. The empirical illustration shows the effect of using a non-Gaussian process, i.e. with non-zero skewness and excess kurtosis, in measuring CVA and DVA. The sensitivity analysis performed with respect to the correlation between the contract seller and the reference name highlights the effect of right-way risk and wrong-way risk on the value adjustments due to counterparty credit risk. Further, the example considered illustrates the problems related to the differences between the reporting rules of Basel III and IAS/IFRS standards, highlighting the need for a consistent reporting framework.

The approach presented in this paper is based on a simple framework à la Merton, in which default is assumed to occur only at maturity. Current research is focusing on extending the model to the case in which the firms can default at any time during the lifetime of the reference contract, in order to gain a more realistic perspective. In this modified framework, the quantification of CVA and DVA involves the calculation of the price of zero-strike calls and puts of exotic nature in that they are activated and paid only in case of default of one of the counterparties. This problem is computationally demanding as it requires in the first place to solve the so-called 'first-to-default' problem; secondly, in the given multivariate construction the barrier determining the default event is stochastic (see Ballotta et al., 2014, for further details).

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A Pricing of the European call option in the NIG economy

Consider a European call option with strike price K and maturity at time T, written on a non dividend paying stock S, whose dynamics under any risk neutral probability measure $\hat{\mathbb{P}}$ is described by

$$S(t) = S(0)e^{(r-\varphi(-i))t+X(t)},$$

where $\varphi(u)$ is the characteristic exponent of the NIG process $X(t)$. By risk neutral valuation, the price of this contract is

$$C_0\left(S(0), K, T\right) = S(0)\mathbb{P}^{(S)}\left(S(T) > K\right) - e^{-rT}K\hat{\mathbb{P}}\left(S(T) > K\right),$$

where $\mathbb{P}^{(S)}$ is the stock-risk-adjusted probability measure defined by the density process

$$\gamma_t^{(S)} \doteq \frac{S(t)}{e^{rt}S(0)} = e^{-\varphi(-i)t+X(t)},$$

(see Geman et al., 1995). As the NIG process is a subordinated Brownian motion, let f_T denote the density function of the subordinator; then it follows that

$$\begin{aligned}
\hat{\mathbb{P}}\left(S_T > K\right) &= \int_0^\infty \hat{\mathbb{P}}\left(X(T) > \ln\frac{K}{S(0)} - \left(r - \varphi(-i)\right)T \middle| G(T) = z\right) f_T(z)dz \\
&= \frac{T}{\sqrt{2\pi k}}e^{T/k}\int_0^\infty \Phi\left(\frac{\ln\frac{S(0)}{K} + \left(r-\varphi(-i)\right)T + \theta z}{\sigma\sqrt{z}}\right) z^{-3/2}e^{-\frac{1}{2}\left(\frac{T^2}{zk}+\frac{z}{k}\right)}dz.
\end{aligned}$$

Let

$$\bar{F}\left(\xi, \chi, a, b\right) \doteq \frac{a}{\sqrt{2\pi}}e^{ab}\int_0^\infty \Phi\left(\frac{\xi}{\sqrt{z}} + \chi\sqrt{z}\right) z^{-3/2}e^{-\frac{1}{2}\left(\frac{a^2}{z}+b^2 z\right)}dz,$$

then

$$\begin{aligned}
\hat{\mathbb{P}}\left(S_T > K\right) &= \bar{F}\left(d, \frac{\theta}{\sigma}, \frac{T}{\sqrt{k}}, \frac{1}{\sqrt{k}}\right), \\
d &= \frac{\ln\frac{S(0)}{K} + \left(r-\varphi(-i)\right)T}{\sigma}.
\end{aligned}$$

Further, we note that, under the probability measure $\mathbb{P}^{(S)}$, the characteristic function of the NIG process $X(t)$ is given by

$$\begin{aligned}
\phi_X^{(S)}\left(u; t\right) &= e^{-\varphi(-i)t}\hat{\mathbb{E}}\left(e^{(iu+1)X(t)}\right) \\
&= e^{t\varphi^{(S)}(u)} \\
\varphi^{(S)}(u) &= \frac{1}{\sqrt{kk^{(S)}}}\left(1 - \sqrt{1 - 2iu\theta^{(S)}k^{(S)} + u^2\sigma^2 k^{(S)}}\right)
\end{aligned}$$

which implies that

$$X(t) = \theta^{(S)} G(t) + \sigma W(G(t)), \tag{A.1}$$

$$\theta^{(S)} = \theta + \sigma^2, \tag{A.2}$$

where $W(t)$ is a $\mathbb{P}^{(S)}$-standard Brownian motion and $G(t)$ is a $\mathbb{P}^{(S)}$-IG process with parameters $\left(t/\sqrt{k}, 1/\sqrt{k^{(S)}}\right)$ for

$$k^{(S)} = \frac{k}{1 - 2\theta k - \sigma^2 k}. \tag{A.3}$$

These results are consistent with the application of the Girsanov theorem (see Jacod and Shiryaev, 1987) to the NIG process. Hence, it follows that

