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A note on diameter-Ramsey sets*

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Abstract

A finite set $A \subset \mathbb{R}^d$ is called *diameter-Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ and a finite set $B \subset \mathbb{R}^n$ with $\operatorname{diam}(A) = \operatorname{diam}(B)$ such that whenever B is coloured with r colours, there is a monochromatic set $A' \subset B$ which is congruent to A. We prove that sets of diameter 1 with circumradius larger than $1/\sqrt{2}$ are not diameter-Ramsey. In particular, we obtain that triangles with an angle larger than $135°$ are not diameter-Ramsey, improving a result of Frankl, Pach, Reiher and Rödl. Furthermore, we deduce that there are simplices which are almost regular but not diameter-Ramsey.

1 Introduction

In this note, we discuss questions related to *Euclidean Ramsey theory*, a field introduced in [1] by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus. A finite set $A \subset \mathbb{R}^d$ is called *Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that in every colouring of $\mathbb{R}^n$ with r colours, there is a monochromatic set $A' \subset \mathbb{R}^n$ which is congruent to A. The problem of classifying which sets are Ramsey has been widely studied and is still open (see [3] for more details).

The *diameter* of a set $P \subset \mathbb{R}^d$ is defined by $\operatorname{diam}(P) := \sup\{\|x - y\| : x, y \in P\}$, where $\|\cdot\|$ denotes the Euclidean norm. Recently, Frankl, Pach, Reiher and Rödl [2] introduced the following stronger property.

Definition 1.1. A finite set $A \subset \mathbb{R}^d$ is called *diameter-Ramsey* if for every $r \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ and a finite set $B \subset \mathbb{R}^n$ with $\operatorname{diam}(A) = \operatorname{diam}(B)$ such that whenever B is coloured with r colours, there is a monochromatic set $A' \subset B$ which is congruent to A.

It follows from the definition that every diameter-Ramsey set is Ramsey. A set $A \subset \mathbb{R}^d$ is called spherical, if it lies on some d-dimensional sphere and the *circumradius* of A,

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denoted by $\mathrm{cr}(A)$, is the radius of the smallest sphere containing A. (Note that if A is spherical and is not contained in a proper subspace of $\mathbb{R}^d$, then there is a unique sphere that contains it.) In [1] it was proved that every Ramsey set must be spherical. Our main result states that every diameter-Ramsey set must also have a small circumradius.

Theorem 1.2. *If $A \subset \mathbb{R}^d$ is a finite, spherical set with circumradius strictly larger than $\mathrm{diam}(A)/\sqrt{2}$, then A is not diameter-Ramsey.*

Frankl, Pach, Reiher and Rödl [2, Theorems 3 and 4] proved that acute and right-angled triangles are diameter-Ramsey, while triangles having an angle larger than 150° are not. Theorem 1.2 implies the following improvement.

Corollary 1.3. *Triangles with an angle larger than 135° are not diameter-Ramsey.*

Let us call a d-simplex $A = \{p_1, \dots, p_{d+1}\}$ *ε-almost regular* if

$$\frac{1}{\binom{d+1}{2}} \sum_{1 \le i < j \le d+1} \mathrm{diam}(A)^2 - \|p_i - p_j\|^2 \le \varepsilon \cdot \mathrm{diam}(A)^2.$$

In [2, Theorem 6, Lemma 4.9] it was further proved that ε-almost regular simplices are diameter-Ramsey for every $\varepsilon \le 1/\binom{d+1}{2}$. This is a rather small class of simplices since $1/\binom{d+1}{2}$ tends to zero, but another corollary of Theorem 1.2 shows that one cannot hope for much more.

Corollary 1.4. *For every $d \in \mathbb{N}$ and every $\varepsilon > \sqrt{d}/\binom{d+1}{2}$, there is an ε-almost regular d-simplex which is not diameter-Ramsey.*

For $d \in \mathbb{N}$ and $r \ge 0$, we denote the closed d-dimensional ball of radius r centred at the origin by $B_d(r)$. We will deduce Theorem 1.2 from the following result.

Theorem 1.5. *For every finite, spherical set $A \subset \mathbb{R}^d$ and every positive number $r < \mathrm{cr}(A)$, there is some $k = k(A, r) \in \mathbb{N}$ such that the following holds. For every $D \in \mathbb{N}$, there is a colouring of $B_D(r)$ with k colours and with no monochromatic, congruent copy of A.*

A result of Matoušek and Rödl [5] shows that the conclusion of Theorem 1.5 does not hold whenever $r > \mathrm{cr}(A)$. We do not know what happens when $r = \mathrm{cr}(A)$.

Remark 1.6. After completing this work, we have learnt that Theorem 1.2 has independently been proved by Frankl, Pach, Reiher and Rödl, with a similar proof (János Pach, private communication).

2 Proofs

2.1 Proof of Theorem 1.5

Fix some finite, spherical $A \subset \mathbb{R}^d$ and some positive number $r < \mathrm{cr}(A)$. The following claim is the key step of the proof.

