Elementary non-Archimedean utility theory

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ABSTRACT. A non-Archimedean utility representation theorem for independendent and transitive preference orderings that are partially continuous on some convex subset and satisfy an axiom of incommensurable preference for elements outside that subset is proven. For complete preference orderings, the theorem is deduced directly from the classical von Neumann-Morgenstern theorem; in the absence of completeness, Aumann's (1962) generalization is utilized.

1. Introduction

Non-Archimedean utility representations are representations of preference orderings by means of utility functions whose range is a lexicographically ordered vector space or a non-Archimedean real-ordered field (e.g. an ultrapower of the reals with respect to a non-principal ultrafilter). Such representations have been studied in remarkable generality, through the theory of mixture spaces, for half a century. Initiated by Hausner [**4**] and Thrall [**11**], the field has seen notable contributions by Fishburn [**2**], Skala [**9, 10**], Fishburn and Lavalle [**3**], Kannai [**6**], and, more recently, Lehmann [**7**], to mention but a few.

It seems that no attempt has been undertaken so far to arrive at non-Archimedean utility functions in a more direct manner, viz. by invoking the classical von Neumann-Morgenstern theorem [**12**] — or, in the absence

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of completeness, Aumann's generalization[1] [**1**] — for the convex subspace of finite-utility lotteries and introducing an incommensurability axiom for infinite values.[2] The present short note fills this gap.

2. Main result

Consider a subset B of a real vector space, and let X be its convex hull. Suppose $\preceq$ is a binary relation $\subseteq X \times X$. For any $x, y \in X$, we shall write $x \prec y$ whenever $x \preceq y$ but $y \npreceq x$, and $x \sim y$ whenever both $x \preceq y$ and $y \preceq x$.

Let $n \in \mathbb{N}$ and suppose that there are $x_1, \ldots, x_n \in B$ such that $x_k \prec x_{k+1}$ for all $k < n$. For every $k \leq n$, we define

$$B_k := B \setminus \{x_{k+1}, \ldots, x_n\},$$

and denote the convex hull of B_k by X_k. For any $x \in X$ and $X' \subseteq X$, we say that x is *incommensurably preferable* to X' if and only if

$$\forall y, z \in X' \quad \forall q \in (0,1] \qquad y \prec qx + (1-q)z.$$

Let V be the real vector space $\mathbb{R}^{n+1}$, and let $<$ be the strict lexicographical linear ordering of V. The unit vectors of the canonical basis of V are denoted $e_0, \ldots, e_n$.

A *utility representation function* for $\preceq$ is an affine map $U : X \to V$ such that for all $x, y \in X$,

- if $x \prec y$ then $U(x) < U(y)$, and
- if $x \sim y$ then $U(x) = U(y)$.

THEOREM 1. *Suppose $\preceq$ satisfies all of the following axioms:*

- **Reflexivity.** *For all $x \in X$, $x \preceq x$.*
- **Transitivity.** *For all $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.*
- **Independence.** *For all $x, y, z \in X$ and $p \in (0,1]$,*
$$x \preceq y \Leftrightarrow px + (1-p)z \preceq py + (1-p)z.$$
- **Partial Continuity.**[3] *For all $x, y, z \in X_0$ with $y \prec z$, there exists some $p \in (0,1)$ such that*
$$z \npreceq px + (1-p)y.$$
- **Incommensurability of Infinite Values.** *For every $k \in \{1, \ldots, n\}$, the outcome x_k is incommensurably preferable to B_{k-1}.*

Then there exists a utility representation function for $\preceq$.

The proof uses Aumann's generalization of the von Neumann-Morgenstern theorem for a partial ordering on a mixture spaces, in the formulation of Aumann [**1**]. A closely related result, which can be proven

[1]The referee suggested the use of Aumann's result [**1**] in order to remove the assumption of completeness. The original submission did not contain Theorem 1.

[2]From a historical vantage point, this is rather surprising, since von Neumann and Morgenstern [**12**] themselves recognized the possibility of non-Archimedean utility functions (cf. Skala [**10**] for a discussion of this point). As the referee pointed out, von Neumann and Morgenstern actually anticipated that in the absence of the completeness axiom, one obtains "a many-dimensional concept of utility" [**12**, 3.7.2].

