The Fishing Optimization Problem: A Tour of Technology in the Teaching of Mathematics Dedicated to Bert Waits and Frank Demana

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Abstract
Relaxation can provide time for reflection. This paper illustrates how relaxation, in the form of fishing, led to reflection involving mathematics and technology – in particular, the calculator technology fostered by Bert Waits and Frank Demana, Ohio's internationally-recognized leaders in the field.

A common form of pier fishing for salmon and trout in Lake Michigan involves repeatedly casting and retrieving an artificial lure. It is usually advantageous to cast the lure as far as possible, so it covers more potential territory and spends more time in the water. This results in the question of how to achieve the maximum distance. It leads to an interesting application of mathematics and illustrates the role of technology in problem solving by:

1. allowing for powerful interactive visualization of problems,
2. finding new ways to solve problems, and
3. performing tedious computations.

The range of technology includes graphing equations in both parametric and function modes and performing difficult computations using a computer algebra system (CAS). All these computations can now be done using any of the several technology platforms which are available. We illustrate it here using Texas Instruments' Nspire CAS.

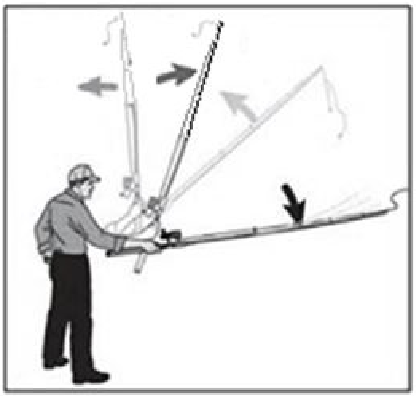

Figure 1. Casting Motion

In the process of casting, as shown in Figure 1, the fisherman swings the rod in an overhead circular motion, releasing the line and the lure at some point in the arc. While many factors, including the weight of the line and the length of the rod are involved, we address this question:

At what point in the motion should the lure be released in order to travel the maximum distance?

Preliminary Problem
A first step in solving the problem is to think of a simpler problem: A projectile is launched with an initial velocity v_0 at ground level at an angle of θ with the horizontal (see Figure 2). Find θ so that the distance

traveled by the projectile before it hits the ground is maximized.

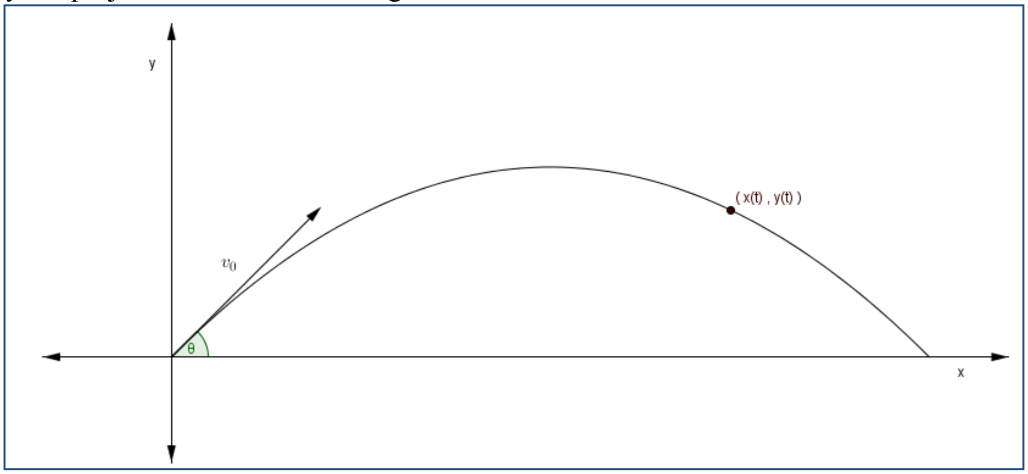

Figure 2. A projectile launched with initial velocity v_0 at an angle θ with the horizontal.

