REPORT ON THE BRAUER-THRALL CONJECTURES:

ROJTER'S THEOREM AND THE THEOREM OF NAZAROVA AND ROJTER

(ON ALGORITHMS FOR SOLVING VECTORSPACE PROBLEMS. I)

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During the preparation of these notes I learned of the death of my admired teacher Reinhold Baer, and I would like to dedicate this paper to his memory. His way of thinking has strongly influenced also the development of the representation theory of rings and algebras: let us mention just one of his concepts, namely that of an injective module.

1.1 Rojter's theorem and the theorem of Nazarova and Rojter

This is a report on two theorems which seem to be the most remarkable ones of the present representation theory of finite dimensional algebras. They are due to Rojter, and to Nazarova and Rojter. Recall that an artinian ring R is said to be of finite representation type provided there is only a finite number of isomorphism classes of indecomposable modules. Note that all modules considered here will be assumed to be finitely generated, and usually will be right modules.

Rojter's theorem. A finite dimensional algebra is either of finite representation type or else there are indecomposable modules of arbitrarily large dimension.

Theorem of Nazarova and Rojter. Let R be a finite dimensional algebra over an algebraically closed field k. Then either R is of finite representation type, or else there is an exact embedding functor $\mathcal{M}_{k[T]} \to \mathcal{M}_R$ which maps the irreducible $k[T]$-modules to R-modules of a fixed dimension, and maps indecomposable modules to indecomposable ones, and non-isomorphic modules to non-isomorphic ones.

Here, $k[T]$ denotes the polynomial ring in one variable T over the field T. As we know from the Jordan canonical form theorem, the indecomposable $k[T]$-modules are of the form $k[T]/(T-\alpha)^n$ for some $\alpha \in k$, $n \in \mathbb{N}$, the irreducible ones being given by $n = 1$. Also, two modules $k[T]/(T-\alpha)^n$, $k[T]/(T-\beta)^m$ are isomorphic if and only if $\alpha = \beta$ and $n = m$. So assume there is given an exact embedding functor $\eta : \mathcal{M}_{k[T]} \to \mathcal{M}_R$ with the properties of the theorem of Nazarova and Rojter. Let $M_\alpha(n) = \eta(k[T]/(T-\alpha)^n)$, and $M_\alpha = M_\alpha(1)$. Then, these modules $M_\alpha(n)$ all are indecomposable, and pairwise non-isomorphic. Also, $M_\alpha(n)$ has a series of submodules of the form

$$M_\alpha(1) \subset M_\alpha(2) \subset \ldots \subset M_\alpha(n),$$

with $M_\alpha(r+1)/M_\alpha(r) \approx M_\alpha$ for all r. In particular, if d is the dimension of M_α, then $M_\alpha(n)$ is of dimension nd. Thus we obtain a family of pairwise non-isomorphic indecomposable modules of dimension nd, indexed by the elements of k.

In this way, Nazarova and Rojter have solved for algebraically closed (and also for perfect) base fields the second "Brauer-Thrall conjecture" which asserts that an algebra over an infinite base field is either of finite representation type, or else there are infinitely many dimensions $d_1, d_2, \ldots$ such that there are infinitely many pairwise non-isomorphic indecomposable modules of dimension d_i, for all i. The first conjecture of Brauer and Thrall was just the assertion of Rojter's theorem.

We will discuss these theorems and some of their generalizations in the sequel, outlining methods which have been used for proofs. But let us first consider one specific example which is rather typical, and which also will have motivated Brauer and Thrall. Let k be an algebraically closed field of characteristic 2, and G the Kleinian four-group, say with generators g_1, g_2. The group algebra $R = kG$ has as a k-basis the elements of G, however we prefer to work with the basis 1, $x_1 = 1+g_1$, $x_2 = 1+g_2$, $s = x_1x_2 = 1+g_1+g_2+g_1g_2$. Now kG is a local algebra, the radical being the subspace generated by x_1, x_2, s, and the socle being the subspace generated by s. For all $(\alpha,\beta) \in k^2 \setminus \{(0,0)\}$, the two-dimensional subspace $I_{\alpha:\beta}$ generated by $\alpha x_1+\beta x_2$ and s is an ideal, and of course $I_{\alpha:\beta} = I_{\lambda\alpha:\lambda\beta}$ for $\lambda \in k \setminus \{0\}$, thus the index $\alpha:\beta$ runs through the projective line $\mathbb{P}_1(k)$. In this way, we obtain two-dimensional R-modules with simple socle; in particular, they are indecomposable. Also, $I_{\alpha:\beta}$ is isomorphic to $I_{\alpha':\beta'}$ only in case they coincide, thus only for $(\alpha:\beta) = (\alpha':\beta')$ in $\mathbb{P}_1(k)$. We see in this example that it may be rather easy to construct infinitely many pairwise non-isomorphic indecomposable R-modules of a fixed dimension, thus showing that R cannot be of finite representation type. We can now apply both Rojter's theorem and the theorem of Nazarova and Rojter in order to conclude that there have to exist indecomposable R-modules of arbitrarily large dimension, and even many infinite families of such modules. In fact, one knows since a long time all indecomposable kG-modules (see for example [5], [22]), the number of isomorphism classes of indecomposable kG-modules of dimension d being as follows: it is 1 for $d = 1$, it is 2 for d odd, $d \neq 1$, and it is infinite, for d even. We obtain a functor $\eta : \mathcal{M}_{k[T]} \to \mathcal{M}_{kG}$ with the properties stated in the theorem of Nazarova and Rojter in the following way: For any $k[T]$-module U, let $\eta(U) = U \oplus U$ with $(u,v)g_1 = (u,u+v)$, $(u,v)g_2 = (u,uT+v)$ for $u,v \in U$, then it is easy to check that in this way $U \oplus U$ actually becomes a kG-module, and also that η has all the properties.

1.2 The origin of the conjectures

M. Auslander once has quoted R. Brauer by saying that the conjectures were just considered as a problem for graduate students. Since there also is no paper published by Brauer and Thrall on this problem, it may seem that the interest of Brauer and Thrall in this question was very limited. However, there does exist rather strong evidence that Brauer and Thrall were working on the problem for some time.

Let us recall what was known around 1940 on indecomposable representations of finite dimensional algebras. For commutative artinian rings Köthe [29] had shown that the uniserial rings are the only ones for which any module is a direct sum of cyclic modules, and he posed the problem to characterize all artinian rings with this property. Soon after, Nakayama investigated serial ("generalized uniserial") rings in great detail [30]. In particular, he showed that any (finitely generated or not) module over a serial ring is a direct sum of indecomposable modules and that the indecomposable ones are serial, thus epimorphic images of indecomposable projective ("principal indecomposable") modules. In particular, in this case, the indecomposable modules are of bounded length. At the same time, Brummund [9] had shown that a non-cyclic p-group always has indecomposable p-modular representations of arbitrarily large length. Nakayama stresses the fact that Brummund's arguments work under more general assumptions and adds: "Now arises the problem to determine general type of rings which possess arbitrarily large directly indecomposable left or right modules. But the author has to leave also this problem open" [30].

There is an abstract of Brauer in the Bulletin of the AMS, 1941, where he announces a paper on indecomposable representations of algebras [7]: "Cases are studied in which A (a finite dimensional algebra) has infinitely many non-equivalent indecomposable representations". Later, Thrall apparently refers to this paper (which never was published) in another abstract in the Bulletin, entitled "On ahdir algebras" [38], where an ahdir algebra is one with indecomposable representations of arbitrarily high degree: "R. Brauer has given three conditions each sufficient to ensure that A is ahdir. These conditions are stated in terms of the Cartan invariants of A, A/N, $A/N^2, \dots$, where N is the radical of A", the conditions being the exclusion of diagrams $\widetilde{A}_1$, $\widetilde{A}_n$ with $n \geq 2$, and $\widetilde{D}_4$. Thrall excludes, in addition, the diagrams $\tilde{D}_n$, $n \geq 5$, and this was worked out in detail by Jans [23]. Thrall had thought to have characterized in this way the radical square zero algebras of finite representation type. Nine years later, Yoshii [42] published his attempt to deal with these algebras, he remarks: "Last autumn Professor Brauer informed me that he and Professor Thrall obtained almost the same results as ours, but their works are not published". In contrast to the extensive use of matrices by Yoshii, Brauer and Thrall apparently soon had realized that the best frame for these problems was the consideration of vectorspaces with subspaces. In one of his papers on modular lattices, Thrall writes: "Detailed study (as yet unpublished) of the representation theory of certain classes of algebras has led me to consider the possibility of searching for connections between representation theory and lattice theory. The present note is devoted to setting up the machinery for certain phases of such investigations" [39]. And in the joint note with Duncan which deals with modular lattices freely generated by at most three chains: "One of these lattices ... is of particular interest as a knowledge of its structure is made use of in the theory of representations of algebras", and he adds the following footnote: "This application occurs in an as yet unpublished investigation into algebras of bounded representation type by R. Brauer and R.M. Thrall" [40].

