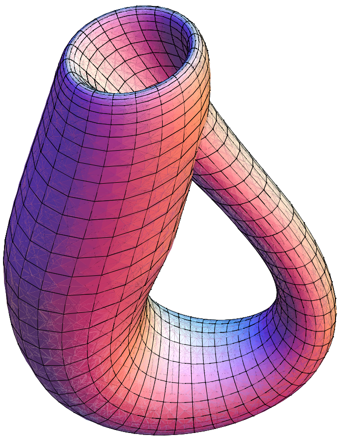
Embeddability of graphs into the Klein surface



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Introduction

Loosely speaking, a graph G is embeddable into a surface $\mathcal{S}$ if G can be drawn on $\mathcal{S}$ in a way that each intersection of edges is a single vertex. Questions which arise from the topic of embeddability are numerous. The most obvious probably is, whether an embedding of a given graph G for a surface $\mathcal{S}$ exists. Another well known problem in graph theory, is the question of possible colorings of a graph, which is known to be embeddable on a surface: The Four-Color-Theorem for $\mathbb{R}^2$ and the Map-Color-Theorem for surfaces of higher genus.

In 1930 Kuratowski answered the question about embeddability of graphs for the plane by proving that each graph, which does not contain a subdivision of a $K_{3,3}$ or a K_5, is embeddable into the plane [Kur30]. A graph G thereby is a subdivision of a graph H if G contains additional vertices of degree two, which divide edges in H. A reformulation of Kuratowski's theorem says that each graph is embeddable into the plane, if it does not have a $K_{3,3}$ or a K_5 as a topological minor. Whereby a graph H is a topological minor of a graph G if and only if H can be obtained from G by a series of contractions of edges with at least one end-vertex of degree less or equal to two and deletion of isolated vertices. Wagner answered the same question, but instead of the topological minor relation he used the minor relation, which allows contractions of edges with end-vertices of every degree. In the case of embeddings into the plane, both relations lead to the same graphs, the $K_{3,3}$ and the K_5, which can never be found in a planar graph.

From the results of Kuratowski and Wagner, we can also conclude, that in order to classify the embeddable graphs for the plane, it is sufficient to characterize the non-embeddable graphs and for this purpose it is even sufficient to characterize the smallest non-embeddable graphs for the plane. If these smallest graphs are no minors in a graph it can be embedded into the plane. These techniques for characterization of embeddability do not only work for the plane but for every surface $\mathcal{S}$. Instead of finding all embeddable graphs it is sufficient to find the smallest non-embeddable graphs, namely the graphs which do not have a minor that is also not embeddable.

The question arises, whether theorems like the ones from Kuratowski and Wagner can also be proven for other surfaces than the plane, and especially if one can find a list of smallest non-embeddable (irreducible) graphs for each surface. The question whether this list is finite or not, is known as Wagner's conjecture, although Wagner insisted that he never conjectured a positive solution, but he discussed this matter with students in the 1960's. Robertson and Seymour worked on this question from 1983 to 2004 and they proved it in a series of papers. The Robertson-Seymour-Theorem (or graph minor theorem) states that the finite graphs are well-quasi-ordered by the minor relation $\preccurlyeq$. From this theorem we can draw the following conclusion: Consider a specific graph property (e.g. embeddability on a given surface), so that each minor of a graph with this property also obeys this property.

Then the Kuratowski set, which represents the smallest minors without this property, is finite. The Kuratowski set of embeddability in the plane is $\{K_{3,3}, K_5\}$. More generally we can say, that for every surface $\mathcal{S}$ there exists a finite set of graphs $H_1, \ldots, H_n$ such that a graph is embeddable in $\mathcal{S}$ if and only if it contains none of the graphs $H_1, \ldots, H_n$ as a minor. ([RS90], [RS04]).

As we will always be speaking about the set of smallest graphs, not embeddable into a given surface $\mathcal{S}$, it is of great use to define $M_1(\mathcal{S})$ as the set of graphs, which are the irreducible graphs with respect to the topological minor-relation and $M_2(\mathcal{S})$ as the set of the irreducible graphs with respect to the minor-relation. The number of graphs in M_1 and M_2 will be very large for surfaces other than the plane. In order to reduce the number of graphs, which have to be found for the characterization of the graphs which are not embeddable on a given surface, Bodendiek, Schumacher and Wagner extended the minor-relations ([BSW81b]). The relations they used when researching the question of embeddability for different surfaces are:

R_0: An edge or an isolated vertex of G is deleted.

R_1: An edge of G is contracted, the degree of at least one vertex incident to this edge equals two.

R_2: An edge of G is contracted, the degree of each of the two vertices incident to this edge is at least three.

R_3: A vertex v of degree three in G is deleted and the three incident edges (v, v_1), (v, v_2), (v, v_3) of G are replaced by the triangle (v_1, v_2), (v_2, v_3), (v_3, v_1).

R_4: An edge (v_1, v_2) in G, where v_1 and v_2 both have degree three is selected. The edge (v_1, v_2) is divided by a new vertex v' and the transformation R_3 is applied to both of the vertices v_1 and v_2.

Obviously the relations R_0 and R_1 represent the topological minor relations and the relations R_0, R_1 and R_2 represent the minor relations. The additional relations also seem to be useful for the reduction of the number of irreducible graphs, as $K_{3,3}$ can be reduced to K_5 by application of R_4.

The graph substitutions associated to the relations R_1, R_3 and R_4 are well known in the theory of electrical networks. The relation R_1 corresponds to the reduction of resistors in series and R_3 to the well known Wye-Delta-transform. The substitution given by R_4 is a special form of applying Wye-Delta twice avoiding prior subdivision. In this context Epifanov proved that every planar network (or graph), satisfying a certain connectivity condition, can be reduced to a single edge (single resistor) by a series of serial, parallel and Wye-Delta transformation [Epi66]. An elementary proof of this theorem was also published by Truemper [Tru89]. If we also allow deletion of leaves, we can even say that each planar graph can be reduced to a single vertex by applying a series of these relations.

The same way the sets $M_1(\mathcal{S})$ and $M_2(\mathcal{S})$ are defined for the minimal bases of irreducible graphs for a surface $\mathcal{S}$ with respect to the relations R_0, R_1 and R_0, R_1, R_2 respectively, we can in general define, that a graph G is an element of $M_i(\mathcal{S})$, for $i \in \{0, \ldots, 4\}$, if G is not embeddable into $\mathcal{S}$ and $R_j(G)$, $j \in \{0, \dots, i\}$ is embeddable. For these minimal bases it is obvious that

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq M_4$$

applies. The basis M_2 is finite as shown by Robertson and Seymour and this can also be shown for M_1; M_0 will always be infinite. For the characterization of the irreducible graphs for a surface $\mathcal{S}$ it is sufficient to find the graphs in $M_4(\mathcal{S})$, as the other bases $M_3(\mathcal{S}), M_2(\mathcal{S})$ and $M_1(\mathcal{S})$ can be reconstructed by application of the inverse relations of R_i, R_i^{-1} for $i \in \{1, 2, 3, 4\}$.

We already know that for the plane the identity

$$M_1(\mathbb{R}^2) = M_2(\mathbb{R}^2) = \{K_{3,3}, K_5\}$$

holds. Now we can show, that the identities

$$M_3(\mathbb{R}^2) = \{K_{3,3}, K_5\} \quad \text{and} \quad M_4(\mathbb{R}^2) = \{K_5\}$$

also hold. These are all results for the plane.

For surfaces of higher genus the question of irreducible graphs is much more complex. Bodendiek, Schumacher and Wagner worked on the torus, the spindle surface and the projective plane. For the torus $\widetilde{\mathcal{S}}_1$, they found 23 graphs in $M_4(\widetilde{\mathcal{S}}_1)$ but could not show that these are all graphs [BW86]. They also published some results on the spindle-surface, but again could not show that they found the complete list of irreducible graphs [BSW85]. For the projective plane they found 12 irreducible graphs in $M_4(\mathcal{S}_1)$ and they have also proven, that this is the complete list of irreducible graphs in $M_4(\mathcal{S}_1)$. Glover, Huneke and Wang also worked on the irreducible graphs of the projective plane and they have constructed 103 irreducible graphs which are in $M_1(\mathcal{S}_1)$ [GHW79]. Archdeacon has later shown that this list of 103 graphs in $M_1(\mathcal{S}_1)$, reduced to 35 graphs in $M_2(\mathcal{S}_1)$, is complete [Arc81].

In this thesis we want to find a class of irreducible graphs for the Klein surface $\mathcal{S}_2$. In [BSW85] Bodendiek, Schumacher and Wagner have shown that the minimal basis $M_1(\mathcal{S}_2)$ can be constructed by subdivision of graphs in $M_1(\mathcal{S}_1)$ and attachment of certain relative components. In our case the subdivisions will be equal to the graphs themselves and the relative components will always be of the form of (a subgraph of) the K_5. As we know that for the characterization of irreducible graphs, it is sufficient to find the minimal basis $M_4(\mathcal{S}_2)$, we will only concentrate on this set of graphs.

Before we immerse into the concrete construction of irreducible graphs of the Klein surface, the theoretical background and the important definitions and theorems are given in Chapter 1. In Chapter 1.3 we look at all 103 graphs from $M_1(\mathcal{S}_1)$ and we show how these graphs are linked to each other and which minimal bases $M_i(\mathcal{S}_1)$, $i \in \{1, 2, 3, 4\}$ they are elements of. After the characterization of these graphs, we start with our constructions.

In Chapter 2 we will begin with the graphs, which consist of graphs from $M_4(\mathcal{S}_1)$ and (a subgraph of) the K_5 as a relative component. Disconnected graphs, as well as graphs with one and two base-points can be construted for $M_4(\mathcal{S}_2)$. We will show, which graphs, constructed in this way, are elements of $M_4(\mathcal{S}_2)$ and that none of the graphs with three base-points is element of $M_4(\mathcal{S}_2)$.

In Chapter–3 we will then only have a closer look at the graphs in $M_3(\mathcal{S}_1)$ which do not already lie in $M_4(\mathcal{S}_1)$. Regarding these graphs we only have to consider attachment of relative components to vertices, which are important for the application of the relation R_4 transforming the graphs into other ones of $M_3(\mathcal{S}_1)$ or $M_4(\mathcal{S}_1)$. The respective graphs and possible attachments of relative components are dealt with in this chapter.

Chapter 4 basically has the same purpose as the previous one, it only deals with the graphs from $M_2(\mathcal{S}_1)$ which are not already elements of $M_3(\mathcal{S}_1)$. The relative component is only attached to vertices which are important for the application of the relation R_3 that transforms the graphs into other ones of $M_2(\mathcal{S}_1)$, $M_3(\mathcal{S}_1)$ or $M_4(\mathcal{S}_1)$. In both chapters we find some graphs, which do lie in $M_4(\mathcal{S}_2)$ and we will show that all other graphs, which are constructed the same way, are not elements of $M_4(\mathcal{S}_2)$.

In Chapter 5 we will consider the growth-rate of the number of irreducible graphs for surfaces of higher genus. We will proof that the number of irreducible graphs for surfaces $\mathcal{S}_g$ of genus g grows exponentially with g, to a basis greater than 2.9.

The main result of this thesis is, that the class of graphs we are interested in, consists of 83 elements. The adjacency-lists of these 83 graphs can be found in the Appendix.

1 Theory

In this chapter we will give the necessary theory about embeddability of graphs, the method of relative components and some theorems which are of importance for the construction of irreducible graphs. We will take a closer look at the 103 irreducible graphs of the projective plane. We will also develop some tools, which will be usefull when construction irreducible graphs in the subsequent chapters. The main result of this thesis will be stated at the end of this chapter.

1.1 Graphs, the Klein surface and embeddability

In this section, we will look at the three important terms in the title of this thesis, *graphs*, the *Klein surface* and *embeddability*.

The graphs we will look at, are simple graphs without loops and parallel edges, which are defined as follows:

Definition
A simple (finite) graph G is an ordered tuple $G = (V, E)$, where V is a (finite) set of vertices and E a (finite) set of edges, with $E = \binom{V}{2}$.

