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Soluzioni di equazioni di Liouville a profilo non banale

Solutions of Liouville equations with non-trivial profile
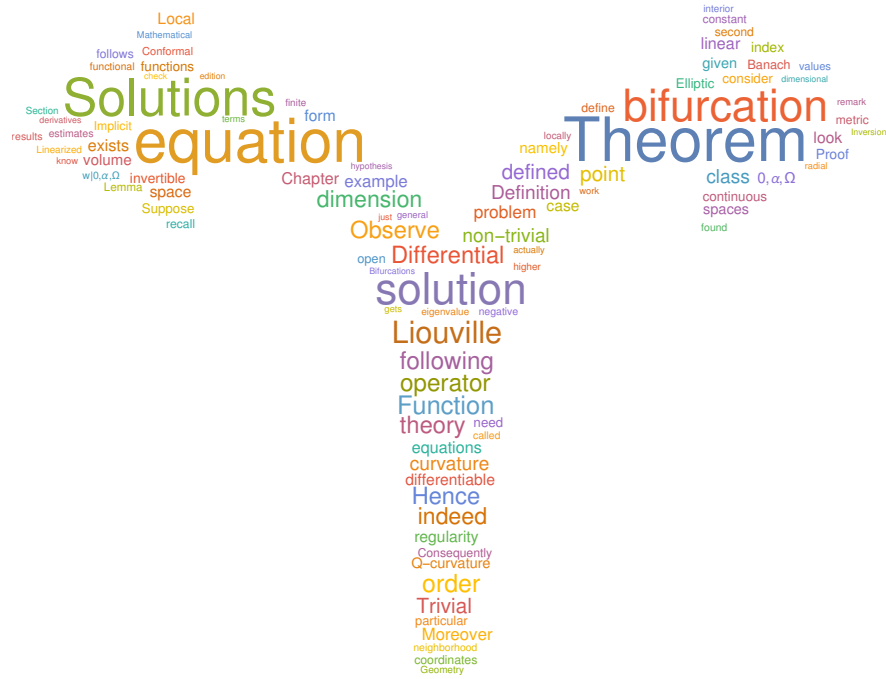
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Solutions of Liouville equations with non-trivial profile

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ABSTRACT. Liouville equations have been widely studied for more than a century. In particular, the interest in this class of PDEs renewed during the last three decades, after the introduction of the so-called Q-curvature and the discovery that they are intimately related to several fundamental concepts both in Analysis and in Geometry. In this work, we will show the existence of a class of non-trivial solutions of the 2D Liouville equation with infinite volume, employing basic tools of bifurcation theory. Using some more advanced techniques of bifurcation theory and Morse theory, we will also lay the groundwork for the study of the same problem in dimension 4.

Contents

CHAPTER 1

Introduction

Liouville equations are a class of elliptic nonlinear partial differential equations of the form

$$(-\Delta)^n \varphi(x) = e^{\varphi(x)}, \quad x \in \mathbb{R}^{2n},$$

for $n \in \mathbb{N}$.[1] This family of equations plays a fundamental role in many problems of Conformal Geometry and Mathematical Physics. As we shall see in the next sections, indeed, Liouville equations govern the transformation laws for some curvatures. For example, the 2-dimensional equation provides the structure of metrics with constant Gaussian curvature which are conformal to the restriction of the Euclidean metric to a 2D surface. In Mathematical Physics, Liouville equations appear for example in the description of mean field vorticity in steady flows ([7], [11]), Chern-Simons vortices in superconductivity or Electroweak theory ([46], [48]). Moreover, they also arise naturally when dealing with functional determinants, which play an essential role in modern Quantum Physics and String theory [38]. The 2-dimensional Liouville equation was also taken as an example by David Hilbert in the formulation of the "nineteenth problem" [26].

The interest in Liouville equations particularly renewed after the introduction of Q-curvature (see Section 2) and many authors studied non-trivial solutions to this class of problems. Classification results for solutions with with finite "volume" $V := \int \exp(u)$ were found in [12] (for the 2D case) and [32] (for the 4D case). Explicitly, solutions with finite volume in $\mathbb{R}^4$ have been constructed in [47] (a generalization of that in which one can fix also the asymptotic behavior of the solution was proved in [33]). The case in which the integral of the solution is not finite, though, is still quite unexplored. In this work we will show the existence of non-trivial solutions with infinite volume for the 2D Liouville equation as perturbations of trivial cylindrical solutions. We will also lay the groundwork for the study of the same problem in $\mathbb{R}^4$.

The intuition behind our quest for this kind of solutions comes as a parallel to what happens with other analogous problems with constant curvature. It is known, indeed, that comparable behaviors appear in the study of solutions to the Yamabe problem, namely: given a conformal class $[g_0]$, finding a representative g such that its scalar curvature R_g is constant (see [42]). Similarly, it is well known that there exist surfaces in $\mathbb{R}^3$ with constant mean curvature that are perturbations of cylinders. These surfaces, which are called *Delaunay unduloids* (after Charles-Eugène Delaunay, who

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[1]We will deal only with spaces of even dimension. For the odd-dimensional case, which is much more difficult as it involves the fractional Laplacian, see [27].
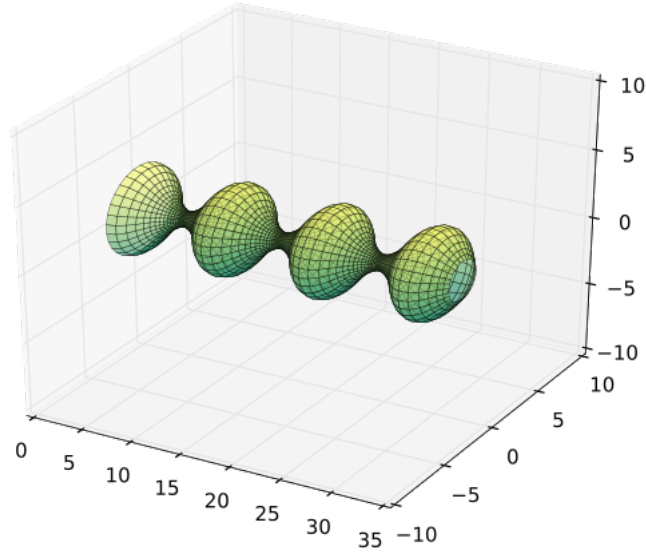
FIGURE 1.1. Delaunay unduloid.

studied them for the first time in 1841 [**15**], see Figure 1.1), are in some sense the analogue of pertubations of cylindrical solutions in our problem. An interesting aspect of these surfaces is that the can be "glued" into "composite surfaces" that still have constant mean curvature (see for example [**34**], [**35**] and Figures 1.2 and 1.3). This phenomenon might happen also with the Liouville equation, but its study is likely to be quite complex and surely goes well beyond the scope of this work.

The outline is then the following. In the rest of this introductory chapter we will explain the origin of the Liouville equation from the point of view of Conformal Geometry. The first section, in particular, will be devoted to the well known 2D case of conformal transformations of the Gaussian curvature, while in the second section we will introduce the notion of Q-curvature and we will see, in dimension 2 and 4, how Liouville equations come out of conformal transformations of this new concept. Specifically, we will also see that the Q-curvature actually encompasses also the notion of Gaussian curvature. Appendix A provides a quick recap of the different notions of curvature in Differential Geometry.

The main tool we will use to look for non-trivial solutions is bifurcation theory: after quickly recalling the fundamentals of infinite-dimensional differential calculus, Chapter 2 will introduce the basic concepts of this theory. Chapter 3 will recall some well-known reguarity results for elliptic equations and will provide a weighted generalization of them. This will be necessary in the following Chapter 4, which will deal with non-trivial solutions of the 2-dimensional Liouville equation: The idea will be to find a solution with finite volume in lower dimension (i.e., invariant in the second coordinate)
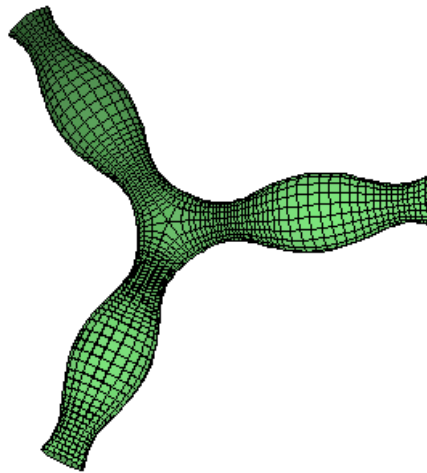
FIGURE 1.2. Three "glued" unduloids forming a *symmetric trinoid* (or, more properly, a *triunduloid*).
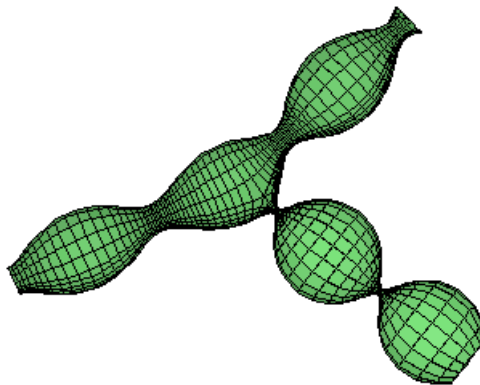
FIGURE 1.3. An *asymmetric triunduloid* with a nodoid end and two slightly unequal unduloid ends.

and then use bifurcation theory to find perturbations along the second coordinate, following the ideas of [**14**]. Finally, in Chapter 5, we will see what might be an approach to the same problem in dimension 4 and we will briefly talk about other future research perspectives. A more in-depth explanation of the state of our work in dimension 4 can be found in Appendix B.

1. Liouville equation in dimension 2

Let us recall first some well-known notions (see for example [**1**] and [**16**]).

DEFINITION 1.1. A connected subset $S \subset \mathbb{R}^3$ is a (*regular* or *embedded*) *surface* if for all $p \in S$ there exists a map $\phi : U \to \mathbb{R}^3$ of class C^∞, where $U \subseteq \mathbb{R}^2$ is an open subset, such that

    (i) $\phi(U) \subseteq S$ is an open neighborhood of p in S,
    (ii) ϕ is an homeomorphism with its image,
    (iii) the differential $\mathrm{d}\phi_x : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $x \in U$.

Such a ϕ, if exists, is called *local parametrization* in p. The inverse map $\phi^{-1} : \phi(U) \to U$ is called *local chart* in p and the coordinates $(u(p), v(p)) = \phi^{-1}(p)$ are called *local coordinates* of p. The curve $t \mapsto \phi(x_0 + te_j)$ is the *j-th coordinate curve* through $\phi(x_0)$.

Given a point $p \in S$, there is an intuitive way of defining a *tangent plane* to S in p: Let u and v be the local coordinates in an open neighborhood $U \subset S$ of p and let ϕ be the local parametrization. A curve $u = u(t)$, $v = v(t)$ in $\phi(U)$ defines a curve $r(t) := \phi(u(t), v(t))$ lying on the surface S. The tangent vector to the curve $\dot{r}(t)$ has the form

$$(1.1) \qquad\qquad\qquad \dot{r}(t) = r_u \dot{u} + r_v \dot{v},$$

with $\phi_u := \frac{\partial \phi}{\partial u}$ and $\phi_v := \frac{\partial \phi}{\partial v}$. By Definition 1.1.(iii), ϕ_u and ϕ_v are linearly independent. Hence, as (1.1) says that every vector tangent to S is a linear combination of ϕ_u and ϕ_v, the totality of vectors tangent to S at a given point p forms a 2-dimensional subspace with basis (ϕ_u, ϕ_v). This subspace is called *tangent plane* to S in p and is written as $T_p S$.

DEFINITION 1.2. The *first fundamental form* is the map associating to each $p \in S$ the restriction of the standard Euclidean product of the ambient space $\mathbb{R}^3$ to $T_p S$, namely

$$g_{11} = x_u x_u + y_u y_u + z_u z_u,$$
$$g_{12} = x_u x_v + y_u y_v + z_u z_v = g_{21},$$
$$g_{22} = x_v x_v + y_v y_v + z_v z_v,$$

where $\phi(u, v) = (x(u,v), y(u,v), z(u,v))$. The Riemannian metric $g = g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j$ obtained in this way is said to be the metric *induced* on the surface S.

A first result is the following [**16**, Theorem 13.1.1].

THEOREM 1.1. *Suppose that g_{11}, g_{12} and g_{22} are real-valued analytic functions. Then there exist new real local coordinates, which we still indicate with u and v, in terms of which the induced metric takes the form*

$$g(u, v) = f(u, v)(\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}v \otimes \mathrm{d}v).$$

Coordinates with this property are called isotermal *or* conformal *coordinates.*

Take now a surface S and a point (x_0, y_0, z_0) on S. Suppose that we can locally write the surface as $z = F(x, y)$, where $z_0 = F(x_0, y_0)$ and $\nabla F(x_0, y_0) = 0$ (thanks to the implicit function theorem we can find coordinates for which this is true). The matrix whose entries are $a_{ij} := \frac{\partial^2 F}{\partial x_i^1 \partial x_j^1}$, where $x_1 = x$ and $x_2 = y$, is known as the *Hessian* of F.

DEFINITION 1.3. Given a surface S which is locally parametrized as $z = F(x, y)$ and a point $(x_0, y_0, z_0) \in S$ at which $\nabla F = 0$, we say that the *principal curvatures* of the surface at that point are the eigenvalues of (a_{ij}) in that point (these eigenvalues are real since (a_{ij}) is symmetric). We call $K := \det(a_{ij})$ the *Gaussian curvature K* and we call $\text{tr}(a_{ij})$ the *mean curvature.*

We can now state a second result [**16**, Theorem 13.1.3].

THEOREM 1.2. *If u and v are conformal coordinates on a surface in an Euclidean 3-dimensional space, in terms of which the induced metric has the form*

$$g(u, v) = f(u, v)(\mathrm{d}u \otimes \mathrm{d}u + \mathrm{d}v \otimes \mathrm{d}v),$$

then the Gaussian curvature of the surface is

$$(1.2) \qquad K(u, v) = -\frac{1}{2f(u, v)} \Delta \log f(u, v),$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplace operator.

Observe that, since $f(u, v) > 0$, we can define a new function $\varphi(u, v)$ such that $f(u, v) = \mathrm{e}^{\varphi(u,v)}$. In terms of φ, then, (1.2) becomes

$$K(u, v) = -\frac{1}{2} \mathrm{e}^{-\varphi(u,v)} \Delta \varphi(u, v).$$

In particular, if K is constant, we finally get the *Liouville equation*

$$(1.3) \qquad \Delta \varphi(u, v) + 2K\mathrm{e}^{\varphi(u,v)} = 0.$$

We remark that in dimension 2 one can find a general solution to the Liouville equation in terms of meromorphic functions. For example, in a simply connected domain Ω, the general solution is given by

$$u(z, \bar{z}) = \log \left(4 \frac{|\partial f(z)/\partial z|}{(1 + K|f(z)|^2)^2} \right),$$

where f is any meromorphic function such that $\frac{\partial f}{\partial z}(z) \neq 0$ for all $z \in \Omega$ and f has at most simple poles in Ω (see [**25**] – see also [**19**] for a classification of solutions with finite volume in the upper half-plane). Observe that this fact is characteristic of dimension 2, because it relies on the identification $\mathbb{R}^2 \simeq \mathbb{C}$.

2. Q-curvature and higher dimensional Liouville equation

Up to now we have only discussed about surfaces of codimension 1 in Euclidean spaces of dimension 3 and we obtained a two dimensional Liouville equation. This equation, which after a rescaling can be written as

$$\Delta u(x, y) + \mathrm{e}^{u(x,y)} = 0, \quad (x, y) \in \mathbb{R}^2,$$

will be the main object of study of this work. Nonetheless, it would be interesting to study also the higher dimensional versions of the Liouville equation, that can be written after a rescaling as

$$(-\Delta)^n u(x) = \mathrm{e}^{u(x)}, \quad x \in \mathbb{R}^{2n}.$$

In this section we will briefly explain how this class of PDEs arises. In 1985 Thomas P. Branson introduced the concept of *Q-curvature* [**3**], a quantity that turned out to be very important in many contexts and that can be regarded as a generalization of the Gaussian curvature. For example, *Q*-curvature appears naturally while studying the *functional determinant* of conformally covariant operators[2], which plays an essential role both in Functional Analysis and in Theoretical Physics. Indeed, for example on a four-manifold, given a conformally covariant operator A_g (like the conformal Laplacian or the Paneitz operator [**39**]) and a conformal factor w, one has

$$\log \frac{\det A_{\hat{g}}}{\det A_g} = \gamma_1(A)F_1[w] + \gamma_2(A)F_2[w] + \gamma_3(A)F_3[w],$$

where $\gamma_1(A)$, $\gamma_2(A)$ and $\gamma_3(A)$ are real numbers (see [**5**]). In particular, $\hat{g} = e^{2w}g$ is a critical point of F_2 if and only if the *Q*-curvature corresponding to $\hat{g}$ is constant (see [**23**] and the references therein).

Q-curvature appears also as the 0-th order term of the GJMS-operator in the ambient metric construction [**17**] and can be related to the Poincaré metric in one higher dimension via an "holographic formula" [**21**]. GJMS-operators, in turn, play an important role in Physics, as their definition extends to Lorentzian manifolds: they are generalizations of the Yamabe operator and the conformally covariant powers of the wave operator on Minkowski space [**28**]. Moreover, the integral of the *Q*-curvature satisfies the so-called Chern-Gauss-Bonnet formula [**28**], which links the integral of some function of the *Q*-curvature to the Euler characteristic of the manifold (as the Gauss-Bonnet formula did with the Gaussian curvature). In $\mathbb{R}^4$, that equation can tell us whether a metric is normal and, in that case, is strictly related to the behavior of the isoperimetric ratios [**10**].

In what follows, we will present only the 2 and 4-dimensional cases. A generic definition of *Q*-curvature can be found in [**4**] and explicit formulas in [**28**]. In dimension 2 the *Q*-curvature is essentially the usual Gaussian curvature (see [**9**] for a more complete introduction in both 2, 4 and higher dimensions – for a quick recap of the basic notions of curvature in Differential Geometry see Appendix A). We just want to point out that in this case, if we conformally rescale the metric, $\hat{g}_{ij} = e^{2\varphi}g_{ij}$ for some smooth function φ on M, then

$$R_{\hat{g}} = e^{-2\varphi}(R_g - 2\Delta f),$$

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[2]Given an operator A with spectrum $\{\lambda_j\}_j$, one can formally define its *determinant* as $\prod_j \lambda_j$. This is divergent, in general, so one should perform some sort of "regularization" of the definition. Define then the *Zeta function* as

$$\zeta(s) := \sum_j \lambda_j^{-s} = \sum_j e^{-s \log \lambda_j}.$$

One can show by means of Weyl's asymptotic law (see for example [**44**, Chapter 11]) that this defines an analytic function for $\Re(s) > n/2$ if A is the Laplace-Beltrami opeator. Moreover, one can meromorphically extend ζ so that it becomes regular at $s = 0$ (see [**41**]). Taking the derivative, one has $\zeta'(0) := -\sum_j \log \lambda_j = -\log \det A$, so that $\det A := \exp(-\zeta'(0))$. For more details we refer for example to [**38**], [**9**], [**23**] and the references therein.

where $R_{\hat{g}}$ and R_g denotes, respectively, the scalar curvatures of $\hat{g}$ and g. Specifically, if g is an Euclidean metric, then we recover Theorem 1.2 and the 2D Liouville equation (1.3).

In dimension 4 things start to become more interesting.

DEFINITION 1.4. Let (M, g) be a 4-dimensional Riemannian manifold. Let Ric_g be its Ricci curvature, R_g its scalar curvature and Δ_g its Laplace-Beltrami operator. The Q-*curvature* of M is defined as

$$Q_g := -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right).$$

Conformally rescaling the metric, $\hat{g}_{ij} = \mathrm{e}^{2\varphi}g_{ij}$ for some smooth function φ on M, then the Q-curvature transforms as follows

(1.4) $$P_g\varphi + 2Q_g = 2Q_{\hat{g}}\mathrm{e}^{4\varphi}$$

(see for example [**8**, Chapter 4]), where P_g is the *Paneitz operator*

$$P_g\varphi := \Delta_g^2\varphi + \mathrm{div}_g\left(\frac{2}{3}R_g g - 2\mathrm{Ric}_g\right)\mathrm{d}\varphi$$

introduced in 1983 by Stephen M. Paneitz [**39**].

Observe that, if we take $M = \mathbb{R}^4$ and g equal to the standard Euclidean metric and consider $\hat{g}$ conformal to g and such that $Q_{\hat{g}} \equiv \bar{Q} \in \mathbb{R}$, then equation (1.4) becomes

$$\Delta_g^2\varphi = 2\bar{Q}\mathrm{e}^{4\varphi}.$$

Setting $u := 4\varphi$ and $\bar{Q} = 2$ and taking into account that the Laplace-Beltrami operator in $\mathbb{R}^4$ endowed with the Euclidean metric is the standard Laplacian, we finally end up with the 4-dimensional Liouville equation

$$\Delta^2 u(x) = \mathrm{e}^{u(x)}, \quad x \in \mathbb{R}^4.$$

A quick look at bifurcation theory

The goal of this chapter is to present all the necessary notions and results of nonlinear functional analysis and bifurcation theory that will be needed to address the problem of finding non-trivial solutions of the planar Liouville equation. The main source for this chapter is [**2**]. Another good reference is [**30**].