The *x*-coordinate at time *t*, represented by **[A]** below, is determined by $(v_0 \cos\theta)t$, the horizontal component of the initial velocity multiplied by *t*. The *y*-coordinate, represented by **[B]** below, is determined by $(v_0 \sin\theta)t$, the vertical component of the initial velocity multiplied by *t*, minus $16t^2$, the distance in feet that an object falls in *t* seconds.

$$\textbf{[A]} \quad x(t) = (v_0 \cos\theta)t$$
$$\textbf{[B]} \quad y(t) = (v_0 \sin\theta)t - 16t^2$$

One of the most significant contributions to learning mathematics, provided by the technology fostered by Waits and Demana, is the power of visualization. A parametric function grapher brings life to the preliminary problem as shown in the following example, in which students can trace the path of the object in specific cases.

Modeling Examples

A parametric function grapher (see Figure 3) is used to determine how far the projectile will travel if v_0 is 100 feet per second and for selected value for θ of 30°, 45°, and 60°.

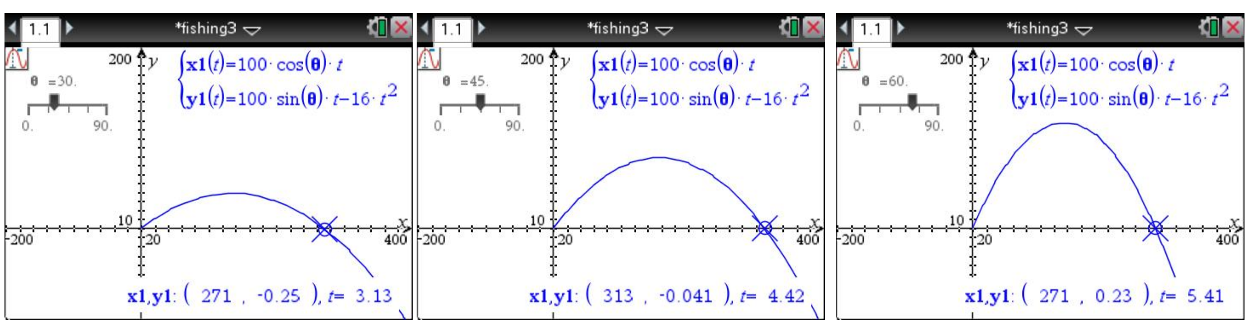

Figure 3. Parametric graphs of **[A]** and **[B]** with canonical values of angle $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$.

Tracing the graphs determines the distances traveled by the projectile are approximately 271 feet, 313 feet, and 271 feet, respectively. An alternate method to tracing the graphs would involve looking at a table of values when *y(t)* is near zero.

After modeling to determine an approximation, finding an exact solution to the preliminary problem involves three steps:

(1) Find the time t when the projectile hits the ground (has height of 0) in terms of θ.
(2) Substitute this value of t into the formula for the distance the projectile travels, so that the distance is expressed solely as a function of θ.
(3) Find θ so that the distance is a maximum.

Step (1): Using **[B]**,

$$0 = (v_0 \sin\theta)t - 16t^2$$
$$0 = t(v_0 \sin\theta - 16t)$$
$$t = 0 \text{ or } v_0 \sin\theta - 16t = 0$$
$$t = 0 \text{ or } t = \tfrac{1}{16} v_0 \sin\theta$$

Step (2): Using **[A]** to find the distance traveled for $t = \frac{1}{16} v_0 \sin\theta$

$$\begin{aligned} x(t) &= (v_0 \cos\theta) \cdot \tfrac{1}{16} v_0 \sin\theta \\ &= \tfrac{1}{16} v_0^2 \sin\theta \cos\theta \\ &= \tfrac{1}{32} v_0^2 \sin 2\theta \end{aligned}$$

Step (3): The maximum occurs when $\sin 2\theta = 1$, so $2\theta = 90^\circ$ or $\theta = 45^\circ$.