Notation 1.1
We will denote vertices $v \in V$ with numbers and edges $e \in E$ between vertices v_1 and v_2 will, in abuse of notation, be written as $e = (v_1, v_2)$.

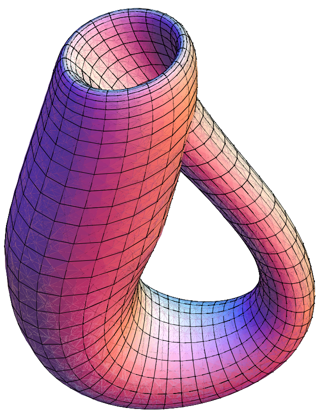


Figure 1.1: The Klein surface

The *Klein surface* is a non-orientable surface of genus two. We will always draw the Klein surface by using the fundamental polygon as drawn in Figure 1.2 just without drawing the orientation of the sides every time. The orientation of the boundaries can also be exchanged.

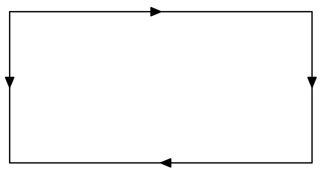


Figure 1.2: The fundamental polygon of the Klein surface

If we cut the Klein surface into two parts, cutting along the dashed lines as ahown in Figure 1.3, we get two Möbius strips. This knowledge is of great use for the construction of irreducible graphs.

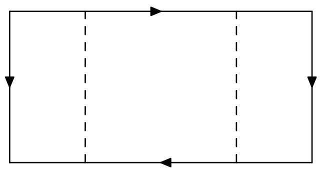


Figure 1.3: The Klein surfaces can be cut into two Möbius strips

Every graph, which is drawn within a rectangle in this thesis, is meant to be drawn on the fundamental polygon of the Klein surface, if not otherwise stated. Oftentimes we will only use the Möbius-structure of the Klein surface, meaning the left and right edges of the fundamental polygon are not passed. Every graph which is not drawn within a rectangle, is meant to be drawn on the projective plane.

The *Embeddability* of graphs is the third term in the title of this thesis and it is defined as follows:

Definition
Let $\mathcal{S}$ be a surface (a connected two-dimensional manifold) and $G = (E, V)$ be a graph. An embedding $\varepsilon : G \to \mathcal{S}$ of a graph G in $\mathcal{S}$ is a pair $\varepsilon = (\varepsilon_V, \varepsilon_E)$ of maps with the following properties:

1. ε_V maps the vertex set V injectively to $\mathcal{S}$.
2. ε_E maps the edge set E onto the set of simple curves $\gamma : [0, 1] \to \mathcal{S}$.
3. The end-vertices of the curve $\varepsilon_E(e)$ are the pictures of the end-vertices of e with $\varepsilon_V : \varepsilon_E(e)(\{0, 1\}) = \varepsilon_V(\Phi(e))$.
4. The curves $\varepsilon_E(E)$ are without crossings: $\varepsilon_E(e_1)((0, 1)) \cap \varepsilon(e_2)((0, 1)) = \emptyset$:

A well known theorem, which is important when working on embeddings of graph is *The Jordan Curve Theorem*:

Theorem 1.2 (Jordan Curve Theorem)
Any simple closed curve C in the plane divides the plane into exactly two arcwise connected components. Both of these regions have C as the boundary.

Although this theorem only applies to embeddings in the plane, it is also of great use for embeddings on surfaces of higher genus, as for each point on these surfaces there exists a neighborhood homeomorphic to the plane.

1.2 Definitions and Theorems

In this section we will give definitions and theorems, which are important for the construction of irreducible graphs. This section is mainly based on [BSW81c], [BSW81a], [BSW81b] and [BW86].

Throughout we will write Γ_0 to denote the class of all simple, finite, and undirected graphs. On the class Γ_0 we define five elementary relations R_i with $R_i \subseteq \Gamma_0 \times \Gamma_0$ and $i \in I = \{0, 1, 2, 3, 4\}$ as follows:

Definition
Let Γ be a subclass of Γ_0 and G, G' be graphs in $\Gamma \subseteq \Gamma_0$ for which we define the following operations:

The ordered pair $(G, G') \in \Gamma \times \Gamma$ belongs to the elementary relation R_i, $i \in I$, if and only if G' results from G through the i-th transformation:

R_0: An edge e or an isolated vertex v of G is deleted.

R_1: An edge e of G is contracted, the degree of at least one vertex of e equals two.

R_2: An edge e of G is contracted, the degree of each of the two vertices of e is at least three.

R_3: A vertex v of degree three in G is deleted and the three to v adjacent edges (v, v_1), (v, v_2), (v, v_3) of G are replaced by the triangle (v_1, v_2), (v_2, v_3), (v_3, v_1).

R_4: An edge (v_1, v_2) in G, v_1 and v_2 both have degree three, is selected. The edge (v_1, v_2) is divided by a new vertex v' and the transformation R_3 is applied to both of the vertices v_1 and v_2.

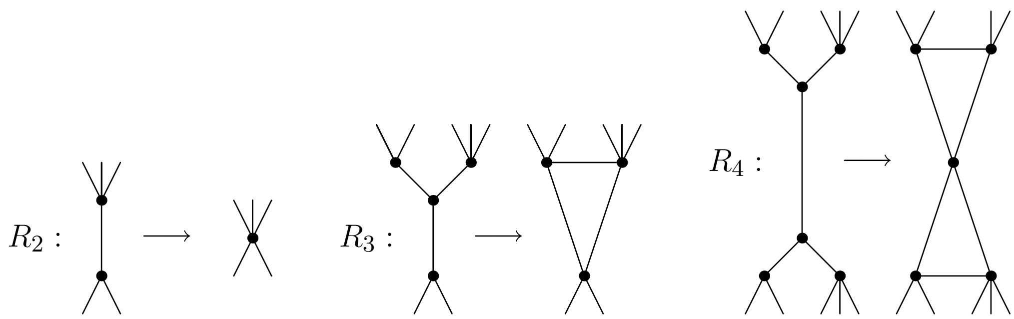


Figure 1.4: Relations R_2, R_3 and R_4

Notation 1.3
When applying these relations on a graph G, we will simply write $R_i(G)$, with $i \in \{0,1,2,3,4\}$. If we want to indicate which vertex v or edge e of G is affected by the relation, we will also write $R_i(G)_{(v)}$ or $R_i(G)_{(e)}$. For the relation R_0 we will also in short use $R_0(G)_{(e)} = G - e$ or $R_0(G)_{(v)} = G - v$. For $R_1(G)_{(e)}$ and $R_2(G)_{(e)}$ we will also write G/e.

On the basis of these five relations R_i, $i \in I$, we define a *partial order* $\succcurlyeq_i$:

Definition
Let G and G' be two graphs in an arbitrary set of graphs $\mathit{\Gamma} \neq \emptyset$. Then $G \succcurlyeq_i G'$, for $i \in I$, holds, if and only if either $G = G'$ already applies or a (finite) sequence of graphs $G_1, G_2, \ldots, G_n$ with $n \in \mathbb{N}$ and $n \geq 2$, $G_1 = G$ or $G_n = G'$ exists and every G_{m+1} for $m = 1, \ldots, n-1$ arises from G_m through application of one of the relations $R_0, \ldots, R_i$.

A sequence $G_1 R_{j_1} G_2, G_2 R_{j_2} G_3, \ldots, G_{n-1} R_{j_{n-1}} G_n$ with $G_1 = G$ and $G_n = G'$ is called a *chain* of $G \succcurlyeq_i G'$ with $i \in I$ and the individual $G_m R_{j_m} G_{m+1}$ are called *links* of this chain. We can always avoid digons in G_{m+1} by adding a link with the relation R_0 to the chain of $G \succcurlyeq_i G'$ and thus deleting one of the edges. If in particular $j_m = i$, then $R_{j_m} = R_i$ for all $m = 1, \ldots, n-1$. We call such a chain R_i*-chain* (from G to G'). We can also refer to this with the equation $G' = R_i^n(G)$, whereas with R_i^n we mean that R_i is applied to G n times.

From the characteristics of the chain, it follows directly:

Remark
Let G and G' be two graphs in $\mathit{\Gamma}_0$. If $G \succcurlyeq_i G'$ for $i \in I$ and $G \neq G'$ applies, then also a chain $G_m R_{j_m} G_{m+1}$, $m = 1, 2 \ldots, n-1$, with $G_1 = G$, $G_n = G'$ for $n \geq 2$ and $j_m \in \{0, 1, \ldots, i\}$ exists. In none of the graphs $G_{m+1} = R_{j_m}(G_m)$ parallel edges appear.

In addition, the notion of minimality is defined as follows:

Definition
Let $\mathit{\Gamma}$ be an arbitrary, non-empty set of graphs. The set $\mathit{\Gamma}$ which is partially ordered by $\succcurlyeq_i$ is written as $(\mathit{\Gamma}, \succcurlyeq_i)$. We will call the minimal graphs of $(\mathit{\Gamma}, \succcurlyeq_i)$ $\succcurlyeq_i$*-minimal* or in short *minimal*. The *minimal basis* of $(\mathit{\Gamma}, \succcurlyeq_i)$ is the set of all minimal graphs of $(\mathit{\Gamma}, \succcurlyeq_i)$, which we will denote with $M_i(\mathit{\Gamma})$ for $i \in I$. The minimal graphs in $M_i(\mathit{\Gamma})$ will also be called *irreducible*.

Proposition 1.4
For the minimal bases, the following relations apply:

$$M_4(\mathit{\Gamma}) \subseteq M_3(\mathit{\Gamma}) \subseteq M_2(\mathit{\Gamma}) \subseteq M_1(\mathit{\Gamma}) \subseteq M_0(\mathit{\Gamma}).$$

In the following $\mathcal{S}$ will stand for a closed, orientable surface $\widetilde{\mathcal{S}}_p$ of order $p \in \mathbb{N}_0$ or a closed, non-orientable surface $\mathcal{S}_q$ of order $q \in \mathbb{N}_0$. Furthermore we will use $\mathit{\Gamma}(\mathcal{S}) \subseteq \mathit{\Gamma}_0$ when talking about the set of all graphs in $\mathit{\Gamma}$ not embeddable in $\mathcal{S}$. Then $\succcurlyeq_i$ $(i \in I)$ also is a partial order on $\mathit{\Gamma}(\mathcal{S})$. The set of all minimal elements of $\mathit{\Gamma}(\mathcal{S})$ with respect to $\succcurlyeq_i$ $(i \in I)$, which is the minimal basis of $\Gamma(\mathcal{S})$ concerning $\succcurlyeq_i$, will be denoted with $M_i(\Gamma(\mathcal{S})) = M_i(\mathcal{S})$.

Analogous to proposition 1.4 the following conclusion holds:

Conclusion 1
Let $\mathcal{S}$ be an arbitrary surface. Then

$$M_4(\mathcal{S}) \subseteq M_3(\mathcal{S}) \subseteq M_2(\mathcal{S}) \subseteq M_1(\mathcal{S}) \subseteq M_0(\mathcal{S})$$

applies for the minimal bases of graphs not embeddable in $\mathcal{S}$.

Theorem 1.5
A graph $G \in \Gamma_0$ is not embeddable on a surface $\mathcal{S}$ if and only if at least one $\succcurlyeq_i$-minimal graph $G' \in M_i(\mathcal{S})$ with $G \succcurlyeq_i G'$ for $i \in \{1, 2, 3, 4\}$ exists.

Theorem 1.6
Let $\mathcal{S}$ be an arbitrary (non-)orientable surface and $\Gamma(\mathcal{S})$ the set of all graphs not embeddable into $\mathcal{S}$, which lie in Γ_0. Then $G \in \Gamma(\mathcal{S})$ and $G \in M_i(\mathcal{S})$ hold for all $i \in I$, if and only if G is not embeddable in $\mathcal{S}$ and all $R_j(G)$, for $j \leq i$, are embeddable in $\mathcal{S}$.

In order to determine elements of $M_i(\mathcal{S})$, $i \in \{1, 2, 3, 4\}$, for an arbitrary surface $\mathcal{S}$ explicitely, Theorem 1.6 is inapplicable. The following theorem is more useful in this context.