1. Differential calculus in Banach spaces

We start our survey of bifurcation theory with the basics of differential calculus in Banach spaces. As we shall see, indeed, all the bifurcation results we will use in the following chapters are essentially applications of the implicit function theorem.

1.1. Fréchet and Gâteau derivatives.

DEFINITION 2.1. Let X and Y be Banach spaces and let $U \subset X$ open. Consider a map $F : U \to Y$ and let $u \in U$. We say that F is *(Fréchet-) differentiable* at u if there exists $A \in L(X, Y)$ such that, if we set

$$R(h) := F(u + h) - F(u) - A(h)$$

it results that

$$R(h) = o(\|h\|),$$

namely

$$\frac{\|R(h)\|}{\|h\|} \to 0 \text{ as } \|h\| \to 0.$$

Such an A is uniquely determined and therefore will be called the *(Fréchet) differential* of F at u and will be denoted as $\mathrm{d}F(u)$. If F is differentiable for all $u \in U$ we will say that F is differentiable in U. When there is no possibility of misunderstanding we will refer to *Fréchet differentiability* simply as *differentiability*.

Observe that, if F is differentiable in U, we have a map

$$\begin{aligned} \mathrm{d}F : U &\longrightarrow L(X, Y) \\ u &\longmapsto \mathrm{d}F(u) \end{aligned} \quad .$$

If the map dF is continuous from U to $L(X, Y)$ we say that $F \in C^1(U, Y)$.

REMARK. If $X = \mathbb{R}$, we can canonically identify $\mathrm{d}F(u)$ with an element of Y and $\mathrm{d}F$ with a map from U to Y simply applying the linear operator to 1.

Verifying that A is unique is straightforward. Suppose indeed that there exists another $B \in L(X, Y)$ satisfying Definition 2.1. Then

$$\frac{\|Ah - Bh\|}{\|h\|} \to 0 \text{ as } \|h\| \to 0.$$

If $A \neq B$ there exists $h^* \in X$ such that $a := \|Ah^* - Bh^*\| \neq 0$. Taking $h = th^*, t \in \mathbb{R} \setminus 0$ one gets

$$\frac{\|A(th^*) - B(th^*)\|}{\|th^*\|} = \frac{\|Ah^* - Bh^*\|}{h^*} = \frac{a}{\|h^*\|}$$

a constant and a contradiction.

As one might expect, the Fréchet differential satisfies differentiation rules similar to those that we have in $\mathbb{R}^n$.

PROPOSITION 2.1. *The following holds.*

(i) *Let $F, G : U \to Y$ be differentiable at $u \in U$, then $aF + bG$ is differentiable at u for any $a, b \in \mathbb{R}$ and*

$$\mathrm{d}(aF + bG)(u)h = a\,\mathrm{d}F(u)h + b\,\mathrm{d}G(u)h.$$

(ii) *Consider $F : U \to Y$ and $G : V \to Z$ with $F(U) \subset V$, U and V open subsets of X and Y, respectively. Consider moreover their composite map $G \circ F : U \to Z$. If F is differentiable at $u \in U$ and G is differentiable at $v := F(u) \in V$, then $G \circ F$ is differentiable at u and*

$$\mathrm{d}(G \circ F)(u)h = \mathrm{d}G(v)[\mathrm{d}F(u)h] = (\mathrm{d}G(v) \circ \mathrm{d}F(u))h.$$

As happens in finite dimension, we have another weaker notion of differentiability.

DEFINITION 2.2. Consider $F : U \to Y$ and let $u \in U$. We say that F is *Gâteaux-differentiable* (or *G-differentiable*) at u if there exists an $A \in L(X, Y)$ such that for all $h \in X$ it results that

$$\frac{F(u + \varepsilon h) - F(u)}{\varepsilon} \to Ah \text{ as } \varepsilon \to 0.$$

Again, the map A is uniquely determined, is called the *Gâteaux differential* of F at u and is denoted by $\mathrm{d}_{\mathrm{G}} F(u)$.

One immediately sees that Fréchet differentiability implies Gâteaux differentiability. Conversely, Gâteaux differentiability does not even imply continuity (see [**2**, p. 13] for a counterexample).

What follows is the generalization of the Mean-Vaulue Theorem. Given $u, v \in U$ denote with $[u, v]$ the segment $\{tu + (1-t)v \mid t \in [0, 1]\}$.

THEOREM 2.2. *Let $F : U \to Y$ be G-differentiable at any point of U. Given $u, v \in U$ such that $[u, v] \subset U$, it follows that*

$$\|F(u) - F(v)\| \leq \sup \left\{ \|\mathrm{d}_{\mathrm{G}} F(w)\| \,\big|\, w \in [u, v] \right\} \|u - v\|.$$

PROOF. The idea of the proof is basically to reduce the problem to a one dimensional one and then apply the standard Mean-Value Theorem.

Of course, if $F(u) = F(v)$ there is nothing to prove, so assume directly that $F(u) \neq F(v)$. By a corollary of the analytic Hahn-Banach Theorem (see

for example Corollary 4 of [**6**, p. 4]), there exists a $\psi \in Y^*$ with $\|\psi\|_{Y^*} = 1$ such that

$$\langle \psi, F(u) - F(v) \rangle = \|F(u) - F(v)\|.$$

Define $\gamma(t) := tu + (1 - t)v$ and consider

$$h : [0, 1] \longrightarrow \mathbb{R}$$
$$t \longmapsto \langle \psi, F(\gamma(t)) \rangle = \langle \psi, F(tu + (1 - t)v) \rangle.$$

Observe that $\gamma(t + \tau) = \gamma(t) + \tau(u - v)$. Thus

$$\frac{h(t + \tau) - h(t)}{\tau} = \left\langle \psi, \frac{F(\gamma(t + \tau)) - F(\gamma(t))}{\tau} \right\rangle$$
$$= \left\langle \psi, \frac{F(\gamma(t) + \tau(u - v)) - F(\gamma(t))}{\tau} \right\rangle.$$

As F is G-differentiable in U, if we let $\tau \to 0$ in this last expression, we get

$$h'(t) = \langle \psi, \mathrm{d_G}\, F(tu + (1 - t)v)(u - v) \rangle.$$

Now simply apply the standard Mean-Value Theorem to h:

$$h(1) - h(0) = h'(\theta) \text{ for some } \theta \in (0, 1).$$

Consequently

$$\|F(u) - F(v)\| = h(1) - h(0) = h'(\theta)$$
$$= \langle \psi, \mathrm{d_G}\, F(\theta u + (1 - \theta)v)(u - v) \rangle$$
$$\leq \|\psi\| \|\mathrm{d_G}\, F(\theta u + (1 - \theta)v)\| \|u - v\|$$

and, as $\|\psi\| = 1$ and $\theta u + (1 - \theta)v \in [u, v]$, the theorem follows. □

An important consequence of Theorem 2.2 is the following result about Fréchet and Gâteaux differentiability.

COROLLARY 2.2.1. *Let $F : U \to Y$ be G-differentiable in U and suppose that the map*

$$F'_{\mathrm{G}} : U \longrightarrow L(X, Y)$$
$$u \longmapsto F'_{\mathrm{G}}(u) = \mathrm{d_G}\, F(u)$$

is continuous at some $u^* \in U$. Then F is Fréchet-differentiable at u^* and $\mathrm{d}F(u^*) = \mathrm{d_G}\, F(u^*)$.

PROOF. Consider

$$R(h) := F(u^* + h) - F(u^*) - \mathrm{d_G}\, F(u^*)h.$$

Our goal is to show that $R(h) = o(\|h\|)$. It is clear that R is G-differentiable in a ball $B_\varepsilon(0)$ with radius $\varepsilon > 0$ sufficiently small and that

$$\mathrm{d_G}\, R(h)[k] = \mathrm{d_G}\, F(u^* + h)[k] - \mathrm{d_G}\, F(u^*)[k].$$

Apply then Theorem 2.2 with $[u, v] = [0, h]$:

$$\|R(h)\| = \|R(h) - R(0)\| \leq \sup_{0 \leq t \leq 1} \|\mathrm{d_G}\, R(th)\| \|h\|$$
$$= \sup_{0 \leq t \leq 1} \|\mathrm{d_G}\, F(u^* + th) - \mathrm{d_G}\, F(u^*)\| \|h\|.$$

Since F'_{G} is continuous

$$\sup_{0 \le t \le 1} \|\mathrm{d}_{\mathrm{G}}\, F(u^* + th) - \mathrm{d}_{\mathrm{G}}\, F(u^*)\| \to\ \text{as}\ \|h\| \to 0$$

and consequently $R(h) = o(\|h\|)$, as wanted. $\qquad\square$

1.2. Higher derivatives.

DEFINITION 2.3. Let X and Y be Banach spaces and $U \subset X$ be open. Take $F \in C(U,Y)$ and consider $\mathrm{d}F : U \to L(X,Y)$. Fix $u^* \in U$. We say that F is *twice (Fréchet-) differentiable* at u^* if $\mathrm{d}F$ is differentiable at u^*. The *second (Fréchet-) differential* of F at u^* is the map

$$\mathrm{d}^2\, F(u^*) \in L(X, L(X,Y))$$

defined as

$$\mathrm{d}^2\, F(u^*) = \mathrm{d}(\mathrm{d}F)(u^*).$$

If F is twice differentiable at all points of U we say that F is *twice (Fréchet-) differentiable* in U.

A good way to see $\mathrm{d}^2\, F(u^*)$ is as a bilinear map on X. This is done in the following canonical way. Let $L_2(X,Y)$ the space of bilinear functions from $X \times X$ to Y. To any $A \in L(X, L(X,Y))$ associate $\Phi_A \in L_2(X,Y)$ defined as $\Phi_A(u,v) := [A(u)](v)$. Conversely, if $\Phi \in L_2(X,Y)$ and $h \in X$, we have the linear map from X to Y defined as $\Phi(h, \cdot) : k \mapsto \Phi(h, k)$. Consequently we can further define the linear and continuous map $\tilde{\Phi} : h \mapsto \Phi(h, \cdot) \in L(X,Y)$. It is easy to see this identification is an isometric isomorphism between $L(X, L(X,Y))$ and $L_2(X,Y)$ (see [**2**, p. 23]). In what follows we will use the same symbol $\mathrm{d}^2\, F(u^*)$ to denote both the element in $L(X, L(X,Y))$ and $L_2(X,Y)$. The value of $\mathrm{d}^2\, F(u^*)$ at the couple $(h,k) \in X \times X$ will be denoted as $\mathrm{d}^2\, F(u^*)[h,k]$.

In a similar fashion to what we did previously, if $\mathrm{d}^2\, F$ is continuous from U to $L_2(X,Y)$ we say that $F \in C^2(X,Y)$.

The following result is useful for explicit computations of the second order differential.

PROPOSITION 2.3. *Let $F : U \to Y$ be twice differentiable at $u^* \in U$. Then for any fixed $h \in X$ the map $F_h : X \to Y$ defined by*

$$F_h(u) := \mathrm{d}F(u)[h]$$

is differentiable at u^* and

$$\mathrm{d}F_h(u^*)k = \mathrm{d}^2\, F(u^*)[h,k].$$

PROOF. We can write F_h as a composition:

$$F_h = \varepsilon_h \circ \mathrm{d}F,$$

where ε_h is the map that associates $A \in L(X,Y)$ to its evaluation $A(h) \in Y$. Since ε_h is linear, the result follows from 2.1.(ii). $\qquad\square$

The map $\mathrm{d}^2\, F(u)$ is actually more than bilinear:

THEOREM 2.4. *If $F : U \to Y$ is twice differentiable at $u \in U$, then $\mathrm{d}^2\, F(u) \in L_2(X,Y)$ is symmetric.*

The proof of this last statement is a bit technical and, therefore, omitted (see [**2**, Theorem 3.4]).

If $n \geq 2$, the $(n+1)-$derivative can be defined by induction. Let indeed $F : U \to Y$ be n times differentiable in U. The n-th differential at $u \in U$ can be identified with a continuous n-linear map from $X \times \cdots \times X$ (n times) to Y with an isometry similar to the one explained before. The $(n+1)$-differential at u^* is defined as the differential of $\mathrm{d}^n F$, namely

$$\mathrm{d}^{n+1} F(u^*) := \mathrm{d}(\mathrm{d}^n F)(u^*) \in L(X, L_n(X, Y)) \simeq L_{n+1}(X, Y).$$

We will say that $F \in C^n(U, Y)$ if F is n times differentiable and the n-th derivative is continuous from U to $L_n(X, Y)$. The value of $\mathrm{d}^n F(u^*)$ at $(h_1, \ldots, h_n)$ will be denoted by $\mathrm{d}^n F(u^*)[h_1, \ldots, h_n]$. If $h_1 = \cdots = h_n = h$ we will write for brevity $\mathrm{d}^n F(u^*)[h]^n$.

THEOREM 2.5. *If $F : U \to Y$ is n times differentiable in U, then the map $(h_1, \ldots, h_n) \mapsto \mathrm{d}^n F(u^*)[h_1, \ldots, h_n]$ is symmetric.*

As before, for the proof we refer to Theorem 3.5 of [**2**].

1.3. Partial derivatives. Let X and Y be Banach spaces and take $(u^*, v^*) \in X \times Y$. Define $\sigma_{v^*} : X \to X \times Y$ and $\tau_{u^*} : Y \to X \times Y$ as

$$\sigma_{v^*}(u) := (u, v^*),$$

$$\tau_{u^*}(v) := (u^*, v).$$

Observe that

$$\sigma := \mathrm{d}\sigma_{v^*} : h \mapsto (h, 0),$$

$$\tau := \mathrm{d}\tau_{u^*} : k \mapsto (0, k).$$

DEFINITION 2.4. Let Z be a Banach space and $Q \subset X \times Y$ open. Take $(u^*, v^*) \in Q$ and $F : Q \to Z$. If the map $F \circ \sigma_{v^*}$ ($F \circ \tau_{u^*}$) is differentiable at u^* (v^*) we say that F is *differentiable with respect to u (v) at (u^*, v^*)*. The linear map $\mathrm{d}[F \circ \sigma_{v^*}](u^*) \in L(X, Z)$ ($\mathrm{d}[F \circ \tau_{u^*}](v^*) \in L(Y, Z)$) is called the *$u$-partial derivative ($v$-partial derivative) of F* at (u^*, v^*) and denoted by $\mathrm{d}_u F(u^*, v^*)$ ($\mathrm{d}_v F(u^*, v^*)$).

Higher order derivatives can be defined as before. By Definition 2.4 and Theorem 2.4 one can see that the Schwartz Theorem holds:

$$\mathrm{d}_{uv} F(u^*, v^*)[h, k] = \mathrm{d}_{vu} F(u^*, v^*)[h, k],$$

namely the order of differentiation does not matter (see [**2**, p. 28]).

2. Local Inversion Theorem and Implicit Function Theorem

As previously said, the machinery at the basis of bifurcation theory is the Implicit Function Theorem. This section will be devoted, therefore, to stating and proving this fundamental result. The first step is to show a version of the Local Inversion Theorem generalized to Banach spaces, then we will be able to move to the Implicit Function Theorem itself.

2.1. Local Inversion Theorem. Let us first fix some notation. In what follows X and Y will always be Banach spaces.

DEFINITION 2.5. Let $A \in L(X, Y)$. We say that A is *invertible* if there exists $B \in L(Y, X)$ such that

$$B \circ A = I_X,$$
$$A \circ B = I_Y.$$

It can be easily seen that B is unique and there will be accordingly denoted as A^{-1}. We also define

$$\text{Inv}(X, Y) := \{A \in L(X, Y) \mid A \text{ is invertible}\}.$$

REMARK. By the Closed Graph Theorem ([**6**, Theorem 2.9]), if $A \in L(X, Y)$ is injective and surjective, then $A \in \text{Inv}(X, Y)$.

LEMMA 2.6. *The following two properties hold.*
  (i) *If $A \in \text{Inv}(X, Y)$ then any $T \in L(X, Y)$ such that*

$$\|T - A\| < \frac{1}{\|A^{-1}\|}$$

   is invertible. Hence, $\text{Inv}(X, Y)$ is an open subset of $L(X, Y)$.
  (ii) *The map $J : \text{Inv}(X, Y) \to L(X, Y)$ defined by $J(A) = A^{-1}$ is of class C^∞.*

This lemma is a well-known result (see for example [**18**, 3.1]).

Take for simplicity of notation $F \in C(X, Y)$ (maps on open subsets of X can be treated analogously).

DEFINITION 2.6. Let U and V be open subsets of X and Y, respectively. We say that $F \in \text{Hom}(U, V)$ if there exists a map $G : V \to U$ such that

$$(2.1) \qquad\qquad\qquad G(F(u)) = u, \quad \forall u \in U,$$

$$(2.2) \qquad\qquad\qquad F(G(v)) = v, \quad \forall v \in V.$$

F is said to be *locally invertible* at $u^* \in X$ if there exist a neighborhood U of u^* and a neighborhood V of $v^* = F(u^*)$ such that $F \in \text{Hom}(U, V)$, namely there exists a map $G : V \to U$ satisfying (2.1) and (2.2). The map G is called *local inverse* and is denoted by F^{-1}.

PROPOSITION 2.7. *Direct consequences of Definition 2.6 are the following two properties*

TRANSITIVITY *If $F \in C(X, Y)$ is locally invertible at $u \in X$ and $G \in C(Y, Z)$ is locally invertible at $v = F(u)$, then $G \circ F$ is locally invertible at u.*
STABILITY *If $F \in C(X, Y)$ is localy invertible at $u \in X$, then there exists a neighborhood of u in which F is locally invertible.*

Moreover, suppose that F is locally invertible at u^* and that F and $G = F^{-1}$ are differentiable, respectively, at u^* and $v^* = F(u^*)$. Differentiating (2.1) and (2.2) at u^* and v^*, respectively, one gets

$$\mathrm{d}G(v^*) \circ \mathrm{d}F(u^*) = I_X,$$
$$\mathrm{d}F(u^*) \circ \mathrm{d}G(v^*) = I_Y,$$

namely: $\mathrm{d}F(u^*) \in \mathrm{Inv}(X,Y)$ with inverse $\mathrm{d}G(v^*) \in \mathrm{Inv}(X,Y)$. The following Local Inversion Theorem gives us condition under which the converse is true as well.

THEOREM 2.8. (LOCAL INVERSION THEOREM)
Consider $F \in C^1(X,Y)$ and suppose that $\mathrm{d}F(u^*) \in \mathrm{Inv}(X,Y)$. Then F is locally invertible at u^* with a C^1 inverse. More precisely, there exist a neighborhood U of u^* and a neighborhood V of $v^* = F(u^*)$ such that

    (i) $F \in \mathrm{Hom}(U,V)$,
    (ii) $F^{-1} \in C^1(V,X)$ and for all $v \in V$ it holds

$$\mathrm{d}F^{-1}(v) = (\mathrm{d}F(u))^{-1}, \; u = F^{-1}(v),$$

    (iii) if $F \in C^k(X,Y)$, $k > 1$, then $F^{-1} \in C^k(V,X)$.

PROOF. (i) Observe first that with a translation we can directly assume $u^* = 0$ and $v^* = F(0) = 0$. Moreover, by transitivity, it is equivalent to show local invertibility of $A \circ F$, with any A linear and invertible. Taking $A = (\mathrm{d}F(0))^{-1}$, we see that it is sufficient to consider the case $F = I_X + \Psi$ with $\Psi \in C^1(X,X)$ and $\mathrm{d}\Psi(0) = 0$. Observe that $\Psi(0) = F(0) - I_X(0) = 0$.

Let $r > 0$ be such that $\|\mathrm{d}\Psi(p)\| < \frac{1}{2}$ for all $\|p\| < r$. By the Mean-Value Theorem 2.2 we have that, for all $p, q \in B(r)$,

$$(2.3) \quad \|\Psi(p) - \Psi(q)\| \leq \sup\left\{\|\mathrm{d}\Psi(w)\| \mid w \in [p,q]\right\} \|p - q\| \leq \frac{1}{2}\|p - q\|.$$

Hence, Ψ is a contraction and $\|\Psi(p)\| \leq \frac{1}{2}\|p\|$ if $\|p\| < r$. Fix $v \in X$ and define

$$\Phi_v(u) := v - \Psi(u).$$

Of course Φ_v is a contraction as well. Moreover

$$\|\Phi_v(u)\| \leq \|v\| + \|\Psi(u)\| \leq r, \quad \forall u \in B(r), \, \forall v \in B(r/2).$$

Thus, if $\|v\| \leq \frac{r}{2}$, Φ_v is a contraction which maps $B(r)$ into itself. Therefore, by the Banach Fixed Point Theorem (see for example Theorem 1.1 in [**22**, p. 10], or Theorem 5.1 in [**20**, p. 74]), Φ_v has a unique fixed point $u \in B(r)$:

$$u = \Phi_v(u) = v - \Psi(u),$$

i.e. $F(u) = v$. That means that we can define the inverse $F^{-1} : B(r/2) \to B(r)$. As we shall immediately see, F^{-1} is Lipschitz with constant 2 and therefore, in particular, it is continuous. Indeed, take $u = F^{-1}(v)$ and $w = F^{-1}(z)$, that is

$$\begin{cases} u + \Psi(u) = v \\ w + \Psi(w) = z \end{cases}.$$

By means of (2.3), we immediately obtain

$$\|u - w\| \leq \|v - z\| + \|\Psi(u) - \Psi(w)\| \leq \|v - z\| + \frac{1}{2}\|u - w\|,$$

which is

$$\left\|F^{-1}(v) - F^{-1}(z)\right\| \leq 2\|v - z\|.$$

Finally, taking $V = B(r/2)$ and $U = B(r) \cap F^{-1}(V)$ we obtain

$$F|_U \in \mathrm{Hom}(U,V).$$

(ii) Taking $u = F^{-1}(v)$ in $u + \Psi(u) = v$ one gets

$$F^{-1}(v) = v - \Psi(F^{-1}(v)).$$

Observe that $\Psi(u) = o(\|u\|)$: as F^{-1} is Lipschitz, it follows that $\Psi(F^{-1}(v)) = o(\|v\|)$. Hence, F^{-1} is differentiable in $v = 0$ with $\mathrm{d}F^{-1}(0) = I_X$. In general, then, if $v \in B(r/2)$ and $u = F^{-1}(v)$, modulo a translation that brings u and v to the origins of X ad Y respectively, one gets that F^{-1} is differentiable at v and that $\mathrm{d}F^{-1}(v) = (\mathrm{d}F(u))^{-1}$.