$$\begin{aligned} \mathbb{P}^{(S)}(S(T) > K) &= \frac{T}{\sqrt{2\pi k}} e^{T/\sqrt{kk^{(S)}}} \int_0^\infty \Phi\left(\frac{\ln\frac{S(0)}{K} + (r - \varphi(-i))\,T + \theta^{(S)} z}{\sigma\sqrt{z}}\right) \\ &\quad \times z^{-3/2} e^{-\frac{1}{2}\left(\frac{T^2}{zk} + \frac{z}{k^{(S)}}\right)} dz \\ &= \bar{F}\left(d, \frac{\theta + \sigma^2}{\sigma}, \frac{T}{\sqrt{k}}, \sqrt{\frac{1-s}{k}}\right), \end{aligned}$$

for $s = k\left(2\theta + \sigma^2\right)$. Therefore, the price of the given option is

$$C_0(S(0), K, T) = S(0)\bar{F}\left(d, \frac{\theta + \sigma^2}{\sigma}, \frac{T}{\sqrt{k}}, \sqrt{\frac{1-s}{k}}\right) - e^{-rT} K \bar{F}\left(d, \frac{\theta}{\sigma}, \frac{T}{\sqrt{k}}, \frac{1}{\sqrt{k}}\right). \tag{A.4}$$

The price of a European call on a dividend paying stock (with dividend yield $q > 0$) follows from the transformation of the option payoff

$$e^{-qT}\left(S(T) - Ke^{qT}\right)^+.$$

Table 1
Term Structure of credit spreads (CS) for ENI and Deutsche Bank (DB). First column: maturities. Second (Fifth) column: ENI (DB) CS computed using default probabilities bootstrapped from CDS quotations. Third (Sixth) column: fitted CS for ENI (DB) in the Gaussian framework. Fourth (Seventh) column: fitted CS for ENI (DB) in the NIG framework. RMSE: root mean squared errors between market and model credit spreads. CDS Data Source: Markit, June 26, 2014. Default probabilities computed using Markit calculator.

		Model CS			Model CS	
Maturity	ENI CS	Gaussian	NIG	DB CS	Gaussian	NIG
6M	0.2739%	0.0013%	0.2584%	0.4423%	0.0016%	0.4108%
1Y	0.4071%	0.1018%	0.4102%	0.5783%	0.1189%	0.5743%
2Y	0.7700%	0.7148%	0.7809%	0.8907%	0.8171%	0.9317%
3Y	1.0189%	1.1522%	1.0814%	1.1637%	1.3263%	1.2280%
4Y	1.2511%	1.3195%	1.2451%	1.4373%	1.5425%	1.4233%
5Y	1.4228%	1.3851%	1.3407%	1.6443%	1.6384%	1.5593%
7Y	1.4853%	1.3913%	1.4136%	1.7715%	1.6748%	1.7106%
10Y	1.5179%	1.5524%	1.6232%	1.8625%	1.8400%	1.9499%
RMSE		0.0565%	0.0206%		0.0846%	0.0200%

Table 2

Option Quotations on Brent Crude Oil. First column: option strike. Second column: option type (1: call, -1 put). Third column: option premium. Fourth (Fifth) column: fitted premium in the Gaussian (NIG) framework. Data Source: Chicago Mercantile Exchange. Settlement date: August, 11 2014. Underlying Futures quotation: 113.76 USD per barrel. RMSE: root mean squared errors between market and model option prices. Quotations refer to June 26, 2014.

Strike	Call/Put	Premium	G	NIG	Strike	Call/Put	Premium	G	NIG
98.5	1	15.32	15.28	15.32					
99	1	14.84	14.79	14.84	109.5	1	5.48	5.52	5.47
99.5	1	14.35	14.30	14.35	110	1	5.13	5.17	5.12
100	1	13.87	13.81	13.86	110.5	1	4.78	4.83	4.78
100.5	1	13.39	13.33	13.38	111	1	4.45	4.51	4.45
101	1	12.9	12.84	12.90	111.5	1	4.14	4.20	4.14
101.5	1	12.43	12.36	12.42	112	1	3.83	3.90	3.84
102	1	11.95	11.89	11.94	112.5	1	3.54	3.62	3.56
102.5	1	11.48	11.42	11.47	113	1	3.27	3.35	3.29
103	1	11.01	10.95	11.00	113.5	1	3.02	3.09	3.04
103.5	1	10.54	10.48	10.53	114	1	2.85	2.85	2.81
104	1	10.03	10.03	10.07	114.5	-1	3.35	3.36	3.33
104.5	1	9.62	9.57	9.61	115	-1	3.65	3.65	3.63
105	1	9.17	9.13	9.16	115.5	-1	3.95	3.94	3.94
105.5	1	8.72	8.69	8.72	116	-1	4.27	4.25	4.26
106	1	8.28	8.26	8.28	116.5	-1	4.61	4.57	4.60
106.5	1	7.84	7.84	7.85	117	-1	4.95	4.91	4.94
107	1	7.41	7.43	7.43	117.5	-1	5.31	5.25	5.30
107.5	1	7.01	7.02	7.02	118	-1	5.67	5.61	5.68
108	1	6.61	6.63	6.61	118.5	-1	6.05	5.98	6.06
108.5	1	6.22	6.25	6.22	119	-1	6.43	6.35	6.45
109	1	5.85	5.88	5.84	119.5	-1	6.83	6.74	6.85
RMSE								0.7601%	0.2179%

Table 3

Calibrated Gaussian model parameters. DB and ENI calibrated to credit spreads data reported in Table 1. Brent Crude Oil calibrated to option prices reported in Table 2.