Claim 2.1. *There exists a constant $c = c(A, r) > 0$ such that for every $D \in \mathbb{N}$ and for every congruent copy A' of A in $B_D(r)$ we have $\max_{x,y \in A'} (\|x\| - \|y\|) \geq c$.*

Proof. First observe that it is sufficient to prove the claim for $D = d+1$. For $D < d+1$, this follows immediately from $B_D(r) \subset B_{d+1}(r)$, and for $D > d+1$ we can consider the at most $(d+1)$-dimensional subspace spanned by the vertices of A' and the origin.

Let $E = \{e : A \to B_D(r)\} \subset B_D(r)^{|A|}$ be the set of all embeddings of A to B_D. It is easy to see that, if $e_1, e_2, \ldots \in E$ and the pointwise limit $e := \lim_n e_n$ exists, then $e \in E$. Therefore, E is a closed subset of a compact metric space and hence E is compact as well. Define $f : E \to \mathbb{R}$ by

$$f(e) := \max_{x,y \in e(A)} (\|x\| - \|y\|) .$$

Clearly, $f(e) \geq 0$ for every $e \in E$, and $f(e) = 0$, if and only if $e(A)$ lies on a sphere around the origin. But since $\mathrm{cr}(e(A)) > r$ for every embedding $e \in E$, this is not the case, and hence $f(e) > 0$ for all $e \in E$. Finally, since f is continuous, there is a constant $c > 0$ such that $f(e) \geq c$ for all $e \in E$. $\square$

Let $k = \lfloor r/c \rfloor + 1$ now, and fix some $D \in \mathbb{N}$. We will colour points in $B_D(r)$ by their distance to the origin: Define $\chi : B_D(r) \to \{0, \ldots, k-1\}$ by $\chi(x) = \lfloor 1/c \cdot \|x\| \rfloor$, and let $A' \subset B_D(r)$ be a congruent copy of A. It follows immediately from Claim 2.1 that there are $x, y \in A'$ with $\|x\| - \|y\| \geq c$, and hence $\chi(x) \neq \chi(y)$. This finishes the proof of Theorem 1.5.

2.2 Implications of Theorem 1.5

In this section we will deduce Theorem 1.2 and then Corollaries 1.3 and 1.4. In order to do so we will use the following classical result.

Theorem 2.2 (Jung's inequality, [4])**.** *Every bounded set $A \subset \mathbb{R}^d$ can be covered by a closed ball of radius $\sqrt{d/(2d+2)} \cdot \mathrm{diam}(A)$.*

In particular, every finite set $B \subset \mathbb{R}^n$ can be covered by a ball of radius $\mathrm{diam}(B)/\sqrt{2}$, and hence Theorem 1.2 follows immediately from Theorems 1.5 and 2.2.

Furthermore, if T is a triangle with an angle $\alpha > 135°$ and diameter a, it is folklore that the circumradius of T is $a/(2 \sin \alpha) > a/\sqrt{2}$. Thus, we obtain Corollary 1.3 as a corollary of Theorem 1.2.

To prove Corollary 1.4, we show that we can move one vertex of the regular d-simplex by just a little bit to obtain a simplex of circumradius strictly larger than $1/\sqrt{2}$. We will use the elementary geometric fact that the circumradius of a d-dimensional unit simplex is $\sqrt{d/(2d+2)}$.

Proof of Corollary 1.4. Let $\delta > 0$, $r^2 = 1/2 + \delta$, $a^2 = 1/(2d) + \delta$, and define $H_a := \{x \in \mathbb{R}^d : x_d = a\}$. Then $B := B_d(r) \cap H_a$ is a $(d-1)$-dimensional ball of radius $\sqrt{r^2 - a^2} = \sqrt{(d-1)/(2d)}$, and hence there is a $(d-1)$-dimensional unit simplex $A' = \{p_1, \ldots, p_d\}$ contained in the boundary of B. Finally, let $A = A' \cup \{p_{d+1}\}$, where $p_{d+1} = (0, \ldots, 0, r)$.

By construction, we have $\mathrm{cr}(A) > 1/\sqrt{2}$, $\|p_i - p_j\|^2 = 1$ for all $1 \leq i < j \leq d$, and $\|p_i - p_{d+1}\|^2 = 1 - 1/\sqrt{d} + O(\delta)$ for all $1 \leq i \leq d$. Hence the theorem follows from Theorem 1.2 after choosing $\delta > 0$ small enough. □

3 Remarks

In [2] it was asked whether there exists an obtuse triangle which is diameter-Ramsey. Although we could not answer this question, we think the answer is no. More generally, we think the following statement is true.

Conjecture 3.1. *A simplex is diameter-Ramsey if and only if its circumcentre is contained in its convex hull.*

Furthermore, it would be interesting to close the gap between Corollary 1.4 and the related result from [2].

Problem 3.2. *For every $d \in \mathbb{N}$, determine the largest $\varepsilon = \varepsilon(d) > 0$, such that every ε-almost-regular simplex is diameter-Ramsey.*

Note that, provided Conjecture 3.1 is true, a similar construction as in the proof of Corollary 1.4 shows that the result in [2] is best possible, i.e. $\varepsilon(d) = 1/\binom{d}{2}$.

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