In order to prove that F^{-1} is of class C^1, just observe that the map $\mathrm{d}F^{-1}$ is the following composition of functions:

$$v \xmapsto{F^{-1}} F^{-1}(v) = u \xmapsto{\mathrm{d}F} \mathrm{d}F(u) \xmapsto{J} J(\mathrm{d}F(u)) = (\mathrm{d}F(u))^{-1}.$$

As F^{-1}, $\mathrm{d}F$ and J are all at least continuous (Lemma 2.6), $F^{-1} \in C^1$.

(iii) Let F be of class C^k. By induction, assume that F^{-1} is of class C^{k-1}. Repeating the last argument in point (ii) and recalling that $J \in C^\infty$ (Lemma 2.6), we get that F^{-1} is of class C^k. □

REMARK. The assumption $F \in C^1$ cannot be dropped. For a counterexample, see Remark 1.3 of [**2**, p. 33].

2.2. Implicit Function Theorem. A generalization of the Local Inversion Theorem is provided by the Implicit Function Theorem. Let T, X and Y be Banach spaces and let $\Lambda \subset T$ and $U \subset X$ be open. Consider a map $F : \Lambda \times U \to Y$.

LEMMA 2.9. *Take* $(\lambda^*, u^*) \in \Lambda \times U$ *and suppose that*

(i) *F is continuous and its u-partial derivative $F_u : \Lambda \times U \to L(X, Y)$ is defined and continuous on the whole $\Lambda \times U$,*
(ii) *$F_u(\lambda^*, u^*) \in L(X, Y)$ is invertible.*

Then $\Psi : \Lambda \times U \to T \times Y$ defined as $\Psi(\lambda, u) := (\lambda, F(\lambda, u))$ is locally invertible at (λ^*, u^*) with continuous inverse Φ. Moreover, if $F \in C^1(\Lambda \times U, Y)$, then Φ is of class C^1.

PROOF. The local invertibility of Ψ at (λ^*, u^*) is obtained in the same way as in the proof of the Local Inversion Theorem 2.8, with clear adjustments.

Suppose then that $F \in C^1(\Lambda \times U, Y)$ and let

$$A = F_\lambda(\lambda^*, u^*) \text{ and } B = F_u(\lambda^*, u^*).$$

Obviously, $\Psi \in C^1(\Lambda \times U, T \times Y)$ and has derivative

$$\mathrm{d}\Psi(\lambda^*, u^*)(\xi, v) = (\xi, A[\xi] + B[v]),$$

which is invertible. Indeed

$$\mathrm{d}\Psi(\lambda^*, u^*)(\xi, v) = (\eta, \nu)$$

implies $\xi = \eta$ and $A[\eta] + B[v] = \nu$. As B is invertible (hypothesis (ii)), we then have a unique solution $v = B^{-1}(\nu - A[\eta])$. Consequently, $\Psi'(\lambda^*, u^*) \in \mathrm{Inv}(T \times Y, T \times Y)$. Applying the Local Inversion Theorem 2.8 to that shows that Ψ is locally invertible at (λ^*, u^*) (which was already known) and that the inverse Φ is of class C^1. □

THEOREM 2.10. (IMPLICIT FUNCTION THEOREM)
Let T, X and Y be Banach spaces and let $\Lambda \subset T$ and $U \subset X$ be open. Take $F \in C^k(\Lambda \times U, Y)$, $k \geq 1$, and suppose that $F(\lambda^*, u^*) = 0$ and that $F_u(\lambda^*, u^*) \in \mathrm{Inv}(X, Y)$. Then there exist neighborhoods Λ^* of λ^* in T and U^* of u^* in X and a map $g \in C^k(\Lambda^*, X)$ such that

(i) $F(\lambda, g(\lambda)) = 0$ for all $\lambda \in \Lambda^*$,
(ii) $F(\lambda, u) = 0$ with $(\lambda, u) \in \Lambda^* \times U^*$ implies $u = g(\lambda)$,
(iii) $\mathrm{d}g(\lambda) = -[F_u(p)]^{-1} \circ F_\lambda(p)$, where $p = (\lambda, g(\lambda))$ and $\lambda \in \Lambda^*$.

PROOF. According to Lemma 2.9, we can associate to F the map Ψ, which is locally invertible at (λ^*, u^*) and $\Psi(\lambda^*, u^*) = (\lambda^*, F(\lambda^*, u^*)) = (\lambda^*, 0)$. In other words, there exists an inverse Φ in a neighborhood $\Lambda^* \times V$ of $(\lambda^*, F(\lambda^*, u^*))$. Because of the definition of Ψ, the first component of Φ is the identity, namely

$$\Phi(\lambda, v) = (\lambda, \phi(\lambda, v))$$

for some $\phi : \Lambda^* \times V \to X$ such that

$$(2.4) \qquad\qquad F(\lambda, \phi(\lambda, v)) = v$$

for all $\lambda \in \Lambda^*$. One can check, by subsequent differentiations of this last identity, that $F \in C^k$ implies $\phi \in C^k$. If we now define $g(\lambda) := \phi(\lambda, 0)$ for $\lambda \in \Lambda^*$ and use (2.4), we obtain

$$F(\lambda, g(\lambda)) = F(\lambda, \phi(\lambda, 0)) = 0, \quad \forall \lambda \in \Lambda^*,$$

proving (i). Since Φ is bijective, (ii) follows as well.

As for the last part of the statement, observe that, differentiating (2.4), one gets

$$F_\lambda + F_u \circ \phi_\lambda = 0,$$

which implies

$$\phi_\lambda = -[F_u]^{-1} F_\lambda$$

and in turn implies (iii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Essential bifurcation theory

In the study of nonlinear functional equation it is quite common to lack unicity of solutions. Bifurcation theory provides tools to study the structure of the set of solutions of such an equation, looking for new solutions generated near a given one after a small perturbation. The main idea is to, in some sense, parametrize the known branch of solutions with some parameter λ and then study the corresponding functional equation $F(\lambda, u) = 0$.

Let X and Y be Banach spaces. We want to study the equation

$$(2.5) \qquad\qquad\qquad F(\lambda, u) = 0,$$

where $F : \mathbb{R} \times X \to Y$. In particular, we require that $F \in C^2(\mathbb{R} \times X, Y)$ and that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Hence, $u = 0$ will be a solution of (2.5) for all λ and will be accordingly called *trivial solution*. What we are interested in is studying for which value of the parameter λ (if any) there are one or more solutions of (2.5) branching off from the trivial one.

DEFINITION 2.7. We say that λ^* is a *bifurcation point* for F (from the trivial solution) if there is a sequence of solutions $(\lambda_n, u_n)_{n \in \mathbb{N}} \subset \mathbb{R} \times X$, with $u_n \neq 0$ for each $n \in \mathbb{N}$, that converges to $(\lambda^*, 0)$.

It follows immediately from the definition and the Implicit Function Theorem 2.10 that

PROPOSITION 2.11. *A necessary condition for λ^* to be a bifurcation point for F is that $F_u(\lambda^*, 0)$ is not invertible.*

PROOF. If we had $F_u(\lambda^*, 0) \in \mathrm{Inv}(X, Y)$, then by the Implicit Function Theorem 2.10 we would get a neighborhood $\Lambda^* \times V$ of $(\lambda^*, 0)$ such that

$$F(\lambda, u) = 0, \ (\lambda, u) \in \Lambda^* \times V \iff u = 0.$$

Consequently, λ^* cannot be a bifurcation point for F. $\qquad\qquad\square$

The goal of this section is to show that, under some additional hypothesis, the condition of Proposition 2.11 is also sufficient.

3.1. Liapunov-Schmidt reduction. We shall first discuss a general method, called *Liapunov-Schmidt reduction*, that allows us to reduce our a priori infinte-dimensional problem to a low-dimensional one. Let $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) = 0$ for each $\lambda \in \mathbb{R}$. Set $L := F_u(\lambda^*, 0)$ and suppose that

(1) $V := \ker(L)$ has a topological complement W in X, namely there exists a closed subspace W of X such that $X = V \oplus W$;
(2) $R = R(L)$ is closed and has a topological complement Z in Y, namely there exists a closed subspace Z of Y such that $Y = Z \oplus R$ and $Z \cap R = \{0\}$.

In order to satisfy these two conditions, it is sufficient that L is a *Fredholm operator*.

DEFINITION 2.8. Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is a *Fredholm operator* if

(i) $\ker T$ is finite dimensional,
(ii) $\mathrm{coker}\, T := \frac{Y}{R(T)}$ is finite dimensional,
(iii) $R(T)$ is closed.

The *index* of T is defined as

$$\mathrm{ind}\, T := \dim \ker T - \mathrm{codim}\, R(T) = \dim \ker T - \dim \mathrm{coker}\, T.$$

REMARK. Actually, one can easily prove that requirement (iii) in the previous definition is redundant and can be therefore omitted.

Let then $P : Y \twoheadrightarrow Z$ and $Q : Y \twoheadrightarrow R$ be the two conjugate projections on Z and R, respectively. Applying P and Q to (2.5) and writing $u \in X$ as $u = v + w$ with $v \in V$ and $w \in W$, one gets the equivalent system

$$(2.6) \qquad \begin{cases} PF(\lambda, v + w) = 0 \\ QF(\lambda, v + w) = 0 \end{cases}$$

Now, recall that $Lv = 0$ and write

$$F(\lambda, u) = Lu + \phi(\lambda, u) = Lw + \phi(\lambda, v + w);$$

then the second in (2.6) becomes

$$(2.7) \qquad \Phi(\lambda, v, w) := Lw + Q\phi(\lambda, v + w) = 0.$$

Notice that $\Phi \in C^2(\mathbb{R} \times V \times W, R)$ and that

$$\Phi_w(\lambda^*, 0, 0)[\tilde{w}] = L\tilde{w} + Q\phi_u(\lambda^*, 0)\tilde{w}.$$

Observe though that, since by definition $\phi(\lambda, u) = F(\lambda, u) - Lu$, it holds

$$\phi_u(\lambda^*, 0) = F_u(\lambda^*, 0) - L = 0$$

and hence $\Phi_w(\lambda^*, 0, 0) = L|_W$. Notice moreover that $L|_W : W \to R$ is injective and surjective. Consequently, as R is closed, by the Closed Graph Theorem $(L|_W)^{-1} : R \to W$ is continuous, i.e. $\Phi_w(\lambda^*, 0, 0) = L|_W \in \text{Iso}(W, R)$. Hence, the Implicit Function Theorem 2.10 applies to Φ and locally (2.7) can be uniquely solved with respect to w. Namely, there exist

  (i) a neighborhood Λ^* of λ,
  (ii) a neighborhood V^* of $v = 0$ in V,
  (iii) a neighborhood W^* of $w = 0$ in W,
  (iv) a function $\gamma \in C^2(\Lambda^* \times V^*, W^*)$

such that the unique solutions of the second entry in (2.6) in $\Lambda^* \times V^* \times W^*$ are given by $(\lambda, v, \gamma(\lambda, v))$:

$$(2.8) \qquad L\gamma(\lambda, v) + Q\phi(\lambda, v + \gamma(\lambda, v)) = 0$$

for all $(\lambda, v) \in \Lambda^* \times V^*$. Observe in particular that $\gamma(\lambda, 0) = 0$ for all $\lambda \in \Lambda$ and that $\gamma_v(\lambda^*, 0) = 0$. Indeed, differentiating (2.8) with respect to v at $(\lambda^*, 0)$ one obtains

$$L\gamma_v(\lambda^*, 0)x + Q\phi_u(\lambda^*, \gamma_v(\lambda^*, 0))[x + \gamma_v(\lambda^*, 0)x] = 0$$

for all $x \in V$. As $\gamma(\lambda^*, 0) = 0$ and $\phi_u(\lambda^*, 0) = 0$, then, we have $L\gamma_v(\lambda^*, 0)x = 0$ for all $x \in V$ and hence $\gamma_v(\lambda^*, 0)x \in V \cap W = \{0\}$ for all $x \in V$.

Summing up, we can write

$$(2.9) \qquad w = \gamma(\lambda, v).$$

Substituting that into the first equation of (2.6) we obtain

$$(2.10) \qquad P(F(\lambda, v + \gamma(\lambda, v))) = 0.$$

Equation (2.10) (in the unknowns $(\lambda, v) \in \Lambda^* \times V^*$) is called *bifurcation equation*. The system (2.9) and (2.10) is equivalent in $\Lambda^* \times V^* \times W^*$ to the initial equation $F(\lambda, u) = 0$.

REMARK. If L is a Fredholm operator, the Lyapunov-Schmidt method allows us to reduce the original infinite-dimensional problem to a finite-dimensional one. Indeed (2.10) is a system of $\dim(\text{coker } L)$ equations in the unknowns $(\lambda, v) \in \mathbb{R} \times \mathbb{R}^{\dim \ker L}$.

3.2. Bifurcation from the simple eigenvalue. We saw in Proposition 2.11 that a necessary condition for λ^* being a bifurcation point of $F(\lambda, u) = 0$ is that $F_u(\lambda^*, 0)$ is not invertible. Actually, this is not a sufficient condition (see for example [**2**, 5.1]). The goal of what follows is to find additional hypothesis that make it a sufficient condition.

Let $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Suppose that $L := F_u(\lambda^*, 0)$ is a Fredholm operator of index 0 and with a one-dimensional kernel. More explicitly, suppose that

(i) there exists $u^* \in X$, $u^* \neq 0$ such that
$$V := \ker L = \langle u^* \rangle := \{ tu^* \mid t \in \mathbb{R} \},$$

(ii) there exists a linear functional $\psi \in Y^*$, $\psi \neq 0$ such that
$$R := R(L) = \{ y \in Y \mid \langle \psi, y \rangle = 0 \}.$$

The bifurcation equation (2.10) then becomes
$$\langle \psi, F(\lambda, tu^* + \gamma(\lambda, tu^*)) \rangle = 0.$$

Set $\mu := \lambda - \lambda^*$ and define
$$\beta(\mu, t) := \langle \psi, F(\lambda^* + \mu, tu^* + \gamma(\lambda^* + \mu, tu^*)) \rangle,$$

which is a real-valued function of class C^2 in a neighborhood U of $(0,0) \in \mathbb{R} \times \mathbb{R}$ (indeed F and γ are C^2).

LEMMA 2.12. *The following are some useful properties of β.*
(i) $\beta(\mu, 0) = 0$ for all μ,
(ii) $\beta_\mu(0,0) = \beta_{\mu\mu}(0,0) = 0$,
(iii) $\beta_t(0,0) = 0$,
(iv) $\beta_{\mu t}(0,0) = \langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle$,
(v) $\beta_{tt}(0,0) = \langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle$.

PROOF. (i) Simply notice that
$$\beta(\mu, 0) = \langle \psi, F(\lambda^*, \gamma(\lambda^* + \mu, 0)) \rangle = \langle \psi, F(\lambda^*, 0) \rangle = \langle \psi, 0 \rangle = 0,$$

where we took into account that $\gamma(\lambda, 0) \equiv 0$ and $F(\lambda^*, 0) = 0$.

(ii) It is an immediate consequence of (i).

(iii) Differentiate β with respect to t
$$\beta_t(\mu, t) = \langle \psi, F_u(\lambda^* + \mu, tu^* + \gamma(\lambda^* + \mu, tu^*))[u^* + \gamma_v(\lambda^* + \mu, tu^*)u^*] \rangle$$

and evaluate that for $t = 0$ and $\mu = 0$:
$$\beta_t(0,0) = \langle \psi, F_u(\lambda^*, \gamma(\lambda^*, 0))[u^* + \gamma_v(\lambda^*, 0)u^*] \rangle$$
$$= \langle \psi, F_u(\lambda^*, 0)u^* \rangle = \langle \psi, Lu^* \rangle = 0$$

(recall that $\gamma_v(\lambda^*, 0) = 0$ and that ψ generates the cokernel of L).

(iv) Differentiating β in t and μ and evaluating that in $t = 0$ and $\mu = 0$ one gets
$$\beta_{\mu t}(0,0) = \langle \psi, F_{\lambda u}(\lambda^*, 0)[u^* + \gamma_v(\lambda^*, 0)u^*] \rangle + \langle \psi, F_u(\lambda^*, 0)\gamma_{\lambda v}(\lambda^*, 0)[u^*] \rangle$$
$$= \langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*]) + \langle \psi, F_u(\lambda^*, 0)\gamma_{\lambda v}(\lambda^*, 0)[u^*] \rangle$$
$$= \langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle$$

(recall again that $\psi|_R = 0$ and that $F_u(\lambda^*, 0) = L$).

(v) It follows again by direct differentiation of β (two times) in u. $\qquad\square$

We can finally state the main theorem we will need to study the 2-dimensional Liouville equation.

THEOREM 2.13. (BIFURCATION FROM THE SIMPLE EIGENVALUE)
Let $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Let λ^* be such that $L = F_u(\lambda^*, 0)$ has one-dimensional kernel $V = \{ tu^* \mid t \in \mathbb{R} \}$ and closed range R with codimension 1. Letting $M := F_{u\lambda}(\lambda^*, 0)$, assume moreover that $M[u^*] \notin R$. Then λ^* is a bifurcation point for F. In addition, the

set of non-trivial solutions of $F = 0$ is, near $(\lambda^*, 0)$, a unique C^1 cartesian curve with parametric representation on V.

PROOF. We need to solve $\beta(\mu, t) = 0$, where we recall that $\beta \in C^2$. Because of (i) in Lemma 2.12, we cannot apply directly the Implicit Function Theorem 2.10 to β. Therefore, define

$$h(\mu, t) := \begin{cases} \frac{\beta(\mu,t)}{t} & \text{if } t \neq 0 \\ \beta_t(\mu, 0) & \text{if } t = 0 \end{cases}.$$

By properties (i) to (v) of Lemma 2.12 and by the hypothesis $Mu^* \notin R$, one can see that $h \in C^1$, $h(0, 0) = 0$ and

$$a := h_\mu(0, 0) = \beta_{\mu t}(0, 0) = \langle \psi, Mu^* \rangle \neq 0,$$

$$b := h_t(0, 0) = \frac{1}{2}\beta_{tt}(0, 0) = \frac{1}{2}\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*]\rangle.$$

Hence, we can apply the Implicit Function Theorem 2.10 to $h(\mu, t) = 0$, getting a neighborhood $(-\varepsilon, \varepsilon)$ of $t = 0$ and a unique function $\mu \in C^1(-\varepsilon, \varepsilon)$ such that $\mu(0) = 0$ and $h(\mu(t), t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Notice that $h(\mu, t) = 0$ is equivalent to $\beta(\mu, t) = 0$ if $t \neq 0$. Hence, the bifurcation equation is solved uniquely by $\mu = \mu(t)$.