Theorem 1.7
A graph $G \in \Gamma_0$ is an element of the minimal basis $M_i(\mathcal{S})$, $i \in \{1, 2, 3, 4\}$, $\mathcal{S}$ an arbitrary surface, if and only if G fulfills the following conditions:

(1) *G is not embeddable in $\mathcal{S}$.*

(2) *Each vertex of G has degree ≥ 3.*

(3) *For each edge e of G the graph $R_0(G) = G - e$ is embeddable in $\mathcal{S}$.*

(4) *For each $j \in \{2, \ldots, i\}$ and each R_j, the graph $R_j(G)$ is embeddable in $\mathcal{S}$.*

Another important theorem, we will base this thesis on, is the following.

Theorem 1.8
If all minimal graphs $G \in M_4(\mathcal{S})$ for a surface $\mathcal{S}$ are explicitely identified, all graphs of the minimal bases $M_i(\mathcal{S})$, $i \in \{1, 2, 3\}$ can be constructed by application of $R_4^{-n_4}$, $R_3^{-n_3}$, $R_2^{-n_2}$ on the graphs $G \in M_4(\mathcal{S})$.

Consequently for the consideration of the minimal bases of $\mathcal{S}$ it is sufficient to identify all graphs of the minimal basis $M_4(\mathcal{S})$. This is very convenient considering the large number of minimal graphs for different surfaces. The minimal basis $M_1(\mathcal{S}_1)$ of the projective plane consists of 103 graphs, $M_4(\mathcal{S}_1)$ however only consists of twelve graphs.

The method of relative components

The method of relative components describes how we can construct irreducible graphs ([BSW85]).

Let G' and G'' be two graphs with $G' \subseteq G$ and G'' subgraph of G, which is spanned by all vertices of G which do not belong to G'. The graph G consequently is composed of G', G'' and certain edges of G. These edges are

(a) edges which themselves do not lie in G', but their end-vertices both lie in G', thus all *chords* of G' and

(b) all edges of G which have one end-vertex in G' and the other end-vertex in G'', thus all *bridges* from G' to G''.

Each component of G'', including its bridges to G' (end-vertices of the bridges in G' included) will be denoted by *relative component* Q of G with respect to G'. The endpoints of the bridges, which lie in G', will be denoted with *base-points* of Q, the respective component of G'' will be denoted with *center of* Q. If the center of Q consists of only one vertex, then Q is a *star*. The number of vertices in the center of Q is also denoted with *order* of Q. The chords, mentioned in (a), thus can also be referred to as relative components of order zero.

$$G = G' \cup Q_1 \cup Q_2 \cup \ldots \cup Q_n \cup \{k_1, k_2, \ldots, k_m\}. \tag{1.1}$$

The $Q_1, Q_2, \ldots, Q_n$ ($n \in \mathbb{N}$) are the relative components of G, relative to G', with order ≥ 1 and the $k_1, k_2, \ldots, k_m$ ($m \in \mathbb{N}$) are the chords of G'.

Let $\mathcal{S}''$ and $\mathcal{S}'$ be two surfaces with the characteristic, that each graph embeddable in $\mathcal{S}''$ is also embeddable in $\mathcal{S}'$, consequently $\mathcal{S}''$ lies below $\mathcal{S}'$ concerning embeddability. This, as an example, applies for $\mathcal{S}' := \mathcal{S}_n$ and $\mathcal{S}'' := \mathcal{S}_m$ with $m \leq n$.

If we assume that all graphs of $M_1(\mathcal{S}'')$ are explicitely known, we could ask ourselves, how the graphs in $M_1(\mathcal{S}')$ can be constructed. We choose an arbitrary graph H from $M_1(\mathcal{S}')$. According to the definition of $M_1(\mathcal{S}')$, H cannot be embedded into $\mathcal{S}'$ and thus, according to the condition for $\mathcal{S}''$, can also not be embedded into $\mathcal{S}''$. Thus from the definition of $M_1(\mathcal{S}'')$ follows that a graph $G \in M_1(\mathcal{S}'')$ with $H \succcurlyeq_1 G$ exists, which means that the graph H contains a subdivision $U(G)$ of G. Consequently, using equation (1.1), we can find a representation of H:

$$H = U(G) \cup Q_1 \cup Q_2 \cup \ldots \cup Q_n \cup \{k_1, k_2, \ldots, k_m\} \tag{1.2}$$

with $m, n \in \mathbb{N}$.

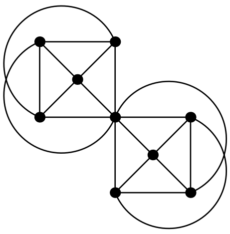
Consequently the following Theorem holds:

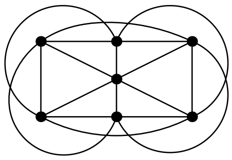
Theorem 1.9
If a surface $\mathcal{S}''$ lies below a surface $\mathcal{S}'$ concerning embeddability, a graph H of $M_1(\mathcal{S}')$ can be constructed by sub-division of a graph G from $M_1(\mathcal{S}'')$ and adjunction of certain relative components of order ≥ 0 onto this $U(G)$, as in equation (1.2).

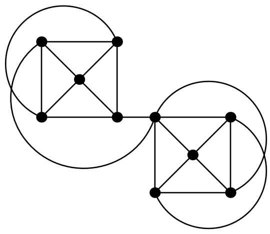
1.3 103 irreducible graphs for the projective plane

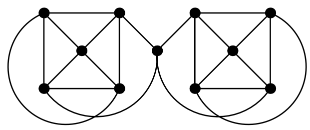
In this section, we will look at the 103 irreducible gaphs for the projective plane as they were published in [GHW79] and we will characterize these concerning the different minimal bases $M_1(\mathcal{S}_1), \ldots, M_4(\mathcal{S}_1)$. We will also give a genealogy of these 103 graphs with regard to their interdependencies.

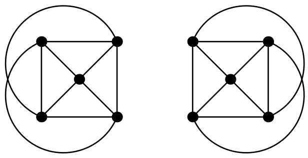
Figure 1.5 shows all 103 irreducible graphs for the projective plane. To simplify matters, the graphs are labeled the same way as done in [GHW79].

