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On Eigenvectors and Adjoints of Modified Matrices

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We study how the adjoint (or adjugate) adj(A) of a real or complex $n\times n$-matrix A behaves under the rank one modification $A \to A+uv^H$. The most interesting cases are rank $A = n-1$, rank$(A+uv^H) = n$ or $n-1$. In the first case relations with generalized inverses can be shown, e.g. that $(\operatorname{adj}(A+uv^H)-\operatorname{adj}(A))/v^H \operatorname{adj}(A)u$ is a $\{1,2\}$-inverse and that conversely any $\{1,2\}$-inverse of A can be expressed in this form. The results in the second case can be interpreted as results on the changes of an eigenvector under this modification and can be carried over to matrices A without rank restrictions. Finally we investigate the adjoint of a bordered matrix.

INTRODUCTION

It is well known how the inverse of a given matrix A behaves under a modification of the form

$$A \to A+uv^H = B$$

where A is an $n\times n$-matrix, u, v are $n\times 1$-matrices and "^{H}" denotes the Hermitian transposed. As an example we mention the Sherman–Morrison formula, e.g. [4], p. 124. In the present paper we derive corresponding relations for adj (A), the adjoint of A (sometimes named the adjugate of A). This is fairly simple for the case that A and B are nonsingular, however, when the rank of A is $n-1$, rather interesting results are obtained and a surprising

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connection to generalized inverses is revealed. The case when A and B are both singular and of rank $n-1$ is of special interest to us here. This situation describes the behaviour of an eigenvector with a corresponding eigenvalue that does not change under the above modification, which can be viewed as dual to the wellknown Wielandt method of deflation where all eigenvalues with the exception of one are invariant, whilst here only one eigenvalue is assumed to be invariant. Such a situation occurs in the investigation of homogeneous Markov chains, when the model is altered in such a way that the difference between the transition matrices describing the models is of rank one. The change of the eigenvector corresponding to the eigenvalue 1 then represents the change in the limiting distribution.

The notation and terminology which will be employed in this paper is introduced in Section 1. This section also contains well known results, as well as several new ones, concerning the adjoint of a matrix which are essential for the sequel.

In Section 2 the case where A and B have ranks $n-1$ and n, respectively, is considered. Our main result here, Theorem 2, is that, up to a scalar, the difference

$$\operatorname{adj}(A+uv^H)-\operatorname{adj}(A)$$

is a $\{1, 2\}$-inverse of A, and conversely, any $\{1, 2\}$-inverse of A can be expressed as a difference of adjoints of the above form.

Section 3 is devoted to the case where the ranks of A and B are both $n-1$, and a detailed investigation of the behaviour of $\operatorname{adj}(A+uv^H)$ is carried out (c.f. Theorem 4).

Theorem 4 suggests an examination of the nullspace of $A+uv^H$ in terms of the nullspace of A. This is done in Section 4 *without* rank restrictions on A and B. Some of the results in this section have been obtained previously, using a different approach, by Egervary [2].

In the final section we investigate the adjoint of a bordered matrix. For that purpose we exploit results obtained in earlier sections, in particular those contained in Theorem 2.

1.

All matrices and vectors have coefficients in F, where $F = R$ or C. Let A be an $n \times n$-matrix. For $1 \leqslant i, j \leqslant n$ denote by A_{ji} the determinant of the matrix which originates from A by deleting the j-th row and the i-th column. The matrix $B = (b_{ij})_{i,j=1,\ldots,n}$, $b_{ij} = (-1)^{i+j}A_{ji}$ is called the adjoint matrix of A and is denoted by $\operatorname{adj}(A)$.

If u, v are $n \times 1$-matrices then uv^H is a matrix of rank 1 and is sometimes called a dyade. We use the notation

$$D(A, uv^H) = \operatorname{adj}(A+uv^H) - \operatorname{adj}(A) \tag{1.1}$$

and for $v^H \operatorname{adj}(A)u \neq 0$

$$C(A, uv^H) = D(A, uv^H)/v^H \operatorname{adj}(A)u. \tag{1.2}$$

I_n denotes the n-dimensional unit matrix, $\det A$ the determinant of A, $R(A) = \{Au, u \in F^n\}$ its range and $N(A) = \{u \in F^n, Au = 0\}$ its nullspace.