Name	K	q	σ
DB	0.3732	0.0056	0.3235
ENI	0.4285	0.0036	0.2765
BRENT	n.a.	0.0018	0.1803

Table 4
Calibrated NIG model parameters. DB and ENI calibrated to credit spreads data reported in Table 1. Brent Crude Oil calibrated to option prices reported in Table 2. Standard deviation, skewness and excess kurtosis calculated using equations (5)-(7) and the reported parameters.

Name	K	q	θ	σ	k	Std. Dev	Skew	Exc. Kurt.
DB	0.2173	0.0060	-0.1204	0.4361	1.0630	0.4534	-0.8471	4.1456
ENI	0.3720	0.0044	-0.0101	0.3112	0.9551	0.3113	-0.0926	2.8766
BRENT	n.a.	0.0016	0.0683	0.1871	0.0796	0.1881	0.0866	0.2487

Table 5
Correlation matrix and parameters of the idiosyncratic and systematic processes. Correlation matrix estimated using historical log-returns of DB, ENI and Brent Crude Oil (spot) over a 2 years period up to and including June 26, 2014. Source: Yahoo! Finance and U.S. Energy Information Administration. Idiosyncratic and systematic components parameters recovered via the procedure described in Sections 3.2.1 - 3.2.2.

	Correlation Matrix			Idiosyncratic and systematic components						
					Gaussian model		NIG model			
Name	DB	ENI	BRENT	Process	γ	a	β	γ	ν	a
DB	1.0000	-	-		0.2317	0.2257	-0.1113	0.2819	2.1023	0.6258
ENI	0.6468	1.0000	-	$Y(t)$	0.1037	0.2563	0.0056	0.1163	4.0226	0.5709
BRENT	0.2151	0.2858	1.0000		0.1715	0.0556	0.0759	0.1776	0.0832	0.1147
				$Z(t)$	1		-0.0221	0.5050	1.1763	

Table 6
Bilateral and unilateral CVA and DVA - expressed in basis points - for a forward contract on Brent Crude Oil entered by DB (contract seller) and ENI (contract buyer). Parameter set: Tables 3, 4, 5. Other parameters: $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = 28.58\%$, $\rho_{13} = 21.51\%$. Forward price K_3: 1.0027 (Gaussian model), 1.0029 (NIG model). NIG model: COS approximation obtained setting $L = 10$, $N = 2^{10}$; Monte Carlo (MC) simulation trials: 10^7.

	Gaussian Model	NIG Model			
		COS	MC	95% C.I.	
CVA_2 (bilateral)	0.4354	4.1031	4.0722	4.1739	4.2757
DVA_2 (bilateral)	2.3791	9.8202	9.6910	9.8477	10.0043
CVA_2 (unilateral)	0.4659	4.2039	4.1722	4.2748	4.3774
DVA_2 (unilateral)	2.8438	14.0070	13.8817	14.0730	14.2643