The first actual formulation of the conjectures is in the paper of Jans [23]: he defines finite, bounded, and strongly unbounded representation types and continues: "Concerning these classes of algebras, R. Brauer and R.M. Thrall have conjectured that algebras of bounded type are actually of finite type and that (over infinite fields) algebras of unbounded type are actually of strongly unbounded type".

1.3 The fertility of the conjectures

The problems posed by Brauer and Thrall were, with no doubt, the most stimulating ones for the development of the representation theory of finite dimensional algebras. In fact, all major advances can be traced back to investigations involved with these questions. Let us mention at least some, and indicate the impact they had.

We have mentioned Yoshii who tried to verify both conjectures for radical square zero algebras over an algebraically closed field [42]. His classification of all such algebras of finite representation type was incorrect, but the correction given by Gabriel [17] showed that these algebras were classified by the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.(Recall that it was Brauer who associated for the first time such types of diagrams to finite dimensional algebras). The appearance of the Dynkin diagrams was considered as an intriguing problem, and an attempt to understand this relation lead to the Coxeter functors of Bernstein-Gelfand-Ponomarev, and to the consideration of quadratic forms and roots in general, but also to the investigation of species (where all Dynkin diagrams, not only just those with single lines, appear [12]).

It seems that Rojter's theorem actually came as a surprise. One of his methods, the "Rojter measure", was formalized by Gabriel in [18]; however, otherwise, it really was not used. Very recently, it stimulated Auslander and Smalø [4] for introducing the concept of preprojective and preinjective modules for general Artin algebras, but the precise relationship between the Auslander-Smalø theory and the Rojter measure is yet not clear. In Auslander's proof of Rojter's theorem [1] (and his generalization to artinian rings), the so-called "almost split sequences" made their first appearance. (They are now called Auslander-Reiten sequences due to the fact that the general existence theorem, as well as all their properties, were established in the joint work of Auslander and Reiten). We will see in part 2 of this report that the methods of Auslander and his school are very well adapted for dealing with the questions raised by the Brauer-Thrall conjectures.

The methods introduced by Nazarova and Rojter for the proof of their theorem were of great importance: As first step, they verified the second Brauer-Thrall conjecture for the category of representations of partially ordered sets [31], and, in this way, developed an elaborate representation theory for partially ordered sets. Kleiner used these methods to give a complete description of the partially ordered sets of finite type and all their representations [25, 26] - this was the prototype of various characterizations of classes of partially ordered sets according to their representation type. The second step then was the consideration of arbitrary vectorspace categories and their subspace categories [32]. We will report on some of these concepts in part 3. Note that the further development of these ideas lead to the introduction of differential graded categories which completely formalizes the matrix calculations.

Finally, let us note that the distinction of finite and infinite representation type quite early was refined to that of finite, tame and wild representation type. This difference of behaviour was clear from the work of Krugliak, Brenner and Butler, and the first systematic use of this distinction seems to have been made by Drozd [15]: he determined completely the commutative algebras according to being of finite, tame or wild type.

In the remaining parts of this report, we will concentrate on two methods: the use of irreducible maps and the vectorspace category approach. These two methods have turned out to be very fruitful for our problem, and it is probable that further investigations will strongly rely on them: there seems to be some evidence that it may be possible to furnish a conceptual proof of the theorem of Nazarova and Rojter using Auslander's approach - on the other hand, a deeper penetration into the classification problem of indecomposable modules will make it necessary to determine the faithful vectorspace categories of finite type in the same way as Kleiner did it with partially ordered sets. Note that in this report the Boston and the Kiev method stand side by side without much interrelation. However, it will be clear from the sequel [35] that in fact both methods fit together rather well.

The presentation of parts 2 and 3 is very different. Whereas in part 2, we try to be as self-contained as possible, presupposing only the existence theorems for Auslander-Reiten sequences (see [3], or also [20]), the part 3 just tries to outline some of the basic concepts and preliminary results which show the direction of the investigation of Nazarova and Rojter.

2. AUSLANDER'S APPROACH

Our aim is to give a survey on the methods and results due to Auslander and his school which provide insight into the relationships between different indecomposable modules. In this way, one obtains a proof of Rojter's theorem even in the case of an arbitrary artinian ring, as well as other types of generalizations of this theorem. Also, the methods furnish proofs of at least some partial results concerning the theorem of Nazarova and Rojter. One may hope that a new proof of the theorem of Nazarova and Rojter may emerge from these methods, perhaps along the lines of the recent investigations of Riedtmann [33].

In this part, we always will assume that R is an artinian ring with finitely generated indecomposable injective modules.

2.1 Irreducible maps

The main notion which we will need is that of an irreducible map as introduced by Auslander and Reiten in [3]. Let R be a ring; X, Y R-modules. A homomorphism $f : X \to Y$ is said to be irreducible provided f is neither a split monomorphism or a split epimorphism, and given any factorization $f = f' \circ f''$, then f' is a split epimorphism or f'' is a split monomorphism. Thus, the irreducible maps are those non-split maps which allow no non-trivial factorization. In case R is an artinian ring of finite representation type, then clearly irreducible maps do exist and any non-invertible map between indecomposable modules is a sum of compositions of irreducible maps. If R is not of finite type, it is much harder to show the existence of irreducible maps, and, in fact, there always will be non-invertible maps between indecomposable modules which cannot be written as sums of compositions of irreducible maps! Let us give the precise description of the irreducible maps between indecomposable modules in case of an artin algebra.

Let R be any artinian ring with finitely generated indecomposable injective modules, and P an indecomposable projective R-module, then the irreducible maps of the form $X \to P$ are easy to determine: they are just the inclusion maps of non-zero direct summands X of the radical $\operatorname{rad}P$ of P. If I is an indecomposable injective R-module, then similarly $f : I \to Y$ is irreducible, if and only if Y is a non-zero direct summand of $I/\operatorname{soc} I$ with f the canonical projection.

A non-split exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ with X, Z indecomposable, is called an <u>Auslander-Reiten sequence</u> provided for any map $\alpha : X \to X'$ which is not a split monomorphism, there exists $\alpha' : Y \to X'$ with $\alpha' \circ f = \alpha$. This property is equivalent to the dual one that for any map $\beta : Z' \to Z$ which is not split epi, there exists $\beta' : Z' \to Y$ with $g \circ \beta' = \beta$. (Auslander and Reiten have called these types of sequences "almost split"). It is easy to see that Auslander-Reiten sequences, if they exist, essentially are unique for given X, or given Z. Of course, given an indecomposable module X, an Auslander-Reiten sequence starting with X can only exist if X is not injective. Similarly, given an indecomposable module Z, an Auslander-Reiten sequence ending in Z can only exist, if Z is not projective. We say that R <u>has Auslander-Reiten sequences</u>, provided for any indecomposable non-injective module X, there exists an Auslander-Reiten sequence starting with X, and to any indecomposable non-projective module Z there exists an Auslander-Reiten sequence ending in Z. One of the main theorems due to Auslander and Reiten [3] asserts that any artin algebra has Auslander-Reiten sequences. Also, any artinian ring of finite representation type has Auslander-Reiten sequences [1].

Given an Auslander-Reiten sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, then the irreducible maps starting with X, or ending in Z, can easily be determined: those starting with X are of the form $f' : X \longrightarrow Y'$, where Y' is a non-zero direct summand of Y, say $Y = Y' \oplus Y''$, and $f = \binom{f'}{f''}$ for some f''; those ending in Z are of the form $g' : Y' \longrightarrow Z$, where again $Y' \neq 0$, $Y = Y' \oplus Y''$, and $g = (g', g'')$, for some g''.

We will denote the length of a module M by $|M|$.

LEMMA. Let R be an artin algebra. Then there exist a constant c (depending only on R) such that for any irreducible map $X \to Y$ with X, Y indecomposable, we have $|X| \leq c\,|Y|$ and $|Y| \leq c\,|X|$.