Therefore, following the Lyapunov-Schmidt reduction method presented in the previous section, one gets

$$F(\lambda^* + \mu(t), tu^* + \gamma(\lambda^* + \mu(t), tu^*)) = 0$$

for all $t \in (-\varepsilon, \varepsilon)$. Observe that $tu^* + \gamma(\lambda^* + \mu(t), tu^*) \neq 0$ if $t \neq 0$. Indeed, γ has values in $W^* \subset W$, which is the complement of $V \ni u^*$. Hence, we found that the set of non-trivial solutions of $F(\lambda, u) = 0$ is given, in a neighborhood of $(\lambda^*, 0)$, by the unique cartesian curve

$$\begin{cases} \lambda = \lambda^* + \mu(t) \\ u = tu^* + \gamma(\lambda^* + \mu(t), tu^*) \end{cases},$$

with $t \in (-\varepsilon, \varepsilon)$, $t \neq 0$. $\square$

3.3. Shape of bifurcation. It would be nice to gain some more information about the type of bifurcation we are encountering. By Theorem 2.13 we know that, in general, the set of non-trivial solutions has the form

$$\begin{cases} \lambda = \lambda^* + \mu(t) \\ u = tu^* + \gamma(\lambda^* + \mu(t), tu^*) \end{cases}, \quad t \in (-\varepsilon, \varepsilon), \quad t \neq 0.$$

This subsection, therefore, will be devoted to the computation of the first terms in the Taylor expansion of $\lambda(t)$ centered in 0. To do so, suppose that F is sufficiently regular (say C^∞) and compute the first terms of the Taylor

expansion of F centered in $(\lambda^*, 0)$: Let $\lambda_0 := \lambda^*$ and $u_1 := u^*$, then

$$
\begin{aligned}
0 = F(\lambda(t), u(t)) &= F(\lambda_0 + \lambda_1 t + \lambda_2 t^2 + O(t^3), u_1 t + u_2 t^2 + O(t^3)) \\
&= F(\lambda_0, 0) + (F_u(\lambda_0, 0)[u_1] + \lambda_1 F_\lambda(\lambda_0, 0))t \\
&\quad + \left(F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_2 F_\lambda(\lambda_0, 0) \right. \\
&\quad \left. + \lambda_1 F_{\lambda u}(\lambda_0, 0)[u_1] + \frac{1}{2}\lambda_1^2 F_{\lambda\lambda}(\lambda_0, 0) \right) t^2 + O(t^3).
\end{aligned}
$$

Hence we have:

$$
F(\lambda_0, 0) = 0,
$$

which is true by hypothesis;

$$
F_u(\lambda_0, 0)[u_1] + \lambda_1 F_\lambda(\lambda_0, 0) = 0,
$$

which is also true because $F_u(\lambda_0, 0)[u_1] = F_u(\lambda_0, 0)[u^*] = 0$ by hypothesis and $F_\lambda(\lambda_0, 0) = 0$ because $F(\lambda, 0) \equiv 0$ for all λ;

$$
\begin{aligned}
0 &= F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_2 F_\lambda(\lambda_0, 0) \\
&\quad + \lambda_1 F_{\lambda u}(\lambda_0, 0)[u_1] + \frac{1}{2}\lambda_1^2 F_{\lambda\lambda}(\lambda_0, 0) \\
&= F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_1 F_{\lambda u}(\lambda_0, 0)[u_1],
\end{aligned}
$$

again because $F(\lambda, 0) \equiv 0$ for all λ. Applying ψ to that last equality and recalling that $\operatorname{coker} F_u(\lambda^*, 0) = \operatorname{coker} L = \langle \psi \rangle$, one gets

$$
0 = \frac{1}{2}\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle + \lambda_1 \langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle,
$$

from which we obtain

$$
\lambda_1 = -\frac{1}{2} \frac{\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle}{\langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle}
$$

(the fraction is well defined because by the hypothesis of Theorem 2.13 we already know that the denominator can't be 0, see also Remark 4.3.iv of [**2**, p. 96] or (I.6.3) of [**30**, p. 21]). If $\lambda_1 \neq 0$ we have a so-called *transcritical bifurcation* (see Figure 2.1).

In case we find $\lambda_1 = 0$, in order to get some knowledge on the type of bifurcation we need to compute higher order terms of the expansion of $\lambda(t)$. Assume then that $\lambda_1 = 0$. Again, in general, one has the following expansion

$$
\begin{aligned}
0 = F(\lambda(t), u(t)) &= F(\lambda_0 + \lambda_2 t^2 + \lambda_3 t^3 + O(t^3), u_1 t + u_2 t^2 + u_3 t^3 + O(t^3)) \\
&= F(\lambda_0, 0) + F_u(\lambda_0, 0)[u_1]t \\
&\quad + \left(F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_2 F_\lambda(\lambda_0, 0) \right) t^2 \\
&\quad + \left(F_u(\lambda_0, 0)[u_3] + F_{uu}(\lambda_0, 0)[u_1, u_2] + \frac{1}{6} F_{uuu}(\lambda_0, 0)[u_1, u_1, u_1] \right. \\
&\quad \left. + \lambda_3 F_\lambda(\lambda_0, 0) + \lambda_2 F_{\lambda, u}(\lambda_0, 0)[u_1] \right) t^3 + O(t^4).
\end{aligned}
$$

As before, the first and the second summands are already known to be 0. The third term gives us the following condition:

$$(2.11) \qquad F_u(\lambda^*, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda^*, 0)[u^*, u^*] = 0.$$

The fourth term, instead, is the one from which we would like to extract the value of λ_2:

$$F_u(\lambda^*, 0)[u_3] + F_{uu}(\lambda^*, 0)[u^*, u_2]$$
$$+ \frac{1}{6} F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] + \lambda_2 F_{\lambda, u}(\lambda^*, 0)[u^*] = 0,$$

which leads to

$$\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u_2] \rangle + \frac{1}{6} \langle \psi, F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] \rangle$$
$$+ \lambda_2 \langle \psi, F_{\lambda, u}(\lambda^*, 0)[u^*] \rangle = 0,$$

namely

$$(2.12) \qquad \lambda_2 = -\frac{\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u_2] \rangle + \frac{1}{6} \langle \psi, F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] \rangle}{\langle \psi, F_{\lambda, u}(\lambda^*, 0)[u^*] \rangle}$$

(again, observe that the fraction is well defined because the denominator is not 0)[3].

At least implicitly, then, one can find λ_2. Indeed, one can restrict $L : X \to Y$ to

$$\tilde{L} : \frac{X}{\langle u^* \rangle} \to \{y \in Y \mid \langle \psi, y \rangle = 0\},$$

which is invertible, and then write

$$u_2 = \tilde{L}^{-1} \left(-\frac{1}{2} F_{uu}(\lambda^*, 0)[u^*, u^*] \right)$$

and substitute it in (2.12). Observe indeed that, applying ψ to (2.11), one immediately gets that

$$F_{uu}(\lambda^*, 0)[u^*, u^*] \in \{y \in Y \mid \langle \psi, y \rangle = 0\}.$$

Moreover, notice that

$$\langle \psi, F_{uu}(\lambda^*, 0)[u^*, v] \rangle = 0, \quad \forall v \in \langle u^* \rangle$$

implies also that the λ_2 obtained in this way is well defined.

If $\lambda_2 > 0$ we say that we have a *supercritical bifurcation*, while if $\lambda_2 < 0$ we have a *subcritical bifurcation* (see again Figure 2.1).

---

[3]It seems then that the formula reported in many books, like formula (I.6.11) in [**30**, p. 23] or formula (4.7) in [**2**, p. 97], is wrong as it misses the term containing u_2. Observe that in general u_2 is different from 0 because of (2.11).
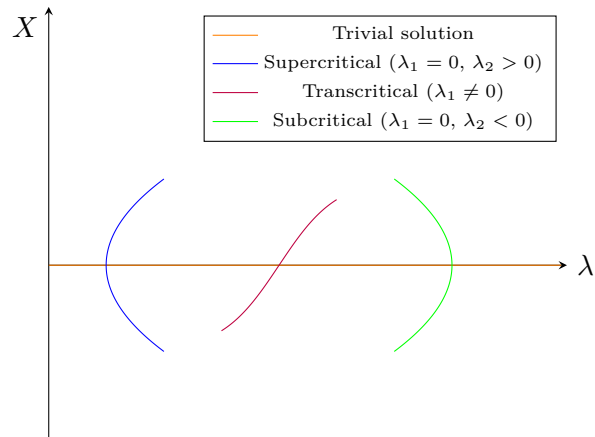
FIGURE 2.1. A qualitative representation of different types of bifurcation.

CHAPTER 3

Some regularity results

In this chapter, we will present some general regularity results that we will need while studying the bifurcations of the Liouville equation. In the first part we will deal with elliptic regularity and we will derive a weighted version of the Schauder interior estimates. In the second part, instead, we will present the bootstrapping technique and we will tackle the problem of extendig solutions defined only on strips of $\mathbb{R}^2$ to the whole plane.

1. Elliptic regularity

Let L be a linear partial differential operator of order 2 defined in an open subset Ω of $\mathbb{R}^n$, $n \geq 2$. Assume that L can be written using the standard cartesian coordinates as follows[4]

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_i u + c(x)u,$$

with $a^{ij} = a^{ji}$ for each $1 \leq i, j \leq n$.

DEFINITION 3.1. L is *elliptic* at a point $x \in \Omega$ if the coefficient matrix $(a^{ij}(x))$ is positive, namely: if $\lambda(x)$ and $\Lambda(x)$ denote, respectively, the minimum and the maximum eigenvalues of $[a^{ij}(x)]_{ij}$, then

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus 0$. If $\lambda > 0$ in Ω then L is *elliptic in Ω*. If moreover there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, we say that L is *strictly elliptic* or *uniformly elliptic*.

A first regularity result is the following (see Theorem 2.1 in [**43**, Chapter 4]).

THEOREM 3.1. *If L is an elliptic operator in $\Omega \subset \mathbb{R}^n$ open and if the coefficients of L are of class C^∞ in Ω, then A is* hypoelliptic *in Ω, namely: If u is a distribution in an open subset Ω_1 of Ω and if Lu is of class C^∞ in Ω_1, then u is of class C^∞ in Ω_1.*

---

[4]We are using here the standard Einstein notation: repeated indexes imply that there is a sum on those indexes (we say that they are *contracted*).

1.1. Schauder interior estimates. Roughly speaking, the *Schauder interior estimates* provide a tool to estimate "higher regularity norms" of solutions of elliptic equations with "lower regularity norms". Let us first introduce the so-called *interior Hölder spaces*.

DEFINITION 3.2. Let $\Omega \subset \mathbb{R}^n$ be open, k be a non-negative integer and $\alpha \in (0,1)$. Set $d_x := \text{dist}(x, \partial\Omega)$ and $d_{x,y} := \min\{d_x, d_y\}$ and consider $u \in C^k(\Omega)$. We say that $u \in C^{k,\alpha}_*(\Omega)$ if its *interior Hölder norm* is finite, namely

$$|u|^*_{k,\alpha,\Omega} := |u|^*_{k,\Omega} + [u]^*_{k,\alpha,\Omega} < +\infty,$$

where

$$|u|^*_{k,\Omega} := \sum_{j=0}^{k} [u]^*_{j,\Omega} := \sum_{j=0}^{k} \sup_{\substack{x \in \Omega \\ |\beta|=j}} d_x^k |D^\beta u(x)|$$

and

$$[u]_{k,\alpha,\Omega} := \sup_{\substack{x,y \in \Omega \\ |\beta|=k}} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

One can prove that $C^{k,\alpha}_*(\Omega)$, equipped with the interior norm, is a Banach space (see for example [**20**, Problem 5.2]). For simplicity of notation, we now drop the subscript $*$ and just write $C^{k,\alpha}(\Omega) := C^{k,\alpha}_*(\Omega)$.

In order to state the Schauder interior estimates we also need to introduce the following norms. Let σ be a real number and define

$$|u|^{(\sigma)}_{k,\alpha,\Omega} := |u|^{(\sigma)}_{k,\Omega} + [u]^{(\sigma)}_{k,\alpha,\Omega} < +\infty,$$

where

$$|u|^{(\sigma)}_{k,\Omega} := \sum_{j=0}^{k} [u]^{(\sigma)}_{j,\Omega} := \sum_{j=0}^{k} \sup_{\substack{x \in \Omega \\ |\beta|=j}} d_x^{j+\sigma} |D^\beta u(x)|$$

and

$$[u]^{(\sigma)}_{k,\alpha,\Omega} := \sup_{\substack{x,y \in \Omega \\ |\beta|=k}} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

One can check the following (see (6.11) on page 90 of [**20**]).

PROPOSITION 3.2. *Let $\sigma + \tau \geq 0$, then*

$$|fg|^{(\sigma+\tau)}_{0,\alpha,\Omega} \leq |f|^{(\sigma)}_{0,\alpha,\Omega} |g|^{(\tau)}_{0,\alpha,\Omega}.$$

The basic Schauder interior estimates are provided by the following

THEOREM 3.3. *Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\Omega)$ be a bounded solution in Ω of*

$$Lu = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u = f,$$

where $f \in C^{0,\alpha}(\Omega)$ and there are positive constants λ, Λ such that the coefficients satisfy

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

and

$$|a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \Lambda.$$

Then there exists a constant $C > 0$ not depending on u and f such that

$$|u|_{2,\alpha,\Omega}^{*} \leq C\left(|u|_{0,\Omega} + |f|_{0,\alpha,\Omega}^{(2)}\right).$$

For the proof of this theorem, we refer to [**20**, Theorem 6.2].

1.2. Weighted Schauder interior estimates. We now want to find a generalization of Theorem 3.3 for the case in which we consider "weighted" Hölder norms. Let $w \in C^{k,\alpha}(\Omega), w > 0$ be the weight. Define, according to the notation explained before, the space

$$C_w^{k,\alpha}(\Omega) := \left\{u \in C^k(\Omega) \middle| |wu|_{k,\alpha,\Omega}^{*} < +\infty\right\}.$$

Let u be a distributional solution of

$$(3.1) \qquad Lu = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u = f,$$

where L is a uniformly elliptic operator. Take $w \in C^{2,\alpha}(\Omega), w > 0$ and set $v := wu$. Our goal, then, is to estimate the $C^{2,\alpha}$ norm of v in terms of the weighted norm of f. The path we are going to follow is to write an elliptic equation for v and then apply Theorem 3.3.

We first compute the (distributional) partial derivatives of v:

$$(3.2) \qquad \partial_i v(x) = \partial_i w(x)u(x) + w(x)\partial_i u(x)$$

and

$$(3.3) \qquad \begin{aligned} \partial_i\partial_j v(x) &= \partial_i\partial_j w(x)u(x) + \partial_i w(x)\partial_j u(x) \\ &\quad + \partial_j w(x)\partial_i u(x) + w(x)\partial_i\partial_j u(x). \end{aligned}$$

Contracting (3.3) with $a^{ij}(x)$ and using (3.1) we get

$$\begin{aligned} a^{ij}(x)\partial_i\partial_j v(x) &- a^{ij}(x)\partial_i\partial_j w(x)u(x) - 2a^{ij}(x)\partial_i w(x)\partial_j u(x) \\ &= w(x)a^{ij}(x)\partial_i\partial_j u(x) \\ &= w(x)f(x) - w(x)b^i(x)\partial_i u(x) - w(x)c(x)u(x), \end{aligned}$$

where we also use the fact that, by hypothesis, $a^{ij} = a^{ji}$. We can rewrite this last equation as follows:

$$(3.4) \qquad \begin{aligned} a^{ij}(x)\partial_i\partial_j v(x) &+ \left(w(x)b^j(x) - 2a^{ij}(x)\partial_i w(x)\right)\partial_j u(x) \\ &+ \left(c(x)w(x) - a^{ij}(x)\partial_i\partial_j w(x)\right)u(x) = w(x)f(x) =: g(x). \end{aligned}$$

We now want to write everything in terms of just v. Observe that, according to the definition of v and to equation (3.2), we know that:

- $u(x) = \frac{v(x)}{w(x)}$ (recall that w is never 0),
- $\partial_i u(x) = \frac{1}{w(x)}\left(\partial_i v(x) - \frac{\partial_i w(x)}{w(x)}v(x)\right)$.

Performing the computations term-by-term we get

$$[\text{2-nd order}] = a^{ij}(x)\partial_i\partial_j v(x),$$

$$[\text{1-st order}] = \left(w(x)b^j(x) - 2a^{ij}(x)\partial_i w(x)\right)\frac{1}{w(x)}\left(\partial_i v(x) - \frac{\partial_i w(x)}{w(x)}v(x)\right)$$

$$= \left(b^j(x) - 2a^{ij}(x)\frac{\partial_i w(x)}{w(x)}\right)\partial_j v(x)$$

$$+ \left(2a^{ij}(x)\frac{\partial_i w(x)\partial_j w(x)}{w^2(x)} - b^j(x)\frac{\partial_j w(x)}{w(x)}\right)v(x)$$

$$= \left(b^j(x) - 2a^{ij}(x)\partial_i \log w(x)\right)\partial_j v(x)$$

$$+ \left(2a^{ij}(x)\partial_i \log w(x)\partial_j \log w(x) - b^j(x)\partial_j \log w(x)\right)v(x),$$

$$[\text{0-th order}] = \left(c(x) - a^{ij}(x)\frac{\partial_i \partial_j w(x)}{w(x)}\right)v(x).$$

Putting all together we finally find

(3.5)
$$a^{ij}(x)\partial_i\partial_j v(x) + \left(b^j(x) - 2a^{ij}(x)\partial_i \log w(x)\right)\partial_j v(x)$$
$$+ \Bigg(2a^{ij}\partial_i \log w(x)\partial_j \log w(x)$$
$$- a^{ij}(x)\frac{\partial_i\partial_j w(x)}{w(x)} - b^j(x)\partial_j \log w(x) + c(x)\Bigg)v(x)$$
$$= w(x)f(x) =: g(x).$$

Observe now that what we got in this way is still a uniformly elliptic equation. The highest order coefficients are indeed the same as those of the equation (3.1).

Assume then that the hypothesis on the coefficients of L given by Theorem 3.3 hold. In order to apply the Schauder estimates on equation (3.5), then, it suffices to check that there exists some positive constant $\tilde{\Lambda}$ such that

$$|\tilde{b}^i|^{(1)}_{0,\alpha,\Omega}, |\tilde{c}|^{(2)}_{0,\alpha,\Omega} \le \tilde{\Lambda},$$

where $\tilde{b}^i$ and $\tilde{c}$ are, respectively, the 1-st and 0-th order coefficients of the new equation (3.5). Moreover, we will need $g \in C^{0,\alpha}(\Omega)$, which is precisely $f \in C^{0,\alpha}_w(\Omega)$.

We first deal with the first order coefficient. By the triangular inequality we have

$$|\tilde{b}^i|^{(1)}_{0,\alpha,\Omega} = |b^i - 2a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega}$$
$$\le |b^i|^{(1)}_{0,\alpha,\Omega} + 2|a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega},$$

hence, it suffices to show that $|a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega}$ is finite. By hypothesis we know that $|a^{ij}|^{(0)}_{0,\alpha,\Omega} \le \Lambda$. Consequently, by Proposition 3.2, we get that

$$|a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le |a^{ij}|^{(0)}_{0,\alpha,\Omega}|\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le \Lambda|\partial_j \log w|^{(1)}_{0,\alpha,\Omega}.$$

Thus, it suffices to require that

$$|\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le C_1 < +\infty, \quad \forall j.$$

We now move to the 0-th order coefficient. Again by the triangular inequality

$$|\tilde{c}|_{0,\alpha,\Omega}^{(2)} = \left| c - b^j \partial_j \log w - a^{ij}\frac{\partial_i \partial_j w}{w} + 2a^{ij}\partial_i \log w \partial_j \log w \right|_{0,\alpha,\Omega}^{(2)}$$

$$\leq |c|_{0,\alpha,\Omega}^{(2)} + |b^j \partial_j \log w|_{0,\alpha,\Omega}^{(2)} + \left| a^{ij}\frac{\partial_i \partial_j w}{w} \right|_{0,\alpha,\Omega}^{(2)} + 2|a^{ij}\partial_i \log w \partial_j \log w|_{0,\alpha,\Omega}^{(2)}.$$

By hypothesis, the first summand in this last expression is bounded by Λ. Moreover, again by Proposition 3.2, we get

$$|b^j \partial_j \log w|_{0,\alpha,\Omega}^{(2)} \leq |b^j|_{0,\alpha,\Omega}^{(1)}|\partial_j \log w|_{0,\alpha,\Omega}^{(1)} \leq \Lambda C_1,$$

$$|a^{ij}\partial_i \log w \partial_j \log w|_{0,\alpha,\Omega}^{(2)} \leq |a^{ij}|_{0,\alpha,\Omega}^{(0)}|\partial_i \log w|_{0,\alpha,\Omega}^{(1)}|\partial_j \log w|_{0,\alpha,\Omega}^{(1)} \leq \Lambda C_1^2$$

and

$$\left| a^{ij}\frac{\partial_i \partial_j w}{w} \right|_{0,\alpha,\Omega}^{(2)} \leq |a^{ij}|_{0,\alpha,\Omega}^{(2)}\left|\frac{\partial_i \partial_j w}{w}\right|_{0,\alpha,\Omega}^{(2)} \leq \Lambda \left|\frac{\partial_i \partial_j w}{w}\right|_{0,\alpha,\Omega}^{(2)},$$

meaning that it suffices to require that

$$\left|\frac{\partial_i \partial_j w}{w}\right|_{0,\alpha,\Omega}^{(2)} \leq C_2 < +\infty, \quad \forall i,j.$$

Summing up, we can state the following

THEOREM 3.4. *Let Ω be an open subset of $\mathbb{R}^n$ and let $w \in C^{2,\alpha}(\Omega), w > 0$ be a weight such that there exists a positive constant K such that*

$$|\partial_j \log w|_{0,\alpha,\Omega}^{(1)}, \left|\frac{\partial_i \partial_j w}{w}\right|_{0,\alpha,\Omega}^{(2)} \leq K, \quad \forall i,j.$$

Let $u \in C_w^{2,\alpha}(\Omega)$ be a solution in Ω of

$$Lu = a^{ij}(x)\partial_i \partial_j u + b^i(x)\partial_i u + c(x)u = f,$$

where $f \in C_w^{0,\alpha}(\Omega)$ and there are positive constants λ, Λ such that the coefficients satisfy

$$a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

and

$$|a^{ij}|_{0,\alpha,\Omega}^{(0)}, |b^i|_{0,\alpha,\Omega}^{(1)}, |c|_{0,\alpha,\Omega}^{(2)} \leq \Lambda.$$

Assume moreover that wu is bounded. Then there exists a constant $C > 0$ not depending on u and f such that

$$|u|_{2,\alpha,\Omega;w}^* \leq C \left(|u|_{0,\Omega;w} + |f|_{0,\alpha,\Omega;w}^{(2)} \right),$$

where $|u|_{2,\alpha,\Omega;w}^* := |wu|_{2,\alpha,\Omega}^*, |u|_{0,\Omega;w} := |wu|_{0,\Omega}$ and $|f|_{0,\alpha,\Omega;w}^{(2)} := |wf|_{0,\alpha,\Omega}^{(2)}$.