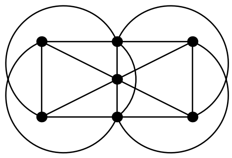

(1) $A_1 \in M_4(\mathcal{S}_1)$

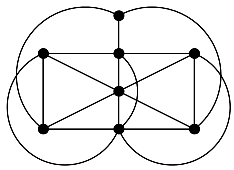

(2) $A_2 \in M_4(\mathcal{S}_1)$

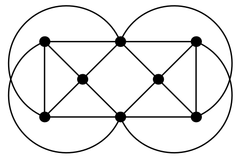

(3) $A_3 \in M_1(\mathcal{S}_1)$

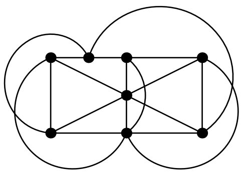

(4) $A_4 \in M_1(\mathcal{S}_1)$

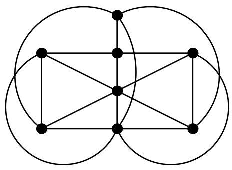

(5) $A_5 \in M_4(\mathcal{S}_1)$

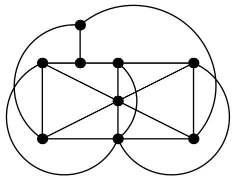

(6) $B_1 \in M_4(\mathcal{S}_1)$

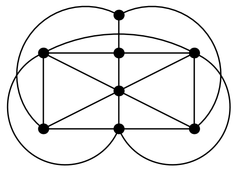

(7) $B_2 \in M_1(\mathcal{S}_1)$

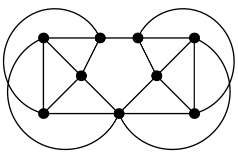

(8) $B_3 \in M_4(\mathcal{S}_1)$

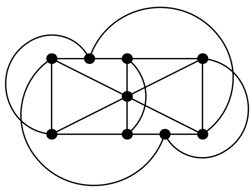

(9) $B_4 \in M_1(\mathcal{S}_1)$

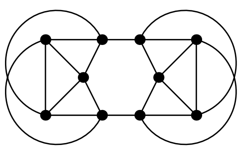

(10) $B_5 \in M_1(\mathcal{S}_1)$

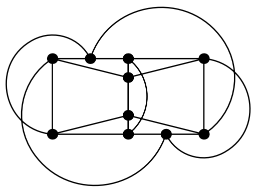

(11) $B_6 \in M_1(\mathcal{S}_1)$

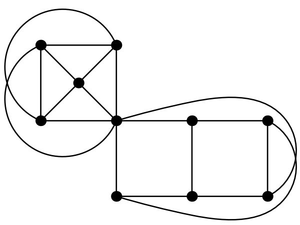

(12) $B_7 \in M_2(\mathcal{S}_1)$

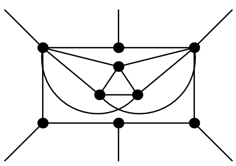

(13) $B_8 \in M_1(\mathcal{S}_1)$

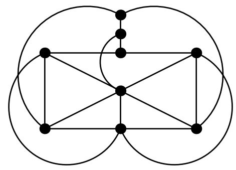

(14) $B_9 \in M_1(\mathcal{S}_1)$

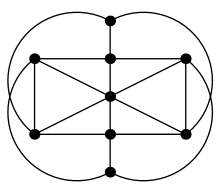

(15) $B_{10} \in M_1(\mathcal{S}_1)$

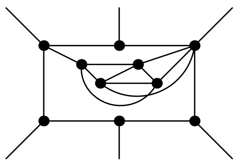

(16) $B_{11} \in M_1(\mathcal{S}_1)$


(17) $C_1 \in M_3(\mathcal{S}_1)$

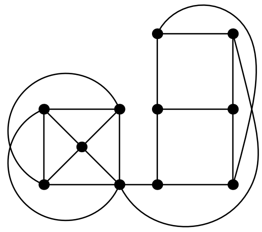

(18) $C_2 \in M_2(\mathcal{S}_1)$


(19) $C_3 \in M_2(\mathcal{S}_1)$

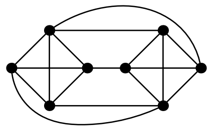

(20) $C_4 \in M_2(\mathcal{S}_1)$



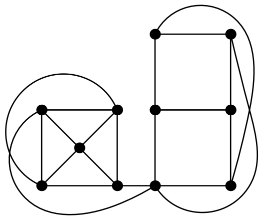
(21) $C_5 \in M_1(\mathcal{S}_1)$



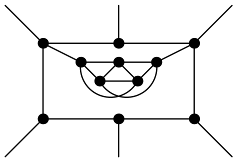
(22) $C_6 \in M_1(\mathcal{S}_1)$



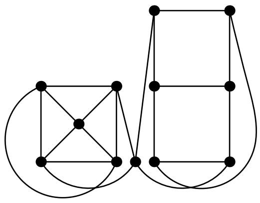
(23) $C_7 \in M_4(\mathcal{S}_1)$



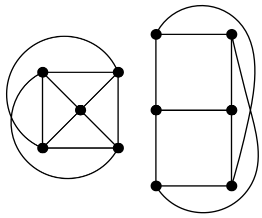
(24) $C_8 \in M_1(\mathcal{S}_1)$



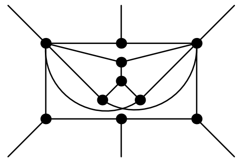
(25) $C_9 \in M_1(\mathcal{S}_1)$



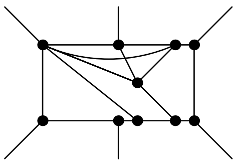
(26) $C_{10} \in M_1(\mathcal{S}_1)$



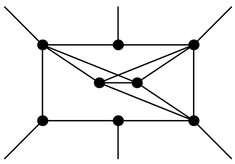
(27) $C_{11} \in M_3(\mathcal{S}_1)$



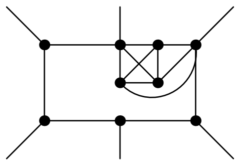
(28) $D_1 \in M_2(\mathcal{S}_1)$



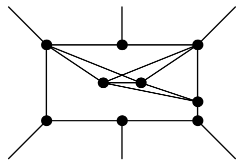
(29) $D_2 \in M_2(\mathcal{S}_1)$



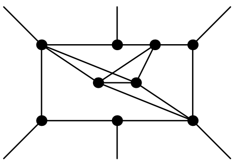
(30) $D_3 \in M_3(\mathcal{S}_1)$



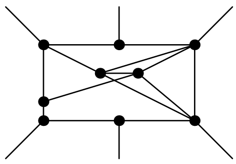
(31) $D_4 \in M_3(\mathcal{S}_1)$



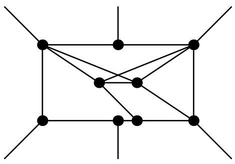
(32) $D_5 \in M_1(\mathcal{S}_1)$



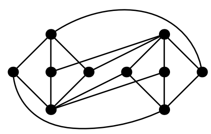
(33) $D_6 \in M_1(\mathcal{S}_1)$



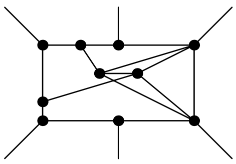
(34) $D_7 \in M_1(\mathcal{S}_1)$



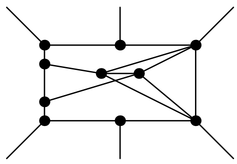
(35) $D_8 \in M_1(\mathcal{S}_1)$



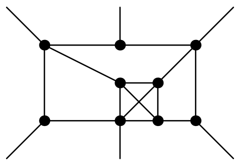
(36) $D_9 \in M_4(\mathcal{S}_1)$



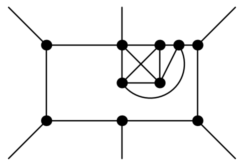
(37) $D_{10} \in M_1(\mathcal{S}_1)$



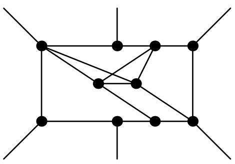
(38) $D_{11} \in M_1(\mathcal{S}_1)$



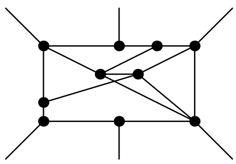
(39) $D_{12} \in M_4(\mathcal{S}_1)$



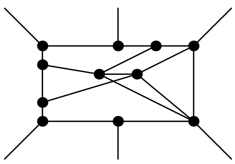
(40) $D_{13} \in M_1(\mathcal{S}_1)$



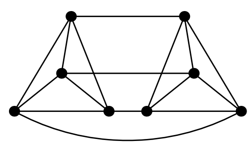
(41) $D_{14} \in M_1(\mathcal{S}_1)$



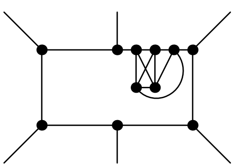
(42) $D_{15} \in M_1(\mathcal{S}_1)$



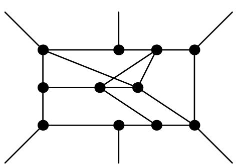
(43) $D_{16} \in M_1(\mathcal{S}_1)$



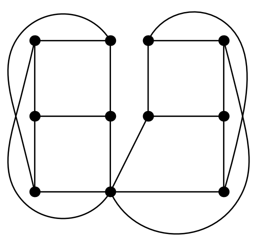
(44) $D_{17} \in M_4(\mathcal{S}_1)$



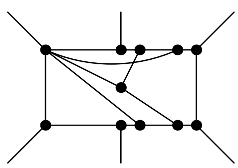
(45) $D_{18} \in M_1(\mathcal{S}_1)$



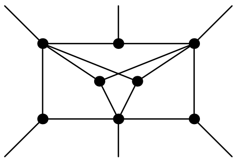
(46) $D_{19} \in M_1(\mathcal{S}_1)$

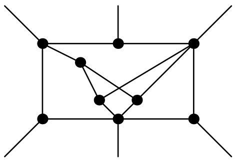


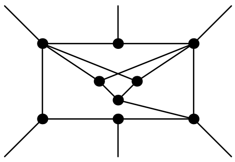
(47) $E_1 \in M_3(\mathcal{S}_1)$

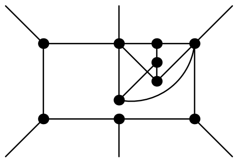


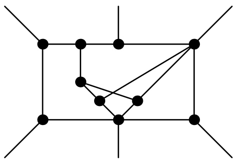
(48) $E_2 \in M_2(\mathcal{S}_1)$

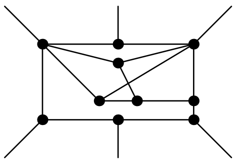

(49) $E_3 \in M_4(\mathcal{S}_1)$

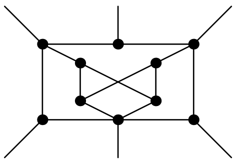

(50) $E_4 \in M_1(\mathcal{S}_1)$

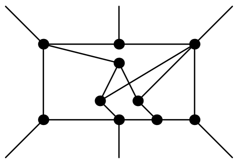

(51) $E_5 \in M_2(\mathcal{S}_1)$

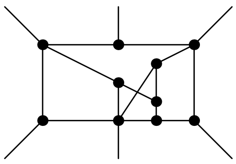

(52) $E_6 \in M_2(\mathcal{S}_1)$

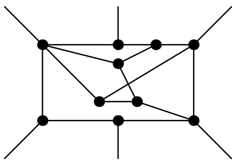

(53) $E_7 \in M_1(\mathcal{S}_1)$

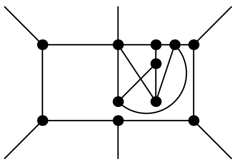

(54) $E_8 \in M_1(\mathcal{S}_1)$

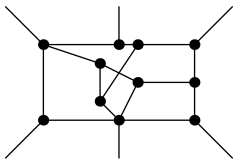

(55) $E_9 \in M_1(\mathcal{S}_1)$

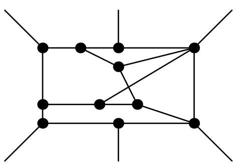

(56) $E_{10} \in M_1(\mathcal{S}_1)$

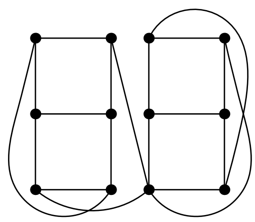

(57) $E_{11} \in M_2(\mathcal{S}_1)$

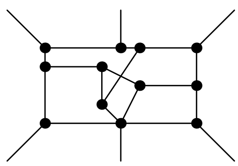

(58) $E_{12} \in M_1(\mathcal{S}_1)$

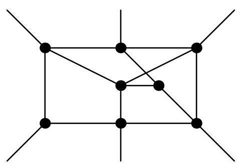

(59) $E_{13} \in M_1(\mathcal{S}_1)$

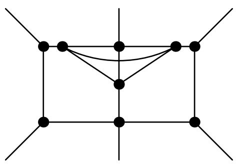

(60) $E_{14} \in M_1(\mathcal{S}_1)$

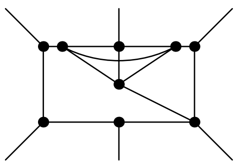

(61) $E_{15} \in M_1(\mathcal{S}_1)$

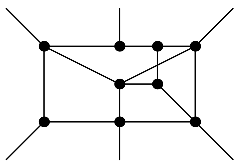

(62) $E_{16} \in M_1(\mathcal{S}_1)$

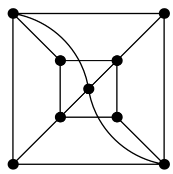

(63) $E_{17} \in M_1(\mathcal{S}_1)$

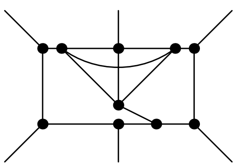

(64) $E_{18} \in M_4(\mathcal{S}_1)$


(65) $E_{19} \in M_3(\mathcal{S}_1)$


(66) $E_{20} \in M_2(\mathcal{S}_1)$


(67) $E_{21} \in M_1(\mathcal{S}_1)$


(68) $E_{22} \in M_4(\mathcal{S}_1)$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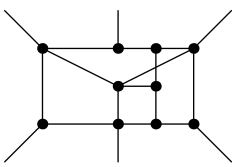
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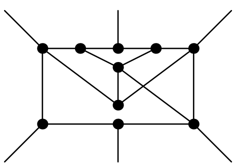
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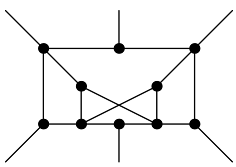
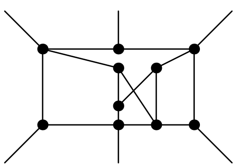

(72) $E_{26} \in M_1(\mathcal{S}_1)$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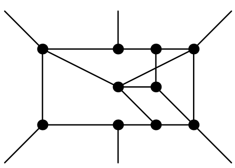
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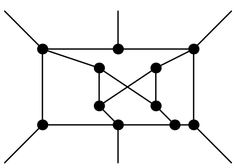
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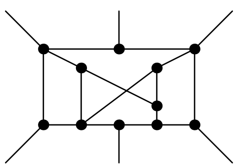
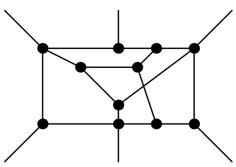
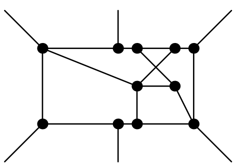
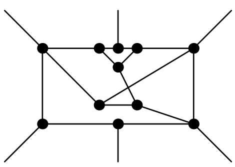
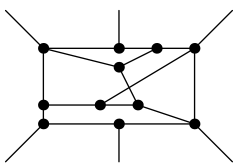
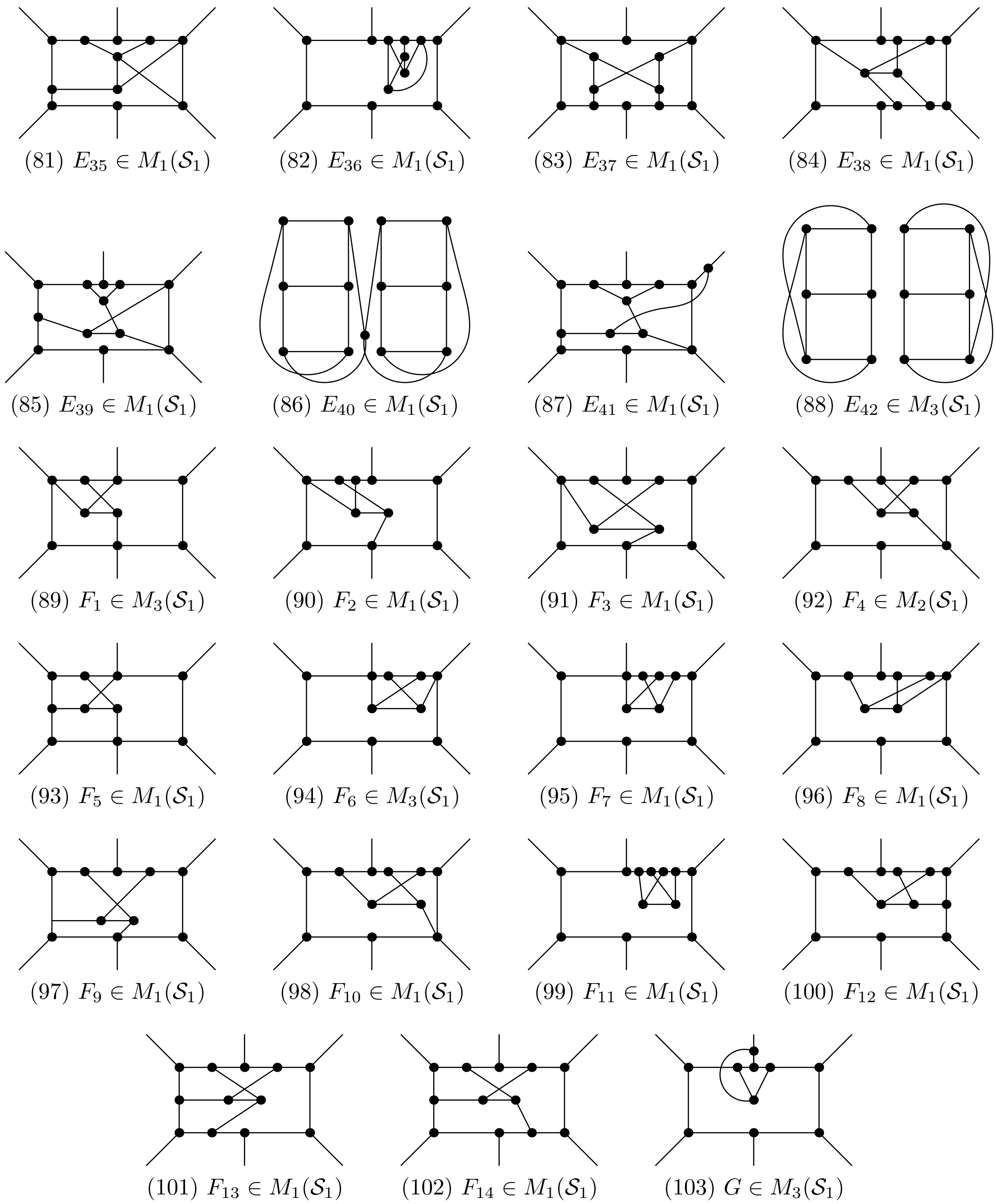