<u>Proof.</u> Let p be the maximal length of an indecomposable projective right module, and q the maximal length of an indecomposable projective left module. Given an Auslander-Reiten sequence

$$0 \to U \to V \to W \to 0,$$

then $U = D\,\mathrm{Tr}\,W$ and $W = \mathrm{Tr}\,D\,U$, see [3]. Now it is easy to see that for any module M, the length of $\mathrm{Tr}\,M$ is bounded by $pq|M|$. For, let

$$Q \to P \to M \to 0$$

be a minimal projective resolution of M, thus $|P| \leq p|M|$, and therefore Q can be written as the direct sum of at most $p|M|$ indecomposable modules. The same is true for $\mathrm{Hom}_R(Q_R,R_R)$, thus the length of $\mathrm{Hom}_R(Q_R,R_R)$ is $\leq qp|M|$. But $\mathrm{Tr}\, M$ is an epimorphic image of $\mathrm{Hom}_I(Q_R,R_R)$, thus also $|\mathrm{Tr}\, M| \leq qp|M|$. We conclude that for our given Auslander-Reiten sequence, $|V| = |U| + |W|$ is bounded both by $(qp+1)|U|$ and $(qp+1)|W|$. Thus, assume X,Y are indecomposable R-modules, with an irreducible map $X \to Y$. If X is injective, then Y is a direct summand of $X/\mathrm{soc}\, X$ (see [3]), thus $|Y| < |X|$. If X is non-injective, then there is an Auslander-Reiten sequence starting with X and Y is a direct summand of the middle term, thus $|Y| \leq (qp+1)|X|$. Similarly, one obtains the other inequality.

In dealing with irreducible maps $X \to Y$, we usually will consider only the case when X and Y both are indecomposable. In particular, in case there are given irreducible maps

$$X_o \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t ,$$

with all X_i indecomposable, then we will call this <u>a chain of irreducible maps of length</u> t. The Auslander-Reiten graph of R has as vertices the isomorphism classes of indecomposable R-modules, and there is an edge between the isomorphism class of X and that of Y provided there is an irreducible map $X \to Y$ or $Y \to X$. The <u>Auslander-Reiten components</u> are, by definition, the connected components of this Auslander-Reiten graph (or the corresponding modules). Note that in case R has Auslander-Reiten sequences, the Auslander-Reiten graph is locally finite.

LEMMA. Assume R has Auslander-Reiten sequences. Let X, Y be indecomposable R-modules with $\mathrm{Hom}(X,Y) \neq 0$, and assume there does not exist a chain of irreducible maps from X to Y of length $< t$. Then

(a) there exists a chain of irreducible maps

$$X = X_o \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \; X_{t-1} \xrightarrow{f_t} X_t$$

and a map $g : X_t \to Y$ with $g \circ f_t \circ \cdots \circ f_1 \neq 0$; and also

(b) there exists a chain of irreducible maps

$$Y_t \xrightarrow{g_t} Y_{t-1} \quad \cdots \quad \xrightarrow{g_2} Y_1 \xrightarrow{g_1} Y_o = Y$$

and a map $f : X \longrightarrow Y_t$ with $g_1 \circ \ldots \circ g_t \circ f \neq 0$.

Proof, by induction on t. For $t = 0$, nothing has to be shown. Now assume X, Y are given, with $\mathrm{Hom}(X,Y) \neq 0$, and no chain of irreducible maps from X to Y of length $< t+1$. First, we consider (a). By induction, we have irreducible maps $f_i : X_{i-1} \longrightarrow X_i$ for $1 \leq i \leq t$, where X_i are indecomposable modules and $X = X_o$, and also a map $g : X_t \longrightarrow Y$ with $g \circ f_t \circ \ldots \circ f_1 \neq 0$. Our assumption implies that g cannot be an isomorphism. We consider two cases: First, let X_t be injective, then g vanishes on the socle $\mathrm{soc} X_t$ of X_t, since otherwise g would be a split monomorphism. Let $X_t/\mathrm{soc} X_t = \bigoplus_{i=1}^r Z_i$ with indecomposable modules Z_i, and with projection maps $\alpha_i : X_t \longrightarrow Z_i$. Note that the α_i are irreducible. We can factor g as follows

$$\begin{tikzcd} X_t \arrow[rr, "g"] \arrow[dr, "\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{pmatrix}"'] & & Y \\ & \bigoplus_{i=1}^r Z_i \arrow[ur, dashed, "(g_1,\ldots,g_r)"'] & \end{tikzcd}$$

In the second case, X_t is not injective, thus there exists an Auslander-Reiten sequence $0 \longrightarrow X_t \xrightarrow{\alpha} Z \longrightarrow Z' \longrightarrow 0$. Let $Z = \bigoplus_{i=1}^r Z_i$ with Z_i indecomposable, and let $\alpha = \begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_r\end{pmatrix}$. Again, the α_i are irreducible. We can extend g to Z, thus g will be factored again in the form $g = \sum_{i=1}^r g_i\alpha_i$. Now, from $0 \neq g \circ f_t \circ \ldots \circ f_1 = \sum_{i=1}^r g_i\alpha_i f_t \circ \ldots \circ f_1$, it follows that one of the summands $g_i\alpha_i f_t \circ \ldots \circ f_1 \neq 0$. Let $X_{t+1} := Z_i$, and $f_{t+1} := \alpha_i$. This finishes the proof of (a).

Similarly, we have by induction irreducible maps $g_i : Y_i \longrightarrow Y_{i-1}$, $1 \leq i \leq t$, with Y_i indecomposable, $Y_o = Y$, and a map $f : X \longrightarrow Y_t$. Note that by assumption f cannot be an isomorphism. Consider first the case where Y_t is projective. Then f maps into the radical $\mathrm{rad} Y_t$ of Y_t. Let $\mathrm{rad} Y_t = \bigoplus_{i=1}^r Z_i$, with Z_i

indecomposable, and with inclusion maps $\beta_i : Z_i \longrightarrow Y_t$. Thus there are maps $f_i : X \longrightarrow Z_i$ with $f = \sum_{i=1}^{r} \beta_i f_i$. In the second case, where Y_t is not projective, we use the Auslander-Reiten sequence $0 \longrightarrow Z' \longrightarrow Z \longrightarrow Y_r \longrightarrow 0$, decompose $Z = \bigoplus_{i=1}^{r} Z_i$ with Z_i indecomposable, and obtain similarly a relation $f = \sum_{i=1}^{r} \beta_i f_i$, where $\beta = (\beta_1,\ldots,\beta_r)$. Note that in both cases, all the β_i are irreducible maps. Now from $0 \neq g_1 \circ \ldots \circ g_t \circ f = \sum_{i=1}^{r} g_1 \circ \ldots \circ g_t \beta_i f_i$ we conclude that at least one $g_1 \circ \ldots \circ g_t \beta_i f_i \neq$. This gives us $Y_{t+1} := Z_i, g_{t+1} := \beta_i$, and finishes the proof of (b).

2.2 <u>The lemma of Harada and Sai</u>

The following lemma is fitted very well to the consideration of chains of irreducible maps, as we will see in the next sections. It is a generalization of the Fitting lemma, and seems to have been noted for the first time by Harada and Sai in [21].

LEMMA. Let M_i, $1 \leq i \leq 2^b$, be indecomposable modules of length $\leq b$. Let $f_i : M_i \longrightarrow M_{i+1}$ be non-invertible. Then the composition $f_{2^b-1} \circ \ldots \circ f_2 \circ f_1$ is zero.

<u>Proof</u>. We show by induction on n the following: Given indecomposable modules M_i, $1 \leq i \leq 2^n$, of length $\leq b$, and non-invertible maps $f_i : M_i \to M_{i+1}$, then either the composition $f_{2^n-1} \circ \ldots \circ f_2 \circ f_1$ is zero or else the length of its image is $\leq b-n$.

If $n = 1$, the image of f_1 cannot have length b, since otherwise f_1 would be invertible.

Assume the assertion is true for n, and there are given M_i, $1 \leq i \leq 2^{n+1}$, and $f_i : M_i \to M_{i+1}$. We consider the compositions $f = f_{2^n-1} \circ \ldots \circ f_2 \circ f_1$ and $h = f_{2^{n+1}-1} \circ \ldots \circ f_{2^n}$. If one of them is zero or the length of its image is $< b-n$, then we are done. Thus, we can assume that the images both of f and h are of length $b-n > 0$. Let $g = f_{2^n}$. We have to show that the image $\operatorname{Im} hgf$ of hgf is of length $\leq b-n-1$. If not, then we clearly must have

$$\operatorname{Im} f \cap \operatorname{Ker} hg = 0, \quad \operatorname{Im} gf \cap \operatorname{Ker} h = 0.$$

Together with $|\mathrm{Im}\ f| = b-n$, $|\mathrm{Ker}\ hg| = |M_{2^n}|-b+n$, the first equality implies that M_{2^n} is the direct sum of $\mathrm{Im}\ f$ and $\mathrm{Ker}\ hg$, thus, since $f \neq 0$ and M_{2^n} is indecomposable, hg is a monomorphism. Similarly, $|\mathrm{Im}\ gf| = b-n$, $|\mathrm{Ker}\ h| = |M_{2^n+1}|-b+n$ show that M_{2^n+1} is the direct sum of $\mathrm{Im}\ gf$ and $\mathrm{Ker}\ h$, thus fg is an epimorphism. As a consequence, g would be both a monomorphism and an epimorphism, contrary to our assumption.