The following results are well known:

$$A^{-1} \det A = \operatorname{adj}(A) \quad \text{if} \quad \det A \neq 0 \tag{1.3}$$

$$A \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det A \cdot I_n \tag{1.4}$$

$$\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A) \tag{1.5}$$

$$\det(A+uv^H) = \det A + v^H \operatorname{adj}(A)u \tag{1.6}$$

If A^{-1} exists and $1+v^H A^{-1}u \neq 0$, then $(A+uv^H)^{-1}$ exists and

$$(A+uv^H)^{-1} = A^{-1} - \frac{A^{-1}uv^H A^{-1}}{1+v^H A^{-1}u}. \tag{1.7}$$

By (1.3), (1.6), (1.7) one gets

$$\begin{aligned}
\operatorname{adj}(A+uv^H) &= (A+uv^H)^{-1} \det(A+uv^H) \\
&= A^{-1} - \frac{A^{-1}uv^H A^{-1}}{1+v^H A^{-1}u} \quad (\det A + v^H \operatorname{adj} Au) \\
&= \operatorname{adj}(A) + v^H \operatorname{adj}(A)uA^{-1} - \operatorname{adj}(A)uv^H A^{-1}
\end{aligned}$$

if $v^H A^{-1}u \neq 1$. Hence

$$D(A, uv^H) = v^H \operatorname{adj}(A)uA^{-1} - \operatorname{adj}(A)uv^H A^{-1} \tag{1.8}$$

if A^{-1} exists, (1.8) holds by continuity also in the case $v^H A^{-1}u + 1 = 0$.

Also

$$C(A, uv^H) = \left[I_n - \frac{\operatorname{adj}(A)uv^H}{v^H \operatorname{adj}(A)u}\right] A^{-1} = A^{-1}\left[I_n - \frac{uv^H \operatorname{adj}(A)}{v^H \operatorname{adj}(A)u}\right] \tag{1.9}$$

if A^{-1} exists and $v^H \operatorname{adj}(A)u \neq 0$.

PROPOSITION 1 *Let* $v^H \operatorname{adj}(A)u \neq 0$ *and* $C = C(A, uv^H)$. *Then*

$$CA = I_n - \frac{\operatorname{adj}(A)uv^H}{v^H \operatorname{adj}(A)u} \tag{1.10}$$

$$AC = I_n - \frac{uv^H \operatorname{adj}(A)}{v^H \operatorname{adj}(A)u} \tag{1.11}$$

$$CAC = C \tag{1.12}$$

$$ACA = A - \det A \cdot \frac{uv^H}{v^H \operatorname{adj}(A)u} \tag{1.13}$$

Proof (1.10) and (1.11) follow from (1.9) if A^{-1} exists and in general by continuity. From (1.10) $CACA = CA$ which implies (1.12) for A non-singular. By continuity (1.12) holds also in the singular case. Post-multiplying (1.11) by A and using (1.4) gives (1.13).

Let us finally recall that the Moore–Penrose-inverse A^+ of A is the unique matrix B satisfying

(i) $ABA = A$

(ii) $BAB = B$

(iii) $AB = (AB)^H$

(iv) $BA = (BA)^H$ (1.14)

and that each matrix satisfying (1.14) (i), (ii) is called a $\{1, 2\}$-inverse of A [1].

2.

In this chapter we assume $\operatorname{rank}(A) = n-1$ throughout. In this case we get from (1.4)

$$\operatorname{adj}(A) = rs^H \neq 0 \tag{2.1}$$

where r and s span $N(A)$ and $N(A^H)$ resp.

$$Ar = 0, \qquad s^H A = 0. \tag{2.2}$$

Hence r is a right-hand and s^H is a left-hand eigenvector of A corresponding to the eigenvalue 0.

PROPOSITION 2 *If* rank $A = n-1$ *and* adj$(A) = rs^H$ *then the following are equivalent*

(a) $A + uv^H$ *nonsingular*

(b) $v^H \operatorname{adj}(A)u \neq 0$ (2.3)

(c) $u \notin R(A), \quad v \notin R(A^H)$.

Proof (a) $\Leftrightarrow$ (b) follows from (1.6). Writing (b) in the form $(v^H r) \cdot (s^H u) \neq 0$ the equivalence with (c) follows from the fact that $u \in R(A)$ iff $s^H u = 0$ and $v \in R(A^H)$ iff $v^H r = 0$. ■

This proposition helps us to prove the following theorems which describe the set of all possible $C(A, uv^H)$ in different ways. Although most of the results are listed in theorem 3 we prefer to present parts of them in theorem 1 and 2 separately, thus underlining their importance.