[3]This formulation of continuity on X_0 follows Aumann [**1**, p. 449, formula (1.2)].

either directly from the von Neumann-Morgenstern theorem or as a corollary of Theorem 1, is the following:

THEOREM 2. *There exists some affine function $U : X \to V$ such that*

$$\forall x, y \in X \qquad x \preceq y \Leftrightarrow U(x) \leq U(y), \qquad \forall k \leq n \qquad U(x_k) = e_k \tag{1}$$

if and only if $\preceq$ satisfies all of the following axioms:

- **Completeness.** *For all $x, y \in X$, either $x \preceq y$ or $y \preceq x$ or both.*
- **Transitivity.** *For all $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.*
- **Independence.** *For all $x, y, z \in X$ and $p \in (0, 1]$,*
$$x \preceq y \Leftrightarrow px + (1-p)z \preceq py + (1-p)z.$$
- **Partial Continuity.** *For all $x, y, z \in X_0$ with $x \prec y \prec z$, there exist $p, q \in (0, 1)$ such that*
$$px + (1-p)z \prec y \prec qx + (1-q)z.$$
- **Incommensurability of Infinite Values.** *For every $k \in \{1, \ldots, n\}$, the outcome x_k is incommensurably preferable to B_{k-1}.*

We have chosen the framework of mixture spaces in this paper because this seems to be the most popular setting for applications of non-Archimedean utility theory in fields such as philosophical decision theory. As an alternative approach, one could assume that X is itself a real vector space, rather than merely a convex subset thereof. In this framework, it would be more natural to employ the theory developed by Kannai [**5**], which also extends to infinite dimensions, rather than Aumann's findings [**1**]. Choosing this route would, in particular, lead to new versions of Theorems 1 and 2 where X may be chosen as $V = \mathbb{R}^{n+1}$ and even the identity map on V could be an admissible choice of U.

As a corollary to Theorem 2, we also obtain a utility representation of $\preceq$ in a non-Archimedean field whenever $\preceq$ satisfies the axioms listed in Theorem 2. In the following, ${}^*\mathbb{R}$ is the non-Archimedean field of hyperreals (in the sense of Robinsonian nonstandard analysis [**8**]),[4] and I denotes an arbitrary, but fixed positive infinite hyperreal (i.e. $I > n$ for all $n \in \mathbb{N}$).

COROLLARY 3. *There exists some affine function $u : X \to {}^*\mathbb{R}$ such that*

$$\forall x, y \in X \qquad x \preceq y \Leftrightarrow u(x) \leq u(y), \qquad \forall k \leq n \qquad u(x_k) = I^k \tag{2}$$

if and only if $\preceq$ satisfies the axioms of completeness, transitivity, independence, partial continuity, and incommensurability of infinite values.

The interpretation of the incommensurability of infinite values is straightforward: For every level $k \leq n$, the slightest positive chance of winning the lottery with pure outcome x_k is still preferable to any lottery with pure outcome from B_{k-1} — and, in fact, to any lottery from X_{k-1} (see Lemma 5).

This means first of all that $x_1, \ldots, x_n$ are incommensurably preferable to X_0. But on X_0, the relation $\preceq$ has a real-valued (von Neumann-Morgenstern) utility representation X_0 (by its continuity on X_0). Therefore, $x_1, \ldots, x_n$ must be infinite-utility outcomes. In addition, the incommensurability of

[4] ${}^*\mathbb{R}$ is the ultrapower of $\mathbb{R}$ with respect to some non-principal ultrafilter.

infinite values asserts that there is a strict hierarchy of incommensurable preference among these outcomes.

For examples and applications, we refer to Thrall [**11**] and only mention that military strategic choices often involve such decisions: Nearly everything should be risked for winning a war, and all supplies should be risked for conserving manpower.

Possible applications might also include economic reasoning in law when legal norms of varying importance are at stake.

From a historical perspective, the prime example would be Pascal's wager: Pascal's wagerer has to choose between searching after the Christian God, which with positive probability leads to salvation and hence infinite utility (in case the Christian God exists and the wagerer ends up wagering for Him) or faith in atheism.

3. Proof

The proofs of Theorems 1 and 2 utilize the following Lemma:

LEMMA 4. *Suppose* $\preceq$ *is transitive and independent, and let* $k \in \{1, \ldots, n\}$. *If* x_k *is incommensurably preferable to* B_{k-1}, *then*

$$\forall y, z \in X_{k-1} \quad \forall p < q \in [0,1] \qquad px_k + (1-p)y \prec qx_k + (1-q)z.$$

The proof of Lemma 4, in turn, requires another result:

LEMMA 5. *Suppose* $\preceq$ *is transitive and independent, and let* $k \in \{1, \ldots, n\}$. *If* x_k *is incommensurably preferable to* B_{k-1}, *then also to* X_{k-1}.