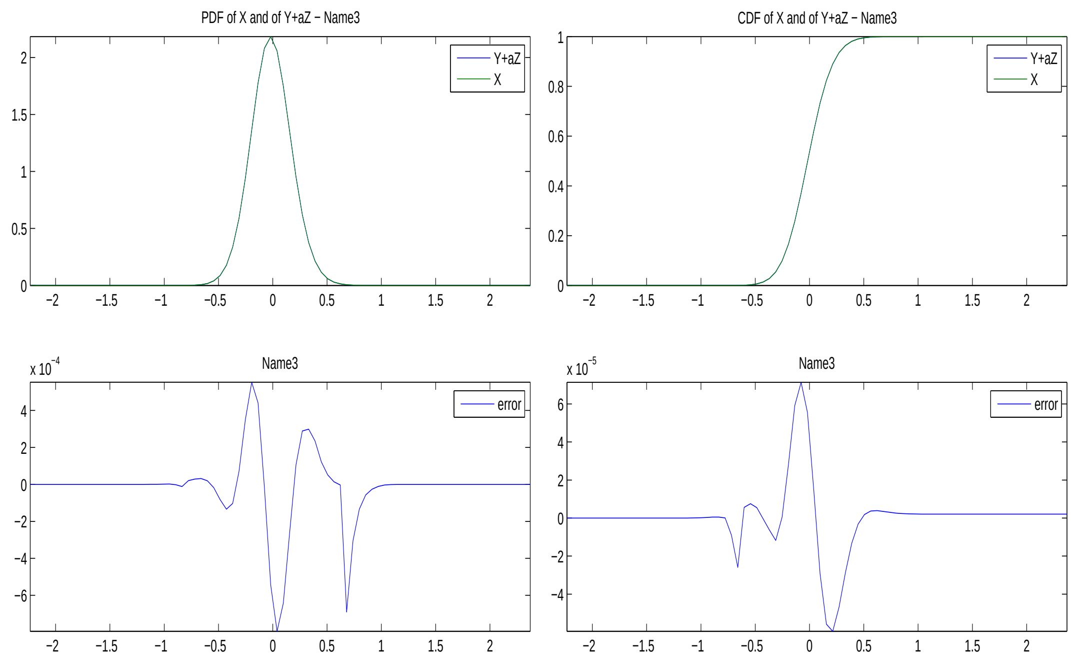

Figure 1
Calibration of Y and Z. Top panels: probability density function (PDF) and cumulative distribution function (CDF) of Asset 3 obtained using the calibrated margin process X and the calibrated linear combination $Y + aZ$. Bottom panels: error is calculated as difference between the corresponding probability functions. Parameter set: Table 5. Other parameters: $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = 28.58\%$, $\rho_{13} = 21.51\%$.

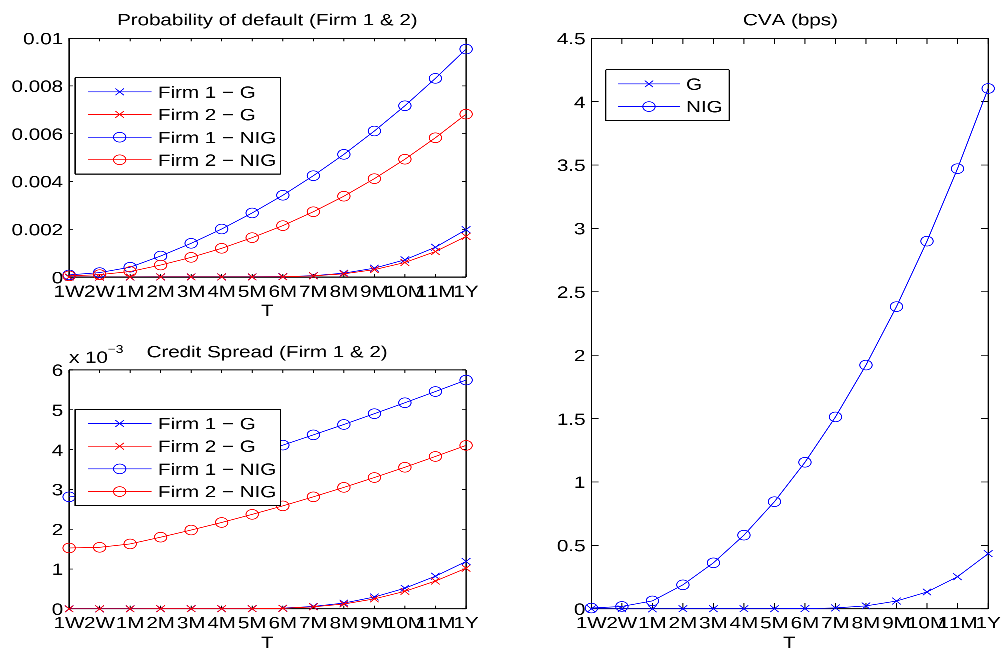

Figure 2
Probability of default, credit spread and CVA under the NIG model and the Gaussian model. Parameter set: Table 5. Other parameters: $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = 28.58\%$, $\rho_{13} = 21.51\%$.