2. Bootstrapping and extension of solutions

Bootstrapping is a very simple technique used to prove regularity of semilinear differential equations. The idea is the following: Let u be a weak solution of some semilinear equation and suppose to know that u has some sort of regularity. Then u is a solution also of a linear equation whose coefficients are functions of u. This linear equation, in turn, may provide an improved regularity estimate for u, in terms of the original regularity estimates of u. If this new regularity estimates are stronger than the original ones, we actually gained a higher regularity for u.

The Liouville equation itself provides a simple example of this technique. Let Ω be an open subset of $\mathbb{R}^n$ and suppose that $u \in C^1(\Omega)$ is a weak solution of

$$\Delta u + \mathrm{e}^u = 0. \tag{3.6}$$

We claim that u is actually a strong solution. Indeed, as the exponential map is of class C^∞, the map $v := -\mathrm{e}^u$ is again of class C^1 and hence, in particular, of class $C^{0,\alpha}$. Moreover, u solves the linear equation

$$\Delta u = v$$

in $\mathbb{R}^n$. Then, by the Schauder interior estimates (Theorem 3.3), one gets

$$|u|_{2,\alpha,\Omega}^* \le C \left(|u|_{0,\Omega} + |v|_{0,\alpha,\Omega}^{(2)} \right)$$
$$= C \left(|u|_{0,\Omega} + |\mathrm{e}^u|_{0,\alpha,\Omega}^{(2)} \right) < +\infty.$$

Therefore, u is a weak solution of class $C^{2,\alpha}$. Hence, u is a strong solution of (3.6).

This procedure allows us to extend solutions defined only on certain subsets of $\mathbb{R}^n$ to the whole space, under some hypotheses. As an example, we again use the Liouville equation. For $\lambda > 0$, define $S_\lambda := \mathbb{R} \times (0, \lambda)$. Suppose to know that $u \in C^{2,\alpha}(S_\lambda)$ is a solution of the Liouville equation (3.6) in S_λ. Suppose moreover that the y-partial derivative of u can be extended up to ∂S_λ and that Neumann conditions hold on the boundary for u. We first construct a solution on $\mathbb{R} \times (-\lambda, \lambda)$, reflecting u along the axis $y = 0$, and we prove that it is actually a strong solution. Then, by induction, it is clear that we can extend u to the whole plane, again by reflecting along the lines $y = k\lambda$ ($k \in \mathbb{Z}$), and that this solution is a strong one. Define

$$\tilde{u}(x, y) := \begin{cases} u(x, y) & \text{if } y \ge 0 \\ u(x, -y) & \text{if } y < 0 \end{cases} \equiv u(x, |y|).$$

Clearly, as u is a strong Neumann solution on S_λ, u is also a weak Neumann solution on S_λ, namely

$$\int_{S_\lambda} (\nabla u \nabla \phi - \mathrm{e}^u \phi) \, \mathrm{d}x \, \mathrm{d}y = 0, \quad \forall \phi \in H^1(S_\lambda).$$

Consequently, $\tilde{u}$ is a weak solution on $\mathbb{R} \times (-\lambda, \lambda)$. Indeed

$$\int_{\mathbb{R} \times (-\lambda, \lambda)} \left(\nabla \tilde{u} \nabla \phi - \mathrm{e}^{\tilde{u}} \phi \right) \mathrm{d}x \, \mathrm{d}y$$

$$= \int_{S_\lambda} \left(\nabla \tilde{u} \nabla \phi - \mathrm{e}^{\tilde{u}} \phi \right) \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R} \times (-\lambda, 0)} \left(\nabla \tilde{u} \nabla \phi - \mathrm{e}^{\tilde{u}} \phi \right) \mathrm{d}x \, \mathrm{d}y$$

$$= 2 \int_{S_\lambda} \left(\nabla u \nabla \phi - \mathrm{e}^{u} \phi \right) \mathrm{d}x \, \mathrm{d}y = 0, \quad \forall \phi \in H^1(S_\lambda).$$

Now, $\tilde{u}$ is of class $C^{2,\alpha}$ in both S_λ and $\mathbb{R} \times (-\lambda, 0)$. Moreover, by construction, it is continuous and has continuous derivatives on $\{y = 0\}$. Thus, $\tilde{u}$ is overall of class C^1 in $\Omega := \mathbb{R} \times (-\lambda, \lambda)$. Following the previous procedure, then, $\tilde{u}$ is of class $C^{2,\alpha}$ in the whole Ω and therefore is a strong solution.

REMARK. The same procedure works also when dealing with weighted Hölder spaces of the kind defined in the previous section.

Bifurcations for the Liouville equation in $\mathbb{R}^2$

The main goal of this chapter is to find some non-trivial solutions of the 2-dimensional Liouville equation

$$\Delta u(x,y) + \mathrm{e}^{u(x,y)} = 0, \qquad \forall (x,y) \in \mathbb{R}^2,$$

using the tools of bifurcation theory. In particular, as anticipaded in the introduction, we plan to start from a finite-volume solution of the same equation in $\mathbb{R}$, extend it on $\mathbb{R}^2$ by invariance in the last variable and then see that there are solutions which are periodic perturbations of the trivial one along the last variable. As we shall see, the two-dimensional case turns out to be quite easy to treat, as we will be able to find explicit solutions of the equations involved. Moreover, it will be possible to show that the linearized operator has kernel of dimension 1, allowing us to use the Theorem 2.13 (Bifurcation from the Simple Eigenvalue).

The first thing we have to do, hence, is to find a trivial solution. We look for cylindrical solutions, i.e. solutions depending only on one variable, say x, and constant in the other. The equation then becomes the ordinary differential equation

$$u''(x) + \mathrm{e}^{u(x)} = 0, \qquad \forall x \in \mathbb{R}^2,$$

which admits the family of solutions

$$\log\left[c_1 - c_1 \tanh^2\left(\frac{1}{2}\sqrt{2c_1\left(c_2 + x\right)^2}\right)\right], \quad c_1 \geq 0, \quad c_2 \in \mathbb{R}.$$

Observe that the two parameters account only for a translation and a dilation of the solution, so we can simply fix them. For our convenience, we choose $c_1 = 2$ and $c_2 = 0$ (namely, we are requiring that the solution is even and we are fixing its volume), getting

$$u_0(x,y) := \log[2(1 - \tanh^2(|x|))] = \log(2\operatorname{sech}^2(x)).$$

Now that we have a trivial solution for our problem, we want to see that there are other non-trivial solutions emanating from that one. In particular we will see that these bifurcating branches have a periodic shape and are oscillating perturbations of the trivial solution.

Before doing that, however, we need to understand what is the proper setting we should work in. The idea is the following: As we know that the trivial solution does not depend on the y variable, we could just choose to restrict our attention to the subsets

$$S_\lambda := \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, y \in (0,\lambda)\}, \quad \lambda > 0$$

In this way, we get a natural parameter for our problem: the witdth λ of the strip S_λ. Now that we have a parametrization, we might wonder whether
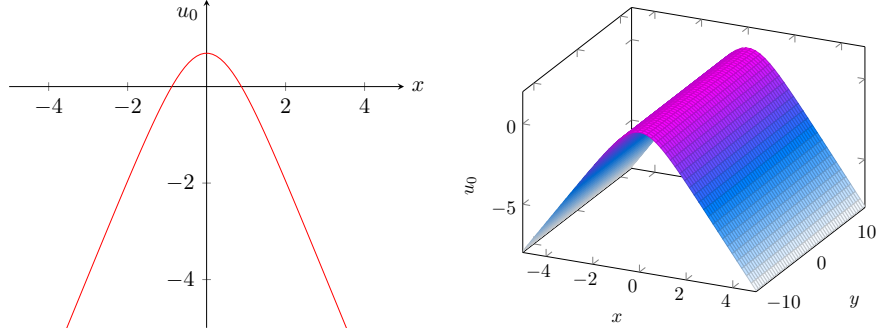
FIGURE 4.1. The trivial solution u_0.

there are some values of λ for which there is a branch of solutions bifurcating from the trivial one, but first we have to make clear which functions are being considered as admissible perturbations. Given the form of the Liouville equation, in order to avoid a finite-time blow-up, we must ask that the solution goes to $-\infty$ as $|x| \to +\infty$. Consequently, again because of the form of the differential equation, the growth must be at most linear (the equation essentially says that at ∞ the second derivative must be 0). In particular, one might ask that the perturbation is bounded by a function that grows slower than $|x|$, like $\sqrt{|x|}$ (and we will see that this choice is general enough). Hence, we will consider the following weighted Hölder space:

$$
X_\lambda := \left\{ u \in C^{2,\alpha}(S_\lambda) \left| \begin{array}{l} \dfrac{\partial}{\partial y} u(x,0) = \dfrac{\partial}{\partial y} u(x,\lambda) = 0 \quad \forall x \in \mathbb{R}, \\[2mm] u(-x,y) = u(x,y) \quad \forall (x,y) \in S_\lambda, \\[2mm] \left| \langle x \rangle^{-\frac{1}{2}} u \right|_{2,\alpha,S_\lambda} + \left| \langle x \rangle^{\frac{3}{2}} \Delta u \right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right. \right\},
$$

where $\langle x \rangle := \sqrt{1 + x^2}$.

Our problem is then finding the zeros of the following function:

$$
\begin{aligned}
\tilde{F} : X_\lambda &\longrightarrow Y_\lambda \\
u &\longmapsto \Delta(u_0 + u) + \mathrm{e}^{u_0 + u} = \Delta u + \mathrm{e}^{u_0}(\mathrm{e}^u - 1),
\end{aligned}
$$

where

$$
Y_\lambda := \left\{ u \in C^{0,\alpha}(S_\lambda) \left| \begin{array}{l} \dfrac{\partial}{\partial y} u(x,0) = \dfrac{\partial}{\partial y} u(x,\lambda) = 0 \quad \forall x \in \mathbb{R}, \\[2mm] u(-x,y) = u(x,y) \quad \forall (x,y) \in S_\lambda, \\[2mm] \left| \langle x \rangle^{\frac{3}{2}} f \right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right. \right\}.
$$

Recalling that the interior Hölder spaces of Definition 3.2 are Banach spaces ([**20**, Problem 5.2]), it can be easily checked that both X_λ and Y_λ are Banach spaces when endowed, respectively, with the norms

$$
\|u\|_{X_\lambda} := \left| \langle x \rangle^{-\frac{1}{2}} u \right|_{2,\alpha,S_\lambda} + \left| \langle x \rangle^{\frac{3}{2}} \Delta u \right|_{0,\alpha,S_\lambda}
$$

(the only point here is to show that $\Delta u_n \to g = \Delta u$, but this true because u_n converges in C^2) and

$$\|f\|_{Y_\lambda} := \left|\langle x\rangle^{\frac{3}{2}} f\right|_{0,\alpha,S_\lambda}.$$

Observe moreover that the functions in X_λ grow at most as $\sqrt{|x|}$, while those in Y_λ grow at most as $|x|^{-\frac{3}{2}}$.

REMARK. The example at the end of Section 2 of Chapter 3 allows us to go back to a strong solution defined on the whole $\mathbb{R}^2$ simply by reflecting along the lines $\{y = k\lambda\}$, with $k \in \mathbb{Z}$. In this way, therefore, if we find non-trivial solutions of the Liouville equation in the strip S_λ, we obtain non-trivial solutions in $\mathbb{R}^2$ with infinite volume.

The main theorem we plan to use is Theorem 2.13. Observe that the problem stated before is not exactly in the form of the theorem, as the parametrization lies in the domain instead of being inside the function. Consequently, we will need to perform some change of variables in order to work this problem out. This will be done later in this chapter. In the following section, instead, we will keep the parameter in the domain, so that it will have a clearer geometric meaning and so that we will be able to work with slightly easier objects.

1. Linearized equation

1.1. Linearization and candidate bifurcation points. Before being able to apply Theorem 2.13, we need to understand what are the values of the parameter for which we can find a bifurcation. Recall that, because of Proposition 2.11, a necessary condition to have a bifurcation on S_λ is that the linearized operator in u_0 on the strip S_λ has a non-trivial kernel. Keeping this in mind, we shall now find the candidate bifurcation points.

Let us first linearize the operator $\tilde{F}(u) = \Delta u + e^{u_0}(e^u - 1)$ in the point 0:

$$L(v) := \tilde{F}_u(0)[v] = \Delta v + e^{u_0} v,$$

which is, explicitly,

$$L(v)(x,y) = \Delta v(x,y) + 2\operatorname{sech}^2(x)v(x,y).$$

We might first want to look for solutions of $Lv = 0$ having the form

$$v(x,y) = w_1(x)w_2(y),$$

hence satisfying

$$w_1''(x)w_2(y) + w_1(x)w_2''(y) + 2\operatorname{sech}^2(x)w_1(x)w_2(y) = 0.$$

By separation of variables then

$$w_2''(y) + \mu^2 w_2(y) = 0,$$

which leads to

$$w_2(y) = A\cos(\mu y) + B\sin(\mu y).$$

Imposing the Neumann boundary conditions we can say that $B = 0$ and that $\mu = \frac{\pi j}{\lambda}$, $j \in \mathbb{Z}$. Hence, we can directly look for solutions of the form

$$v_j(x, y) = \cos\left(\frac{\pi j}{\lambda} y\right) \tilde{v}_j(x),$$

where $j \in \mathbb{N}$ is fixed. Hence:

$$\frac{\partial^2 v_j}{\partial x^2}(x, y) = \cos\left(\frac{\pi j}{\lambda} y\right) \tilde{v}_j''(x),$$

$$\frac{\partial^2 v_j}{\partial y^2}(x, y) = -\left(\frac{\pi j}{\lambda}\right)^2 \cos\left(\frac{\pi j}{\lambda} y\right) \tilde{v}_j(x),$$

which means that we need to solve the problem:

$$-\tilde{v}_j''(x) - 2\operatorname{sech}^2(x)\tilde{v}_j(x) = -\left(\frac{\pi j}{\lambda}\right)^2 \tilde{v}_j(x), \quad \forall x \in \mathbb{R}, \forall j \in \mathbb{N}.$$

Observe that the last equation is a stationary Schrödinger equation, with a so-called *Pöschl-Teller potential* (introduced for the first time in [**40**]). We now solve this last equation, for fixed $j \in \mathbb{N}$. First, if $j = 0$ we have the equation

$$\tilde{v}_0''(x) + 2\operatorname{sech}^2(x)\tilde{v}_0(x) = 0,$$

which has general solution

$$\tilde{v}_0(x) = c_1 \tanh(x) + c_2\left(-\frac{1}{2}\tanh(x)\log\frac{1 - \tanh(x)}{1 + \tanh(x)} - 1\right).$$

Now, notice that the first summand is odd[5] and the second is even. Thus, $c_1 = 0$. Moreover, we also have $c_2 = 0$, as the second summand grows linearly at infinity (i.e., faster than the requirements). Hence, we can already exclude the possibility of having elements in $\ker L$ with $j = 0$.

Let then $j > 0$ and make the substitution $y = \tanh(x)$:

$$\left[\left(1 - y^2\right)\tilde{v}_j'(y)\right]' + 2\tilde{v}_j(y) - \frac{1}{1 - y^2}\left(\frac{\pi j}{\lambda}\right)^2 \tilde{v}_j(y) = 0.$$

We get then a Legendre equation with integer degree $l = 1$ and with order $\mu = \frac{2\pi j}{\lambda}$. A general solution is then given by a linear combination of first and second order Legendre functions:

$$\tilde{v}_j(x) = A\, P_1^{\frac{\pi j}{\lambda}}(\tanh(x)) + B\, Q_1^{\frac{\pi j}{\lambda}}(\tanh(x)).$$

Actually, not all the values of A and B are admissible, as we shall immediately see. The following expansions can be found, for instance, on [**37**].

---

[5]Indeed, it is the x derivative of the trivial solution u_0, which is even. Observe that the x derivative of a trivial solution is always a solution of the linearized equation. In fact, as u_0 is a solution of the original equation

$$\Delta u_0(x, y) + e^{u_0(x,y)} = 0,$$

taking the derivative in x of this expression one gets

$$\Delta \frac{\partial u_0}{\partial x}(x, y) + e^{u_0(x,y)}\frac{\partial u_0}{\partial x}(x, y) = 0,$$

which means that $\frac{\partial u_0}{\partial x}$ is a solution of the linearized problem in u_0.

Suppose first that $B = 0$ and consider thus $P_1^{\frac{\pi j}{\lambda}}$ only. It is known that

$$P_1^\mu(y) \underset{y \to 1^-}{\sim} \frac{1}{\Gamma(1 - \mu)} \left(\frac{2}{1 - y} \right)^{\frac{\mu}{2}}$$

for $\mu \notin \mathbb{N}$. Therefore, for such values of μ, $P_1^\mu(\tanh(x)) \sim_{x \to +\infty} Ce^{\mu x}$, meaning that such a solution cannot lead to functions in the space X_λ. Hence, we know that, if $B = 0$, μ must be an integer. We now recall that, if μ is an integer and $\mu > l$, then $P_l^\mu \equiv 0$. Consequently, if $B = 0$, the only non-trivial solution is the one with $\mu = 1$, namely $1 = \frac{\pi j}{\lambda}$. Explicitly:

$$\tilde{v}_j(x) = \begin{cases} A \operatorname{sech}(x) & \text{if } j = \frac{\lambda}{\pi} \\ 0 & \text{otherwise} \end{cases}$$

(notice that $\operatorname{sech}(x)$ is even).

Suppose now that $A = 0$ and consider $Q_1^{\frac{\pi j}{\lambda}}$ only. It is known that

$$Q_1^\mu(y) \underset{y \to 1^-}{\sim} \frac{1}{2} \cos(\mu\pi)\Gamma(\mu) \left(\frac{2}{1 - y} \right)^{\frac{\mu}{2}}$$

for $\mu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. As before, then, for such values of μ we have that $Q_1^\mu(\tanh(x)) \sim_{x \to +\infty} Ce^{\mu x}$. Hence, in order to have functions in X_λ, we must require that $\mu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. In this case the expansion at 1^- becomes

$$Q_1^\mu(y) \underset{y \to 1^-}{\sim} (-1)^{\mu + \frac{1}{2}} \frac{\pi \Gamma(\mu + 2)}{2\Gamma(\mu + 1)\Gamma(2 - \mu)} \left(\frac{1 - y}{2} \right)^{\frac{\mu}{2}}$$

if $1 \pm \mu = l \pm \mu \neq -1, -2, -3, \ldots$ (which is trivially true). Therefore, the behavior for $x \to +\infty$ is sufficiently good. Nonetheless,

$$Q_1^\mu(y) = -\cos((1 + \mu)\pi)Q_1^\mu(-y) - \frac{\pi}{2} \sin((1 + \mu)\pi)P_1^\mu(-y)$$

immediately shows that the function blows-up as $y \to -1^+$, i.e. as $x \to -\infty$. In this way we have excluded all the possible μ and we can therefore assess that, in order to have a non-trivial $v_j \in X_\lambda$, it must be $A \neq 0$.