In the proof of the Lemmas and the Theorem, we will use that

$$\forall x, y, z \in X \quad \forall p \in (0,1] \qquad x \prec y \Leftrightarrow px + (1-p)z \prec py + (1-p)z$$

(which is a consequence of $\preceq$'s independence in general and even equivalent to independence for complete $\preceq$).

PROOF OF LEMMA 5. Define

$$Z_{k-1} := \{z \in X_{k-1} \ : \ \forall y \in X_{k-1} \quad \forall q \in (0,1] \qquad y \prec qx_k + (1-q)z\}.$$

First we shall prove that $B_{k-1} \subseteq Z_{k-1}$. Consider any $z \in B_{k-1}$. Let $q \in (0,1]$, and define

$$Y := \{y \in X_{k-1} \ : \ y \prec qx_k + (1-q)z\}.$$

Note that Y is convex: For every $y, y' \in Y$ and $p \in (0,1)$, independence yields

$$\begin{aligned} py + (1-p)\,y' &\prec p\,(qx_k + (1-q)z) + (1-p)y' \\ &\prec p\,(qx_k + (1-q)z) + (1-p)\,(qx_k + (1-q)z) = qx_k + (1-q)z, \end{aligned}$$

hence by transitivity, $py + (1-p)y' \prec qx_k + (1-q)z$, so $py + (1-p)y' \in Y$.

On the other hand, $B_{k-1} \subseteq Y$ by assumption, so X_{k-1}, the convex hull of B_{k-1}, must also be $\subseteq Y$. Therefore, $y \prec qx_k + (1-q)\,z$ for all $y \in X_{k-1}$. Since $q \in (0,1]$ was arbitrary, we even get that

$$\forall y \in X_{k-1} \quad \forall q \in (0,1] \qquad y \prec qx_k + (1-q)z,$$

whence already $z \in Z_{k-1}$. Therefore $B_{k-1} \subseteq Z_{k-1}$, as claimed.

Next we prove that Z_{k-1} is convex. Consider any $w, z \in Z_{k-1}$ and $p \in (0,1)$. We shall prove that $pw + (1-p)z \in Z_{k-1}$. For all $y \in X_{k-1}$ and $q \in (0,1]$, independence gives us

$$\begin{aligned} y &= py + (1-p)y \prec py + (1-p)\left(qx_k + (1-q)z\right) \\ &\prec p\left(qx_k + (1-q)w\right) + (1-p)\left(qx_k + (1-q)z\right) \\ &= qx_k + p(1-q)w + (1-p)(1-q)z = qx_k + (1-q)\left(pw + (1-p)z\right). \end{aligned}$$

By transitivity, we obtain

$$\forall y \in X_{k-1} \quad \forall q \in (0,1] \qquad y \prec qx_k + (1-q)\left(pw + (1-p)z\right),$$

in other words: $pw + (1-p)z \in Z_{k-1}$. Therefore, Z_{k-1} is convex, as claimed.

But $B_{k-1} \subseteq Z_{k-1}$, as was shown in the first part of this proof. Therefore, the set X_{k-1}, the convex hull of B_{k-1}, must also be $\subseteq Z_{k-1}$. (In fact, $X_{k-1} = Z_{k-1}$.) Thus,

$$\forall z \in X_{k-1} \quad \forall q \in (0,1] \quad \forall y \in X_{k-1} \qquad y \prec qx_k + (1-q)z.$$

□

Proof of Lemma 4. Let $y, z \in X_{k-1}$ and $p < q \in [0,1]$. We need to prove that $px_k + (1-p)\,y \prec qx_k + (1-q)\,z = px_k + (q-p)\,x_k + (1-q)\,z$, which is, by independence, equivalent to $y \prec \frac{q-p}{1-p}x_k + \frac{1-q}{1-p}z$. This last assertion, however, is true by Lemma 5. □

In particular, the formula in Lemma 4 is valid for all $k \in \{1, \ldots, n\}$ if $\preceq$ satisfies the conditions in Theorem 1.

Proof of Theorem 1. Assume that $\preceq$ satisfies the axioms listed in the Theorem. Recursively in k, we shall construct an affine function $U_k : X_k \to \mathbb{R}^{k+1}$ by

$$\text{(3)} \quad \forall p \in [0,1] \quad \forall y \in X_{k-1} \qquad U_k\left(px_k + (1-p)y\right) = \langle p, (1-p)U_{k-1}(y)\rangle.$$

U_0 is chosen as Aumann's utility representation [**1**, Theorem A] of the restriction of $\preceq$ to X_0 (here we use the reflexivity and continuity of $\preceq$ on X_0).