THEOREM 1 *Let* rank $A = n-1$, adj$(A) = rs^H$. *Then*

$$C(A, sr^H) = A^+. \tag{2.4}$$

If $A + uv^H$ *is nonsingular then*

$$C(A, uv^H) = \left(I - \frac{rv^H}{v^H r}\right) A^+ \left(I - \frac{us^H}{s^H u}\right). \tag{2.5}$$

Proof As $r^H \operatorname{adj}(A)s \neq 0$, $B = C(A, sr^H)$ is defined. We show that $B = C(A, sr^H)$ satisfies the equations (1.14): (1.12) implies (ii), (1.13) and det $A = 0$ gives (i), by (1.10) $BA = I - rr^H/r^H r = (BA)^H$ and by (1.11) $AB = I - ss^H/s^H s = (AB)^H$, hence also (iii) and (iv) hold. For $C = C(A, uv^H)$ we get using (1.12), (1.14) (i), (1.10) and (1.11)

$$C = CAC = CAA^+AC = \left(I - \frac{rv^H}{v^H r}\right) A^+ \left(I - \frac{us^H}{s^H u}\right) \quad ■$$

In the situation of this chapter any $C = C(A, uv^H)$ satisfies (1.12) and by (1.13) also $ACA = A$, it is hence a $\{1, 2\}$-inverse of A. But also the converse is true. This is the result of

THEOREM 2 *Let* rank $A = n-1$, adj$(A) = rs^H$. *Then a given matrix C is a $\{1, 2\}$-inverse of A, i.e.*

$$ACA = A, \qquad CAC = C \tag{2.6}$$

iff there exist vectors u, v such that $v^H \operatorname{adj}(A)u \neq 0$ *and*

$$C = C(A, uv^H)$$

u, v are up to a factor uniquely determined.

Proof In view of the preceding remark it remains to show that a $\{1, 2\}$-inverse C of A is of the form $C = C(A, uv^H)$ for some appropriate u, v. We start by observing the following fact: If Q is a projection of rank 1, $Qx = x$, $y^HQ = y^H$, then $Q = xy^H/y^Hx$. Hence if P is a projection of rank $n-1$, $Px = 0$, $y^HP = 0$, then $Q = I-P$ satisfies the assumptions above, giving $P = I-Q = I-xy^H/y^Hx$. From (2.6) follows that $P = CA$ is a projection. Furthermore from $\operatorname{rank}(A) = \operatorname{rank}(ACA) \leqslant \operatorname{rank}(CA) \leqslant \operatorname{rank}(A)$ we infer $\operatorname{rank}(CA) = n-1$. Define $v \neq 0$ by $v^HCA = 0$. In addition $CAr = 0$. The observation above gives

$$CA = I - rv^H/v^Hr. \tag{2.7}$$

Similarly

$$AC = I - us^H/s^Hu \tag{2.8}$$

where $u \neq 0$ satisfies $ACu = 0$. Now by (2.6), (1.14), (2.7), (2.8), (2.5) $C = CAC = CAA^+AC = (I-rv^H/v^Hr)A^+(I-us^H/s^Hu) = C(A, uv^H)$. Let us finally remark that we can recover u and v from $C(A, uv^H)$ by (1.11) and (1.10) as $AC = I - us^H/s^Hu$ and $CA = I-rv^H/v^Hr$. Hence u and v are uniquely determined up to a factor. ∎

In the following theorem different descriptions of the set of all $C(A, uv^H)$ are listed. An indication of the proofs is given below:

THEOREM 3 *Let* $\operatorname{rank} A = n-1$, $\operatorname{adj}(A) = rs^H$ *and*

$$C_1 = \{C(A, uv^H): u, v \in F^n, v^H \operatorname{adj}(A)u \neq 0\}$$

$$C_2 = \{C: CAC = C, ACA = A\}$$

$$C_3 = \{(I-rv^H/v^Hr)A^+(I-us^H/s^Hu): v^Hr \neq 0, s^Hu \neq 0\}$$

$$C_4 = \{A^+ - rg^H - hs^H + g^HAh \cdot rs^H : g^Hs = r^Hh = 0\}$$

$$C_5 = \{(A^+ - rg^H)A(A^+ - hs^H): g^Hs = r^Hh = 0\}$$

$$C_6 = \{A^+ - rg^H - hs^H : g^Hs/s^Hs + r^Hh/r^Hr + g^HAh = 0\}$$

Then $C_1 = C_2 = C_3 = C_4 = C_5 = C_6$.