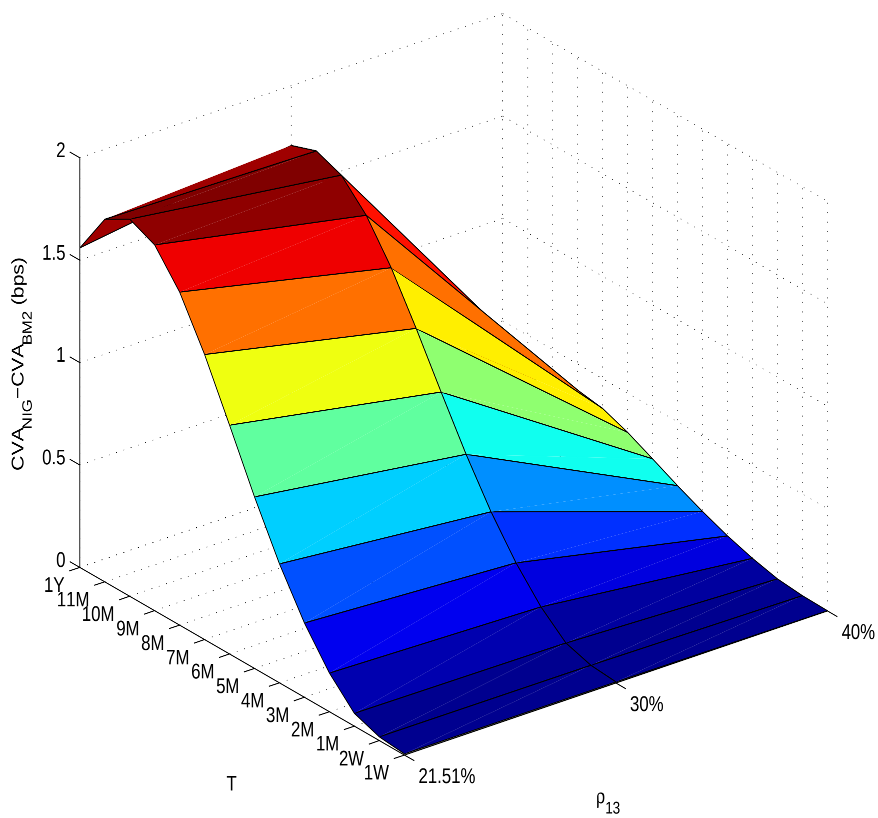

Figure 3
An "inflated" Gaussian model. Change in the CVA calculated as the difference between the CVA under the NIG model and the modified Gaussian model with volatility parameters $\sigma_1 = 38.79\%$, $\sigma_2 = 32.28\%$, $\sigma_3 = 18.72\%$.

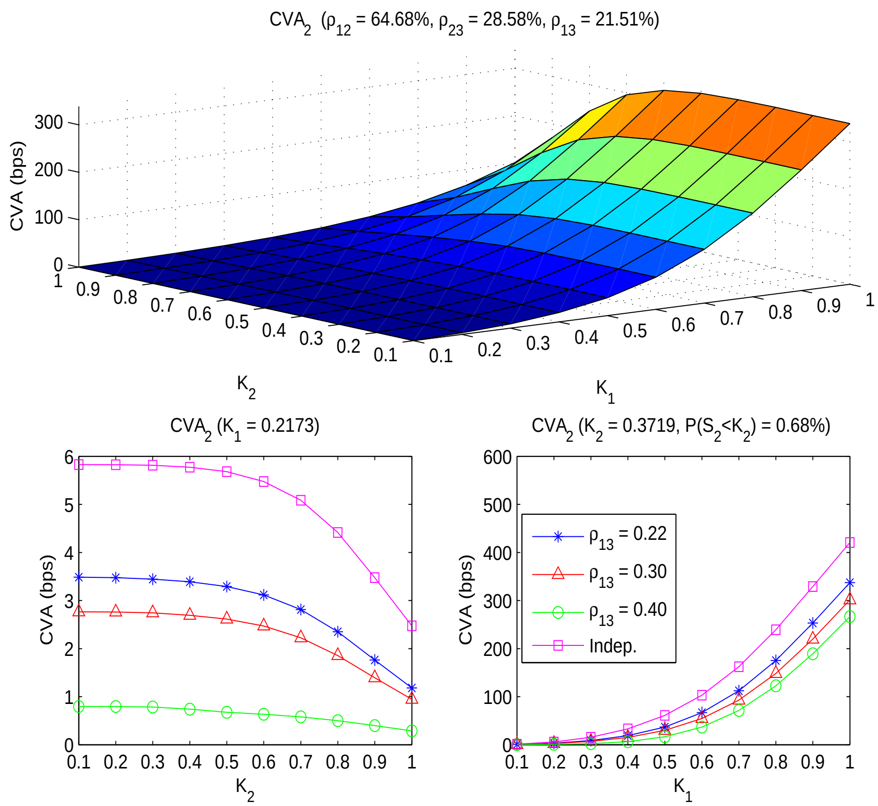

Figure 4
CVA for forward contract on Brent Crude Oil. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Case of positive correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = 28.58\%$.