(76) $E_{30} \in M_1(\mathcal{S}_1)$


(77) $E_{31} \in M_1(\mathcal{S}_1)$


(78) $E_{32} \in M_1(\mathcal{S}_1)$


(79) $E_{33} \in M_1(\mathcal{S}_1)$


(80) $E_{34} \in M_1(\mathcal{S}_1)$



(81) $E_{35} \in M_1(\mathcal{S}_1)$ (82) $E_{36} \in M_1(\mathcal{S}_1)$ (83) $E_{37} \in M_1(\mathcal{S}_1)$ (84) $E_{38} \in M_1(\mathcal{S}_1)$

(85) $E_{39} \in M_1(\mathcal{S}_1)$ (86) $E_{40} \in M_1(\mathcal{S}_1)$ (87) $E_{41} \in M_1(\mathcal{S}_1)$ (88) $E_{42} \in M_3(\mathcal{S}_1)$

(89) $F_1 \in M_3(\mathcal{S}_1)$ (90) $F_2 \in M_1(\mathcal{S}_1)$ (91) $F_3 \in M_1(\mathcal{S}_1)$ (92) $F_4 \in M_2(\mathcal{S}_1)$

(93) $F_5 \in M_1(\mathcal{S}_1)$ (94) $F_6 \in M_3(\mathcal{S}_1)$ (95) $F_7 \in M_1(\mathcal{S}_1)$ (96) $F_8 \in M_1(\mathcal{S}_1)$

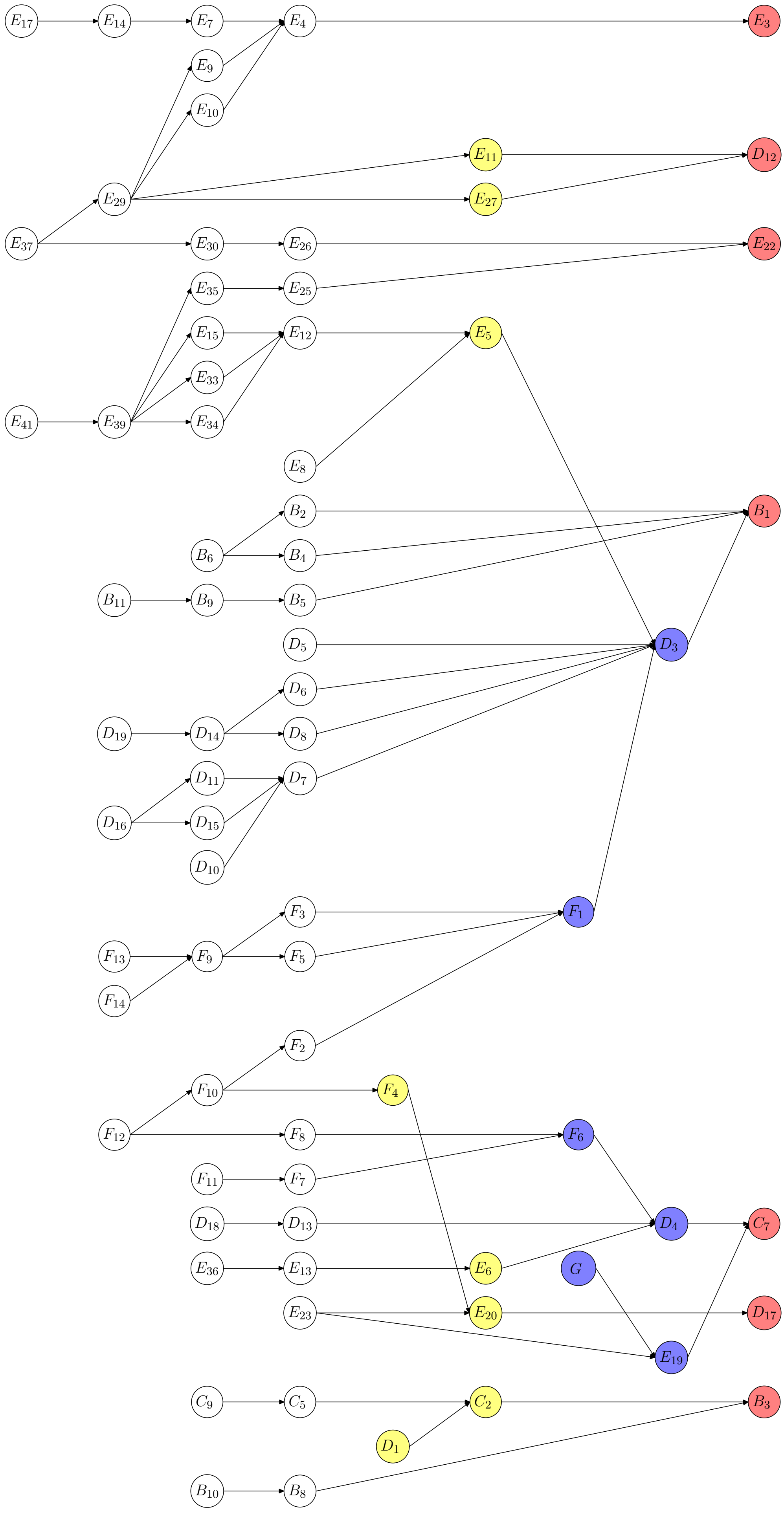
(97) $F_9 \in M_1(\mathcal{S}_1)$ (98) $F_{10} \in M_1(\mathcal{S}_1)$ (99) $F_{11} \in M_1(\mathcal{S}_1)$ (100) $F_{12} \in M_1(\mathcal{S}_1)$

(101) $F_{13} \in M_1(\mathcal{S}_1)$ (102) $F_{14} \in M_1(\mathcal{S}_1)$ (103) $G \in M_3(\mathcal{S}_1)$

Figure 1.5: All graphs in $M_1(\mathcal{S}_1)$ and their minimal bases $M_i(\mathcal{S}_1)$, $i \in \{1, 2, 3, 4\}$

For the construction of irreducible graphs of the Klein surface, it is of great use to know which of these 103 graphs lie in which of the minimal bases for the projective plane, as stated in figure 1.5.

Figure 1.6 in addition shows a genealogy of these graphs. The arrows represent a relation which transforms one graph into another. If the relation R_i for $i \in \{1, 2, 3, 4\}$ is applied to a graph in M_i which is not also an element of M_{i+1} (in short $M_i - M_{i+1}$), for $i \in \{1, 2, 3\}$, the resulting graph is either also an element of $M_i - M_{i+1}$ or M_k for $k > i$.


E_{17}
E_{14}
E_7
E_4
E_3
E_9
E_{10}
E_{11}
D_{12}
E_{29}
E_{27}
E_{37}
E_{30}
E_{26}
E_{22}
E_{35}
E_{25}
E_{15}
E_{12}
E_5
E_{33}
E_{41}
E_{39}
E_{34}
E_8
B_2
B_1
B_6
B_4
B_{11}
B_9
B_5
D_5
D_3
D_6
D_{19}
D_{14}
D_8
D_{11}
D_7
D_{16}
D_{15}
D_{10}
F_3
F_1
F_{13}
F_9
F_5
F_{14}
F_2
F_{10}
F_4
F_{12}
F_8
F_6
F_{11}
F_7
D_{18}
D_{13}
D_4
C_7
E_{36}
E_{13}
E_6
G
E_{23}
E_{20}
D_{17}
E_{19}
C_9
C_5
C_2
B_3
D_1
B_{10}
B_8

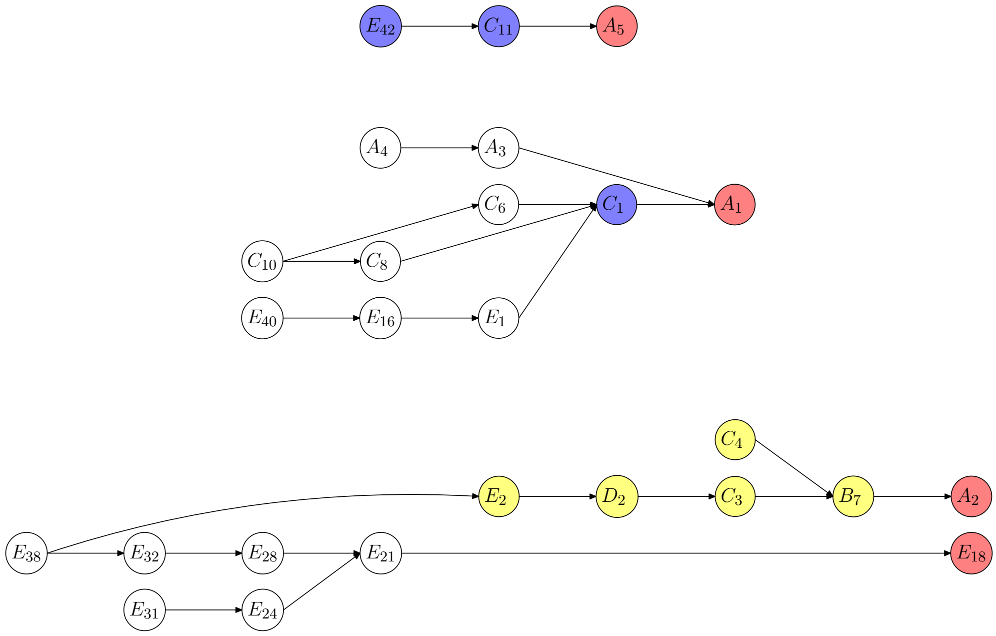


Figure 1.6: Genealogy of graphs in $M_1(\mathcal{S}_1)$

Figure 1.7: Color-coding for Figure 1.6

1.4 Toolbox

In this section, we will develop some generel criteria, which help with the construction of graphs in $M_4(\mathcal{S}_2)$. We will also find possibilities to reduce the number of cases, we have to consider.

The Möbius strip is obviously homeomorphic to the punctured projective plane (an arbitrary point of the projective plane is removed). If we want to show that a certain face is the outer face of the Möbius strip, it is consequently enough to show that this face exists on the projective plane. If we delete one point within this face, this face is homeomorphic to the outer face of the Möbius strip.

We can easily reduce the number of graphs, we have to consider for the construction of irreducible graphs by using the following lemma:

Lemma 1.10
For the construction of graphs in $M_4(\mathcal{S}_2)$, we will not have to consider any graphs in $M_1(\mathcal{S}_1)$, which are not also elements of $M_2(\mathcal{S}_1)$.