We finally have to check that there are no combinations of $A, B \neq 0$ that lead to solutions in X_λ. Observe first that, according to what we said before

- $\mu \in \mathbb{N}$ implies that $A P_1^\mu(\tanh(x)) + B Q_1^\mu(\tanh(x))$ blows-up exponentially at both $+\infty$ and $-\infty$ (P is finite and Q blows-up as before);
- $\mu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ implies that $A P_1^\mu(\tanh(x)) + B Q_1^\mu(\tanh(x))$ blows-up exponentially at $-\infty$ (Q is finite and P blows-up as before);

so that we can choose from the beginning $\mu \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$. From the expansion for $y \to 1^-$ we know that:

$$A P_1^\mu(y) + B Q_1^\mu(y) \underset{y \to 1^-}{\sim} \left[\frac{A}{\Gamma(1 - \mu)} + \frac{B}{2} \cos(\mu\pi)\Gamma(\mu) \right] \left(\frac{2}{1 - y} \right)^{\frac{\mu}{2}},$$

so we need

$$A = -\frac{\Gamma(\mu)\Gamma(1 - \mu)}{2} \cos(\mu\pi)B.$$

We now turn to the expansions for $y \to -1^+$. We have that

$$Q_1^\mu(y) = -\cos((1+\pi)\pi)Q_1^\mu(-y) - \frac{\pi}{2}\sin((1+\mu)\pi)P_1^\mu(-y)$$

$$\underset{y \to -1^+}{\sim} -\frac{1}{2}\cos((1+\mu)\pi)\cos(\mu\pi)\Gamma(\mu)\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}} +$$

$$-\frac{\pi}{2}\sin((1+\mu)\pi)\frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}}$$

and

$$P_1^\mu(y) = -\frac{2}{\pi}\sin((1+\mu)\pi)Q_1^\mu(-y) + \cos((1+\mu)\pi)P_1^\mu(-y)$$

$$\underset{y \to -1^+}{\sim} -\frac{2}{\pi}\sin((1+\mu)\pi)\frac{1}{2}\cos(\mu\pi)\Gamma(\mu)\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}} +$$

$$+\cos((1+\mu)\pi)\frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}}.$$

We hence need to check whether it is possible to have (writing the full expansion for $AP_1^\mu + BQ_1^\mu$ and substituting the value of A we found before)

$$-\frac{\Gamma(\mu)\Gamma(1-\mu)}{2}\cos(\mu\pi)\left[-\frac{\Gamma(\mu)}{\pi}\sin((1+\mu)\pi)\cos(\mu\pi) + \frac{\cos((1+\mu)\pi)}{\Gamma(1-\mu)}\right] +$$

$$+\left[-\frac{\Gamma(\mu)}{2}\cos((1+\mu)\pi)\cos(\mu\pi) - \frac{\pi}{2}\frac{\sin((1+\mu)\pi)}{\Gamma(1-\mu)}\right] = 0$$

for some μ. This is the only case, indeed, for which the solution does not grow exponentially as $x \to -\infty$ ($y \to -1^+$). This equation in μ can be simplified to

$$-\Gamma(1-\mu)\Gamma(\mu)\cos(\pi\mu)\cot(\pi\mu)\left[\Gamma(\mu)\Gamma(1-\mu)\sin((1+\mu)\pi) + 2\pi\right] = \pi^2$$

and one can check that it does not exist a $\mu \in \mathbb{R}_{>0}$ that satisfies this last expression.

We can now go back to our linearized equation and to the family of solutions $(v_j)_j$ we were examining before. According to the discussion we made about the Pöschl-Teller potential, it is clear then that

$$\tilde{v}_j \not\equiv 0 \iff -\left(\frac{\pi j}{\lambda}\right)^2 = -1 \iff j = \frac{\lambda}{\pi}.$$

In such a case, the only element of the family which is nonzero is

$$\tilde{v}_{\frac{\lambda}{\pi}}(x) = A\operatorname{sech}(x)$$

and, accordingly,

$$v_{\frac{\lambda}{\pi}}(x,y) = A\operatorname{sech}(x)\cos(y).$$

The Fourier series in y, hence, has only one nonzero summand. Consequently, both the Fourier series and its series of second derivatives trivially converge.

Summing up, we found that the only values of λ for which we can expect to have a bifurcation are the points πj, with $j \in \mathbb{N}_{>0}$. For these values of λ, in particular, the operator L has one dimensional kernel:

$$\ker L = \{t \operatorname{sech}(x) \cos(y) \mid t \in \mathbb{R}\}.$$

1.2. The linearized operator is Fredholm. We want now to expand what we have just found in order to show that the linearized operator L is Fredholm of index 0. This is needed in order to apply the Simple Eigenvalue Bifurcation Theorem.

We have already shown that $L : X_\lambda \to Y_\lambda$ has a one dimensional kernel. Therefore, we just need to prove that it also has a one dimensional cokernel. To this end, observe first that $Y_\lambda \subset L^2(S_\lambda)$, so that we can make use of the L^2 product in Y_λ:

$$\operatorname{codim} L = \dim\{f \in Y_\lambda \mid \langle Lu, f\rangle_{L^2} = 0 \quad \forall u \in X_\lambda\}.$$

Indeed, if $[f] \in \operatorname{coker} L$, then $[f]$ has a representative which is perpendicular to $R(L)$ (just take its projection on $R(L)^\perp$), while if f is in the set on the right hand side then f cannot be parallel to any element of $R(L)$ (otherwise the scalar product with such an element would not be zero) and thus cannot be in $R(L)$, which is a vector subspace of Y_λ because L is linear.

Let then $f \in Y_\lambda$ be such that $\langle Lu, f\rangle_{L^2} = 0$ for each $u \in X_\lambda$. Observe that, if we knew that f is at least of class C^2, we would fall back in the previous case and we would get that the only solution is $f = v_0$. Indeed, we shall show that in such a case the equation $\langle Lu, f\rangle_{L^2} = 0, \quad \forall u \in X_\lambda$ is equivalent to $Lf = 0$ strongly.

Let us first show that f is sufficiently regular. It is immediate to check that all the coefficients are of class C^∞. Moreover, it is obvious that $Lu = 0 \in C^\infty$. Therefore, the hypothesis of Theorem 3.1 are verified and, hence, f is of class C^∞ in S_λ.

We now prove that L is self-adjoint on the elements f in Y_λ such that $\langle Lu, f\rangle_{L^2} = 0$ for each $u \in X_\lambda$. Take $u \in X_\lambda$ and $f \in Y_\lambda \cap C^2 \subset L^2$ such that $Lf = 0$ (actually we have just shown that we have more: $f \in Y_\lambda \cap C^\infty$). We want to see that then $\langle Lu, f\rangle_{L^2} = \langle u, Lf\rangle_{L^2}$. Indeed

(4.1)

$$\langle Lu, f\rangle_{L^2} = \int_{S_\lambda} (\Delta u + e^{u_0} u) f$$

$$= \int_{(0,\lambda)\times\mathbb{R}} \left(\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + 2\operatorname{sech}^2(x)u(x,y) \right) f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Of course, the last summand inside the integral is clearly self-adjoint. Thus, we can just look at the first two. Observe preliminarily that, according to the weighted Schauder estimates found in Chapter 3 (Theorem 3.4) applied to the equation $Lf = 0$, we know that

(4.2)
$$\left| \langle x \rangle^{\frac{3}{2}} f \right|_{2,\alpha,S_\lambda} \leq C' \left| \langle x \rangle^{\frac{3}{2}} f \right|_{0,\alpha,S_\lambda} \leq C$$

for some constant $C > 0$ (recall that $f \in Y_\lambda$). Hence, both the first order and the second order partial derivatives in x decay as $|x|^{-\frac{3}{2}}$ as $|x| \to \infty$.

Indeed

$$\frac{\partial}{\partial x}\left(\langle x\rangle^{\frac{3}{2}}f(x,y)\right) = \frac{3}{2}x\langle x\rangle^{-\frac{1}{2}}f(x,y) + \langle x\rangle^{\frac{3}{2}}\frac{\partial f}{\partial x}(x,y)$$

and consequently, making use of (4.2),

$$(4.3)\qquad \begin{aligned} \sup_{(x,y)\in S_\lambda}\left|\langle x\rangle^{\frac{3}{2}}\frac{\partial f}{\partial x}(x,y)\right| &\leq C + \sup_{(x,y)\in S_\lambda}\left|\frac{3}{2}x\langle x\rangle^{-\frac{1}{2}}f(x,y)\right| \\ &\leq C + \frac{3}{2}\sup_{(x,y)\in S_\lambda}\left|\langle x\rangle^{\frac{3}{2}}f(x,y)\right| \leq C_1 \end{aligned}$$

for some $C_1 > 0$ (which can be computed explicitly). Analogously, the second derivative turns out to be

$$\frac{\partial^2}{\partial x^2}\left(\langle x\rangle^{\frac{3}{2}}f(x,y)\right)$$
$$= \frac{3}{4}(x^2+2)\langle x\rangle^{-\frac{5}{2}}f(x,y) + 3x\langle x\rangle^{-\frac{1}{2}}\frac{\partial f}{\partial x}(x,y) + \langle x\rangle^{\frac{3}{2}}\frac{\partial^2 f}{\partial x^2}(x,y).$$

With estimates similar to the above and using (4.3) we then find

$$(4.4)\qquad \sup_{(x,y)\in S_\lambda}\left|\langle x\rangle^{\frac{3}{2}}\frac{\partial^2 f}{\partial x^2}(x,y)\right| \leq C_2$$

for some $C_2 > 0$ (which, again, can be computed explicitly).

A first consequence of (4.3) and (4.4), together with the Hölder inequality, is that in what follows we can always apply Fubini's Theorem. Let us look then at the first summand in (4.1).

$$\int_{S_\lambda}\frac{\partial^2 u}{\partial x^2}(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^\lambda\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(x,y)f(x,y)\right) - \frac{\partial u}{\partial x}(x,y)\frac{\partial f}{\partial x}(x,y)\right]\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^\lambda\left[\frac{\partial u}{\partial x}(x,y)f(x,y)\right]_{x=-\infty}^{+\infty}\mathrm{d}x - \int_{S_\lambda}\frac{\partial u}{\partial x}(x,y)\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y \ ,$$
$$= -\int_0^\lambda\left[u(x,y)\frac{\partial f}{\partial x}(x,y)\right]_{x=-\infty}^{+\infty}\mathrm{d}x + \int_{S_\lambda}u(x,y)\frac{\partial^2 f}{\partial x^2}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{S_\lambda}u(x,y)\frac{\partial^2 f}{\partial x^2}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

where, in order to pass from the 3-rd to the 4-th to the 5-th line, we used the fact that both $\frac{\partial u}{\partial x}f$ and $u\frac{\partial f}{\partial x}$ decay as $|x|^{-2}$ as $|x|\to\infty$ (the estimate for $\frac{\partial u}{\partial x}f$ is obtained in the same way as in (4.3)).

As for the second summand, the computations are similar but easier, as we can directly exploit the fact that we are imposing Neumann conditions on the boundary of S_λ (which, recall, is $\mathbb{R}\times(0,\lambda)$). Hence, again the boundary terms go away while integrating by parts and so, putting all together, we obtain that L is self-adjoint.

At this point, therefore, we can repeat the computations of the previous section and find that there exists only one family of solution of $\langle Lu, f\rangle = 0, \forall u \in X_\lambda$ that satisfy the Neumann condition on ∂S_λ and belong to Y_λ,

namely $\langle u^* \rangle \subset X_\lambda \cap Y_\lambda$. Consequently, $\operatorname{codim} L = 1$ and therefore L is Freholm of index 0, as wanted.

2. Bifurcation

This section is devoted to showing that all the candidate points found in Subsection 1.1 are actually real bifurcation points. In order to do that we will need Theorem 2.13, so now we finally have to perform the change of variables that removes the parameter from the domain. Consider then the following map:

$$R_\lambda : \quad \mathcal{F}_\lambda \longrightarrow \mathcal{F}_1$$
$$u(x, y) \longmapsto u(x, \lambda y),$$

where $\mathcal{F}_\lambda$ is the set of all functions defined on S_λ. Clearly, R_λ is a linear and bijective map such that $R_\lambda(X_\lambda) = X_1$ and $R_\lambda(Y_\lambda) = Y_1$. Observe moreover that the restrictions of R_λ to X_λ and Y_λ are bounded maps.

We need to find an operator $F^{(\lambda)} : X_1 \longrightarrow Y_1$ such that the following diagram commutes

$$
\begin{array}{ccc}
X_\lambda & \xrightarrow{\tilde{F}} & Y_\lambda \\
{\scriptstyle R_\lambda} \downarrow & & \downarrow {\scriptstyle R_\lambda} \\
X_1 & \xrightarrow{F(\lambda, \cdot)} & Y_1
\end{array},
$$

i.e. $R_\lambda \circ \tilde{F} = F(\lambda, \cdot) \circ R_\lambda$. It can be easily checked that

$$F(\lambda, u) = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\lambda^2} \frac{\partial^2 u}{\partial y^2} + \mathrm{e}^{u_0} \left(\mathrm{e}^u - 1 \right),$$

does the job (where we recall that $u_0(x, y) = \log(2 \operatorname{sech}^2(x))$). Indeed

$$\left(R_\lambda \circ \tilde{F}(u) \right)(x, y) = R_\lambda \left(\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) + \mathrm{e}^{u_0(x, y)}(\mathrm{e}^{u(x, y)} - 1) \right)$$
$$= \left(\frac{\partial^2 u}{\partial x^2} \right)(x, \lambda y) + \left(\frac{\partial^2 u}{\partial y^2} \right)(x, \lambda y) + \mathrm{e}^{u_0(x, \lambda y)}(\mathrm{e}^{u(x, \lambda y)} - 1)$$

and

$$\left(F(\lambda, u) \circ R_\lambda \right)(x, y) = F(\lambda, u(x, \lambda y))$$

$$= \frac{\partial^2}{\partial x^2}(u(x, \lambda y)) + \frac{1}{\lambda^2} \frac{\partial^2}{\partial y^2}(u(x, \lambda y)) + \mathrm{e}^{u_0(x, \lambda y)}(\mathrm{e}^{u(x, \lambda y)} - 1)$$

$$= \left(\frac{\partial^2 u}{\partial x^2} \right)(x, \lambda y) + \frac{1}{\lambda^2} \left(\frac{\partial^2 u}{\partial y^2} \right)(x, \lambda y)\lambda^2 + \mathrm{e}^{u_0(x, \lambda y)}(\mathrm{e}^{u(x, \lambda y)} - 1).$$

Take then $\lambda^* = \pi j$, with $j \in \mathbb{N}_{>0}$. First,

$$F_u(\lambda^*, 0)[v] = \frac{\partial^2 v}{\partial x^2} + \frac{1}{(\lambda^*)^2} \frac{\partial^2 v}{\partial y^2} + \mathrm{e}^{u_0} v$$

is Fredholm. Indeed

$$F_u(\lambda^*, 0) = \mathrm{d}_u \left(\underbrace{R_{\lambda^*}}_{\text{lin. \& cont.}} \circ \tilde{F} \circ \underbrace{R_{(\lambda^*)^{-1}}}_{\text{lin. \& cont.}} \right) = R_{\lambda^*} \circ \mathrm{d}_u \tilde{F}(\lambda^*, 0) \circ R_{(\lambda^*)^{-1}}$$
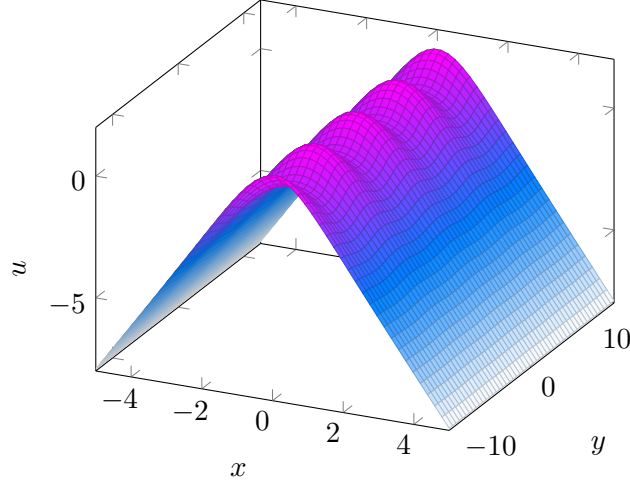
FIGURE 4.2. A qualitative representation of the perturbed solution (first order perturbation, $\lambda^* = 2\pi$).

and we have already shown that $\mathrm{d}_u\tilde{F}(\lambda^*, 0)$ is Fredholm (that is sufficient because R_{λ^*} and $R_{(\lambda^*)^{-1}}$ are linear, bijective and continuous). Secondly,

$$M[v] := F_{u,\lambda}(\lambda^*, 0)[1, v] = -\frac{1}{(\lambda^*)^3}\frac{\partial^2 v}{\partial y^2}$$

and, as we know that

$$v_0(x, y) = \operatorname{sech}(x)\cos(y) \in \ker L,$$

we have that

$$\ker F_u(\lambda^*, 0) = \langle u^* \rangle$$

with

$$u^*(x, y) := R_{\lambda^*}(v_0)(x, y) = \operatorname{sech}(x)\cos(\lambda^* y) \in X_1.$$

Hence

$$M[u^*] = \frac{2}{\lambda^*}u^*(x, y) \in \langle u^* \rangle,$$

so that $M[u^*] \notin R$.

According to Theorem 2.13, then, $\lambda^* = \pi j$ for $j \in \mathbb{N}_{>0}$ are bifurcation points for F and, consequently, also for $\tilde{F}$ (which is nothing less than F written using different coordinates).

2.1. Shape of bifurcation. Now that we know what are the points of bifurcation, we might wonder whether the bifurcations we are encountering are transcritical, subcritical or supercritical. As F is a C^∞ operator, we can employ the formulas found in 3.3 of Chapter 2. In our case equation (2.11) becomes

(4.5)
$$\frac{\partial^2 u_2}{\partial x^2}(x, y) + \frac{1}{\pi^2}\frac{\partial^2 u_2}{\partial y^2}(x, y) + 2\operatorname{sech}^2(x)u_2(x, y)$$
$$= -\operatorname{sech}^4(x)\cos^2(\pi y) = -\frac{1}{2}\operatorname{sech}^4(x)(1 + \cos(2\pi y)).$$

We just need to find a particular solution which is not in the kernel of L. Therefore, we look for solutions of the form

$$u_2(x, y) = v_1(x) + v_2(x) \cos(2\pi y).$$

Hence, (4.5) becomes

$$v_1''(x) + v_2''(x) \cos(2\pi y) - \frac{1}{\pi^2} v_2(x) 4\pi^2 \cos(2\pi y) + 2 \operatorname{sech}^2(x) v_1(x)$$

$$+ 2 \operatorname{sech}^2(x) \cos(2\pi y) v_2(x) = -\frac{1}{2} \operatorname{sech}^4(x)(1 + \cos(2\pi y)),$$

which in turn is

$$\begin{cases} v_1''(x) + 2 \operatorname{sech}^2(x) v_1(x) = -\frac{1}{2} \operatorname{sech}^4(x) \\ v_2''(x) + (2 \operatorname{sech}^2(x) - 4) v_2(x) = -\frac{1}{2} \operatorname{sech}^4(x) \end{cases}.$$

One can check that a particular solution is given by

$$\begin{cases} v_1(x) = -\frac{1}{8}(1 + \tanh^2(x)) \\ v_2(x) = \frac{1}{8}[\sinh^2(x)(\tanh^2(x) - 2) + \cosh^2(x)] \end{cases},$$

namely

$$u_2(x, y) = -\frac{1}{8}(1 + \tanh^2(x)) + \frac{1}{8}[\sinh^2(x)(\tanh^2(x) - 2) + \cosh^2(x)] \cos(2\pi y).$$

Now recall that in our case ψ is the linear operator given by the L^2 product with u^*. Thus we need to compute

$$\langle u^*, F_{uu}(\lambda^*, 0)[u^*, u_2] \rangle_{L^2}$$

$$= - \int_{S_\lambda} \frac{\operatorname{sech}^4(x) \cos^2(\pi y)}{4} (1 + \tanh^2(x)) \, \mathrm{d}x \, \mathrm{d}y$$

$$+ \int_{S_\lambda} \frac{\operatorname{sech}^4(x) \cos^2(\pi y)}{4} (\sinh^2(x)(\tanh^2(x) - 2) + \cosh^2(x)) \cos(2\pi y) \, \mathrm{d}x \, \mathrm{d}y$$

$$= -\frac{1}{3} + \frac{1}{15} = -\frac{4}{15}.$$

Moreover, we also need

$$\langle u^*, F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] \rangle_{L^2} = \frac{2}{15}$$

and

$$\langle u^*, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle_{L^2} = \frac{2}{\pi}.$$

Putting these three values together as in (2.12) we get then

$$\lambda_2 = \frac{\pi}{15} > 0,$$

which amounts to a *supercritical bifurcation*, i.e. on the right (see Figure 4.3).
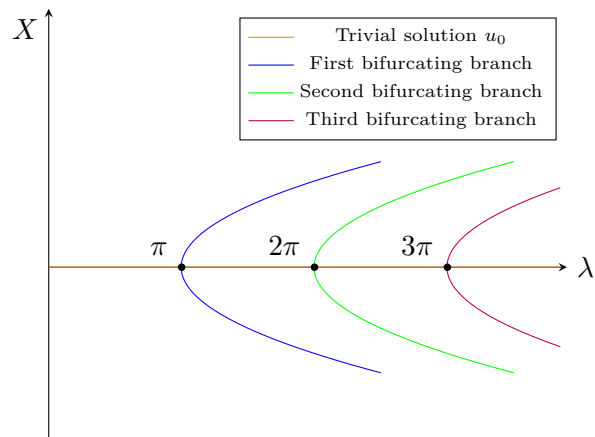
FIGURE 4.3. A qualitative representation of the bifurcations of the Liouville equation.

Perspectives

As we saw, we found non-trivial solutions of the Lioville equation in $\mathbb{R}^2$ with infinite volume using results of bifurcation theory that, in the end, are not much more than an application of the Implicit Function Theorem. This is a consequence of the fact that all the differential equations we meet while studying the 2D problem can be solved explicitly. Unfortunately, though, that is not the case with the four-dimensional Liouville equation

$$(5.1) \qquad \Delta^2 u(x) = \mathrm{e}^{u(x)}, \quad \forall x \in \mathbb{R}^4.$$

As in the 2D case, the idea is again to start from a solution of the same equation in some lower dimension, extend it on a 4D strip adding enough "dummy variables" and finding the bifurcations with varying dimensions of the strip. The first step then is to find a trivial solution in lower dimension, or at least prove its existence. Specifically, inspired by the trivial solution we had in dimension 2, we can look for radial solutions with finite volume and some sort of decay at infinity. Now, different paths lay in front of us: we could choose to look for trivial solutions in dimension 1 and use 3 parameters, or trivial solutions in dimension 2 and use 2 parameters, or 3-dimensional trivial solutions and 1 parameter. We observe, though, that the first choice leads nowhere. Indeed, we would be looking for solutions of the ODE

$$(5.2) \qquad u^{(4)}(x) = \mathrm{e}^{u(x)}, \quad \forall x \in \mathbb{R}.$$

Integrating (5.2) one immediately finds

$$u^{(3)}(y) - u^{(3)}(x) = \int_x^y \mathrm{e}^{u(s)}\,\mathrm{d}s > 0, \quad \forall x, y \in \mathbb{R}, x \neq y,$$

which means that $\lim_{x \to -\infty} u^{(3)}(x) < \lim_{x \to +\infty} u^{(3)}(x)$. Notice that, as we are requiring that u goes to $-\infty$ both at $-\infty$ and $+\infty$, we should have $\lim_{x \to -\infty} u^{(3)}(x) \geq 0$ and $\lim_{x \to +\infty} u^{(3)}(x) \leq 0$, which is a contradiction.

