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Comparisons of Weak Regular Splittings and Multisplitting Methods

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Dedicated to the memory of Peter Henrici

Summary. Comparison results for weak regular splittings of monotone matrices are derived. As an application we get upper and lower bounds for the convergence rate of iterative procedures based on multisplittings. This yields a very simple proof of results of Neumann-Plemmons on upper bounds, and establishes lower bounds, which has in special cases been conjectured by these authors.

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1 Introduction, Definitions

Let A be a real $n \times n$-matrix. A splitting

$$A = M - N \tag{1.1}$$

is called *regular*, if M^{-1} exists, $M^{-1} \geqq 0$ and $N \geqq 0$ and *weak regular*, if M^{-1} exists, $M^{-1} \geqq 0$, $M^{-1}N \geqq 0$. Here we use the notation $B \geqq 0$ for a matrix B to be nonnegative, i.e. that all entries are nonnegative. Let $\rho(B)$ denote the spectral radius of B. It is well known (see [Va; BP]) that for a weak regular splitting (1.1)

$$A^{-1} \geqq 0 \Leftrightarrow \rho(M^{-1}N) < 1.$$

In the course of constructing iterative methods for solving the linear system $Ax = b$ which are suitable for parallel execution, O'Leary-White introduced in [OW] the concept of multisplittings. Let

$$A = M_l - M_l, \qquad l = 1, \ldots, k \tag{1.2}$$

be k weak regular splittings of A, and let E_l, $l=1, \ldots, k$ satisfy

$$E_l \geqq 0, \qquad \sum_{l=1}^{k} E_l = I. \tag{1.3}$$

The iteration

$$x_{i+1} = \sum_{l=1}^{k} E_l M_l^{-1} (N_l x_i + b) = H x_i + S b \tag{1.4}$$

where

$$H = \sum_{l=1}^{k} E_l M_l^{-1} N_l, \qquad S = \sum_{l=1}^{k} E_l M_l^{-1}$$

is called a *multisplitting method.*

In [OW] it is shown that for $A^{-1} \geqq 0$ the iteration (1.4) converges for any x_0 to the solution $x = A^{-1} b$ of $Ax = b$.

In particular

$$\rho(H) < 1$$

and S nonsingular.

In this note we consider different multisplittings. In order to compare the asymptotic convergence rate of (1.4) we compare $\rho(H)$ for different sets of splittings. The following question was raised in [NP].

Consider for a matrix A, where $A^{-1} \geqq 0$, weak regular splittings

$$A = M_l - N_l = \bar{M}_l - \bar{N}_l, \qquad l = 1, \ldots, k \tag{1.5}$$

such that

$$M_l \leqq \bar{M}_l, \qquad l = 1, \ldots, k \tag{1.6}$$

and E_l, $l = 1, \ldots, k$ that satisfy (1.3).

Let

$$H = \sum_{l=1}^{k} E_l M_l^{-1} N_l, \qquad \bar{H} = \sum_{l=1}^{k} E_l \bar{M}_l^{-1} \bar{N}_l. \tag{1.7}$$

Is it true that

$$\rho(H) \leqq \rho(\bar{H}). \tag{1.8}$$

It was shown in [NP] that in the case $\bar{M}_l = \bar{M}$ $(l = 1, \ldots, k)$ (1.8) holds. We will show that (1.8) is also true if $M_l = M$ $(l = 1, \ldots, k)$ and point out other sufficient conditions. The following example shows that without additional assumptions (1.8) is not true.

Consider

$$A=\begin{pmatrix} 1 & -1 \\ -\frac{1}{2} & 1 \end{pmatrix}=M_1-N_1=M_2-N_2$$

where

$$M_1=\begin{pmatrix} 1+\varepsilon & -1+\sigma \\ -\frac{1}{2} & 1 \end{pmatrix}, \quad N_1=\begin{pmatrix} \varepsilon & \sigma \\ 0 & 0 \end{pmatrix}, \quad M_2=A, \quad N_2=0,$$

and $E_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Here $\varepsilon, \sigma \geqq 0$, $\sigma<1$. An easy calculation gives

$$S(\varepsilon,\sigma)=E_1 M_1^{-1}+E_2 M_2^{-1}=\begin{pmatrix} \frac{2}{1+\sigma+2\varepsilon} & \frac{2(1-\sigma)}{1+\sigma+2\varepsilon} \\ 1 & 2 \end{pmatrix}$$

$$H(\varepsilon,\sigma)=E_1 M_1^{-1} N_1+E_2 M_2^{-1} N_2=\begin{pmatrix} \frac{2\varepsilon}{1+\sigma+2\varepsilon} & \frac{2\sigma}{1+\sigma+2\varepsilon} \\ 0 & 0 \end{pmatrix}. \tag{1.9}$$

Also

$$S^{-1}-A=\begin{pmatrix} 2\varepsilon & 2\sigma \\ -\varepsilon & -\sigma \end{pmatrix}\frac{1}{1+\sigma}. \tag{1.10}$$