Through a simultaneous induction in k, we shall prove that for all $k \in \{1, \ldots, n\}$,

$$\text{(4)} \qquad U_k \text{ is well-defined}$$

and

$$\text{(5)} \quad \forall p \in [0,1) \quad \forall y, z \in X_{k-1} \quad \left(\begin{array}{c} px_k + (1-p)y \prec px_k + (1-p)z \\ \Rightarrow U_{k-1}(y) < U_{k-1}(z) \end{array}\right),$$

wherein $<$ denotes the lexicographical ordering on $\mathbb{R}^k$.

For the rest of the proof, it will be helpful to recall that

$$\text{(6)} \quad \forall p < q \in [0,1] \quad \forall y, z \in X_{k-1} \qquad px_k + (1-p)y \prec qx_k + (1-q)z,$$

which is just a consequence of Lemma 4 (since $\preceq$ satisfies the axiom of incommensurability of infinite values). Furthermore, because of recursion formula (3) (combined with the fact that $<$ is the lexicographical ordering on $\mathbb{R}^k$), one can summarize formulae (6) and (5) as follows:

$$\forall p,q \in [0,1] \quad \forall y,z \in X_{k-1} \left(\begin{array}{c} px_k + (1-p)y \prec qx_k + (1-q)z \\ \Rightarrow U_k\left(px_k + (1-p)y\right) < U_k\left(qx_k + (1-q)z\right) \end{array} \right),$$

in other words,

$$\forall x,y \in X_k \quad \left(x \prec y \Rightarrow U_k(x) < U_k(y)\right). \tag{7}$$

Let us now present the details of the inductive proof for assertions (5) and (4).

Base step for assertion (5). Suppose that $px_1 + (1-p)\,y \prec px_1 + (1-p)\,z$. The independence of $\preceq$ means that

$$px_1{+}(1-p)\,y \preceq px_1{+}(1-p)\,z \Leftrightarrow y \preceq z, \quad px_1{+}(1-p)\,z \npreceq px_1{+}(1-p)\,y \Leftrightarrow z \npreceq y,$$

whence

$$px_1 + (1-p)\,y \prec px_1 + (1-p)\,z \Leftrightarrow y \prec z.$$

Thus, $y \prec z$. Recalling that U_0 was chosen as Aumann's utility representation of $\preceq$ on X_0, it follows that $U_0(y) < U_0(z)$.

Base step for assertion (4). Consider two elements $\xi = px_1 + (1-p)y$ and $\xi' = qx_1 + (1-q)z$ of X_1, wherein $y,z \in X_0$, and suppose that $\xi = \xi'$. Then $p = q$, because otherwise formula (6) would yield that either $\xi \prec \xi'$ (in case $p < q$) or $\xi' \prec \xi$ (in case $q < p$), both of which contradicts $\xi = \xi'$ since $\preceq$ is reflexive. However, from $p = q$ and $px_1 + (1-p)y = \xi = \xi' = qx_1 + (1-q)z$, we may deduce that either $p = q = 1$ or $y = z$ (or both). In either case, both $(1-p)U_0(y) = (1-q)U_0(z)$ and $p = q$, and therefore finally $U_1(\xi) = U_1(\xi')$ by the recursive definition of U_1 in Equation (3).

Simultaneous induction step for assertions (5) *and* (4). Suppose that assertions (5) and (4) hold for $k-1$ instead of k.

- *Proof of assertion* (5) *from the induction hypothesis.* As in the base step for assertion (5), the independence of $\preceq$ yields that

  $$px_k + (1-p)\,y \prec px_k + (1-p)\,z \Leftrightarrow y \prec z.$$

  On the other hand, the induction hypothesis implies assertion (5) for $k-1$ instead of k and therefore also formula (7) for $k-1$ instead of k. Combining this, we get

  $$px_k + (1-p)\,y \prec px_k + (1-p)\,z \Rightarrow U_{k-1}(y) < U_{k-1}(z).$$

  Note that $p < 1$, because otherwise the antecedens in the previous implication means $x_k \prec x_k$, contradicting the reflexivity of $\preceq$. Therefore, we finally obtain

$$px_k{+}(1-p)\,y \prec px_k{+}(1-p)\,z \Rightarrow \underbrace{\langle p,(1-p)U_{k-1}(y)\rangle}_{=U_k(px_k+(1-p)y)} < \underbrace{\langle p,(1-p)U_{k-1}(z)\rangle}_{=U_k(px_k+(1-p)z)}.$$