[Note that the answer is independent of the initial velocity, v_0.]

Fishing Problem

Solving the fishing problem adds several new considerations, including the initial height from which the lure is released (dependent upon the height of the pier and the height of the fisherman), the length of the rod (including the lower arm which swings with the rod), and the angular velocity of the casting process. The problem is solved here using data that fit the situation, shown in Figure 1, with some assumptions:

- the height of the pier is 5 feet,
- the center of rotation of the rod is an additional 5 feet (shoulder height on a 6-foot fisherman),
- R, the length of the rod (including the lower part of the arm that is part of the casting motion), is 8 feet,
- the angular velocity in radians is 2π per second, based on an estimate that the casting motion is a semicircle which takes half a second,
- the initial velocity perpendicular to the rod at the time of release is $v_0 = 2\pi R$ feet per second, or 16π feet per second, with a horizontal component of $16\pi \sin\theta$ feet per second and a vertical component of $16\pi \cos\theta$ feet per second,
- the solution is independent of the properties of the line and the lure,
- there is no drag on the line, making the cast free of friction, and
- the fisherman's physical motion is optimal and is described as in the diagram.

The relationships among the variables are summarized in Figure 4, where point E is the fisherman's elbow and F is the point the lure is released.

The horizontal distance $x(t)$ traveled by the lure t seconds after release is $(16\pi\sin\theta)t$, the horizontal component of the velocity, multiplied by the time t. From this subtract $8\cos\theta$, the horizontal distance the lure is behind the fisherman's elbow at the time of release:

[C] $x(t)=(16\pi\sin\theta)t-8\cos\theta$.

At the point of release the height of the lure is $5+5+p=10+8\sin\theta$.

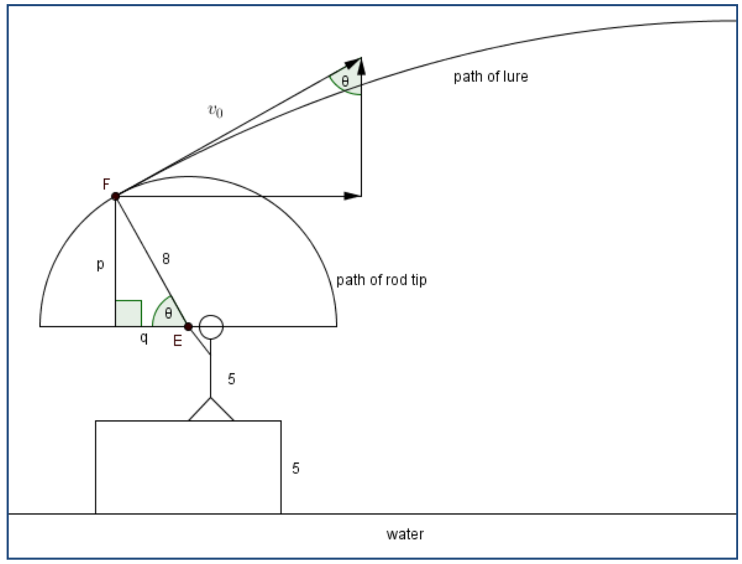

Figure 4. Illustration of the *Fishing Optimization Problem*, including the variables and constants used to solve the problem.

The height of the lure in feet after t seconds, represented by **[D]** below, consists of the sum of $(10+8\sin\theta)$, the height at the time of release, and $(16\pi\cos\theta)t$, the height gained by the casting motion (the vertical component of the velocity times t), minus $16t^2$ (the distance the lure falls in t seconds due to gravity):

$$\textbf{[D]}\ \ y(t)=(10+8\sin\theta)+(16\pi\cos\theta)t-16t^2.$$

Modeling Example

A parametric function grapher is used to determine how far the lure will travel given the above parameters. Graphs (see Figure 5) of parametric equations **[C]** and **[D]** are shown in the case where θ is 45°, which is the solution to the preliminary problem. When θ is 45°, tracing the graph (or displaying a table of values) shows that the lure is released approximately 5.7 feet behind and 15.7 feet above the water level, reaches a height of about 35.4 feet in 1.1 seconds, and hits the water about 86.8 feet away in about 2.6 seconds.