Note that the given bounds in the lemma are best possible. For example, for $R = k[X,Y]/(X^2,Y^2,XY)$, it is easy to construct a sequence of seven indecomposable R-modules of length ≤ 3 and non-invertible maps

$$M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_6 \longrightarrow M_7$$

with non-zero composition, starting with M_1 the indecomposable projective module and ending with the indecomposable injective module M_7.

2.3 Bounded length components

As a first application of the existence of chains of irreducible maps and the Harada-Sai lemma, we prove Auslander's result that a connected artinian ring R with Auslander-Reiten sequences can have a component with modules of bounded length only in case R is of finite representation type. Note that any artin algebra, in particular any finite dimensional algebra over some field, has Auslander-Reiten sequences, thus this result is a direct generalization of Rojter's theorem.

PROPOSITION (Auslander [2]). Let R be a connected artinian ring with Auslander-Reiten sequences. If R has an Auslander-Reiten component with modules of bounded length, then these are all indecomposable modules and R is of finite representation type.

Proof. Let $\mathcal{C}$ be an Auslander-Reiten component such that all indecomposable modules in $\mathcal{C}$ are of length $\leq b$. Let M, N be two indecomposable modules with $\mathrm{Hom}(M,N) \neq 0$. If M belongs to $\mathcal{C}$, then there is a chain of irreducible maps from M to N of length $< 2^b-1 := s$, and thus also N is in $\mathcal{C}$. For, otherwise there is a chain of irreducible maps

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots M_{s-1} \xrightarrow{f_s} M_s$$

and a map $g : M_s \longrightarrow N$ with $g \circ f_s \circ \ldots \circ f_1 \neq 0$. However, by the Harada-Sai lemma, $f_s \circ \ldots \circ f_1 = 0$. Conversely, if N is in C, then similarly there is a chain of irreducible maps from M to N of length $< s$, since otherwise there would exist a chain of irreducible maps

$$N_s \xrightarrow{g_s} N_{s-1} \longrightarrow \ldots \longrightarrow N_1 \xrightarrow{g_1} N_o = N$$

and a map $f : M \longrightarrow N_s$ with $g_1 \circ \ldots \circ g_s \circ f \neq 0$, again a contradiction to the Harada-Sai lemma.

Thus, let M be any indecomposable module in C. There is an indecomposable projective module P with $\mathrm{Hom}(P,M) \neq 0$, thus also P belongs to C. Since R is connected, for any other indecomposable projective module P', there is a sequence $P = P_o, P_1, \ldots, P_t = P'$ with $\mathrm{Hom}(P_{i-1}, P_i) \neq 0$ or $\mathrm{Hom}(P_i, P_{i-1}) \neq 0$ for all $1 \leq i \leq t$, thus also P' belongs to C. Finally, we conclude that all indecomposable R-modules belong to C, since for any such module N, there is an indecomposable projective module P' with $\mathrm{Hom}(P',N) \neq 0$. Also, we have seen that there exists a chain of $< s$ irreducible maps starting with P' and ending with N. Since there are only finitely many indecomposable projective modules, there can exist only finitely many indecomposable modules. This finishes the proof.

We have noted above that this proposition is a direct generalization of Rojter's theorem. In fact, Auslander also has shown that an artinian ring R with a bound on the length of the indecomposable R-modules always has Auslander-Reiten sequences ([1], see also [41]), thus it follows that such a ring has to be of finite representation type:

<u>Corollary</u> (Auslander [1]). An artinian ring is either of finite representation type or else there are indecomposable modules of arbitrarily large length.

2.4 <u>The induction step</u>

The second application of the existence of chains of irreducible maps and the Harada-Sai lemma is a result due to Smalø which establishes the induction step of the second Brauer-Thrall conjecture in a way which is different from that of Nazarova and Rojter, and which also applies immediately to all artin algebras. To see the difference between the assertions of Nazarova-Rojter and Smalø, let R be a finite-dimensional algebra over some algebraically closed field, then Nazarova-Rojter construct one particular one-parameter family of indecomposable modules M_i such that for any natural n, there is an indecomposable module $M_i(n)$ with a chain of inclusions

$$M_i = M_i(1) \subset M_i(2) \subset \ldots \subset M_i(n)$$

such that all quotients $M_i(r+1)/M_i(r)$ are isomorphic to M_i, and with $M_i(n)$ isomorphic to $M_j(n)$ only for $M_i \approx M_j$. In particular, the different one-parameter families which we obtain all have as dimensions multiples of the dimension of the modules in the family M_i. In contrast, Smalø's assertion is that given any infinite family $\{M_i | i \in I\}$ of indecomposable modules, say of length b, we find a subset J of I of the same cardinality, and for some $b' > b$ a family $\{N_j | j \in J\}$ of indecomposable modules of length b' such that there is a chain of irreducible maps of a fixed length t starting with M_j and ending with N_j, for every $j \in J$. Thus, here we do not know much about the relation of the dimensions b and b', except that $b < b'$, and, in particular, we cannot build up N_j from copies of M_j using extensions, as examples show. On the other hand we know that the module N_j belongs to the same Auslander-Reiten component as M_j.

<u>PROPOSITION</u> (Smalø [37]). Let R be an artin algebra, and assume there is given an infinite family $\{M_i | i \in I\}$ of pairwise non-isomorphic indecomposable modules of length b. Then there exists $b' > b$, some t, and a subset J of I of the same cardinality as I, such that for every $j \in J$, there is a non-zero map $M_j \to N_j$ which is the composition of a chain of t irreducible maps and with a family $\{N_j | j \in J\}$ of pairwise non-isomorphic, indecomposable modules of length b'.

<u>Proof.</u> Let $\{M_i | i \in I\}$ be the given family of indecomposable modules of length b. There are only finitely many M_i which may have a chain of irreducible maps of length $< 2^b-1 =: s$ starting with M_i and ending in an indecomposable injective module, thus we may delete those, retaining as index set a subset J' of J. To any M_i, $i \in J'$, there exists an indecomposable injective module Q_i with $\mathrm{Hom}(M_i, Q_i) \neq 0$. As a consequence, there is a chain of irreducible maps

$$M_i = M_{i0} \xrightarrow{f_{i1}} M_{i1} \longrightarrow \cdots M_{i,s-1} \xrightarrow{f_{is}} M_{is}$$

and a map $g_i : M_{is} \longrightarrow Q_i$ such that $g_i f_{is} \circ \ldots \circ f_{i1} \neq 0$. The Harada-Sai lemma immediately implies that not all M_{ij} can have length $\leq b$, thus there exists a subset J'' of J' of the same cardinality, and some t with $1 \leq t \leq s$ such that for all $i \in J''$, we have $|M_{it}| > b$. Now, $|M_{it}|$ takes only finitely many different values, since $|M_{it}| \leq c^t |M_i| = c^t b$, thus there is again a subset J''' of J'' of the same cardinality such that all $|M_{it}|$, $i \in J'''$, are equal. Any isomorphism class can contain only finitely many M_{it}, since for any module N, there is only a finite number of modules M_i with a chain of irreducible maps of length t, starting with M_i and ending with N. Thus choosing one module out of any such isomorphism class, we obtain a subset I of J''' of the same cardinality such that for $i \neq j$ in I, the modules M_{it}, M_{jt} are non-isomorphic. This finishes the proof.

In order to be able to apply the result of Smalø, we need to have found in some different way at least one infinite family of indecomposable modules (of a fixed dimension). In the next section we will see that, at least sometimes, this can be read off from properties of the bimodules of irreducible maps.