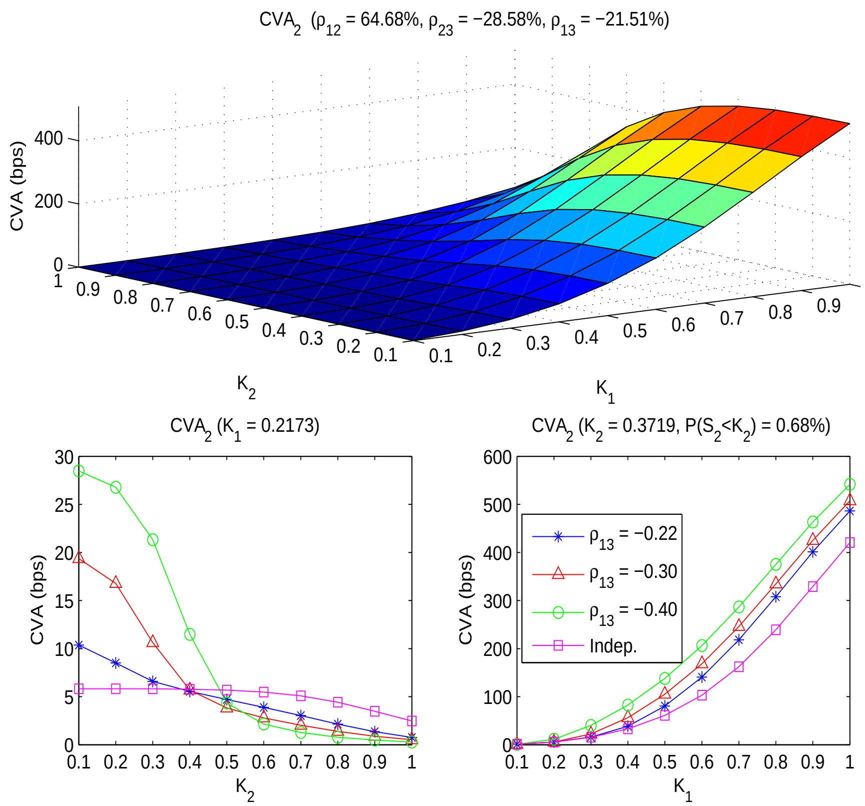

Figure 5
CVA for forward contract on Brent Crude Oil. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Case of negative correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = -28.58\%$.

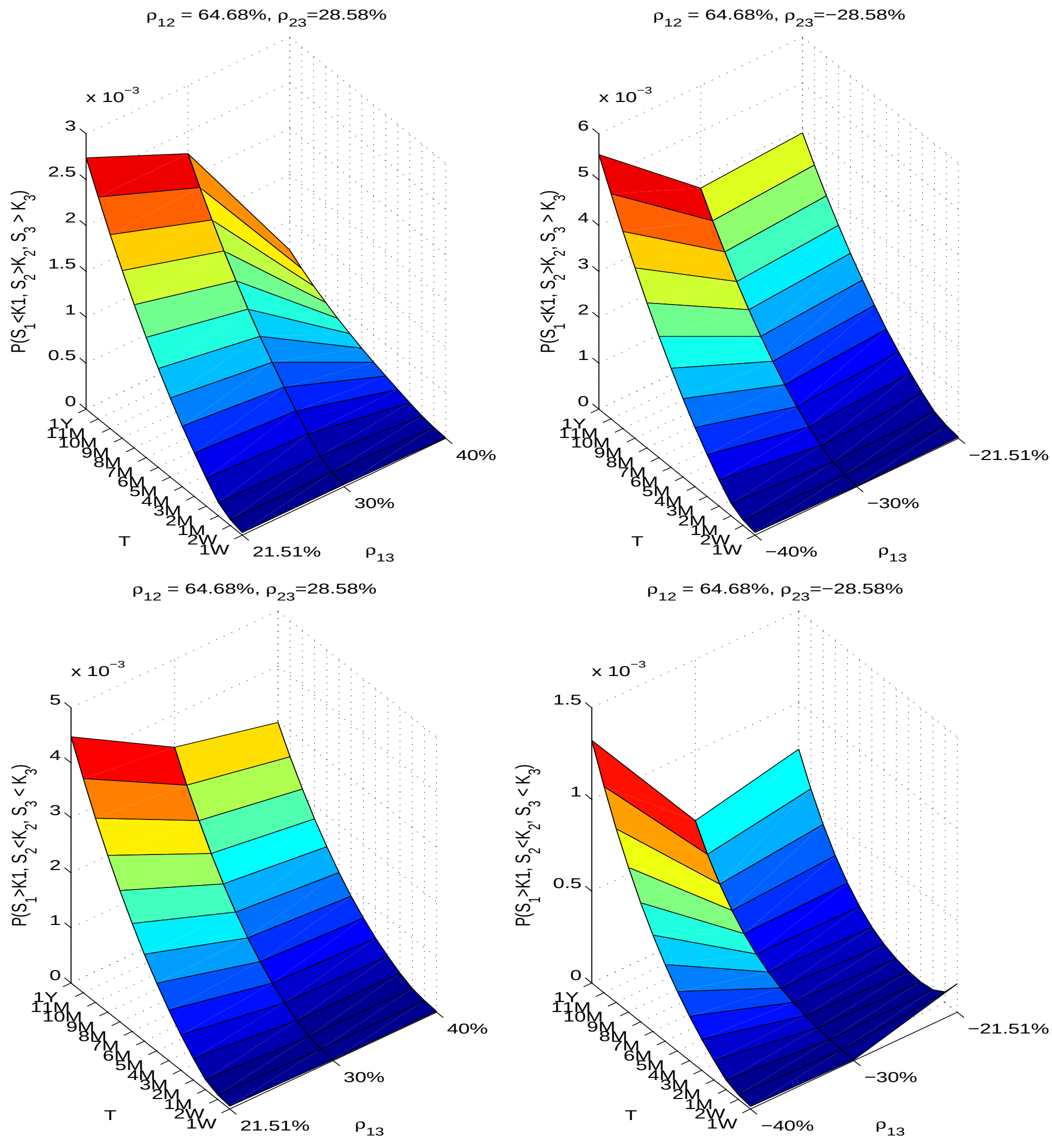

Figure 6
Joint probability of S_1, S_2 and S_3 under the NIG model. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$.