Proof
The relation R_2 and the attachment of a relative component are commutable, meaning that it does not make a difference whether we attach the relative component first and

apply R_2 in a second step, or if we do it the other way round. Consequently none of the graphs construted by attaching a relative component to a graph in $M_1(\mathcal{S}_1) - M_2(\mathcal{S}_1)$ can be an element of $M_4(\mathcal{S}_2)$, as the attachment of a relative component can never foreclose the transformation of a graph in $M_1(\mathcal{S}_1) - M_2(\mathcal{S}_1)$ into another (smaller) irreducible graph of the projective-plane. □

A further help for reduction of the number of cases which have to be considered, is that we use vertex- and edge-orbits. When applying one relation $R_0, \ldots, R_4$ on a graph, it is sufficient to apply it to one representative of each vertex- or edge-orbit. The vertex- and edge-orbits of the graphs in $M_2(\mathcal{S}_1)$ are listed in the appendix.

We can also show some more characteristics, which we will use to show whether a graph is an element of $M_4(\mathcal{S}_2)$ or not.

Lemma 1.11
Let C_1, C_2 be two cycles in a graph G with the property that $C_1 - e$ is a path in C_2. If G is not embeddable on a surface $\mathcal{S}$ with the condition, that C_1 is the boundary of one face, it is also not possible for C_2.

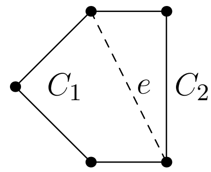


Figure 1.8: C_1 and C_2

Proof
It is ovious that, if G can be embedded with the property that the vertices and edges of C_2 lie on the boundary of one face, C_1 can also function as the boundary of a face, as the edge e can be drawn within the face C_2 is a boundary of. □

Lemma 1.12
Let G be a graph with $G - e$ embeddable into the projective plane, $e = (v_1, v_2)$. Let H be a graph constructed by attachment of a $K_5 - e'$, $e' = (v_i, v_j)$, to G with $v_1 = v_i$ and $v_2 = v_j$. These identified vertices thus are the base points of the relative component $K_5 - e'$. If H is irreducible for the Klein surface, G is embeddable into the projective plane.

Proof
If H is irreducible for the Klein surface, $H - e$ is embeddable into the Klein surface for an arbitrary edge e. Obviously v_1 and v_2 have to lie in the same face for every embedding of $G - e$ into the projective plane and thus the edge e can be added to the embedding of $G - e$ and G itself is embeddable into the projective plane. □

Corollary 1.13
Let G be a graph which is irreducible for the projective plane and $e = (v_1, v_2)$ an edge in G. Let H be a graph which consists of G and a relative component $K_5 - e'$, $e' = (v_i, v_j)$), with v_1, v_2 base-points of the relative component, which are identified with vertices v_i and v_j respectively. As a consequence of Lemma 1.12 the graph H cannot be irreducible for the Klein surface.

Corollary 1.14
As a consequence of Corollary 1.13, the attachment of relative components to vertices which are adjacent in irreducible graphs of the projective plane, do not have to be considered when constructing irreducible graphs of the Klein surface.

It is known, that for or a non-orientable surface $\mathcal{S}_g$ with genus g, $\chi(\mathcal{S}) = 2 - g$ is the Euler-characteristic and that the following lemma holds ([MT01]):

Lemma 1.15
Let $\mathcal{S}$ be a non-orientable surface and G a graph embeddable on $\mathcal{S}$. Then

$$\chi(\mathcal{S}) = f - v + e$$

holds, with v number of vertices, e number of edges and f number of faces.

The Euler-characteristic for the projective-plane is $\chi(\mathcal{S}_1) = 1$ and for the Klein surface it is $\chi(\mathcal{S}_2) = 0$.

Corollary 1.16
The Euler-characteristic bounds face sizes of an embedding of a graph.

Example 1
If we want to embed the $K_{3,4}$ into the projective plane, we can use the Euler-characteristic to calculate the number of faces the embedding of $K_{3,4}$ has.

$$\begin{aligned} \chi(\mathcal{S}_1) = 1 &= f - 12 + 7 \\ \Rightarrow \qquad f &= 6 \end{aligned}$$

As the bipartite graph $K_{3,4}$ does not have any cycles of length three, each face of the embedding has to have at least size four. Also each edge can at most lie on the boundary of two faces. Consequently the six faces all have size 4.

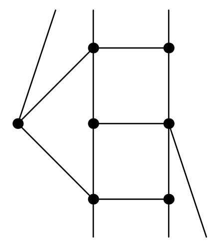


Figure 1.9: $K_{3,4}$ embeddeded into the projective plane

Throughout this thesis, we will regularly use colored vertices and edges to illustrate certain information. If we draw a graph with its vertex-orbits, the vertices in one orbit will be drawn in the same color. The black vertices will always indicate individual orbits.

An example of a graph and its vertex orbits, is illustrated in Figure 1.10. The set of vertex-orbits of this graph is: $\big\{\{1,3\},\{2,4\},\{5\},\{6\},\{7\},\{8,9\}\big\}$

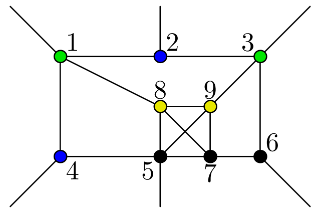


Figure 1.10: A graph and its vertex-orbits

If we want to proof that a certain graph is not in the minimal basis of irreducible graphs for the Klein surface, we will most of the times try to find a cycle including the required vertices (base points for the relative component) and show that the graph is not embeddable having this cycle as the boundary of one face. To show which cycle we are looking at, we will also use colors for the required vertices, necessary edges and the ones which cannot be included.

Example 2
A cycle in a graph will be drawn like this:

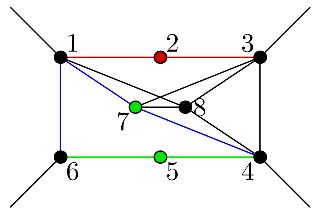


Figure 1.11: A cycle in $D_3 - (2, 5)$

In this case the edge $(2, 5)$ was deleted. As we want to find a cycle including vertices 5 and 7, these vertices are colored green. We already know from Corollary 1.13 that the vertices 2 and 5 cannot lie on the boundary of one face. Consequently vertex 2 cannot be included in the cycle. This vertex is colored red. In the next step the edges, which have to be included in the cycle, in this case $(4, 5)$ and $(5, 6)$, are colored green, and the edges which cannot be included, in this case $(1, 2)$ and $(2, 3)$, are colored red. The remaining edges of the cycle we are looking at, will always be colored blue.

1.5 The result

In this thesis, we want to construct graphs in $M_4(\mathcal{S}_2)$, with the property that these graphs consist of a graph of $M_1(\mathcal{S}_1)$ and a subgraph of the K_5 as a relative component. The following theorem summarizes the results of the subsequent chapters:

Theorem 1.17
The graphs $G_1, \ldots, G_{83}$, which are constructed using the method of relative components with graphs of $M_1(\mathcal{S}_1)$ and a subgraph of the K_5 as a relative component, are elements of $M_4(\mathcal{S}_2)$.

At this point we will only proof that the graphs $G_1, \ldots, G_{83}$ are not embeddable into the Klein surface:

Proof
It is easy to show that the graphs $G_1, \ldots, G_{83}$ are not embeddable into the Klein surface. The graphs in $M_4(\mathcal{S}_1)$ are irreducible for the projective plane and thus only leave an elementary face, when embedded into the Klein surface. The K_5 is irreducible for the plane and consequently cannot be embedded in the elementary face. Thus the graphs $G_1, \ldots, G_{83}$ cannot be embedded into the Klein surface.

The second part of the proof, namely the minimality of the graphs $G_1, \ldots, G_{83}$ will be done in the subsequent chapters. □

The adjacency-lists of the graphs $G_1, \ldots, G_{83}$ can be found in the Appendix.

In the subsequent chapters we will proof that the graphs $G_1, \ldots, G_{83}$ are irreducible for the Klein surface. In addition we will also show that the following theorem holds:

Theorem 1.18
The graphs $G_1, \ldots, G_{83}$ are the only graphs in $M_4(\mathcal{S}_2)$, which consist of a graph in $M_1(\mathcal{S}_1)$ and a subgraph of the K_5 as a relative component.

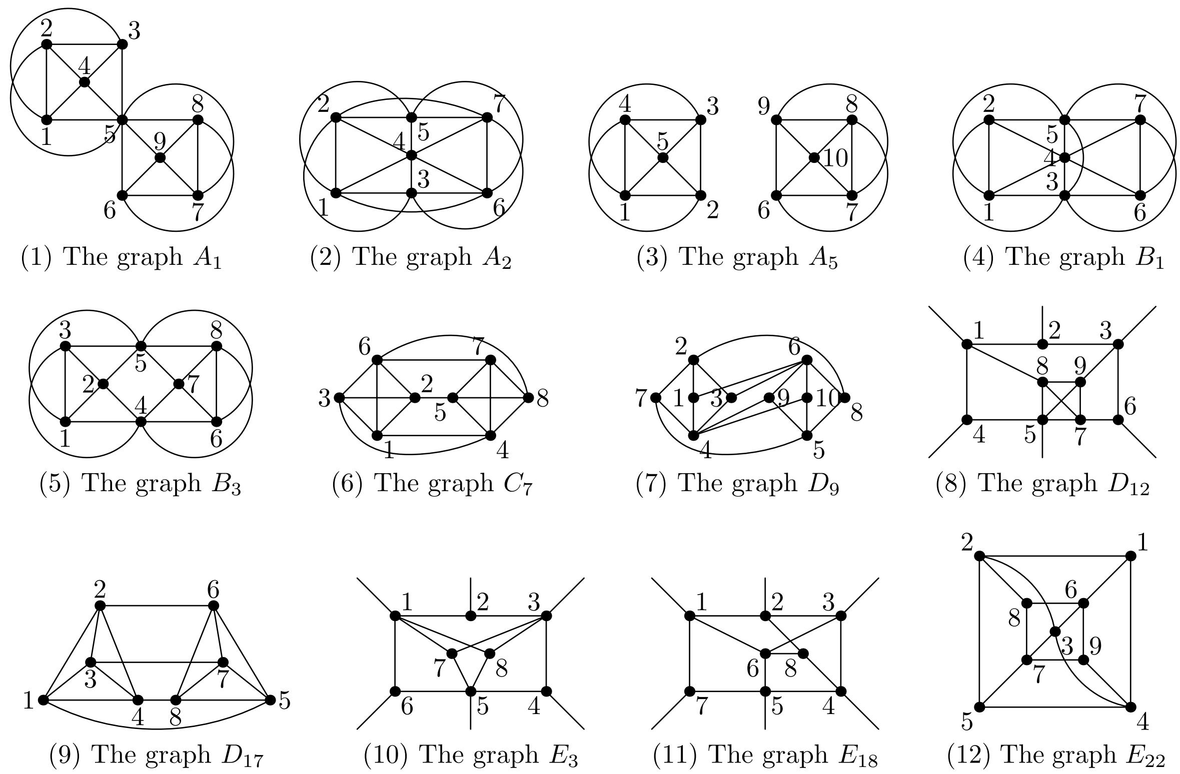
This theorem will be a side-result of the research done in the subsequent chapters. As we will study every possible attachment of a subgraph of the K_5 to each graph in $M_1(\mathcal{S}_1)$ and show which of these are elements of $M_4(\mathcal{S}_2)$ and which are not irreducible, no other graph constructed the same way, can be element of $M_4(\mathcal{S}_2)$. As shown in Lemma 1.10 it is enough to consider attachments of relative components to graphs in $M_2(\mathcal{S}_1)$, as will be done in the subsequent chapters.

2 $M_4(\mathcal{S}_1)$ to $M_4(\mathcal{S}_2)$

In this chapter we will construct all graphs in $M_4(\mathcal{S}_2)$, which consist of a graph from $M_4(\mathcal{S}_1)$ and a relative component, (a subgraph of) a K_5. In the different sections we will look at different constructions of these graphs, namely the number of base-points the subgraph of the K_5 is attached to.

2.1 The minimal basis $M_4(\mathcal{S}_1)$ of the projective plane

Before we start with the construction of irreducible graphs in $M_4(\mathcal{S}_2)$, we will take a closer look at the graphs of the minimal basis $M_4(\mathcal{S}_1)$.