2.5 The bimodules of irreducible maps

Let X, Y be indecomposable R-modules. Clearly, the set of irreducible maps $f : X \longrightarrow Y$ is not closed under addition, but actually the set of non-irreducible maps is. Let us consider the corresponding factor module. To be more precise, denote by $\mathrm{rad}(X,Y)$ the set of non-invertible homomorphisms. If X, Y are not necessarily indecomposable, say with decompositions $X = \oplus X_i$, $Y = \oplus Y_j$ with indecomposable modules X_i, Y_j, define $\mathrm{rad}(X,Y) = \oplus_{i,j} \mathrm{rad}(X_i,Y_j)$, where we use the identification $\mathrm{Hom}(X,Y) = \oplus_{i,j} \mathrm{Hom}(X_i,Y_j)$. In this way, we define an ideal in the category of R-modules which is the obvious generalization of the Jacobson radical of a ring, and which may be called <u>the Jacobson radical</u> of the category of R-modules [24]. As in the case of a ring, let $\mathrm{rad}^2(X,Y)$ be the set of all homomorphisms $f : X \longrightarrow Y$ with $f = f'f''$, where $f'' \in \mathrm{rad}(X,I)$, $f' \in \mathrm{rad}(I,Y)$ for some R-module I. This again is an ideal in the category of R-modules. We only have to check the additivity. Assume $f = f'f''$, $g = g'g''$ both belong to $\mathrm{rad}^2(X,Y)$, say with $f' \in \mathrm{rad}(I,Y)$, $g' \in \mathrm{rad}(J,Y)$. Then $(f',g') \in \mathrm{rad}(I \oplus J, Y)$, $\begin{pmatrix} f'' \\ g'' \end{pmatrix} \in \mathrm{rad}(X, I \oplus J)$, and $(f'g')\begin{pmatrix} f'' \\ g'' \end{pmatrix} = f'f''+g'g''= f+g$. Note that for X, Y indecomposable, $\mathrm{rad}(X,Y) \setminus \mathrm{rad}^2(X,Y)$ is just the set of irreducible homomorphisms from X to Y.

Let $N(X,Y) = \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$. For an indecomposable module X, let $F(X) = \mathrm{End}(X)/\mathrm{rad}(X,X)$, this is a division ring. Note that $\mathrm{rad}^2(X,Y)$ is an $\mathrm{End}(Y)$-$\mathrm{End}(X)$-submodule of the $\mathrm{End}(Y)$-$\mathrm{End}(X)$-bimodule $\mathrm{rad}(X,Y)$, and that $N(X,Y)$ is annihilated on the left by $\mathrm{rad}(Y,Y)$, on the right by $\mathrm{rad}(X,X)$, thus $N(X,Y)$ is, in fact, an $F(Y)$-$F(X)$-bimodule, the <u>bimodule of irreducible maps.</u> We are interested in the dimensions of the vectorspaces ${}_{F(Y)}N(X,Y)$ and $N(X,Y)_{F(X)}$.

<u>LEMMA.</u> Let $0 \longrightarrow X \xrightarrow{f} \bigoplus_{i=1}^{s} Y_i \xrightarrow{g} Z \longrightarrow 0$ be an Auslander-Reiten sequence, with indecomposable modules Y_i. Let Y be a fixed indecomposable module, and suppose $Y \approx Y_i$ iff $1 \leq i \leq d$, for some $d \leq s$. Then the vectorspaces ${}_{F(Y)}N(X,Y)$ and $N(Y,Z)_{F(Y)}$ both are d-dimensional.

<u>Proof.</u> We may assume $Y = Y_i$ for $1 \leq i \leq d$, thus $f = \begin{pmatrix} f_1 \\ \vdots \\ f_s \end{pmatrix}$ gives d homomorphisms $f_i : X \longrightarrow Y$, $1 \leq i \leq d$. They are linearly independent in ${}_{F(Y)}N(X,Y)$. For, assume there are given endomorphisms α_i of Y, $1 \leq i \leq d$. If at least one of the α_i is an automorphism, then $(\alpha_1,\ldots,\alpha_d) : \bigoplus_{i=1}^{d} Y \longrightarrow Y$ is a split epimorphism, and therefore the composition

$$X \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix}} \bigoplus_{i=1}^{d} Y \xrightarrow{(\alpha_1\ldots\alpha_d)} Y$$

is an irreducible homomorphism, thus $\sum_{i=1}^{s} \alpha_i f_i \notin \mathrm{rad}^2(X,Y)$. On the other hand, let $f' : X \longrightarrow Y$ be any homomorphism. Then by the defining property of an Auslander-Reiten sequence, there is $\alpha = (\alpha_1\ldots\alpha_s) : \bigoplus_{i=1}^{s} Y_i \longrightarrow Y$ with $f' = \alpha f = \sum_{i=1}^{s} \alpha_i f_i$. Now $f_i \in \mathrm{rad}(X,Y_i)$ for all i, since f_i is irreducible. Also, $\alpha_i \in \mathrm{rad}(Y_i,Y)$, for $i > d$, since these Y_i are indecomposable and not isomorphic to Y. Thus $\sum_{i=d+1}^{s} \alpha_i f_i \in \mathrm{rad}^2(X,Y)$, and therefore f' coincides with $\sum_{i=1}^{d} \alpha_i f_i$ modulo $\mathrm{rad}^2(X,Y)$. This shows that $f_1,\dots,f_d$ is a basis of ${}_{F(Y)}N(X,Y)$. Similarly, one shows that $g_1,\dots,g_d$ is a basis of $N(X,Y)_{F(X)}$.

For the remainder of this section, we will need the following property of irreducible monomorphisms (which actually gives rise to a characterization [3]): if $f : X \to Y$ is an irreducible monomorphism, and $\pi : Y \to Q$ is the cokernel of f, then for any map $\eta : Q' \to Q$ there is either $\eta' : Q' \to Y$ with $f\eta' = \eta$, or $\eta'' : Y \to Q'$ with $\eta\eta'' = f$. It is easy to conclude from this that Q has to be indecomposable. Thus we see that irreducible monomorphisms are a good source for obtaining indecomposable modules! Similarly, the kernel of an irreducible epimorphism has the dual factorization property, and, in particular, is indecomposable, so that in the same way irreducible epimorphisms give rise to indecomposable modules. We will use these properties to obtain, in fact, series of indecomposable modules.

We assume that R is an artin algebra with center C. Note that if R is connected, then C is a local ring, so the residue ring $C/\mathrm{rad}\, C$ is a field. The following proposition is a direct consequence of the properties of irreducible maps as outlined above, due to Auslander and Reiten, and some known facts on the number of orbits in certain bimodules which were considered by Dlab and the author in [12]; it also can be found in a forthcoming paper by Bautista [6].

PROPOSITION. Let R be a connected artin algebra with infinite residue field $k = C/\mathrm{rad}\, C$. Assume there exist indecomposable R-modules X, Y, with

$$\dim_{F(Y)} N(X,Y) \cdot \dim N(X,Y)_{F(X)} \geq 4.$$

Then there is a natural number d such that the number of indecomposable R-modules of length d equals the cardinality of k.

Proof. The division rings $F(X)$, $F(Y)$ are finite dimensional k-algebras, and k operates centrally on $N(X,Y)$. Assume first that $\dim_{F(Y)} N(X,Y) \geq 2$, $\dim(X,Y)_{F(X)} \geq 2$. Consider the following algebraic group $G = (F(X) \setminus \{0\}) \times (F(Y) \setminus \{0\})$ operating on $N(X,Y)$ as follows: if $\alpha \in F(X) \setminus \{0\}$, $\beta \in F(Y) \setminus \{0\}$, and $\varphi \in N(X,Y)$, let $(\alpha,\beta) * \varphi = \beta\varphi\alpha^{-1}$. It is easy to see that the number of orbits is precisely $|k|$, calculating the k-dimension both of $N(X,Y)$ and G. Now the orbits give rise to indecomposable modules of length $||X| - |Y||$ as follows: Assume $|X| \leq |Y|$. If $\varphi : X \to Y$ is irreducible, then it has to be a proper monomorphism, and we denote by Q_φ its cokernel, with $\pi_\varphi : Y \to Q_\varphi$ the projection map. Now Q_φ is indecomposable, and we claim that for irreducible maps $\varphi, \psi : X \to Y$, the modules Q_φ and Q_ψ only can be isomorphic in case $\bar{\varphi} = \bar{\psi}$. Namely, assume there is given an isomorphism $\eta : Q_\varphi \to Q_\psi$, then there exists either $\beta : Y \to Y$ with $\pi_\psi \beta = \eta\pi_\varphi$, or else $\beta' : Y \to Y$ with $\pi_\psi = \eta\pi_\varphi\beta'$. Replacing, if necessary, η by η^{-1} and interchanging φ and ψ, we can assume that we have $\pi_\psi\beta = \eta\pi_\varphi$, thus we have a commutating diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\varphi} & Y & \xrightarrow{\pi_\varphi} & Q_\varphi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\eta} & & \\
0 & \longrightarrow & X & \xrightarrow{\psi} & Y & \xrightarrow{\pi_\psi} & Q_\psi & \longrightarrow & 0
\end{array}$$

Now β cannot be nilpotent, since η is an isomorphism. Thus, since Y is indecomposable, β is an automorphism, and therefore also α is an automorphism, thus $\psi = \beta\varphi\alpha^{-1} = (\alpha,\beta)*\varphi$. Similarly, in case $|X| > |Y|$, we consider the kernels of the irreducible maps.