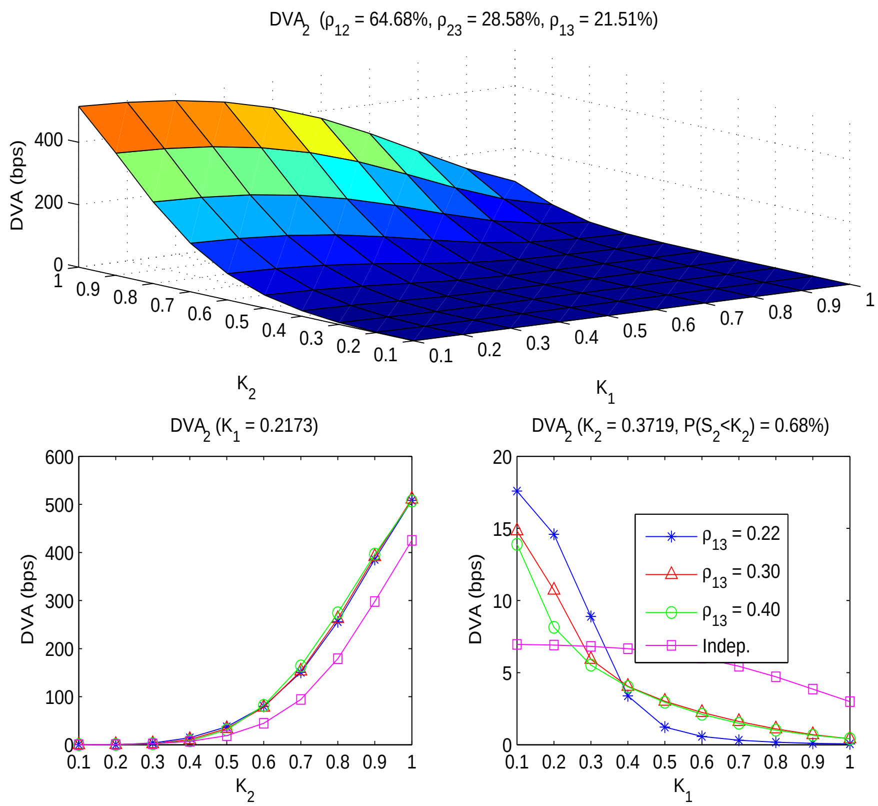

Figure 7
DVA for forward contract on Brent Crude Oil. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Case of positive correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = 28.58\%$.

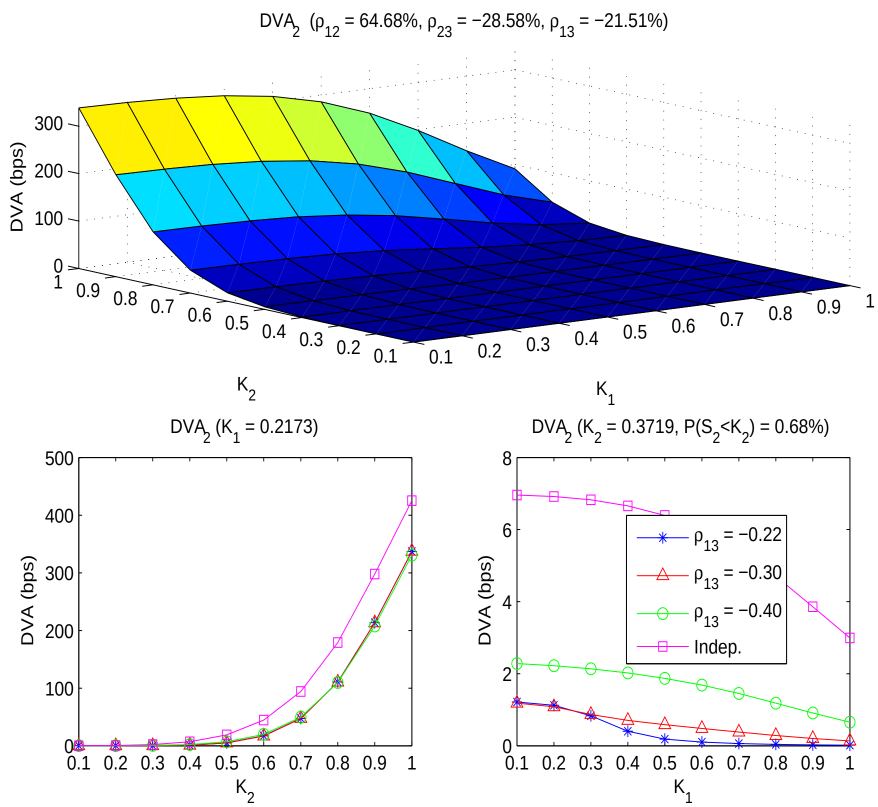

Figure 8
DVA for forward contract on Brent Crude Oil. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Case of negative correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = -28.58\%$.