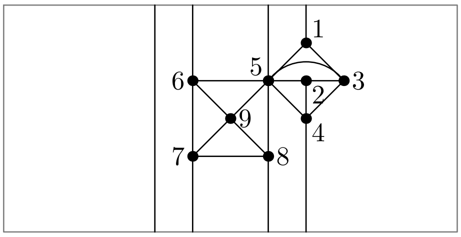
(1) The graph A_1 (2) The graph A_2 (3) The graph A_5 (4) The graph B_1

(5) The graph B_3 (6) The graph C_7 (7) The graph D_9 (8) The graph D_{12}

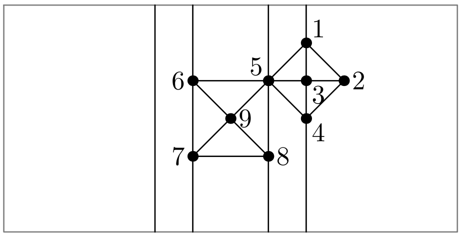
(9) The graph D_{17} (10) The graph E_3 (11) The graph E_{18} (12) The graph E_{22}

Figure 2.1: The twelve graphs in $M_4(\mathcal{S}_1)$

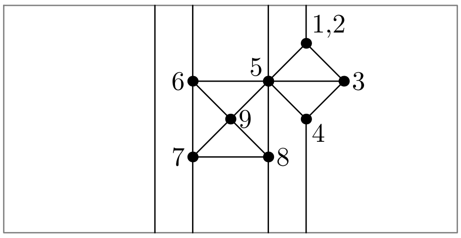
As already known from Chapter 1.3 there are 12 graphs in $M_4(\mathcal{S}_1)$, which are shown in Figure 2.1. As these graphs are minimally not embeddable into the projective plane, they are also minimally not embeddable into the Möbius strip. Thus, as soon as one of the relations $R_0, \ldots, R_4$ is applied, these graphs are embeddable into the Möbius strip, as can be seen in Figures 2.2 - 2.13.



(1) $A_1 - (1, 2)$

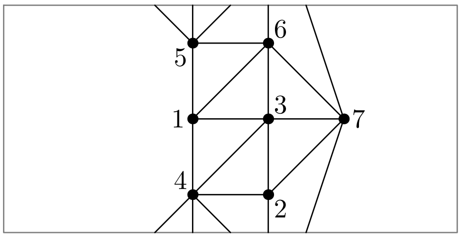


(2) $A_1 - (2, 5)$

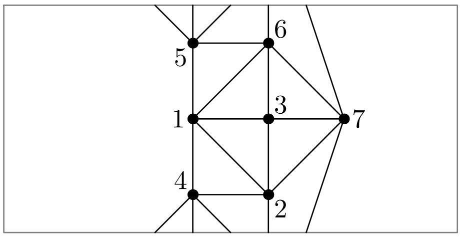


(3) $A_1/(1, 2)$

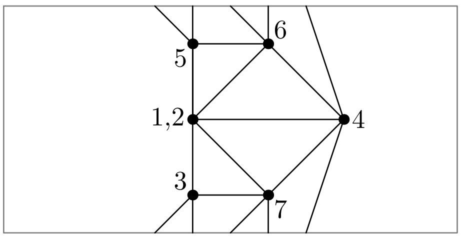
Figure 2.2: Embeddings of $R_i(A_1)$, $i \in \{0, 2\}$, into the Klein surface



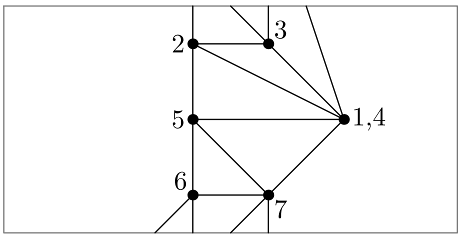
(1) $A_2 - (1, 2)$



(2) $A_2 - (3, 4)$

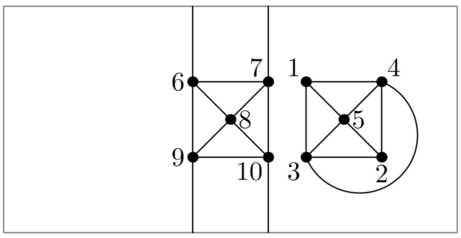


(3) $A_2/(1, 2)$

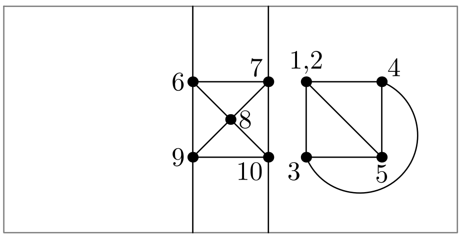


(4) $A_2/(1, 4)$

Figure 2.3: Embeddings of $R_i(A_2)$, $i \in \{0, 2\}$, into the Klein surface

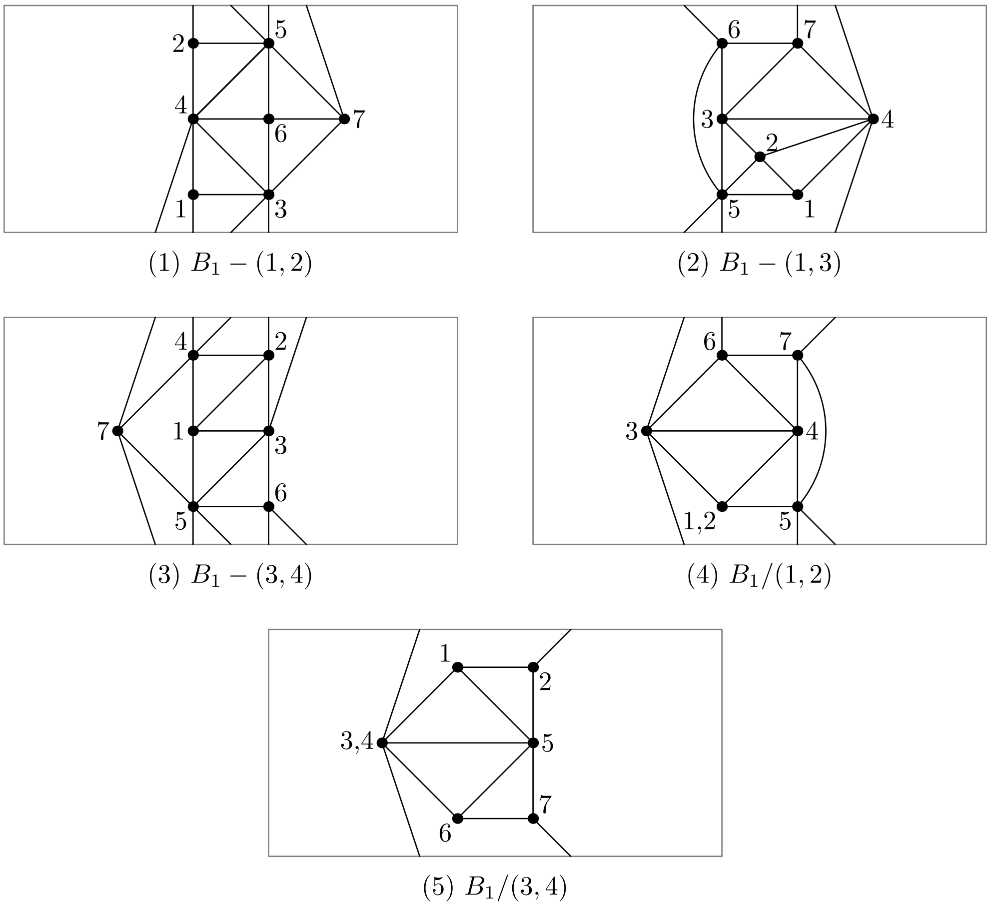


(1) $A_5 - (1, 2)$



(2) $A_5/(1, 2)$

Figure 2.4: Embeddings of $R_i(A_5)$, $i \in \{0, 2\}$, into the Klein surface

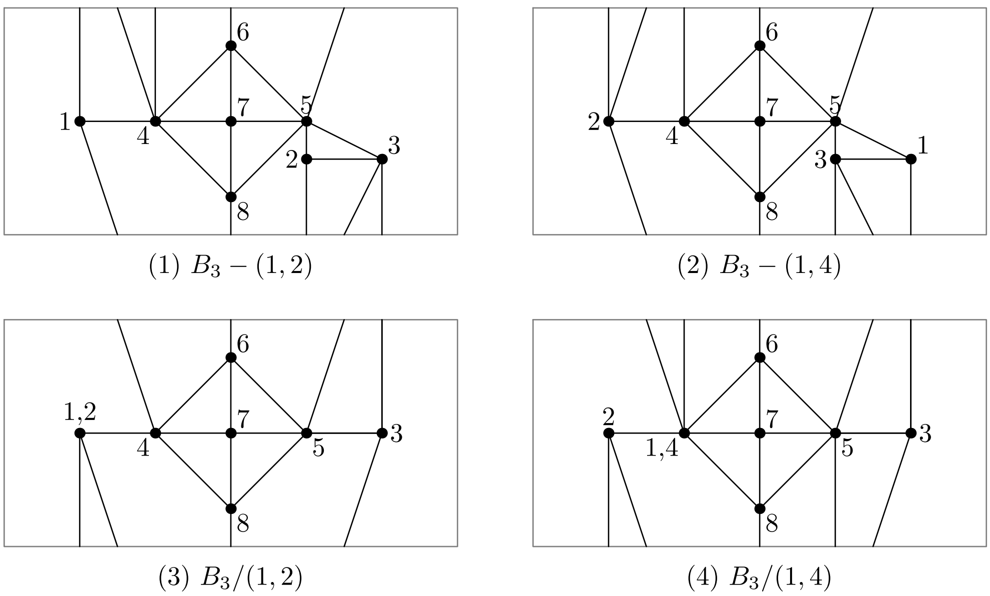


(1) $B_1 - (1,2)$ (2) $B_1 - (1,3)$

(3) $B_1 - (3,4)$ (4) $B_1/(1,2)$

(5) $B_1/(3,4)$

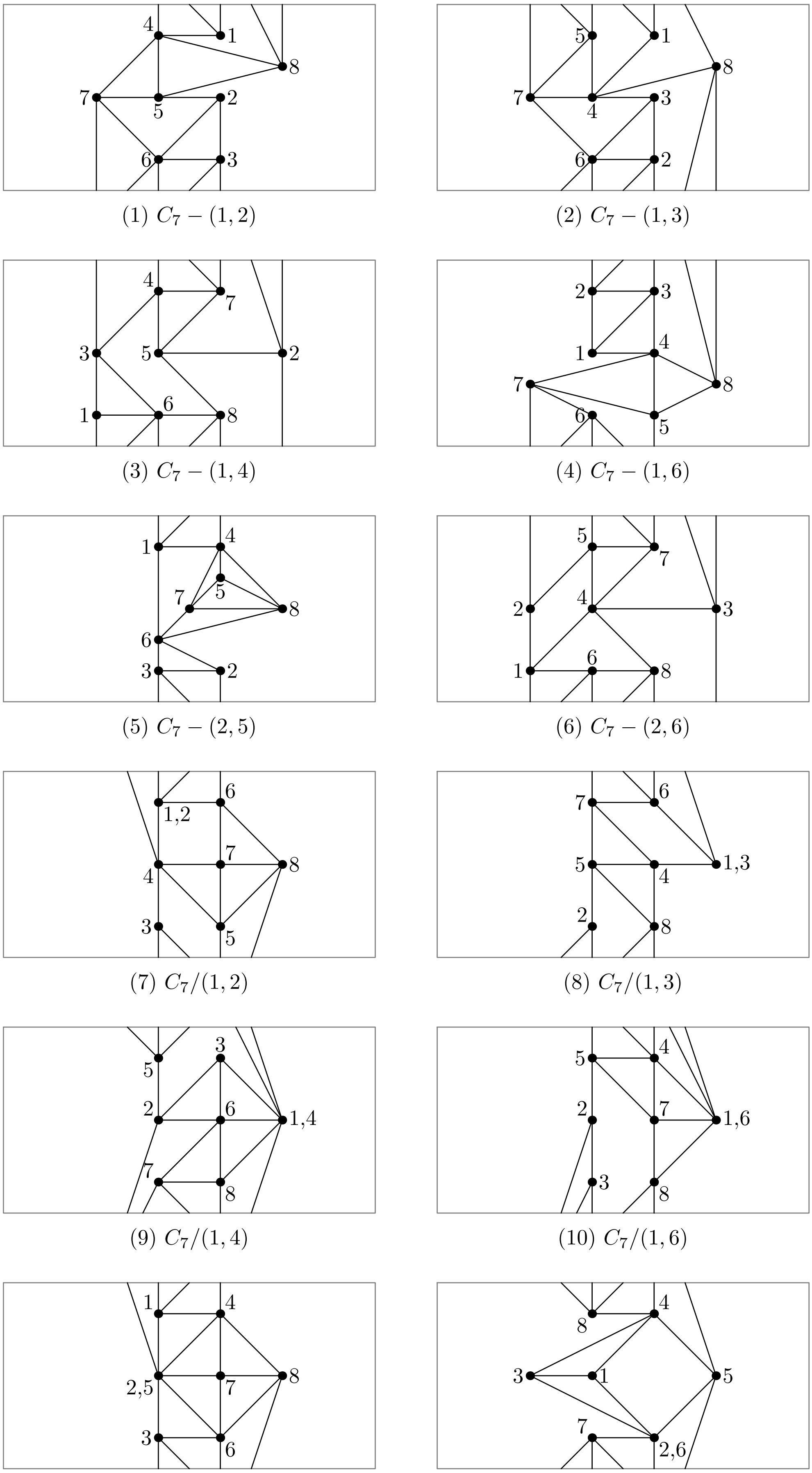
Figure 2.5: Embeddings of $R_i(B_1)$, $i \in \{0, 2\}$, into the Klein surface