Next, assume $\dim_{F(Y)} N(X,Y) = 1$, $\dim N(X,Y)_{F(X)} \geq 4$. The previous lemma, together with the characterization of irreducible maps using Auslander-Reiten sequences, shows that linearly independent elements φ_1, φ_2 of $N(X,Y)_{F(X)}$ give rise to irreducible maps $\varphi = (\varphi_1, \varphi_2) : X \oplus X \to Y$. Thus, we consider the abelian group $N(X \oplus X, Y) = \mathrm{Hom}(X \oplus X, Y)/\mathrm{rad}^2(X \oplus X, Y)$ which is a bimodule with respect to the action of $F(Y)$ from the left, and the 2×2-matrices over $F(X)$ from the right. Thus, the group $GL_2(F(X)) \times (F(Y) \setminus \{0\})$ operates on $N(X \oplus X, Y)$ via $(\alpha,\beta)*\varphi = \beta\varphi\alpha^{-1}$, where $\varphi \in N(X \oplus X, Y)$, $\alpha \in GL_2(F(X))$, $0 \neq \beta \in F(Y)$. Again, the number of orbits is equal to $|k|$, and we want to construct, for every orbit, an indecomposable module of length $|2|X| - |Y||$ as follows: First, assume $2|X| \leq |Y|$. In this case, we consider the cokernel Q_φ of $\varphi : X \oplus X \to Y$ and conclude that for Q_φ, Q_ψ isomorphic, $\bar{\varphi}$, and $\bar{\psi}$ belong to the same orbit, using the indecomposibility of Y. Next, assume $2|X| > |Y|$. In this case, we have to consider the kernel K_φ of $\varphi : X \oplus X \to Y$, say with inclusion map μ_φ. Let K_φ, K_ψ be isomorphic, say with isomorphism $\eta : K_\varphi \to K_\psi$. We obtain an endomorphism α of $X \oplus X$ with either $\alpha\mu_\varphi = \mu_\psi\eta$, or $\mu_\varphi = \alpha\mu_\psi\eta$. Replacing, if necessary, η by η^{-1} and exchanging φ and

ψ, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_\varphi \arrow[r, "\mu_\varphi"] \arrow[d, "\eta"'] & X \oplus X \arrow[r, "\varphi"] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & K_\psi \arrow[r, "\mu_\psi"] & X \oplus X \arrow[r, "\psi"] & Y \arrow[r] & 0 .
\end{tikzcd}$$

We claim that β cannot be nilpotent. For, otherwise, we even may assume $\beta = 0$, replacing η by some power. But then we obtain $\gamma : X \oplus X \longrightarrow K_\psi$ with $\alpha = \mu_\psi \gamma$, and therefore $\eta^{-1}\gamma\mu_\varphi = 1_{K_\alpha}$. This shows that μ_φ, and thus also φ, splits, impossible. Since β is not nilpotent, and Y is indecomposable, we see that β, and therefore also α, is an isomorphism. Thus $\bar{\varphi}$ and $\bar{\psi}$ belong to the same orbit.

2.6 Examples

Let us consider rings of the form

$$R = R({}_F M_G) = \begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$$

where F, G are division rings and ${}_F M_G$ is a bimodule (this means that R consists of all 2×2-matrices of the form $\begin{pmatrix} f & m \\ 0 & g \end{pmatrix}$ with $f \in F$, $m \in M$, $g \in G$, with component-wise addition, and the usual matrix-multiplication — which makes sense as one verifies easily). The R-modules can be written in the form $(U_F, V_G, \eta : U_F \otimes_F M_G \longrightarrow V_G)$, and we call $(\dim U_F, \dim V_G)$ its dimension type.

Now, there are two indecomposable projective R-modules, namely $P_1 = (0, G_G, o)$ and $P_2 = (F_F, M_G, \mathrm{id})$, with endomorphism rings $\mathrm{End}(P_1) = G$, $\mathrm{End}(P_2) = F$. It is easy to check that we obtain as bimodule of irreducible maps

$$N(P_1,P_2) = \mathrm{Hom}(P_1,P_2) \approx \mathrm{Hom}_G({}_G G_G, {}_F M_G) \approx {}_F M_G .$$

First, assume that R is a finite-dimensional k-algebra, for some infinite field k. Thus, F,G contain k in the center, k operates centrally on M, and all $\dim_k F$, $\dim_k G$, $\dim_k M$ are finite. It is easy to see ([12], [13]) that in case

$$\dim {}_F M \cdot \dim M_G \leq 3,$$

the algebra R is of finite representation type. On the other hand, it follows from the previous section (and has been shown in [12]) that for

$$\dim {}_F M \cdot \dim M_G \geq 4$$

there are infinitely many pairwise non-isomorphic R-modules X_i with a minimal projective resolution of one of the forms

$$0 \longrightarrow P' \xrightarrow{\varphi_i} P \longrightarrow X_i \longrightarrow 0,$$

where $P' = P_1$ or $P_1 \oplus P_1$, $P = P_2$ or $P_2 \oplus P_2$, and at least one of them indecomposable. This last condition immediately shows that for all i, $\mathrm{End}(X_i)$ is a subring either of F or G, thus a division ring. Also, calculating dimensions, one obtains easily

$$\dim_k \mathrm{Hom}(P,X_i) \leq \dim_k \mathrm{Hom}(P',X_i).$$

Applying the long exact sequence for $\mathrm{Hom}(-,X_i)$ to the projective resolution of X_i, we obtain

$$0 \to \mathrm{Hom}(X_i,X_i) \to \mathrm{Hom}(P,X_i) \to \mathrm{Hom}(P',X_i) \to \mathrm{Ext}^1(X_i,X_i) \to 0,$$

thus, since $\mathrm{Hom}(X_i,X_i) \neq 0$, we also have $\mathrm{Ext}^1(X_i,X_i) \neq 0$. We therefore can construct inductively indecomposable modules $X_i(n)$ with a chain of submodules

$$X_i = X_i(1) \subset X_i(2) \subset \ldots \subset X_i(n-1) \subset X_i(n),$$

such that all factors $X_i(r+1)/X_i(r)$ are isomorphic to X_i. In particular, $\dim_k X_i(n) = n \cdot \dim X_i$, and, for different i, j, the modules $X_i(n)$ and $X_j(n)$ are non-isomorphic. (See [34] for the process of "simplification" which can be applied in our situation for obtaining the modules $X_i(n)$, and also for a general argument for deriving the assertion $\mathrm{Ext}^1(X_i,X_i) \neq 0$ using quadratic forms). Thus, we see that the finite dimensional k-algebras of the form $R({}_F M_G)$ satisfy the second Brauer-Thrall conjecture. In part 3 we will see that these algebras are rather typical for the general situation.

Next, let us consider examples where $R({}_F M_G)$ is not necessarily a finite dimensional algebra. Let F be a division ring with an automorphism ε and an ε-1-derivation δ, (thus $\delta : F \to F$ is a map which satisfies

$$\delta(f_1+f_2) = \delta(f_1) + \delta(f_2), \quad \delta(f_1 f_2) = \varepsilon(f_1)\delta(f_2) + \delta(f_1)f_2,$$

see [10]). Then $F \oplus F$ can be made into an F-F-bimodule, denoted by $M(\varepsilon,\delta)$ by using the canonical left operation, but the following right operation

$$(1,0)f = (f,0), \quad (0,1)f = (\delta(f), \varepsilon(f)).$$

We also can form the twisted polynomial ring $F[T;\varepsilon,\delta]$, the elements being the polynomials $\sum_{i=0}^{n} f_i T^i$ with $f_i \in F$ (and T an indeterminant) and with multiplication defined by the multiplication of F and the rule $Tf = \delta(f) + \varepsilon(f)T$.

<u>PROPOSITION.</u> Let $R = R(M(\varepsilon,\delta))$. For $n \in \mathbb{N}$, there is precisely one indecomposable module P_n of dimension type $(n-1,n)$, one indecomposable module I_n of dimension type $(n,n-1)$. Also there is precisely one indecomposable module Z_n of dimension type (n,n) with $\mathrm{Hom}(Z,Z_n) \neq 0$, where $Z = Z_1 = (F_F, M/(1,0)F, \pi)$ with canonical projection π. The modules P_n form an Auslander-Reiten component

$$P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow \cdots$$

The modules I_n form an Auslander-Reiten component

$$I_1 \longleftarrow I_2 \longleftarrow I_3 \longleftarrow \cdots$$

The modules Z_n form an Auslander-Reiten component

$$Z_1 \rightleftarrows Z_2 \rightleftarrows Z_3 \rightleftarrows \cdots$$

The full subcategory of R-modules without direct summand of the form P_n, I_n, or Z_n is equivalent to $\mathcal{M}_{F[T;\varepsilon,\delta]}$.