Obviously

$$\rho(H(\varepsilon,\sigma))=\frac{2\varepsilon}{1+\sigma+2\varepsilon} \tag{1.11}$$

is a nonincreasing function of σ. Hence M_1 is increased by increasing σ, but $\rho(H(\varepsilon,\sigma)$ is decreased, (1.8) is hence not satisfied. This example serves at the same time as a counterexample on two further occasions. Firstly it shows that even if all splittings (1.5) are regular, the splitting leading to the iteration (1.4) needs not to be regular, it is only weak regular (see end of Chapter 2). Secondly it shows that a wellknown comparison result for regular splittings does not carry over to the weak regular case, see the remark after the proof of the Lemma in Chapter 3.

2 Multisplittings and Weak Regular Splittings

We consider the iteration (1.4) and assume $A^{-1}\geqq 0$. From $A=M_i-N_i$ we have $I-M_i^{-1}N_i=M_i^{-1}A$ and hence

$$I-H=SA. \tag{2.1}$$

As $\rho(H)<1$, S is nonsingular and we have

$$A=M-N, \qquad M=S^{-1}, \qquad N=S^{-1}-A \tag{2.2}$$

and $M^{-1}N=S(S^{-1}-A)=I-SA=H$ by (2.1). Hence the iteration (1.4) is induced by a weak regular splitting, as $M^{-1}=S\geqq 0$ and $M^{-1}N=H\geqq 0$.

Thus we might use comparison results for weak regular splittings to give answers to the question addressed in (1.5)–(1.8). Observe that the splittings leading to H and $\bar{H}$ in (1.7) are

$$A=\tilde{M}-\tilde{N}=\tilde{\bar{M}}-\tilde{\bar{N}} \tag{2.3}$$

where $\tilde{M}^{-1}=\sum E_l M_l^{-1}$, $\tilde{\bar{M}}^{-1}=\sum E_l \bar{M}_l^{-1}$. Now from (1.6) we have $M_l^{-1}\geqq \bar{M}_l^{-1}$, $l=1, \ldots, k$ and hence

$$\tilde{M}^{-1}\geqq \tilde{\bar{M}}^{-1}. \tag{2.4}$$

If the splittings (2.3) both are regular then (see [CV, MN]) (2.4) implies $\rho(H)\leqq\rho(\bar{H})$. However these splittings are not regular in general, even if the splittings (1.5) are. See the example in 1.), (1.10).

3 Comparison Results for Weak Regular Splittings

We try here to generalize comparison results for regular splittings to the case of weak regular splittings.

Lemma. *Let* $A^{-1}\geqq 0$ *and*

$$A=\tilde{M}_1-\tilde{N}_1=\tilde{M}_2-\tilde{N}_2 \tag{3.1}$$

weak regular splittings. In either of the following cases

a) $\tilde{N}_1\leqq\tilde{N}_2$
b) $\tilde{M}_1^{-1}\geqq\tilde{M}_2^{-1}$, $\tilde{N}_1\geqq 0$
c) $\tilde{M}_1^{-1}\geqq\tilde{M}_2^{-1}$, $\tilde{N}_2\geqq 0$

the inequality

$$\rho(\tilde{M}_1^{-1}\tilde{N}_1)\leqq\rho(\tilde{M}_2^{-1}\tilde{N}_2) \tag{3.2}$$

holds.

Proof. a) We have, as $\rho(\tilde{M}_1^{-1}\tilde{N}_1)<1$

$$0\leqq(I-\tilde{M}_1^{-1}\tilde{N}_1)^{-1}\tilde{M}_1^{-1}\tilde{N}_1=A^{-1}\tilde{N}_1\leqq A^{-1}\tilde{N}_2$$

hence $\rho(A^{-1}\tilde{N}_1)\leqq\rho(A^{-1}\tilde{N}_2)$ and

$$\rho(\tilde{M}_1^{-1}\tilde{N}_1)=\rho(A^{-1}\tilde{N}_1)/(1+\rho(A^{-1}\tilde{N}_1))\leqq\rho(A^{-1}\tilde{N}_2)/(1+\rho(A^{-1}\tilde{N}_2))=\rho(\tilde{M}_2^{-1}\tilde{N}_2).$$

In b), c) we follow closely a reasoning used in [NP]. If $x\geqq 0$ and $Ax\geqq 0$ then

$$\tilde{M}_1^{-1}Ax\geqq\tilde{M}_2^{-1}Ax$$

and by (3.1)

$$\tilde{M}_1^{-1}\tilde{N}_1 x \leqq \tilde{M}_2^{-1}\tilde{N}_2 x. \tag{3.3}$$

Now in the case b) choose x as a Perron vector of $\tilde{M}_1^{-1}\tilde{N}_1$. If $\mu x = \rho(\tilde{M}_1^{-1}\tilde{N}_1)x = \tilde{M}_1^{-1}\tilde{N}_1 x$, then $Ax = (\tilde{M}_1 - \tilde{N}_1)x = \left(\frac{1}{\mu} - 1\right)\tilde{N}_1 x \geqq 0$, as $\tilde{N}_1 \geqq 0$ and $\mu < 1$. Hence by (3.3)

$$\rho(\tilde{M}_1^{-1}\tilde{N}_1)x \leqq \tilde{M}_2^{-1}\tilde{N}_2 x$$

and (3.2) follows.