Proof Theorems 1 and 2 show $C_1 = C_2 = C_3$. If $C \in C_3$, then it has the form $A^+ - rg^H - hs^H + g^HAh \cdot rs^H$ where $g^H = v^HA^+/v^Hr$, $h = A^+u/s^Hu$. But $A^+s = 0$, $r^HA^+ = 0$ show $g^Hs = r^Hh = 0$, hence $C_3 \subset C_4$. On the other hand, each g^H such that $g^Hs = 0$ is of the form $g^H = v^HA^+$ for a suitable v, which in addition can be chosen to satisfy $v^Hr = 1$. Similarly $r^Hh = 0$ implies the existence of u such that $s^Hu = 1$, $h = A^+u$. This shows $C_4 = C_3$.

Observing that for h with $r^Hh = 0$

$$A^+Ah = (I - rr^H/r^Hr)h = h$$

and $g^H AA^+ = g^H(I - ss^H/s^H s) = g^H$ if $g^H s = 0$, we have for $g^H s = r^H h = 0$

$$(A^+ - rg^H)A(A^+ - hs^H) = A^+ - rg^H - hs^H + (g^H Ah)rs^H$$

and hence $C_4 = C_5$.

If $C \in C_6$, $C = A^+ - rg^H - hs^H$, then because of

$$\frac{g^H s}{s^H s} + \frac{r^H h}{r^H r} = -g^H Ah$$

$$C = A^+ - r\left(g^H - \frac{g^H s}{s^H s} s^H\right) - \left(h - \frac{r^H h}{r^H r} r\right) s^H + g^H Ah \cdot rs^H$$

$$= A^+ - r\tilde{g}^H - \tilde{h}s^H + \tilde{g}^H A\tilde{h}\, rs^H$$

which shows that $C \in C_4$. On the other hand, writing a $C \in C_4$ in the form $C = A^+ - r[g^H - g^H Ah\, s^H] - hs^H = A^+ - r\tilde{g}^H - hs^H$ it is readily established that $C \in C_6$, hence $C_4 = C_6$. ■

3.

We shall now examine the case where $B = A + uv^H$ is singular, too.

THEOREM 4 *Let* $\operatorname{rank}(A) = n-1$, $\operatorname{adj}(A) = rs^H$, $\operatorname{rank}(A + uv^H) < n$. *Then either* (1) $\operatorname{rank}(A + uv^H) = n-2$. *In this case* $u = Aw$, $v^H = x^H A$, $x^H Aw = -1$, $\operatorname{adj}(A + uv^H) = 0$

or

(2) $\operatorname{rank}(A + uv^H) = n-1$. *Here the following cases are possible*

(a) $u = Aw$, $v \notin R(A^H)$. *Then* $v^H r \neq 0$ *and*

$$\operatorname{adj}(A + uv^H) = ((1 + v^H w)r - v^H rw)s^H \tag{3.1}$$

(b) $u \notin R(A)$, $v^H = x^H A$. *Then* $s^H u \neq 0$ *and*

$$\operatorname{adj}(A + uv^H) = r((1 + x^H u)s^H - s^H ux^H) \tag{3.2}$$

(c) $u = Aw$, $v^H = x^H A$, $x^H Aw + 1 \neq 0$. *Then*

$$\operatorname{adj}(A + uv^H) = (1 + x^H Aw)rs^H. \tag{3.3}$$

Proof We start from the relation

$$\operatorname{adj}(A + uv^H) - \operatorname{adj}(A) = D(A, uv^H) = (v^H rI_n - rv^H)A^+(s^H uI_n - us^H). \tag{3.4}$$

It follows from (2.5) in case of $v^H rs^H u \neq 0$ and in general by continuity. As

$A + uv^H$ is singular, by proposition 2 exactly one of the following cases occur

(i) $u \in R(A), \quad v \notin R(A^H)$

(ii) $u \notin R(A), \quad v \in R(A^H)$

(iii) $u \in R(A), \quad v \in R(A^H)$.