This has been proved essentially in [34]. The assertions concerning Auslander-Reiten components follow directly from the fact that we can construct without difficulty the corresponding Auslander-Reiten sequences.

Note that the embedding $\mathcal{M}_{F[T;\varepsilon,\delta]}$ into $\mathcal{M}_R$ is given as follows: The $F[T;\varepsilon,\delta]$-module X is sent to (X_F, X_F, η) with $\eta(x \otimes (1,0)) = x$, $(x \otimes (0,1)) = xT$ for all $x \in X$. In particular, the indecomposable R-modules of finite length which are not of the form P_n, I_n, or Z_n correspond to the indecomposable $F[T;\varepsilon,\delta]$-modules of finite length. Since there exist rings of the form $F[T;\varepsilon,\delta]$ with precisely one indecomposable module of finite length ([11], [29]: take F a differentially closed field with derivation δ, and $\varepsilon = \mathrm{id}$), we obtain in this way an artinian ring R with length of R_R and length of ${}_RR$ both equal 4, such that there are precisely two indecomposable modules of any odd length, also two indecomposable modules of length 2, and just one indecomposable module for any other even length. A similar example is obtained by taking for F the algebraic closure of the prime field $\mathbb{Z}/p\mathbb{Z}$ (p a prime), for ε the Frobenius automorphism of F (thus $\varepsilon(f) = f^p$), and $\delta = 0$, see [11]. Note that these examples are also of interest from the point of view that they are rather well-behaved artinian rings which do not have Auslander-Reiten sequences (some modules have, some not).

3. THE METHODS OF NAZAROVA AND ROJTER

In this second part, we want to indicate a few of the methods of proof used by Nazarova and Rojter, indicating at the same time some of the changes which are necessary in order to obtain the corresponding result for algebras over an arbitrary base field, and, in fact, even for arbitrary artin algebras. We touch only a very small part of the work of Nazarova and Rojter and we have to refer to the original paper [32] for a deeper penetration.

3.1 Vectorspace categories

We assume throughout that k is a commutative field and D a division ring which is finite dimensional over k.

By a vectorspace category $\mathcal{K} = {}_D\mathcal{K}$ we mean a k-additive category together with a faithful functor from $\mathcal{K}$ to the category ${}_D\mathcal{M}$ of finite-dimensional left D-vectorspaces, usually we will denote this functor by $|\cdot|$, and for X in $\mathcal{K}$, the D-vectorspace $|X|$ will be called the underlying vectorspace. Using $|\cdot|$, we may consider $\mathcal{K}$ as a (usually not full) subcategory, thus a vectorspace category may be considered in the following way: there are given certain finite-dimensional left D-vectorspaces, called the objects of $\mathcal{K}$, and for any two such objects X,Y, there is given a k-subspace $\mathcal{K}(X,Y)$ of the set of D-linear transformations from ${}_DX$ to ${}_DY$ (or better, from ${}_D|X|$ to ${}_D|Y|$) such that $\mathcal{K}$ becomes a category. We always will assume that $\mathcal{K}$ has split idempotents, thus the endomorphism ring of an indecomposable object of $\mathcal{K}$ (= one which is not isomorphic in $\mathcal{K}$ to a direct sum of two non-zero objects) is a local algebra. Also, usually we will assume that $\mathcal{K}$ has only finitely many isomorphism classes of indecomposable objects.

Given a vectorspace category ${}_D\mathcal{K}$, we denote by $\mathcal{U}({}_D\mathcal{K})$ the subspace category of ${}_D\mathcal{K}$, defined as follows: its objects are of the form (U,X,φ) with U a (finite dimensional) D-vectorspace, X an object of $\mathcal{K}$, and $\varphi : {}_DU \to {}_D|X|$ a D-linear transformation, and the maps from (U,X,φ) to (U',X',φ') are of the form (α,β) with $\alpha : {}_DU \to {}_DU'$ a D-linear transformation, $\beta : X \to X'$ a map in the category $\mathcal{K}$, such that $\varphi|\beta| = \alpha\varphi'$. Note that for all the indecomposable objects (U,X,φ) of $\mathcal{U}_D(\mathcal{K})$, but $({}_DD,0,o)$, the map φ is a monomorphism, thus we may assume an inclusion. This explains why $\mathcal{U}({}_D\mathcal{K})$ is called the subspace category of ${}_D\mathcal{K}$.

In the case when k is an algebraically closed field, these notions where introduced by Nazarova and Rojter [32]. Of course, in this case, we must have $k = D$, and for any indecomposable object X in $\mathcal{K}$, also the factor ring of End(X) modulo its radical equals k.

The vectorspace category $\mathcal{K}$ is said to be of finite representation type provided $\mathcal{U}(\mathcal{K})$ has only finitely many isomorphism classes of indecomposable objects. We will see in which way we can use the following result for the proof of the theorem of Nazarova and Rojter. It confirms the second Brauer-Thrall conjecture for vectorspace categories. In the case of k algebraically closed, it is due to Nazarova and Rojter [32], and, as we want to indicate, the general result follows with similar considerations.

PROPOSITION: Let ${}_D\mathcal{K}$ be a vectorspace category which is not of finite representation type. Then there exists an algebraic bimodule ${}_FM_G$, with F,G division rings, $\dim {}_FM \cdot \dim M_G \geq 4$, and a full subcategory $\mathcal{V}$ of $\mathcal{U}({}_D\mathcal{K})$ which is representation equivalent to $\mathcal{M}_{R({}_FM_G)}$.

Here, a bimodule ${}_FM_G$ is called algebraic, in case the ring $R({}_FM_G)$ is a finite dimensional algebra over some field.

We will outline the first steps of the proof of this proposition. Before we do this, let us indicate the typical situation in which this result can be applied for the study of module categories over finite dimensional algebras.

3.2 The use of subspace categories

If we want to prove a result for all finite dimensional algebras, we may use induction on the dimension, thus one may assume that the result is true for all proper factor algebras of the given algebra. For example, in establishing the theorem of Nazarova and Rojter for a particular algebra R, we may assume that all the factor algebras R/I, with I a non-zero (twosided) ideal, are of finite representation type.

PROPOSITION: Let R be a finite-dimensional algebra. Let S be a simple (right) R-module, with endomorphism ring D, and assume

$\mathrm{Ext}^1(S,S) = 0$. Let K be the full subcategory of all R-modules M with $\mathrm{Hom}(S,M) = 0$, and L the full subcategory of all R-modules which are direct sums of indecomposable modules M with $\mathrm{Hom}(S,M) \neq 0$ or $\mathrm{Ext}^1(M,S) \neq 0$. Then L is representation equivalent to $U({}_D\mathrm{Ext}^1(K,S))$.

We apply this in the situation where in addition to $\mathrm{Ext}^1(S,S) = 0$, we also have $\mathrm{Hom}(S,R_R) \neq 0$. Then, let I be the S-socle of R_R, the sum of all simple submodules of R_R isomorphic to S. This is a nonzero twosided ideal, and we can assume that R/I is of finite representation type. Now the R-modules in K are annihilated by I, thus K consists of R/I-modules and has therefore only finitely many indecomposable objects. Thus, we conclude that the subspace category $U({}_D\mathrm{Ext}^1(K,S))$ either has also only finitely many indecomposable objects, or else there is a full subcategory which is representation equivalent to some $M_{R({}_FM_G)}$ with $\dim {}_FM \cdot \dim M_G \geq 4$. Note that L is a cofinite subcategory of M_R, since the indecomposable R-modules not in L belong to K.