In case c) choose x as Perron vector of $\tilde{M}_2^{-1}\tilde{N}_2$ and get

$$\tilde{M}_1^{-1}\tilde{N}_1 x \leqq \rho(\tilde{M}_2^{-1}\tilde{N}_2)x. \tag{3.4}$$

By eventually replacing A by $A - \varepsilon J$, where the entries of J are all 1, $\varepsilon > 0$, small, and $\tilde{N}_2$ by $\tilde{N}_2 + \varepsilon J$, we can assume that all entries of x are positive. Then (3.2) follows from (3.4) by the Collatz quotient theorem. □

Let us compare the results of the Lemma with known results for regular splittings. Suppose that the splittings (3.1) both are regular. In this case Varga has shown (see [Va]) that $\tilde{N}_1 \leqq \tilde{N}_2$ implies (3.2), while Woźnicky has proved that (3.2) follows already from the weaker condition $\tilde{M}_1^{-1} \geqq M_2^{-1}$. For an overview and other still weaker conditions implying (3.2), see [CV] and [MN]. Vargas result carries over to the weak regular case (see the Lemma), while the example shows that Woźnicki's does not. Hence it seems to be superfluous to discuss the weaker conditions in [CV] and [MN] for the weak regular case.

4 Comparison Results for Multisplitting

We return now to the multisplitting iterations.

Theorem. *Let $A^{-1} \geqq 0$ and*

$$A = M_l - N_l, \qquad l = 1, \ldots, k \tag{4.1}$$

be weak regular splittings, $E_i \geqq 0$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} E_i = I$ and

$$H = \sum_{l=1}^{k} E_l M_l^{-1} N_l$$

and $\underline{M}$, $\bar{M}$ such that

$$\underline{M} \leqq M_l \leqq \bar{M}, \qquad l = 1, \ldots, k. \tag{4.2}$$

If

$$A=\bar{M}-\bar{N} \tag{4.3}$$

is a regular splitting then

$$\rho(H)\leqq\rho(\bar{M}^{-1}\bar{N}) \tag{4.4}$$

and if

$$A=\underline{M}-\underline{N} \tag{4.5}$$

is a regular splitting then

$$\rho(\underline{M}^{-1}\underline{N})\leqq\rho(H). \tag{4.6}$$

Proof. In the first case, apply the Lemma to

$$\tilde{M}_1=\left(\sum_{l=1}^{k} E_l M_l^{-1}\right)^{-1}, \qquad \tilde{M}_2=\bar{M}, \qquad \tilde{N}_2=\bar{N}.$$

As $M_l^{-1}\geqq\bar{M}^{-1}$, we have $\tilde{M}_1^{-1}\geqq\tilde{M}_2^{-1}$, $\tilde{N}_2\geqq 0$. Case c) of the Lemma gives (4.4).

In the second case take $\tilde{M}_1=M$, $\tilde{N}_1=\underline{N}$, $\tilde{M}_2=\left(\sum_{l=1}^{k} E_l M_l^{-1}\right)^{-1}$ and apply the Lemma, case b). □

It is clear that the Lemma can lead to other conditions under which in the situation (1.5)–(1.7) the inequality (1.8) holds. We see however besides the cases treated in the Theorem no other natural conditions, and have thus refrained from elaborating this point.

As special cases of the Theorem we get the upper bounds (3.16), (3.17) of [NP] and the conjectured lower bounds, see last sentence of [NP]: Let $A=I-L-U$ be the standard decomposition of A and

$$\left.\begin{aligned} A&=(I-L_i)-(U+L-L_i)\\ &=(I-L_i)(I-U_i)(U+L+L_i\,U_i-L_i-U_i)\end{aligned}\right\} i=1,\ldots,k$$

where

$$0\leqq L_i\leqq L\, 0\leqq U_i\leqq U, \qquad i=1,\ldots,k$$

and E_i satisfy (1.3). Then

$$\rho((I-L)^{-1}U)\leqq\rho\left(\sum_{i=1}^{k} E_i(I-L_i)^{-1}(U+L-L_i)\right)\leqq\rho(L+U)$$

and

$$\rho((I-U)^{-1}(I-L)^{-1}LU)\leqq\rho\left(\sum_{i=1}^{k} E_i(I-U_i)^{-1}(I-L_i)^{-1}(U+L+U_i\,L_i-U_i-L_i)\right)$$
$$\leqq\rho(L+U).$$

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