For simplicity, we might for example choose to follow the last path (3D trivial solution plus 1 parameter). In particular, we can look for radial solutions in dimension 3 with linear decay at infinity. In polar coordinates, indeed, equation (5.1) becomes

$$u^{(4)}(r) - \frac{4}{r} u^{(3)}(r) = \mathrm{e}^{u(x)}$$

(remember that we are looking only for radial solutions) and therefore solutions with a linear decay are a priori acceptable.

Roughly speaking, the steps one could try to follow are:

(1) Show the existence of a radial solution u_0 for (5.1) in $\mathbb{R}^3$ with linear decay at infinity. This will be the trivial solution.

(2) Notice that equation (5.1) is a variational problem, being the Euler-Lagrange equation for the following functional

$$I(u) := \int_{\mathbb{R}^4} \left[\frac{1}{2} (\Delta u(x))^2 - e^{u(x)} \right] dx.$$

(3) Restrict the problem to the strip $S_\lambda = \mathbb{R}^3 \times (0, \lambda)$. It is important now to choose the right space of functions. In particular, we must require
   - enough regularity to have well defined differential operators and nice regularity results (we might try something like some weighted interior Hölder space $C_w^{4,\alpha}(S_\lambda)$, in analogy to what we did in dimension 2);
   - Neumann conditions on the boundary of S_λ, in order to be able to "glue" the solution on each strip to form a solution on the whole $\mathbb{R}^4$;
   - radial symmetry in the first three variables with a sufficiently slow growth at infinity, so that we might hope to have simple eigenvalues, vanishing border terms while integrating by parts and some nice scalar product (the ideal would be the L^2 scalar product).

(4) Construct a function on S_λ on which the bilinear form

$$d^2 I_\lambda(u_0)[v_1, v_2] = \int_{S_\lambda} \left[\Delta v_1(x) \Delta v_2(x) - e^{u_0(x)} v_1(x) v_2(x) \right] dx,$$

is negatively defined and observe that

$$d^2 I_\lambda(u_0)[v_1, v_1] = \langle L[v_1], v_1 \rangle_{L^2},$$

where $L[v] = \Delta^2 v - e^{u_0} v$ is the linearized operator. Then, if we are able to prove that the linearized operator L is semibounded and self-adjoint, we can use the Rayleigh Min-Max Theorem (see for example [**24**, Theorem 11.4]) to see that the first eigenvalue ν_0 of L is negative.

(5) Show that if v_0 is an eigenfunction of the first eigenvector ν_0, then it does not depend on the fourth variable $x_4 \in (0, \lambda)$. Then show that the first eigenspace is one-dimensional.

(6) Consider that the family of functions

$$v_k(x_1, x_2, x_3, x_4) := v_0(x_1, x_2, x_3) \cos \left(\frac{\pi k}{\lambda} x_4 \right).$$

and show that if $d^2 I_\lambda(u_0)$ is negatively defined on both v_{k_1} and v_{k_2}, then for each $\alpha, \beta \in \mathbb{R}$ it is negatively defined on $\alpha v_{k_1} + \beta v_{k_2}$, i.e. v_{k_1} and v_{k_2} are independent generators of the negative space of $d^2 I_\lambda(u_0)$ (even if they are not eigenfunctions).

(7) Show that there are values of λ for which the number of functions of the form v_k that belong to the negative space of $d^2 I_\lambda(u_0)$ increase by one, namely: there are values of λ for which the Morse index M of I_λ increases by one. Consequently, the index of the original equation, which is $(-1)^M$, changes for such λ's (for more details on Morse index, see for example [**36**]).

(8) Show that we can put the original Liouville equation (5.1) in the form $A(\lambda, u) = 0$, with A having some kind of compactness property. Like in the $\mathbb{R}^2$ case, we might try to perform a change of variables to transform the strip S_λ into S_1. Once more, it is essential to choose the right function spaces.

(9) Adapt the Krasnosel'skii Index Bifurcation Theorem to our scenario, recalling that we are dealing with a variational problem and that we can work with the Morse index of the functional instead of the index of the differential operator.

THEOREM 5.1. [**31**, Theorem 56.2] *Let A be a completely continuous operator and assume that λ^* is a point of changing index for the operator A. Then λ^* is a bifurcation point for equation $u = A(\lambda, u)$.*

Conclude that the values of λ found before are indeed bifurcation points. In this way, we should be able then to show the existence of non-trivial solutions for the four-dimensional Liouville equation with infinite volume.

As one can see, the idea is quite straightforward, but there are a lot of technical details that must be worked out. Up to now, thanks to some precious hints from Ali Hyder, item (1) is almost completed (the idea is to look for solutions that can be written using Green's formula through a Schaefer's Fixed Point argument).

Item (2) is obvious, while (4) and (6) are quite easy computations and are done, provided that item (3), which is likely the most important and the trickiest one, is worked out.

As for item (5), the independence on x_4 might be obtained by rearranging $v(x_1, x_2, x_3, x_4)$ and using something similar to the Pólya-Szegö inequality (see [**29**]). The fact that the eigenspace is one-dimensional might then follow from (3) with something like a shooting method: As the linearized equation in radial coordinates is a fourth order ODE, we should have a space of solutions of dimension 4. We could then reduce the space of solutions to dimension 2 by requiring regularity in the origin (i.e., the first and third derivatives of the solution in 0 must vanish). Finally, we can hope to get only two kind of solutions, one of which grows to fast to be in our function space and another that respects our requests.

Item (7) should then follow from (5) and (6): when λ increases, there are more values of k for which the harmonics v_k make the bilinear form $\mathrm{d}^2 I_\lambda(u_0)$ negative (the oscillating part, roughly speaking, adds a term that goes like $\frac{\pi^2 k^2}{\lambda^2} > 0$ to the originally negative value of $\mathrm{d}^2 I_\lambda(u_0)[v_0, v_0]$: if λ is bigger, there are more values of k for which what we obtain is still negative). The last two hurdles are (8), which again should come from a wise choice of the space of functions, and adapting the Krasnosel'skii Index Theorem in (9).

For more details on this, see Appendix B.

Of course, one could then try to follow a similar argument also for the case of a two-dimensional trivial equation and a two-parameters strip. Observe, in any case, that the non-trivial solutions with infinite volume we found and we can find through bifurcation theory are likely far from being

the most general solutions with infinite volume we can aim for. Indeed, the procedure we employed to find them imposes a very strong constriction on the shape that these solutions can have, namely: perturbations of some trivial solution not depending on at least one variable (which is a very rigid requirement). Hence, apart from going into higher dimensions, other lines of research aimed at finding more general non-trivial solutions with infinite volume are possible. As mentioned in the introduction, for example, we might be able to "glue" the oscillating solutions obtained from the bifurcation into more complex solutions, in a similar manner to the one used to construct Delaunay k-noids starting from Delaunay unduloids and nodoids. In this way we could then obtain non-trivial solutions with infinite volume that are not a direct result of a bifurcation from a cylindrical solution.

Another aspect that might be worth investigating, even if sligtly apart from the main line of this work but still with the aim of finding more general solutions, could also be to see if it is possible to use the characterization of the problem in $\mathbb{R}^2$ through meromorphic functions to get some non-trivial solution in $\mathbb{R}^2$ without resorting to bifurcation theory.

Finally, following the ideas presented in [**14**], one might try to use some global bifurcation results to see if, following the bifurcating branches, one can retrieve a spherical solution of the Liouville equation.

Riemannian manifolds and curvature

The aim of this appendix is to recall some basic definitions in Riemannian geometry and to fix some notation. It is assumed that the reader is familiar with the basics of differential geometry (i.e., knows what are smooth manifolds, tangent and cotangent bundles, tensors and k-forms – see for example [**45**]). In what follows, we will denote with $\Gamma(TM)$ the vector space of vector fields on a smooth manifold M.

DEFINITION A.1. Let M be a smooth manifold of dimension n. A *Riemannian metric* on M is a smooth and positive definite section of the bundle $S^2(T^*M)$ of the symmetric bilinear 2-forms on M. A manifold M endowed with a metric g is called *Riemannian manifold* and is indicated as (M, g).

In local coordinates around a point $p \in M$, given two vectors

$$u = \sum_{i=1}^{n} u^i \left. \frac{\partial}{\partial x^i} \right|_p \quad \text{and} \quad v = \sum_{i=1}^{n} v^i \left. \frac{\partial}{\partial x^i} \right|_p,$$

one has

$$g(u, v) = \sum_{i,j=1}^{n} g_{ij}(p) u^i v^j,$$

with

$$g_{ij}(p) := g \left(\left. \frac{\partial}{\partial x^i} \right|_p, \left. \frac{\partial}{\partial x^j} \right|_p \right).$$

Hence, locally we can write

$$g = \sum_{i,j=1}^{n} g_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j =: \sum_{i,j=1}^{n} g_{ij} \, \mathrm{d}x^i \, \mathrm{d}x^j.$$

REMARK. Given any smooth manifold M there always exists at least one Riemannian metric (see for example [**45**, Theorem 2.2]).

DEFINITION A.2. Given a smooth manifold M, a *connection* on the tangent bundle TM is an $\mathbb{R}$-bilinear map D from $\Gamma(TM) \times \Gamma(TM)$ to $\Gamma(TM)$ such that, for any $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$, one has

$$D_{fX}Y = f D_X Y$$

and

$$D_X(fY) = (Xf)Y + f D_X Y.$$

A connection is *torsion-free* if

$$D_X Y - D_Y X = [X, Y]$$

for any $X, Y \in \Gamma(TM)$.

THEOREM A.1. [**45**, Theorem 2.51] *Given any Riemannian manifold* (M, g), *there exists a unique torsion-free connection* consistent with the metric, *i.e. such that*

$$Xg(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z)$$

for each $X, Y, Z \in \Gamma(TM)$. *Such connection is called* Levi-Civita connection *(or* canonical connection*) of the metric g and is usually denoted with* ∇_g.

The notion of connection can be expanded further, as the following result shows ([**45**, Proposition 2.58]).

PROPOSITION A.2. *Let X be a vector field on a smooth manifold M. The endomorphism D_X of $\Gamma(TM)$ has a unique extension as an endomorphism of the space of tensors, still denoted by D_X, which is type-preserving and satisfies the following conditions*

    (i) *for any tensor $S \in \Gamma(T_k^h M)$ (with $h, k \in \mathbb{N}$) and any contraction c on $T_k^h M$, then $D_X(c(S)) = c(D_X S)$,*

    (ii) $D_X(S \otimes T) = (D_X S) \otimes T + S \otimes (D_X T)$ *for any tensors S and T.*

DEFINITION A.3. Let D be a connection on a smooth manifold M. Given $X, Y \in \Gamma(TM)$ and $h, k \in \mathbb{N}$, the *curvature endomorphism* $R_{XY} : \Gamma(T_k^h M) \to \Gamma(T_k^h M)$ of the connection is defined as

$$R_{XY} = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

The *curvature tensor* of D is the tensor field $\mathcal{R} \in \Gamma(T_3^1 M)$ defined as

$$\mathcal{R}(X, Y, Z) := R_{XY} Z$$

for any $X, Y, Z \in \Gamma(TM)$. If D is the Levi-Civita connection ∇_g we will say that $\mathcal{R}$ is the *curvature tensor of the manifold* and we will also consider the tensor field $\hat{R} \in \Gamma(T_4^0 M)$ given by

$$\hat{R}(X, Y, Z, W) := g(R_{XY} Z, W)$$

for any $X, Y, Z, W \in \Gamma(TM)$. The *Ricci curvature* $\mathrm{Ric} \in \Gamma(T_2^0 M)$ is defined saying that $\mathrm{Ric}_g(X, Y)$ is the trace of the linear operator $Z \mapsto R_{ZX} Y$ (in local coordinates, $\mathrm{Ric}_g = R_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$ with $R_{ij} = g^{hk} R_{hikj}$). Finally, the *scalar curvature* R_g is the trace of the Ricci curvature (in local coordinates, $R_g = g^{ij} R_{ij}$).

REMARK. In dimension 2 the scalar curvature R is linked to the Gaussian curvature K by the simple relation $R = 2K$.

A path toward non-trivial solutions in dimension 4

In this appendix we present what we have done so far to tackle the problem of finding non-trivial solutions to the Liouville equation in $\mathbb{R}^4$

(B.1) $$\Delta^2 u(x) = \mathrm{e}^{u(x)}, \quad \forall x \in \mathbb{R}^4$$

using bifurcation theory. The idea is to start from a "trivial" solution with finite volume and prescribed asymptotic behavior in $\mathbb{R}^3$, extend it to $S_\lambda := \mathbb{R}^3 \times (0, \lambda)$ and use such a λ as a parameter for the bifurcation (similarly to what we did in $\mathbb{R}^2$). As one can see, what follows has several missing steps and should not be regarded as completed and totally rigorous.

1. Trivial solution

Our first goal is to show that there exists at least one solution of

(B.2) $$\begin{cases} \Delta^2 u = \mathrm{e}^u \text{ in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} \mathrm{e}^{u(x)} \, \mathrm{d}x < +\infty \end{cases}.$$

The proof will be done in two steps and will look in particular for solutions of the integral form of (B.2), namely solutions of the form

$$u(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{u(y)} \, \mathrm{d}y.$$

Observe that a u with finite volume satisfying this last expression is a solution of (B.2). Indeed, a fundamental solution to Δ^2 is $G(x) = -\frac{1}{8\pi}|x|$ (see [**13**]). I am much in debt with Ali Hyder for the big suggestions he gave me for this part.

LEMMA B.1. *Let*
$$X := \left\{ u \in C^0(\mathbb{R}^3) \,\middle|\, u \text{ is radially symmetric and } \|u\| < +\infty \right\},$$

where $\|u\| := \sup_{x \in \mathbb{R}^3} \frac{|u(x)|}{1+|x|}$. *Then for every* $\varepsilon > 0$ *there exist* $u_\varepsilon \in X$ *such that*

(B.3) $$u_\varepsilon(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, \mathrm{d}y.$$

PROOF. First of all, observe that X, endowed with the norm $\|\cdot\|$, is a well-defined Banach space. Define then

$$T_\varepsilon : X \longrightarrow X$$

$$u \longmapsto \bar{u}_\varepsilon, \quad \bar{u}_\varepsilon(x) := -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)} \, \mathrm{d}y.$$

T_ε is well defined. Take in fact $u \in X$, then $\bar{u}_\varepsilon \in C^0(\mathbb{R}^3)$ thanks to the Lebesgue Dominated Convergence Theorem. Moreover, $\bar{u}_\varepsilon$ is clearly radial

because of the radial invariance of the Lebesgue integral: indeed, if $A \in SO(3)$, then

$$\bar{u}_\varepsilon(Ax) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |Ax - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)} \, \mathrm{d}y$$

$$= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |A(x - y)| \, \mathrm{e}^{-\varepsilon|Ay|^2} \mathrm{e}^{u(Ay)} |\det A| \, \mathrm{d}y$$

$$= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)} \, \mathrm{d}y = \bar{u}_\varepsilon(x).$$

Finally,

$$|\bar{u}_\varepsilon(x)| = \frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(x)} \, \mathrm{d}y$$

$$\leq \frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{\|u\|(1+|y|)} \, \mathrm{d}y$$

$$\leq \frac{1}{8\pi} \int_{\mathbb{R}^3} |y| \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{\|u\|(1+|y|)} \, \mathrm{d}y + |x| \frac{1}{8\pi} \int_{\mathbb{R}^3} \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{\|u\|(1+|y|)} \, \mathrm{d}y$$

$$\leq C_1 + C_2|x| \leq \bar{C}(1 + |x|),$$

so that $\|\bar{u}_\varepsilon\| < +\infty$. Hence $T_\varepsilon(u) = \bar{u}_\varepsilon \in X$.

We now show that T_ε is compact. Take a bounded sequence $\{u_n\}_n \subset X$, $\|u_n\| \leq C < +\infty$ for all $n \in \mathbb{N}$. We want to show that then $\{T_\varepsilon(u_n)\}_n$ admits a converging subsequence. The idea is to use Arzelà-Ascoli's Theorem on the sequence $\left\{ \frac{T_\varepsilon(u_n)}{1+|x|} \right\}_n$. First,

$$\frac{|T_\varepsilon(u_n)|}{1 + |x|} = \frac{1}{8\pi(1 + |x|)} \int_{\mathbb{R}^3} |x - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u_n(y)} \, \mathrm{d}y$$

$$\leq \frac{1}{8\pi(1 + |x|)} \int_{\mathbb{R}^3} |x - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{C(1+|x|)} \, \mathrm{d}y$$

$$\leq \frac{C_1 + C_2|x|}{8\pi(1 + |x|)} \leq \bar{C} < +\infty,$$

for any $x \in \mathbb{R}^3$ and $n \in \mathbb{N}$, so that $\left\{ \frac{T_\varepsilon(u_n)}{1+|x|} \right\}_n$ is equibounded. Moreover,

$$\left| \frac{T_\varepsilon(u_n(x))}{1 + |x|} - \frac{T_\varepsilon(u_n(y))}{1 + |y|} \right| = \frac{1}{8\pi} \left| \int_{\mathbb{R}^3} \left(\frac{|x - z|}{1 + |x|} - \frac{|y - z|}{1 + |y|} \right) \mathrm{e}^{-\varepsilon|z|^2} \mathrm{e}^{u_n(z)} \, \mathrm{d}z \right|$$

$$\leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \left| \frac{|x - z|}{1 + |x|} - \frac{|y - z|}{1 + |y|} \right| \mathrm{e}^{-\varepsilon|z|^2} \mathrm{e}^{u_n(z)} \, \mathrm{d}z$$

$$\leq \left(\frac{1}{8\pi} \int_{\mathbb{R}^3} (2 + |z|) \mathrm{e}^{-\varepsilon|z|^2} \mathrm{e}^{C(1+|z|)} \, \mathrm{d}z \right) |x - y|$$

for any $x, y \in \mathbb{R}^3$ and $n \in \mathbb{N}$, so that $\left\{ \frac{T_\varepsilon(u_n(x))}{1+|x|} \right\}_n$ is uniformly equibounded. The last inequality, in particular follows from the triangular inequality. As one clearly has

$$|y| \leq |x| + |y - x|,$$
$$|x - y| \leq |x - y| + |y - z|,$$
$$|z| \leq |x| + |z - x|,$$
$$|x - z| \leq |x| + |z|,$$

we obtain indeed

$$
\begin{aligned}
\left| \frac{|x-z|}{1+|x|} - \frac{|y-z|}{1+|y|} \right| &= \left| \frac{|x-z| - |y-z| + |y||x-z| - |x||y-z|}{(1+|x|)(1+|y|)} \right| \\
&\leq \frac{|x-y| + (|x|+|y-x|)|x-y| - |x||y-z|}{(1+|x|)(1+|y|)} \\
&\leq \frac{|x-y| + |x||x-y| + |x-z||x-y|}{(1+|x|)(1+|y|)} \\
&\leq \left(\frac{1+2|x|}{(1+|x|)(1+|y|)} + \frac{|z|}{(1+|x|)(1+|y|)} \right) |x-y| \\
&\leq \left(\frac{2}{(1+|y|)} + \frac{|z|}{(1+|x|)(1+|y|)} \right) |x-y| \\
&\leq (2+|z|)|x-y|.
\end{aligned}
$$

By Arzelà-Ascoli's Theorem, then, $\left\{ \frac{T_\varepsilon(u_n(x))}{1+|x|} \right\}_n$ admits a subsequence which converges uniformly. Thus, $\{T_\varepsilon(u_n)\}_n$ admits a converging subsequence in $(X, \|\cdot\|)$ and therefore T is a compact operator.

Next, we prove that T has a fixed point using Schaefer's Fixed Point Theorem (see for example [**49**]). Let $u \in X$ satisfy $u = tT_\varepsilon(u)$ for some $0 \leq t \leq 1$, then

$$
u(x) = -\frac{t}{8\pi} \int_{\mathbb{R}^3} |x-y| e^{-\varepsilon|y|^2} e^{u(y)} \, dy \leq 0.
$$

Consequently

$$
|u(x)| \leq \frac{t}{8\pi} \int_{\mathbb{R}^3} |x-y| e^{-\varepsilon|y|^2} \, dy \leq C(1+|x|)
$$

and therefore $\|u\| \leq C$. That means that the set $\{u \in X \mid u = tT_\varepsilon(u), 0 \leq t \leq 1\}$ is bounded in $(X, \|\cdot\|)$: by Schaefer's Theorem then T_ε has a fixed point in X. □

THEOREM B.2. u_ε converges to some u in $(X, \|\cdot\|)$ as ε goes to 0, with u satisfying

$$
u(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y| e^{u(y)} \, dy
$$

(hence being a solution to (B.2) with the desired properties).