In case (i) $u = Aw, s^Hu = 0, v^Hr \neq 0$. We may assume, by eventually replacing w by $w+tr$, that $r^Hw = 0$. In this case $A^+u = A^+Aw = w$. Now (3.4) gives $D(A, uv^H) = -v^HrA^+us^H + v^HA^+urs^H = ((v^Hw)r-(v^Hr)w)s^H$ from which (3.1) follows. As $v^Hr \neq 0$, $\operatorname{adj}(A+uv^H) \neq 0$ and $\operatorname{rank}(A+uv^H) = n-1$.

Similarly case (ii) is treated, it corresponds to case (2)(b). In case (iii) one has $v^Hr = s^Hu = 0, u = Aw, v^H = x^HA$.

From (3.4)

$$\operatorname{adj}(A+uv^H) = rs^H+(v^HA^+u)rs^H = (1+v^HAw)rs^H.$$

If $1+x^HAw \neq 0$, $\operatorname{rank}(A+uv^H) = n-1$, this is case (2)(c). Otherwise $\operatorname{adj}(A+uv^H) = 0$, $\operatorname{rank}(A+uv^H) \leqslant n-2$ and by the well-known relation $\operatorname{rank}(X+Y) \geqslant \operatorname{rank}(X)-\operatorname{rank}(Y)$ we get $\operatorname{rank}(B) = n-2$. This corresponds to case (1). ■

Let us indicate a different proof of theorem 4, which does not use theorem 1 and the relation (3.4).

We start from

$$\operatorname{adj}(I_n+xy^H) = (1+y^Hx)I_n - xy^H \tag{3.5}$$

which follows easily from (1.6), (1.7) applied to $A = I_n$. Now in case (i) $u = Aw$ and hence $\operatorname{adj}(A+uv^H) = \operatorname{adj}(A(I_n+wv^H)) = \operatorname{adj}(I_n+wv^H)\operatorname{adj}(A) = [(1+v^Hw)I_n-wv^H]\cdot rs^H = [(1+v^Hw)r-(v^Hr)w]s^H$. Similarly the remaining cases can be treated.

Using the remark following (2.2) we may interpret the results of theorem 4 in terms of eigenvectors:

2(a): If $u = Aw$, $v \notin R(A^H)$ then under the modification $A \to A+uv^H$ the eigenvalue 0 and the corresponding left-hand eigenvector stay invariant while the right-hand eigenvector r is replaced by $(1+v^Hw)r-(v^Hr)w$.

2(b): If $u \notin R(A)$, $v^H = x^HA$, the eigenvalue 0 and the corresponding right-hand eigenvector stay invariant, while the left-hand eigenvector s^H is replaced by $(1+x^Hu)s^H-(s^Hu)\cdot x^H$.

2(c): If $u = Aw$, $v^H = x^HA$ and $x^HAw+1 \neq 0$, the eigenvalue 0 and the corresponding right- and left-hand eigenvectors stay invariant.

We illustrate the results of theorem 4 by some simple examples. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $r^H = (1, 0, 0)$, $s^H = (0, 0, 1)$. Then $u = (u_1, u_2, u_3)^H \in R(A)$ iff $u_3 = 0$. In this case $u = Aw(t)$ for $w(t) = (t, u_1, u_2)^H$ where t is a free parameter. Similarly $v = (v_1, v_2, v_3)^H \in R(A^H)$ iff $v_1 = 0$, $v^H = x(t)^H A$, $x^H(t) = (v_2, v_3, t)$. The four cases in theorem 4 are

(1) $u_3 = v_1 = 0, \quad \bar{u}_1 v_2 + \bar{u}_2 v_3 + 1 = 0$

(2a) $u_3 = 0, \quad v_1 \neq 0$

(2b) $u_3 \neq 0, \quad v_1 = 0$

(2c) $u_3 = v_1 = 0, \quad \bar{u}_1 v_2 + \bar{u}_2 v_3 + 1 \neq 0.$

In the case (2a) the new right-hand eigenvector $\tilde{r}$ is given by

$$\tilde{r} = (1 + \bar{v}_2 u_1 + \bar{v}_3 u_2, \ -\bar{v}_1 u_1, \ -\bar{v}_1 u_2)^H$$

