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A Note on Optimal Block-Scaling of Matrices

Dedicated to Prof. Dr. F.L. Bauer on the occasion of his 60th birthday

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Summary. After pointing out that two recent results on optimal block-scaling are equivalent, a new short and simple proof of both results is given.

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We use the following notation: ρ is the spectralradius and $\| \ \|_2$ the spectralnorm, $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$ the condition of A with respect to the spectralnorm and $C^{n,k}$ the set of all complex $n \times k$ matrices. Let X_1, X_2 be linear subspaces of C^n, $\dim X_i = n_i$ and $X_1 \oplus X_2 = C^n$. Let

$$\mathcal{M} = \{A = (A_1, A_2) \colon A_i \in C^{n, n_i}, \ \operatorname{Im}(A_i) = X_i, \ i = 1, 2\}.$$

Demmel [3] and Bart et al. [2] have proved

Theorem 1. *Let $A = (A_1, A_2) \in \mathcal{M}$ and $A_i^H A_i = I_{n_i}$. Then*

$$\kappa_2(A) = \min\{\kappa_2(X), X \in \mathcal{M}\}.$$

In other words, if the columns of A_i are orthonormal bases of X_i then A has optimal condition.

Eisenstat et al. [4] have proved

Theorem 2. *Let $B = \begin{pmatrix} I_{n_1} & X \\ X^H & I_{n_2} \end{pmatrix}$ be positive definite. Then*

$$\kappa_2(B) = \min\{\kappa_2(D^H B D) \colon D \in \mathcal{D}\},$$

where

$$\mathcal{D} = \{D = D_1 \oplus D_2 \colon D_i \in C^{n_i, n_i}, \text{ nonsingular}\}.$$

By observing that $(\kappa_2(A))^2 = \kappa_2(A^H A)$ and that $C \in \mathcal{M} \Leftrightarrow C^H C = D^H B D$ for some $D \in \mathcal{D}$ and B is a positive definite matrix of the form of Theorem 2, we see at once that Theorem 1 is an easy consequence of Theorem 2. Similarly by taking Cholesky decomposition of B, Theorem 2 can be derived from Theorem 1.

We give now a proof of Theorem 2 (and hence of Theorem 1) which is simpler than the proofs given before. The main tool is a trick used by F.L. Bauer in [1].

From $B\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}$ we get $Xy=(\lambda-1)x$, $X^H x=(\lambda-1)y$ and $(\lambda-1)^2 x = XX^H x$. This shows that the eigenvalues of B are given by $1\pm\mu_i$, μ_i singular values of X. Also, if $z=\begin{pmatrix}x\\y\end{pmatrix}$ is an eigenvector belonging to the maximal eigenvalue $1+\|X\|_2$, then $\tilde{z}=\begin{pmatrix}x\\-y\end{pmatrix}=Sz$, $S=I_{n_1}\oplus -I_{n_2}$ is an eigenvector belonging to the minimal eigenvalue $1-\|X\|_2$. This gives

$$SB^{-1}SBz=\frac{1+\|X\|_2}{1-\|X\|_2}z=\kappa_2(B)\cdot z$$

hence

$$\rho(SB^{-1}SB)\geqq\kappa_2(B).$$

Now for any $D\in\mathscr{D}$, as $\|S\|_2=1$ and S commutes with D,

$$\begin{aligned}\kappa_2(B)&\leqq\rho(SB^{-1}SB)\leqq\|D^{-1}(SB^{-1}SB)D\|_2\\&=\|S(D^{-1}B^{-1}D^{-H})S(D^H BD)\|_2\leqq\kappa_2(D^H BD).\end{aligned}$$

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