Proof: Define a functor from L to $U({}_D\mathrm{Ext}^1(K,S))$ as follows: given M in L, let S(M) be the S-socle of M, the submodule of M generated by all simple submodules of M isomorphic to S. Let $S(M) \approx \bigoplus_{i=1}^{m} S$, and $\mu : \bigoplus_{i=1}^{m} S \to M$ the inclusion map . Applying to the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{m} S \xrightarrow{\mu} M \longrightarrow M/S(M) \longrightarrow 0$$

the functor $\mathrm{Hom}(-,S)$, we obtain the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(M/S(M),S) \longrightarrow \mathrm{Hom}(M,S) \longrightarrow \mathrm{Hom}(\bigoplus_{i=1}^{m} S,S)$$

$$\xrightarrow{\delta_M} \mathrm{Ext}^1(M/S(M),S) \to \mathrm{Ext}^1(M,S) \to \mathrm{Ext}^1(\bigoplus_{i=1}^{m} S,S),$$

and we are interested in the map δ_M. In fact,

$\mathrm{Hom}(\bigoplus_{i=1}^{m} S, {}_DS) \approx \bigoplus_{i=1}^{m} \mathrm{Hom}(S,{}_DS) = \bigoplus_{i=1}^{m} {}_DD$ is a left D-vectorspace, and δ_M is a D-linear transformation from $\mathrm{Hom}(\bigoplus_{i=1}^{m} S,{}_DS)$ to $\mathrm{Ext}^1(M/S(M),{}_DS)$. Since $\mathrm{Ext}^1(S,S) = 0$, we have $\mathrm{Hom}(S,M/S(M)) = 0$, thus M/S(M) is a module in K, and therefore we obtain a functor

$$\eta : \mathcal{L} \longrightarrow \mathcal{U}({}_D\mathrm{Ext}^1(\mathcal{K},S))$$

by

$$\eta(M) = (\mathrm{Hom}(S(M),{}_DS),\ \mathrm{Ext}^1(M/S(M),S),\ \delta_M).$$

It is not difficult to verify that η is, in fact, a representation equivalence. Note that given an object $({}_DU,\ \mathrm{Ext}^1(X,S),\varphi)$ of $\mathcal{U}({}_D\mathrm{Ext}^1(\mathcal{K},S))$, we may write ${}_DU$ in the form $\bigoplus_{i=1}^{m} {}_DD$, and $\varphi = \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_m\end{pmatrix}$ with $\varphi_i : {}_DD \to \mathrm{Ext}^1(X,S)$ where $X \in \mathcal{K}$.
In this way, we obtain a map $\tilde{\varphi} : {}_DD \to \mathrm{Ext}^1(X, \bigoplus_{i=1}^{m} S)$ with $\tilde{\varphi} = (\varphi_1,\ldots,\varphi_m)$, and the image of $1 \in D$ under $\tilde{\varphi}$ gives an element in $\mathrm{Ext}^1(X, \bigoplus_{i=1}^{m} S)$, thus to an extension of the form

$$0 \longrightarrow \bigoplus_{i=1}^{m} S \longrightarrow Y \longrightarrow X \longrightarrow 0,$$

and it follows that $\eta(Y)$ is isomorphic to $({}_DU,\ \mathrm{Ext}^1(X,S),\varphi)$. In this way, we show that the functor η is dense.

3.3 Vectorspace categories of finite representation type.

The proof of the theorem of Nazarova and Rojter rests on a good knowledge of the structure of vectorspace categories of finite representation type. But also in many other situations, it may be helpful to have insight into the structure of these vectorspace categories. In contrast to the special case of the additive category of a partially ordered set, no characterization of the vectorspace categories of finite representation type is known, however, the work of Nazarova and Rojter provides a large amount of information on these categories.

Let us write down at least the structure of the indecomposable objects of such a category. In fact, we only will assume that $\mathcal{K}$ is a vectorspace category satisfying the following property:

(*) There is no algebraic bimodule ${}_FM_G$ with F,G division rings, $(\dim {}_FM)(\dim M_G) \geq 4$, such that a full exact subcategory $\mathcal{V}$ of $\mathcal{U}({}_D\mathcal{K})$ is representation equivalent to $\mathcal{M}_{R({}_FM_G)}$.

In considering an object X of a vectorspace category ${}_D\mathcal{K}$, we always will identify X and the underlying vectorspace $|X|$. If E is its endomorphism ring in $\mathcal{K}$, then X becomes a D-E-module ${}_DX_E$.

<u>Lemma</u>: Let ${}_D\mathcal{K}$ be a vectorspace category satisfying (*). Let X be an indecomposable object of ${}_D\mathcal{K}$, with $E = \mathrm{End}(X)$. Then either E is a division ring and $(\dim {}_DX)(\dim X_E) \leq 3$, or else E is a local ring with factor ring D, and X_E is a uniserial E-module of length ≤ 3.

Thus, we have the following possibilities for the bimodule ${}_DX_E$:

(i) E is a division ring and is a subring of D of index ≤ 3, and ${}_DX_E$ is canonically isomorphic to ${}_DD_E$,

(ii) E is a division ring, D is a subring of E of index ≤ 3, and ${}_DX_E$ is canonically isomorphic to ${}_DE_E$,

(iii) ${}_DX = {}_D(D \oplus D)$, and E is a subring of the 2×2-matrix ring $M_2(D)$ of the form $E = \left\{\begin{pmatrix} x & y \\ 0 & \alpha(x)\end{pmatrix} \middle| \ x,y \in D\right\}$, where α is an automorphism of D,

(iv) ${}_DX = {}_D(D \oplus D \oplus D)$, and E is a subring of $M_3(D)$ of the form

$$E = \left\{\begin{pmatrix} x & y_1 & z \\ 0 & \alpha(x) & y_2 \\ 0 & 0 & \beta(x)\end{pmatrix} \middle| \ x,y_1,y_2,z \in D\right\}$$

where α,β are automorphisms of D, and finally

(v) ${}_DX = {}_D(D \oplus D \oplus D)$, and E is a subring of $M_3(D)$ of the form

$$E = \left\{\begin{pmatrix} x & y & z \\ 0 & \alpha(x) & \alpha(y)t \\ 0 & 0 & t^{-1}\alpha^2(x)t\end{pmatrix} \middle| x,y,z \in D\right\}$$

where α is an automorphism of D, and $t \in D$ is a fixed element.

A vectorspace category $\mathcal{K}$ is called <u>Schurian</u> provided the endomorphism ring of any indecomposable object in $\mathcal{K}$ is a division ring. The Schurian vectorspace categories play a dominant role in the investigation, for two reasons: On the one hand, proposition 3.1 first is established for Schurian vectorspace categories, and then the general case is reduced to this special case. On the other hand, the reduction will be possible only in case we have a rather large amount of information concerning a given vectorspace category, and this information will be obtained by comparing K with a Schurian vectorspace category $\bar{K}$ derived from K. Schurian vectorspace categories of finite type are very well behaved and, in fact, one can write down esplicitly all indecomposable subspaces.

With any vectorspace category ${}_DK$, one associates a Schurian vectorspace category ${}_D\bar{K}$ as follows: for an indecomposable object X of ${}_DK$, let $X_i = X \cdot \mathrm{rad}^i\mathrm{End}(X)$, thus we obtain a filtration

$$X = X_o \supset X_1 \supseteq X_2 \supseteq \cdots ,$$

and we use as indecomposable objects of $\bar{K}$ the non-zero factors X_i/X_{i+1}, and as homomorphisms the induced maps. It is easy to see that many properties of ${}_DK$ can be transfered to ${}_D\bar{K}$. In particular, with ${}_DK$ also ${}_D\bar{K}$ satisfies the property (*).

Let K be a vectorspace category. A subspace (U,X,φ) of K is called faithful provided $U \neq 0$ and any indecomposable object of K appears as a direct summand of X. The vectorspace category K is called faithful provided there exists an indecomposable subspace of K which is faithful. Of course, in dealing with an individual subspace (U,X,φ) of an arbitrary vectorspace category K, we always may assume (U,X,φ) to be faithful, replacing K by the smallest subcategory of K containing X and closed under direct sums and direct summands.

The main result on Schurian vectorspace categories asserts that there are only finitely many faithful ones and gives a precise list of all faithful subspaces. In case of the additive category of a partially ordered set, this result is due to Kleiner [26].

The list of the possible faithful subspaces is one of the main working tool in the proof of proposition 3.1. Namely, one always compares a given subspace of a vectorspace category with the corresponding one of the derived Schurian vectorspace category: the last one can be decomposed according to the list, and this clearly gives a large amount of information on the given subspace.

Let us finish by stressing the fact that for vectorspace categories which are not Schurian, a corresponding classification of the faithful ones of finite type is known only in a very special situation (even in case K is defined over an algebraically closed field) [27]. Namely, one has to assume that dim $|X| \leq 2$ for X indecomposable, and that for X,Y indecomposable with dim $|X|$ = dim $|Y| = 2$, not both ${}_{\mathrm{End}(X)}\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(X,Y)_{\mathrm{End}(Y)}$ are indecomposable faithful modules. Even in this case, it is no longer true that there are only finitely many faithful categories: actually, it is rather easy to construct faithful categories of this form with an arbitrary large number of indecomposable objects.

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