0 is a defective eigenvalue of $A + uv^H$ too, if $s^H \tilde{r} = 0$, i.e. $u_2 = 0$ or equivalently $u \in R(A^2)$. Otherwise $\tilde{r}$ can be normalized such that $s^H \tilde{r} = 1$. This can be done by dividing by $1 + v^H w(t)$ for a suitable t, i.e. by writing

$$\tilde{r}(t) = r - \frac{v^H r w(t)}{1 + v^H w(t)}. \tag{3.6}$$

For $\bar{t} = -v_1^{-1}(1 + \bar{u}_2 v_1 + \bar{u}_2 v_3 + \bar{u}_1 v_2) \qquad s^H \tilde{r}(t) = 1.$

For

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

$r = s = (1, 0, 0)^H$. $u = (u_1, u_2, u_3)^H \in R(A)$ iff $u_1 = 0$, and then $u = Aw(t)$, where $w(t) = (t, u_2 \bar{d}_2^{-1}, u_3 \bar{d}_3^{-1})$. Let $v^H = (v_1, v_2, v_3)$.

We discuss case (2a) only, given by $u_1 = 0$, $v_1 \neq 0$.

$$\begin{aligned} \tilde{r} &= (1 + v^H w(t)) r - (v^H r) w(t) \\ &= (1 + \bar{v}_2 u_2 \bar{d}_3^{-1} + \bar{v}_3 u_3 \bar{d}_3^{-1}, \ -\bar{v}_1 u_2 \bar{d}_2^{-1}, \ -\bar{v}_1 u_3 \bar{d}_3^{-1})^H. \end{aligned}$$

0 is a defective eigenvalue of $A + uv^H$ if

$$\tilde{r}^H s = 1 + \bar{v}_2 u_2 \bar{d}_3^{-1} + \bar{v}_3 u_3 \bar{d}_3^{-1} = 0$$

otherwise $\tilde{r}(t)$ as defined in (3.6) satisfies $\tilde{r}^H(t) s = 1$ for $t = 0$.

4.

Theorem 4 provides an analysis of the behaviour of the eigenvectors under rank one-modifications only in the special case that the invariant eigenvalue is of geometric multiplicity 1 for A. We study now more general settings.

THEOREM 5 *Let A and B be $n \times n$-matrices, $B = A + uv^H \neq A$. Then*

(1) $\operatorname{rank}(B) = \operatorname{rank}(A) - 1$ *iff* $u = Aw$, $v^H = x^H A$ *and* $x^H A w + 1 = 0$.

Here

$$N(B) = L(N(A) \cup w) \qquad N(B^H) = L(N(A^H) \cup x).$$

(If S is a set, $L(S)$ denotes the smallest linear subspace containing S)

(2) $\operatorname{rank}(B) = \operatorname{rank}(A)$ iff

either (i) $u = Aw$, $v \notin R(A^H)$.

In this case it is possible to choose w such that $v^H w = 0$. For this w

$$N(B) = (I - wv^H)N(A), \quad N(B^H) = N(A^H)$$

or (ii) $u \notin R(A)$ $v^H = x^H A$.

In this case it is possible to choose x such that $x^H u = 0$. For this x

$$N(B) = N(A), \quad N(B^H) = (1 - xu^H)N(A^H)$$

or (iii) $u = Aw$, $v^H = x^H A$, $x^H A w + 1 \neq 0$.

Here $N(B) = N(A)$, $N(B^H) = N(A^H)$.

(3) $\operatorname{rank}(B) = \operatorname{rank}(A) + 1$ iff $u \notin R(A)$, $v \notin R(A^H)$. Here

$$N(B) = N(A) \cap v^\perp$$

$N(B^H) = N(A^H) \cap u^\perp$. Here $v^\perp = \{x : v^H x = 0\}$.

Proof (a) Let $u \notin R(A)$, $v \notin R(A^H)$.

If $y \in N(B)$ then $Ay + (v^H y)u = 0$. As $u \notin R(A)$, $v^H y = 0$ and $Ay = 0$. Hence $N(B) \subset N(A) \cap v^\perp$. It is obvious that $N(A) \cap v^\perp \subset N(B)$. This shows $N(B) = N(A) \cap v^\perp$. Similarly $N(B^H) = N(A^H) \cap u^\perp$. As $v \notin R(A^H)$, $v^\perp \not\supset N(A)$, $\dim N(B) = \dim N(A) - 1$, or $\operatorname{rank}(B) = \operatorname{rank}(A) + 1$.

(b) Let $u = Aw$, $v \notin R(A^H)$.