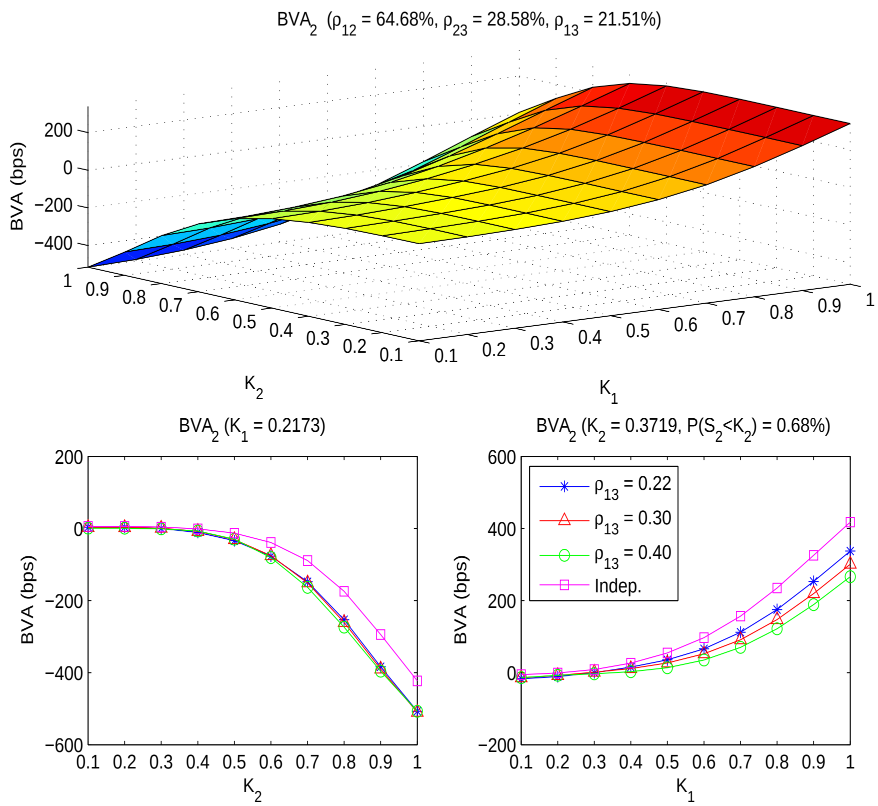

Figure 9
Bilateral counterparty value adjustment for forward contract on Brent Crude Oil. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Case of positive correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = 28.58\%$.

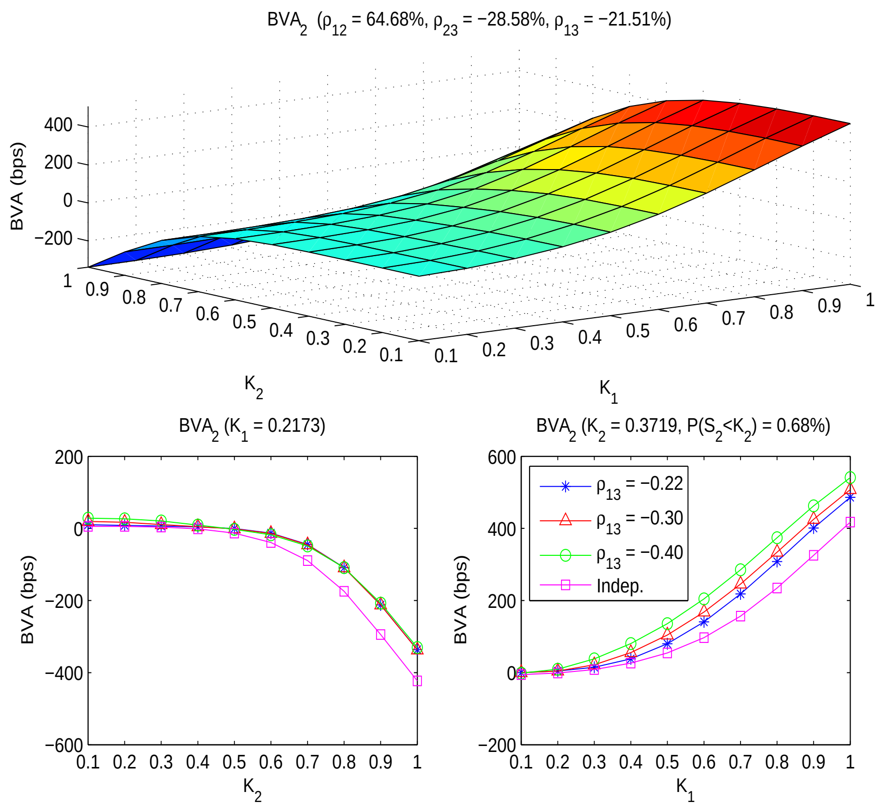

Figure 10
Bilateral counterparty value adjustment for forward contract on Brent Crude Oil. $S_1(0) = S_2(0) = S_3(0) = 1$, $T = 1$ year, $r = 0.45\%$. Case of negative correlation: $\rho_{12} = 64.68\%$, $\rho_{23} = -28.58\%$.