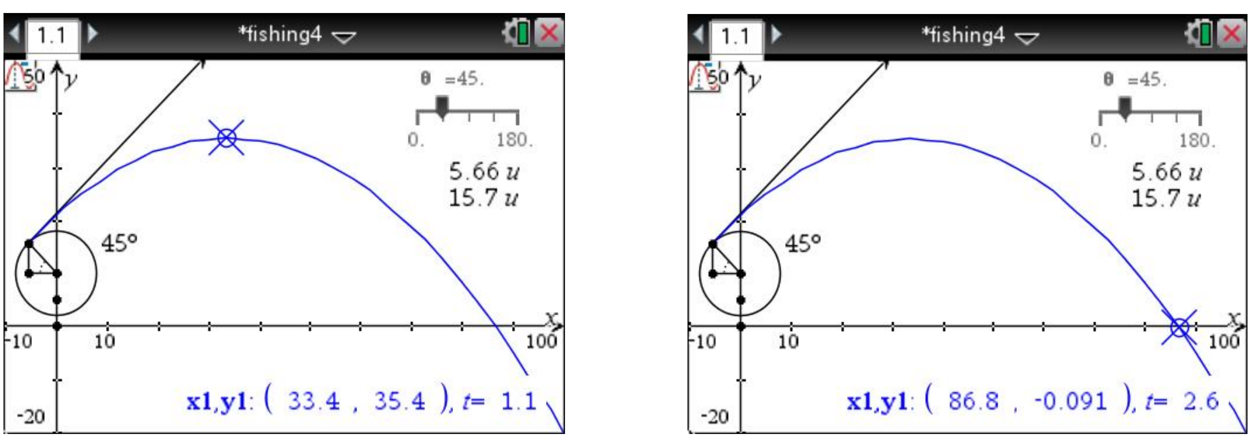

Figure 5. Parametric graphs of **[C]** and **[D]** for $\theta=45°$ illustrating the height and distance of the lure.

While it is quite easy to find an exact solution to the preliminary problem, this is not the case for the fishing optimization problem. Here function graphing technology is used to provide a new approach to finding an approximate solution. To find the value of θ which results in the maximum distance, the same three steps are used that were used to solve the preliminary problem, this time with the aid of function graphing technology as employed by Waits and Demana:

Step (1): Using **[D]**,

$$0 = \left(10 + 8\sin\theta\right) + \left(16\pi\cos\theta\right)t - 16t^2$$

Using the quadratic formula to solve for t in terms of θ and simplifying gives

$$t = \frac{\sqrt{4\pi^2\cos^2\theta + 8\sin\theta + 10} + 2\pi\cos\theta}{4}$$

Note that the other solution gives t as a negative value, which is discarded.
Step (2): Using **[C]** and expressing the distance as a function of θ,

$$\textbf{[E]}\quad x(\theta) = \left(16\pi\sin\theta\right)\frac{\sqrt{4\pi^2\cos^2\theta + 8\sin\theta + 10} + 2\pi\cos\theta}{4} - 8\cos\theta\,.$$

Step (3): Putting **[E]** in a form compatible with a function grapher and zooming in, or using the calculate maximum feature of the technology, shows that the lure attains the maximum distance of about 88.8 feet, which occurs when the angle of release is about 52° (see Figure 6). In fishing terms, this may translate to "just a bit more than" 45°.