There is $y \in N(A)$, $v^H y \neq 0$. Hence we may assume, by eventually replacing w by $w + ty$, that $v^H w = 0$. Now $B = A(I + wv^H)$, $(I + wv^H)^{-1} = I - wv^H$. Hence $y \in N(B) \Leftrightarrow A(I + wv^H)y = 0 \Leftrightarrow (1 + wv^H)y \in N(A) \Leftrightarrow y \in (I - wv^H)N(A)$.

This shows $N(B) = (I - wv^H)N(A)$, similarly $N(B^H) = N(A^H)$. Hence $\text{rank}(B) = \text{rank}(A)$.

(c) The case $u \notin R(A)$, $v = x^H A$ is treated by applying (b) to A^H and B^H. Hence $N(B^H) = (1 - xu^H)N(A^H)$, $u^H x = 0$, $N(A) = N(B)$, $\text{rank}(A) = \text{rank}(B)$.

(d) Let $u = Aw$, $v^H = x^H A$. Then

$$B = A + Awx^H A = A(I + wx^H A) = (I + Awx^H)A. \tag{4.1}$$

(i) If $x^H Aw + 1 \neq 0$, then by (1.6) $I + wx^H A$ and $I + Awx^H$ are nonsingular. As in case (b) and (c) we infer from the two representations of B given in (4.1) that $N(B) = N(A)$, $N(B^H) = N(A^H)$, $\text{rank}(A) = \text{rank}(B)$.

(ii) If $x^H Aw + 1 = 0$, then obviously $Bw = 0$, but $w \notin N(A)$, as $Aw = u \neq 0$. Hence $N(B) \supset L(N(A) \cup w)$, and $\dim N(B) \geqslant 1 + \dim N(A)$. Hence $\text{rank}(A) = n - \dim N(A) \geqslant n + 1 - \dim N(B) = 1 + \text{rank}(B) \geqslant \text{rank}(A)$ the last inequality following from $\text{rank}(X + Y) \geqslant \text{rank}(X) + \text{rank}(Y)$. Hence $\text{rank}(B) = \text{rank}(A) - 1$, $N(B) = L(N(A) \cup w)$. Similarly $N(B^H) = L(N(A^H) \cup x)$.

Regrouping of the proved results yields theorem 5. ∎

Remark A particular result of theorem 5, namely that $\text{rank}(A + uv^H) = \text{rank}(A) - 1$ iff $u = Aw$, $v^H = x^H A$ and $x^H Aw + 1 = 0$, has been proved by Egervary [2].

The following corollary is an easy consequence of theorem 5.

Corollary *Let $n \geqslant 2$. Let A be a $n \times n$-complex matrix, u, $v \in C^n$ and $B = A + uv^H$. Then the following are equivalent.*

(1) A and B have no common eigenvalues.

(2) A and B have no common right-hand and no common left-hand eigenvectors.

(3) A and B have no common right-hand eigenvectors in $v^\perp$ and no common left-hand eigenvectors in $u^\perp$.

Proof Let A and B have a common eigenvalue μ. Theorem 5 applied to $A - \mu I$ implies that one of the three cases occur:

(i) $u = (A - \mu I)w$,

(ii) $v^H = x^H(A - \mu I)$,

(iii) $N(B - \mu I) = N(A - \mu I) \cap v^\perp \neq \{0\}$.

(3) does not hold, as in (i) any left-hand eigenvector z of A is a left-hand eigenvector of B and satisfies $z^H u = u^H z = 0$. In (ii) any right-hand eigenvector z of A is a right-hand eigenvector of B and satisfies $v^H z = 0$ and in

(iii) any right-hand eigenvector of B is a right-hand eigenvector of A and orthogonal to v. Hence (3) ⇒ (1). As (2) ⇒ (3) trivially, it remains to show that (1) ⇒ (2). If A and B have a common, say, right-hand eigenvector r, $Ar = \mu r$, $(A+uv^H)r = \nu r$, then $(\nu-\mu)r = u\cdot(v^H r)$. If $\nu-\mu = 0$ then (1) does not hold. Otherwise $v^H r \neq 0$ and u is a multiple of r. As $n \geqslant 2$ there exists always a left-hand eigenvector y^H of A such that $y^H r = 0$. But then $y^H u = 0$, too, and $y^H B = y^H A = \kappa y^H$. Hence (1) ⇒ (2). ■

5.