PROOF. First of all, we check that u_ε is monotone decreasing for all $\varepsilon > 0$. Indeed, write the integral in u_ε in polar coordinates (with a slight abuse of notation)

$$
\begin{aligned}
u_\varepsilon(r) &= -\frac{1}{8\pi} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{s=0}^{+\infty} \sqrt{r^2 - 2rs\cos\theta + s^2} e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s^2 \sin\theta \, ds \, d\theta \, d\varphi \\
&= -\frac{1}{12r} \int_0^{+\infty} (r^2 - 2rs\cos\theta + s^2)^{\frac{3}{2}} \Big|_{\theta=0}^{\theta=\pi} e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s \, ds \\
&= -\frac{1}{12r} \int_0^{+\infty} \left[(r+s)^3 - |r-s|^3 \right] e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s \, ds \\
&= -\frac{1}{6r} \int_0^r s^2(3r^2 + s^2) e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds - \frac{1}{6} \int_r^{+\infty} s(r^2 + 3s^2) e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds.
\end{aligned}
$$

In the previous computation we set $y = (s\sin\theta\cos\varphi, s\sin\theta\sin\varphi, s\cos\theta)$ and we chose $x = (r, 0, 0)$ (recall that we have already checked the radial invariance). Now take a derivative in r:

$$
\begin{aligned}
u'_\varepsilon(r) = {} & \frac{1}{6r^2}\int_0^r s^2(3r^2 + s^2)\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \frac{2}{3}r^3\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} \\
& - \int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s + \frac{2}{3}r^3\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} \\
& - \frac{1}{3}\int_r^{+\infty} rs\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s \\
= {} & \int_0^r \underbrace{\frac{s^2 - 3r^2}{6r^2}}_{<0} s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \frac{r}{3}\int_r^{+\infty} s\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)} < 0.
\end{aligned}
$$

Hence u_ε is monotone decreasing for all $\varepsilon > 0$.

Applying a Pohozaev-like identity to (B.3) one gets

$$
\int_{\mathbb{R}^3}\left(u_\varepsilon(x) + 6 - 4\varepsilon|x|^2\right)\mathrm{e}^{-\varepsilon|x|^2}\mathrm{e}^{u_\varepsilon(x)}\,\mathrm{d}x = 0.
$$

Since u_ε is monotone decreasing and continuous, we must have $u_\varepsilon(0) > -6$ (otherwise the previous integral would be strictly negative). Hence $-6 < u_\varepsilon(0) < 0$: applying that to (B.3) we get

$$
\left|\int_{\mathbb{R}^3}|y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y\right| \le 6
$$

and thus

(B.4) $$\left|\Delta u_\varepsilon(0)\right| = \frac{1}{4\pi}\left|\int_{\mathbb{R}^3}\frac{1}{|y|}\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y\right| \le C < +\infty.$$

By Green's formula indeed

$$
\Delta u_\varepsilon(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}.
$$

We now check that Δu_ε is monotone increasing for each $\varepsilon > 0$. Indeed, using again polar coordinates as before,

$$
\begin{aligned}
(\Delta u_\varepsilon)(r) = {} & -\frac{1}{4\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{s=0}^{+\infty}\frac{\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}s^2}{\sqrt{r^2 - 2rs\cos\theta + s^2}}\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
= {} & -\frac{1}{2r}\int_0^{+\infty}\sqrt{r^2 - 2rs\cos\theta + s^2}\Big|_{\theta=0}^{\theta=\pi} s\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s \\
= {} & -\frac{1}{2r}\int_0^{+\infty}\left[(r + s) - |r - s|\right] s\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s \\
= {} & -\frac{1}{r}\int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \int_r^{+\infty} s\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s,
\end{aligned}
$$

one sees that the derivative in r is positive:

$$
\begin{aligned}
(\Delta u_\varepsilon)'(r) = {} & \frac{1}{r^2}\int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - r\,\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} + r\,\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} \\
= {} & \frac{1}{r^2}\int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s > 0.
\end{aligned}
$$

Now, by monotonicity of Δu_ε and because $\Delta u_\varepsilon < 0$ and (B.4) hold, we have $\|\Delta u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq C < \infty$. Therefore u_ε goes to some radial u in $C^4_{\text{loc}}(\mathbb{R}^3)$, because of elliptic estimates.

At this point it suffices to check that there exists some $\delta > 0$, independent on ε, such that $u_\varepsilon(x) \leq \delta(1 - |x|)$ for all $\varepsilon > 0$. Indeed, that shows that the limit grows at most linearly and that

$$u(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| e^{u(y)} \, dy.$$

Indeed,

$$\left| |x - y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \right| \leq |x - y| e^{\delta(1-|y|)} \in L^1(\mathbb{R}^3),$$

so that by Lebesgue's Dominated Convergence Theorem

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = -\frac{1}{8\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy$$

$$= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| e^{u_\varepsilon(y)} \, dy.$$

Let us check then that such a $\delta > 0$ exists. Observe preliminarly that $|u_\varepsilon(x)| \leq \|u_\varepsilon\| (1 + |x|)$ and $u_\varepsilon(x) < 0$ for all $x \in \mathbb{R}^3$ imply that $u_\varepsilon(x) \geq -\|u_\varepsilon\| (1 + |x|)$ for all $x \in \mathbb{R}^3$. Therefore

$$-\|u_\varepsilon\| (1 + |x|) \leq u_\varepsilon(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy$$

$$\leq -\frac{1}{8\pi} \int_{|y|<1} |x - y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy$$

$$\leq -\frac{1}{8\pi} \int_{|y|<1} |x - y| e^{-\varepsilon|y|^2} e^{\|u_\varepsilon\|(1+|y|)} \, dy$$

$$\leq -\frac{1}{8\pi} \left(\int_{|y|<1} |x - y| \, dy \right) e^{-2\|u_\varepsilon\|-1}.$$

Now, if by absurd $\|u_\varepsilon\|$ went to zero, on the left hand side we would have something going pointwise to zero, while on the right hand side we would have something going poinwise to some strictly negative function of x, which is a contradiction. Hence it is true that there exists some $C > 0$ such that $\|u_\varepsilon\| \leq C$ for all $\varepsilon > 0$. Thus

$$u_\varepsilon(x) \leq -\frac{C}{8\pi} \int_{|y|<1} |x - y| \, dy \leq \delta(1 - |x|),$$

for some $\delta > 0$. This completes the proof. $\qquad \square$

To sum up, in this section we showed the existence of a solution u_0 to the Liouville equation in $\mathbb{R}^4$ which is radial with linear decay in the first three coordinates and does not depend on the last one. This is the trivial solution.

REMARK. Observe that, if $u_1(x)$ is a solution of (B.1), then

$$u_\mu(x) = u_1(\mu x) + 4 \log \mu$$

is a solution as well. Therefore, actually, we have shown the existence of a whole family of trivial solutions. Once a trivial solution with the aforementioned properties u_1 is fixed, u_λ can be characterized equivalently by its volume $\int_{\mathbb{R}^3} \mathrm{e}^{u_\lambda}$, its value in 0 or its asymptotic behavior.

2. First eigenvalue and finiteness of index

Now that we have obtained the existence of a family of trivial solutions we can start to look for bifurcations. As we did in the 2-dimensional case, we will first restrict the trivial solution u_0 to the strip $\mathbb{R}^3 \times (0, \lambda)$ and then we see for which values of λ we miss unicity of solutions. To this end, we plan to adapt Krasnosel'skii Index Theorem

THEOREM B.3. [**31**, Theorem 56.2] *Let A be a completely continuous operator and assume that λ^* is a point of changing index for the operator A. Then λ^* is a bifurcation point for equation $u = A(\lambda, u)$.*

Again, we will first keep the parameter in the domain and then we will move it explicitly to the operator. In order to be able to glue the solutions as we did in the $\mathbb{R}^2$ case, we ask that Neumann conditions are satisfied on the lines $w = 0$ and $w = \lambda$.

To start, it is useful to observe that equation (B.1) is actually the Euler-Lagrange equation of the following functional

$$I(u) := \int_{\mathbb{R}^4} \left[\frac{1}{2}(\Delta u(x))^2 - \mathrm{e}^{u(x)} \right] \mathrm{d}x.$$

Notice indeed that its first variation is

$$\mathrm{d}I(u)[v] = \int_{\mathbb{R}^4} \left[\Delta u \Delta v - \mathrm{e}^u v \right] \mathrm{d}x.$$

Therefore, in order to compute the index of the originary equation on the strip $S_\lambda := \mathbb{R}^3 \times (0, \lambda)$, it will be enough to calculate the Morse index (see for instance [**36**]) of the functional

$$I_\lambda(u) := \int_{S_\lambda} \left[\frac{1}{2}(\Delta u(x))^2 - \mathrm{e}^{u(x)} \right] \mathrm{d}x.$$

We then have to compute the dimension of the negative space of the following bilinear form of v_1 and v_2

$$\mathrm{d}^2 I_\lambda(u_0)[v_1, v_2] = \int_{S_\lambda} \left[\Delta v_1(x) \Delta v_2(x) - \mathrm{e}^{u_0(x)} v_1(x) v_2(x) \right] \mathrm{d}x,$$

where u_0 is the trivial solution.

The first step is to show that

LEMMA B.4. *The linearized operator*

$$L[v] := \Delta^2 v - e^{u_0} v$$

admits a negative eigenvalue.

PROOF. We construct a compactly supported function v on which the bilinear form $\mathrm{d}^2 I_\lambda(u_0)$ is negatively defined, namely $\mathrm{d}^2 I_\lambda(u_0)[v, v] < 0$. In particular we will look for v radial in the first three coordinates and independent on the last one.

Define

$$
f(r) := \begin{cases} 0 & \text{if } r \leq 1 \\ \mathrm{e}^{-\frac{1}{(r-1)^2}} \mathrm{e}^{-\frac{1}{(r-2)^2}} & \text{if } 1 < r < 2 \\ 0 & \text{if } r \geq 2 \end{cases}
$$

and take

$$
v(x_1, x_2, x_3, x_4) := \frac{1}{A} \int_{\sqrt{x_1^2 + x_2^2 + x_3^2}}^{+\infty} f(s)\,\mathrm{d}s,
$$

with

$$
A := \int_0^{+\infty} f(s)\,\mathrm{d}s.
$$

Observe that v is constantly equal to 1 if $r := \sqrt{x_1^2 + x_2^2 + x_3^2} < 1$ and is identically 0 outside $B_2^3(0) \times (0, \lambda)$, so that its Laplacian is different from zero only in the annulus $1 \leq r \leq 2$. Therefore, setting

$$
V\left(\sqrt{x_1^2 + x_2^2 + x_3^2} \right) := v(x_1, x_2, x_3, x_4),
$$

one gets

$$
\int_{S_\lambda} (\Delta v)^2\,\mathrm{d}x = \lambda \int_1^2 \left(V''(r) + \frac{2}{r} V'(r) \right)^2 4\pi r^2\,\mathrm{d}r = C < +\infty
$$

Fix now a trivial solution u_1, as found in Section 1. Recall that we thus have the family $\{u_\mu\}_\mu$ of trivial solutions. We will then show that we can choose $u_0 \in \{u_\mu\}_\mu$ so that

$$
\int_{S_\lambda} \mathrm{e}^{u_0(x)} [v(x)]^2\,\mathrm{d}x
$$

is sufficiently large. In fact

$$
\int_{S_\lambda} \mathrm{e}^{u_\mu(x)} [v(x)]^2\,\mathrm{d}x = \lambda \int_{\mathbb{R}^3} \mu^4 \mathrm{e}^{u_1(\mu x)}\,\mathrm{d}x
$$

$$
= \lambda \int_{\mathbb{R}^3} \mu^4 \mathrm{e}^{u_1(y)} v^2\left(\frac{y}{\mu}\right) \frac{\mathrm{d}y}{\mu^3} \geq \lambda\mu \int_{|y|\leq\mu} \mathrm{e}^{u_1(y)} \underbrace{v^2\left(\frac{y}{\mu}\right)}_{1}\,\mathrm{d}y
$$

$$
= \lambda\mu \int_{|y|\leq\mu} \mathrm{e}^{u_1(y)}\,\mathrm{d}y \underset{\mu\to+\infty}{\longrightarrow} +\infty
$$

Summing up, if we fix $u_0 = u_\mu$ with μ sufficiently large, then the bilinear form $\mathrm{d}^2 I_\lambda(u_0)[v_1, v_2]$ is negatively defined on the function v defined before.

Now, provided that we choose the right function space[6], L is self-adjoint and semibounded:

$$
\begin{aligned}
\langle Lv, v\rangle_{L^2(S_\lambda)} &= \int_{S_\lambda} \left[\left(\Delta^2 v(x)\right) v(x) - \mathrm{e}^{u_0(x)} v^2(x) \right] \mathrm{d}x \\
&= \int_{S_\lambda} \left[\left(\Delta v(x)\right)^2 - \mathrm{e}^{u_0(x)} v^2(x) \right] \mathrm{d}x \\
&\geq - \int_{S_\lambda} \mathrm{e}^{u_0(x)} v^2(x)\, \mathrm{d}x \geq -\mathrm{e}^{u_0(0)} \|v\|_{L^2(S_\lambda)}.
\end{aligned}
$$

Hence, by the Rayleigh Min-Max Theorem (see for example [**24**, Theorem 11.4]) we have that the lowest eigenvalue for L is

$$
\nu_0 = \min_{u \neq 0} \frac{\langle Lu, u\rangle_{L^2(S_\lambda)}}{\|u\|^2_{L^2(S_\lambda)}} \leq \frac{\mathrm{d}^2\, I_\lambda(u_0)[v,v]}{\|v\|^2_{L^2(S_\lambda)}} < 0.
$$

$\square$

The following two lemmas are still to be proved.

LEMMA B.5. *If v_0 is an eigenfunction of L relative to the first eigenvalue ν_0, then v_0 does not depend on the last coordinate x_4.*

The idea of the proof could be to transform v_0 in order to have no dependence in x_4 and the same L^2 norm. The positive addendum in the bilinear form should then decrease, because depends only on the Laplacian of v_0 (which should decrease after the transformation because the new function should "vary less"). In other words, one could try to write an appropriate rearrangement of v_0 independent on x_4 and use some result of the kind of Pólya-Szegö inequality [**29**].

LEMMA B.6. *The first eigenspace is one-dimensional.*

That might be a consequence of the following argument: Functions in the first eigenspace are solutions of the following linear fourth-order differential equation (recall that we fixed radial symmetry in the first three variables at the beginning and suppose that Lemma B.5 is proved)

$$
v^{(4)}(r) + \frac{4}{r} v^{(3)}(r) - \mathrm{e}^{v(r)} = \nu_0 v(r), \quad r > 0.
$$

Therefore, a priori, we have a four-dimensional eigenspace. Nonetheless, as we need regularity in the origin, we have the further conditions $v'(0) = v'''(r) = 0$, so it should reduce to a two-dimensional eigenspace. Now, the idea would be to find, for instance by means of a shooting method, at least one solution that does not respect the growth we imposed and one that respects it. Consequently, the eigenspace would be one-dimensional.

Suppose then that we proved Lemma B.5 and Lemma B.6. Let v_0 be the eigenfunction of L relative to the first eigenvalue $\nu_0 < 0$ and consider the family of functions

$$
v_k(x_1, x_2, x_3, x_4) := v_0(x_1, x_2, x_3) \cos\left(\frac{2\pi k}{\lambda} x_4\right).
$$

---

[6]Functions must go to zero sufficiently quickly to be able to integrate by parts and have zero border terms. Moreover, we also need to have function spaces which are immersed in L^2 in order to have a nice scalar product.

Observe that v_k satisfies Neumann conditions on ∂S_λ as well, for each $k \in \mathbb{N}$.

LEMMA B.7. *If* $\mathrm{d}^2 I_\lambda(u_0)$ *is negatively defined on both* v_{k_1} *and* v_{k_2}, *then for each* $\alpha, \beta \in \mathbb{R}$ *it is negatively defined on* $\alpha v_{k_1} + \beta v_{k_2}$.

REMARK. The lemma means that v_{k_1} and v_{k_2} are independent generators of the negative space of $\mathrm{d}^2 I_\lambda(u_0)$ (even if they are not eigenfunctions).

PROOF.

$$\int_{S_\lambda} \left\{ \left[\Delta(\alpha v_{k_1}(x) + \beta v_{k_2}(x)) \right]^2 - \mathrm{e}^{u_0(x)} (\alpha v_{k_1}(x) + \beta v_{k_2}(x))^2 \right\} \mathrm{d}x$$

$$= \int_{S_\lambda} \left[\alpha^2 \left(\Delta v_{k_1}(x) \right)^2 + \beta^2 \left(\Delta v_{k_2}(x) \right)^2 + 2\alpha\beta \Delta v_{k_1}(x) \Delta v_{k_2}(x) \right] \mathrm{d}x$$

$$- \int_{S_\lambda} \mathrm{e}^{u_0(x)} \left(\alpha^2 v_{k_1}^2(x) + \beta^2 v_{k_2}^2(x) + 2\alpha\beta v_{k_1}(x) v_{k_2}(x) \right) \mathrm{d}x$$

$$= \alpha^2 \, \mathrm{d}^2 I_\lambda(u_0)[v_{k_1}, v_{k_1}] + \beta^2 \, \mathrm{d}^2 I_\lambda(u_0)[v_{k_2}, v_{k_2}]$$

$$+ 2\alpha\beta \int_{S_\lambda} \left[\Delta v_{k_1}(x) \Delta v_{k_2}(x) - \mathrm{e}^{u_0(x)} v_{k_1}(x) v_{k_2}(x) \right] \mathrm{d}x$$

$$\leq 2\alpha\beta \int_{S_\lambda} \left(\Delta v_0(x_1, x_2, x_3) - \frac{4\pi^2 k_1^2}{\lambda^2} v_0(x_1, x_2, x_3) \right) \cos\left(\frac{2\pi k_1}{\lambda} x_4 \right) \times$$

$$\left(\Delta v_0(x_1, x_2, x_3) - \frac{4\pi^2 k_2^2}{\lambda^2} v_0(x_1, x_2, x_3) \right) \cos\left(\frac{2\pi k_2}{\lambda} x_4 \right) \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}x_4$$

$$- 2\alpha\beta \int_{S_\lambda} \mathrm{e}^{u_0(x_1, x_2, x_3)} v_0(x_1, x_2, x_3)^2 \cos\left(\frac{2\pi k_1}{\lambda} x_4 \right) \times$$

$$\times \cos\left(\frac{2\pi k_2}{\lambda} x_4 \right) \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}x_4 = 0.$$

$\square$

Therefore, we can just count the number of functions of the form v_k to know what is the dimension of the negative eigenspace of $\mathrm{d}^2 I_\lambda(u_0)$, i.e. what is the Morse index of I. Specifically:

$$(\Delta v_k)^2 = \left[(\Delta v_0)^2 - \frac{8\pi^2 k^2}{\lambda^2} v_0 \Delta v_0 + \frac{16\pi^4 k^4}{\lambda^4} v_0^4 \right] \cos^2\left(\frac{2\pi k}{\lambda} x_4 \right).$$

Recall that v_0 does not depend on the last coordinate, so

$$\Delta v_0 := \frac{\partial^2 v_0}{\partial x_1^2} + \frac{\partial^2 v_0}{\partial x_2^2} + \frac{\partial^2 v_0}{\partial x_3^2}.$$

Hence:

$$\mathrm{d}^2 I_\lambda(u_0)[v_k, v_k]$$

$$= \int_{S_\lambda} \left[(\Delta v_0)^2 - \frac{8\pi^2 k^2}{\lambda^2} v_0 \Delta v_0 + \frac{16\pi^4 k^4}{\lambda^4} v_0^4 - \mathrm{e}^{u_0} v_0^2 \right] \cos^2\left(\frac{2\pi k}{\lambda} x_4 \right) \mathrm{d}x$$

$$= \frac{\lambda}{2\pi k} \int_{S_\lambda} \left[(\Delta v_0)^2 - \mathrm{e}^{u_0} v_0^2 \right] \cos^2(s) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}s$$

$$+ \int_{S_\lambda} \left(\frac{8\pi^2}{\lambda} |\nabla v_0|^2 + \frac{16\pi^4}{\lambda^3} \right) \cos^2(s) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 \, \mathrm{d}s.$$

Roughly speaking, upon showing that the first addendum is actually negative (as one might expect, because apart from $\cos^2(s)$ it is the bilinear form on v_0, which is negative), we can see that for λ sufficiently big the number of the different values of k for which $\mathrm{d}^2 I_\lambda(u_0)[v_k, v_k]$ is negative grows. In particular, by continuity, there will exist values of λ at which that number increases exactly by 1, namely values at which the index of the operator describing our equation changes. An adaption of Krasnosel'skii Index Theorem should then show that these values of λ are points of bifurcation, proving that there exist non-trivial solutions of the 4D Liouville equation with infinite volume.

Acknowledgements

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