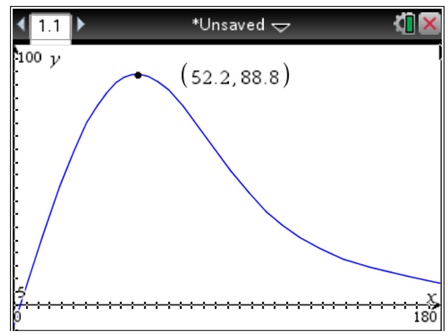

Figure 6. Graph of **[E]** with θ being the independent variable and *x*, the dependent variable.

As a check on this result given parametrically by **[C]** and **[D]**, tracing the graph (or displaying a table of values) when θ is 52° shows that the lure is released approximately 4.9 feet behind and 16.3 feet above the water level, reaches a height of about 31.3 feet in 1.0 seconds, and hits the water about 88.7 feet away in about 2.4 seconds, slightly better than releasing when θ is 45° (see Figure 7).

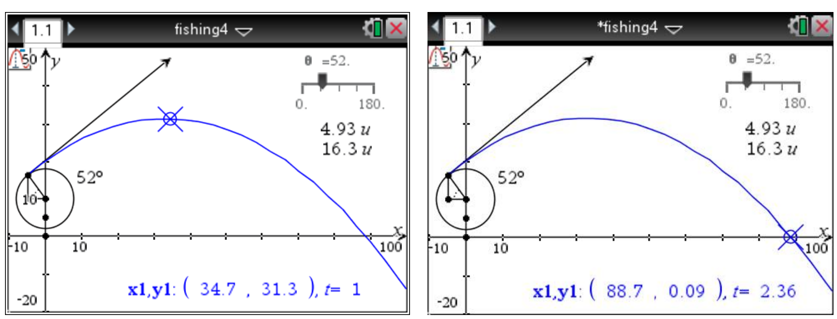

Figure 7. Parametric graphs of **[C]** and **[D]** for $\theta = 52°$, illustrating the optimal solution.

A third important use of technology in the teaching of mathematics is to use technology to perform tedious computations. Just as decades ago when calculators replaced tedious computations done by hand to find approximations for square roots and values of trigonometric functions (like sin 23°), computer algebra systems (CAS) can now replace more complex computations.

For students familiar with calculus, finding a maximum for **[E]** involves computing the derivative, setting it equal to zero, and solving the resulting equation. A formidable task to do by hand, finding derivatives and finding approximate solutions to equations can be readily done using a computer algebra system, technology promoted by Waits and Demana late in their careers. An approximate result of $\theta = 52^\circ$ with a corresponding maximum value of about 88.8 feet is obtained using the TI-Nspire.

Having himself visited Sheboygan's South Pier, Bert Waits was delighted to hear about this unexpected application of the Demana/Waits technology shortly before the events which led to his unfortunate passing. When asked, while discussing the fishing problem with the authors, to cite an article he felt best represented the work he did with Frank Demana, he suggested *A Call for Action!* (Waits, B. & Demana, F., n.d.).

Jim Schultz, his wife Donna Menzer, with Barb and Bert Waits near the South Pier in Sheboygan, WI, in 2007.

The authors proudly dedicate this paper to "Hank" and Frank as a token of their appreciation of the outstanding contribution they provided to the appropriate use of technology in the teaching of mathematics.

The following **related problem** is offered in closing:
Use a function grapher to simultaneously graph the two functions:
using the window: $x$min = 0, $x$max = 18, $y$min = -10, and $y$max = 1.

$$y_1 = \sqrt{36 - x^2 + 16x + 64} - 15$$

$$y_2 = -\sqrt{36 - x^2 + 16x + 64} + 5$$

Reference

Waits, B. & Demana, F. (n.d.). A call for action! Retrieved from http://mathforum.org/technology/papers/papers/waits/waits.html

James Schultz, a long-time colleague of Bert Waits and Frank Demana in the Department of Mathematics at The Ohio State University, is the Robert L. Morton Emeritus Professor of Mathematics Education at Ohio University. He is now retired and living in Wisconsin.

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