We study for completeness a further modification, the bordering

$$A \to B = \begin{pmatrix} A & u \\ v^H & \alpha \end{pmatrix} \tag{5.1}$$

where A is $n\times n$, u and v are $n\times 1$ and $\alpha \in R$, B is $n+1\times n+1$. Using the well-known formula for the inverse of B (e.g. [3], p. 113), using

$$\det B = \alpha \det A - v^H \operatorname{adj}(A)u, \tag{5.2}$$

(1.18) and the continuity of adj(B), we get

$$\operatorname{adj}(B) = \begin{pmatrix} (\alpha+1)\operatorname{adj} A - \operatorname{adj}(A+uv^H) & -\operatorname{adj}(A)u \\ -v^H \operatorname{adj}(A) & \det A \end{pmatrix} \tag{5.3}$$

By applying the results in 1 or 2 numerous other representations of adj(B) can be given.

In the spirit of chapter 3 we can get explicit formulas for adj(B) in terms of adj(A), which as before can also be viewed as results on the eigenvectors of B. We collect some of the results in the following theorem 6, the proof of which proceeds along similar lines as theorem 4 and is therefore omitted.

THEOREM 6 *Let* $\det B = \alpha \det A - v^H \operatorname{adj}(A)u = 0$. *Then* $\operatorname{rank}(B) = n$ *iff one of the following cases holds*:

(1) $\det A \neq 0$, $\alpha = v^H A^{-1}u$. Here

$$\operatorname{adj}(B) = \det A\cdot \tilde{r}\tilde{s}^H$$

where $\tilde{r} = (u^H(A^{-1})^H, 1)^H$, $\tilde{s}^H = (v^H A^{-1}, 1)$.

(2) $\operatorname{rank}(A) = n-1$, $Ar = 0$, $s^H A = 0$, $\operatorname{adj}(A) = rs^H$

and either

(i) $u = Aw$, $v \notin R(A^H)$. Then $\operatorname{adj}(B) = \tilde{r}\tilde{s}^H$ where

$$\tilde{r} = ((\alpha - v^H w)r^H + (v^H r)w^H, -v^H r)^H, \tilde{s}^H = (s^H, 0)$$

or

(ii) $v^H = x^H A$, $u \notin R(A)$. Then $\operatorname{adj}(B) = \tilde{r}\tilde{s}^H$ where

$$\tilde{r} = (r^H, 0)^H, \quad \tilde{s}^H = ((\alpha - u^H x)s^H + u^H s x^H, \ -u^H s)$$

or

(iii) $u = Aw$, $v^H = x^H A$, $x^H A w \neq \alpha$. Then $\operatorname{adj}(A) = (\alpha - x^H A w)\tilde{r}\tilde{s}^H$

$$\tilde{r} = (r^H, 0)^H, \quad \tilde{s}^H = (s^H, 0).$$

(3) $\operatorname{rank}(A) = n-2$, $u \notin R(A)$, $v \notin R(A^H)$. In this case $\operatorname{rank}(A + uv^H) = n-1$, $\operatorname{adj}(A + uv^H) = r_1 s_1^H \neq 0$ for some vectors r_1, s_1 and $\operatorname{adj}(B) = -\tilde{r}\tilde{s}^H$, where

$$\tilde{r} = (r_1^H, 0)^H, \quad \tilde{s}^H = (s_1^H, 0).$$

In all other cases $\operatorname{adj}(B) = 0$.

We remark finally that by using a bordering of A it is possible to give another proof of (2.5).

Assuming $\operatorname{rank}(A) = n-1$, $\operatorname{adj} A = rs^H$, $\alpha = 0$, $v^H \operatorname{adj}(A)u \neq 0$ define B by (5.1). B is nonsingular by (5.2) and by (5.3)

$$B^{-1} = \frac{1}{v^H \operatorname{adj} Au} \begin{pmatrix} \operatorname{adj}(A + uv^H) - \operatorname{adj}(A) & \operatorname{adj}(A)u \\ v^H \operatorname{adj}(A) & 0 \end{pmatrix}$$

On the other hand it is easy to verify (see also [1], p. 231) that

$$B^{-1} = \begin{pmatrix} \left(I - \frac{rv^H}{v^H r}\right) A^+ \left(I - \frac{us^H}{s^H u}\right) & \frac{r}{v^H r} \\ \frac{s^H}{s^H u} & 0 \end{pmatrix}$$

Comparison of the upper left corner gives (2.5).

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