- *Proof of assertion* (4) *from the induction hypothesis.* The reasoning is analogous to the base step for assertion (4): Consider two elements $\xi = px_k + (1-p)y$ and $\xi' = qx_k + (1-q)z$ of X_k, wherein $y,z \in X_{k-1}$, and suppose that $\xi = \xi'$. Then $p = q$, because otherwise formula (6) would yield that either $\xi \prec \xi'$ (in case $p < q$) or $\xi' \prec \xi$ (in case $q < p$), both of which contradicts $\xi = \xi'$ since $\preceq$ is reflexive. However, from $p = q$ and $px_k + (1-p)y = \xi = \xi' = qx_k + (1-q)z$, we may deduce that either $p = q = 1$ or $y = z$ (or both). In either case, both $(1-p)U_{k-1}(y) = (1-q)U_{k-1}(z)$ and $p = q$, and therefore finally $U_k(\xi) = U_k(\xi')$ by the recursive definition of U_k in Equation (3).

Hence, we have completed the inductive proof of assertions (5) and (4) for all $k \in \{1, \ldots, n\}$ and as a consequence also obtain formula (7) for all k.

Finally, we shall prove, again by an inductive argument in k, that

$$\forall x, y \in X_k \quad (x \sim y \Rightarrow U_k(x) = U_k(y)) \tag{8}$$

holds for all $k \in \{1, \ldots, n\}$. Consider two elements $\xi = px_k + (1-p)y$ and $\xi' = qx_k + (1-q)z$ of X_k. Formula (6) implies that

$$px_k + (1-p)y \sim qx_k + (1-q)z \Leftrightarrow p = q,$$

hence by the independence axiom

$$px_k + (1-p)y \sim qx_k + (1-q)z \Leftrightarrow px_k + (1-p)y \sim px_k + (1-p)z \Leftrightarrow y \sim z.$$

However, $y \sim z$ implies $U_{k-1}(y) = U_{k-1}(z)$ (by induction hypothesis in case $k > 1$, and by the choice of U_0 as Aumann's utility representation function in case $k = 1$), hence

$$px_k + (1-p)y \sim qx_k + (1-q)z \Rightarrow (U_{k-1}(y) = U_{k-1}(z), \quad p = q)\,.$$

Using the recursive definition of U_k in Equation (3), this can be expressed as

$$px_k + (1-p)y \sim qx_k + (1-q)z \Rightarrow \underbrace{\langle p, (1-p)U_{k-1}(y)\rangle}_{=U_k(px_k+(1-p)y)} = \underbrace{\langle q, (1-q)U_{k-1}(z)\rangle}_{=U_k(qx_k+(1-q)z)}\,.$$

Therefore,

$$\xi \sim \xi' \Rightarrow U_k(\xi) = U_k(\xi')$$

for all $\xi, \xi' \in U_k$. This completes the proof of assertion (8).

Combinining assertions (8) and (7) for $k = n$, we arrive at

$$\forall x, y \in X_n \quad (x \prec y \Rightarrow U_n(x) < U_n(y), \quad x \sim y \Rightarrow U_n(x) = U_n(y))\,. \tag{9}$$

Moreover, $B_n = B$ and hence $X_n = X$. Also, a simple induction shows that $U_n : X \to V$ is affine. Hence U_n is a utility representation function for $\preceq$. □

Proof of Theorem 2. Clearly, if there exists such a function as in Equation (1), then $\preceq$ must have the properties listed in the Theorem.

The proof of the converse implication is identical to the proof of Theorem 1, except that the classical von Neumann-Morgenstern theorem is used instead of Aumann's result [**1**]. □

Proof of Corollary 3. Again it is clear that any relation $\preceq$ with a utility representation as in formula (2) will have the properties listed in the Corollary.

The utility representation formula (2), however, follows from Theorem 2: For, there exists a canonical linear order-preserving embedding $\iota : V \to {}^*\mathbb{R}$ of V into the hyperreals, defined through $\iota\,(e_k) := I^k$ for all $k \leq n$. Thus, if we define $u := \iota \circ U$, then not only $u(x_k) = \iota(U(x_k)) = \iota(e_k) = I^k$ for all $k \leq n$, but also

$$\forall x, y \in X \qquad x \preceq y \Leftrightarrow U(x) \leq U(y) \Leftrightarrow \iota(U(x)) \leq \iota(U(y)) \Leftrightarrow u(x) \leq u(y).$$

□

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