(1) $B_3 - (1,2)$ (2) $B_3 - (1,4)$

(3) $B_3/(1,2)$ (4) $B_3/(1,4)$

Figure 2.6: Embeddings of $R_i(B_3)$, $i \in \{0, 2\}$, into the Klein surface



(1) $C_7 - (1,2)$ (2) $C_7 - (1,3)$

(3) $C_7 - (1,4)$ (4) $C_7 - (1,6)$

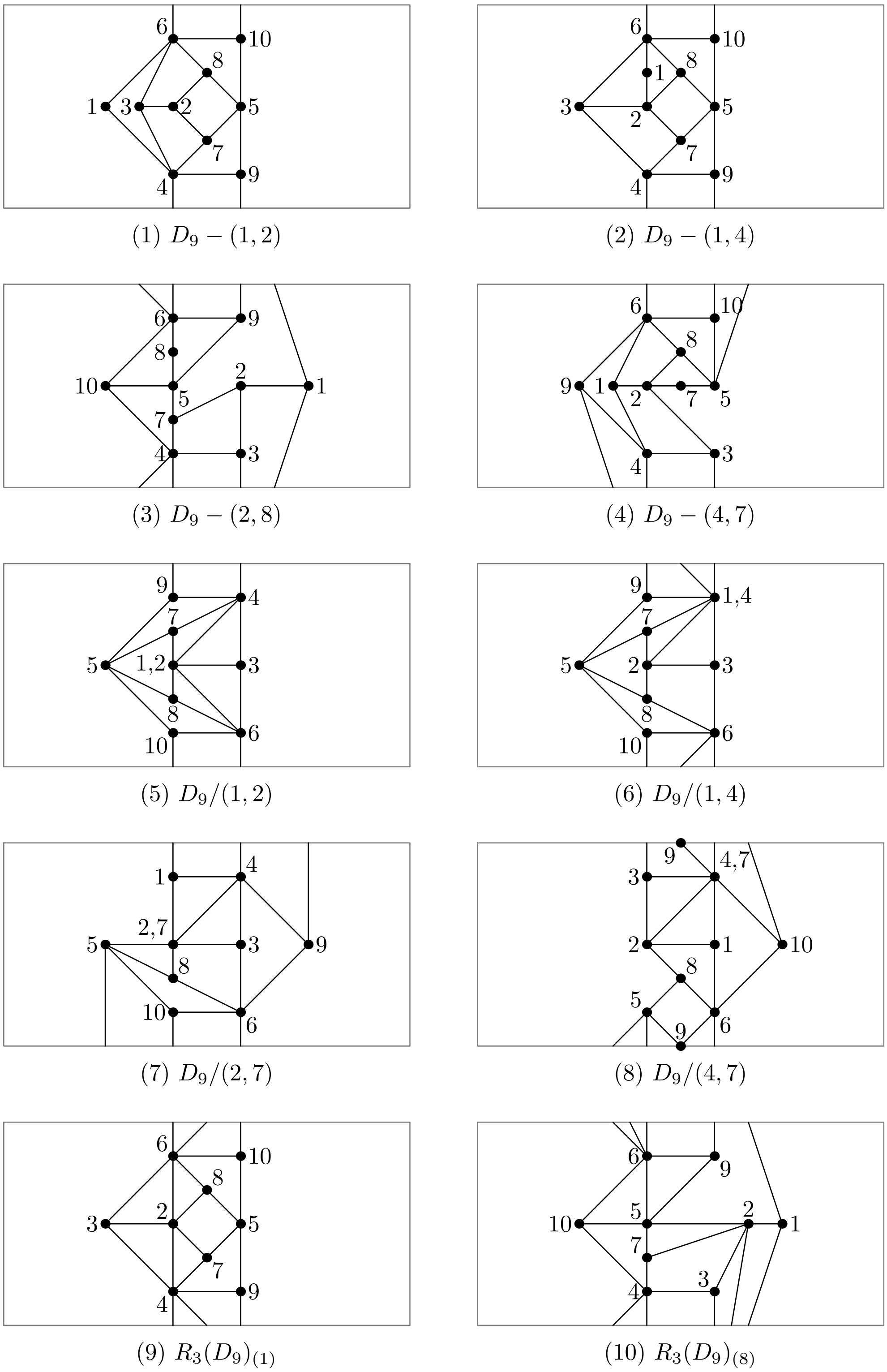
(5) $C_7 - (2,5)$ (6) $C_7 - (2,6)$

(7) $C_7/(1,2)$ (8) $C_7/(1,3)$

(9) $C_7/(1,4)$ (10) $C_7/(1,6)$

(11) $C_7/(2,5)$ (12) $C_7/(2,6)$

Figure 2.7: Embeddings of $R_i(C_7)$, $i \in \{0,2\}$, into the Klein surface



(1) $D_9 - (1,2)$ (2) $D_9 - (1,4)$

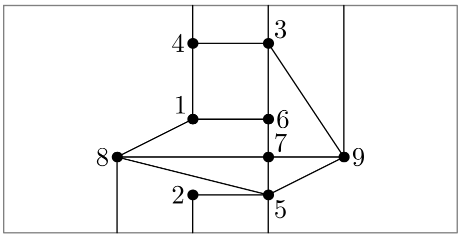
(3) $D_9 - (2,8)$ (4) $D_9 - (4,7)$

(5) $D_9/(1,2)$ (6) $D_9/(1,4)$

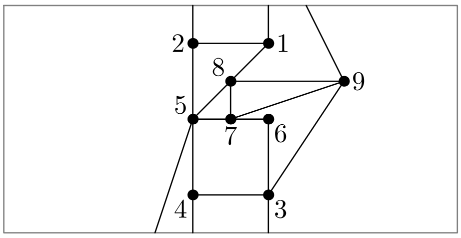
(7) $D_9/(2,7)$ (8) $D_9/(4,7)$

(9) $R_3(D_9)_{(1)}$ (10) $R_3(D_9)_{(8)}$

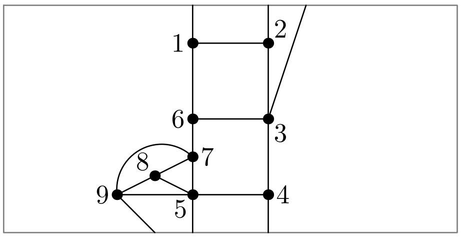
Figure 2.8: Embeddings of $R_i(D_9)$, $i \in \{0, 2, 3\}$, into the Klein surface



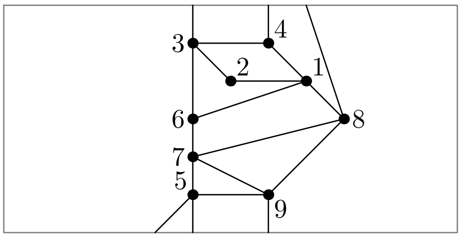
(1) $D_{12} - (1, 2)$



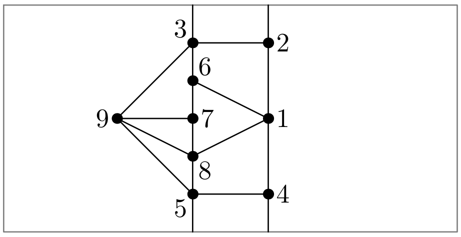
(2) $D_{12} - (1, 6)$



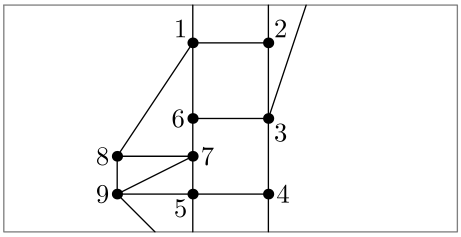
(3) $D_{12} - (1, 8)$



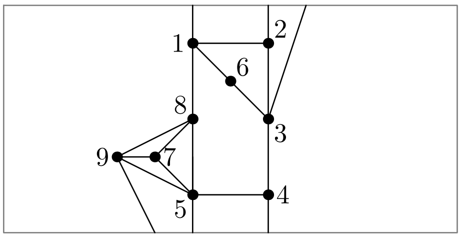
(4) $D_{12} - (2, 5)$



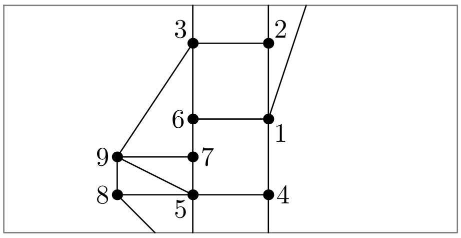
(5) $D_{12} - (5, 7)$



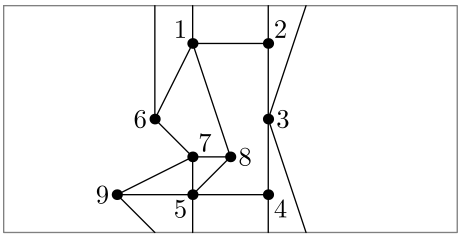
(6) $D_{12} - (5, 8)$



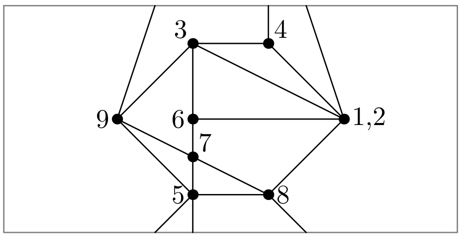
(7) $D_{12} - (6, 7)$



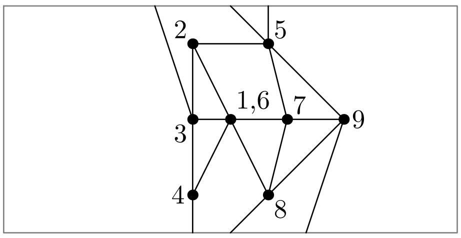
(8) $D_{12} - (7, 8)$



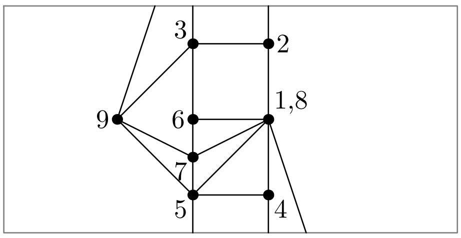
(9) $D_{12} - (8, 9)$



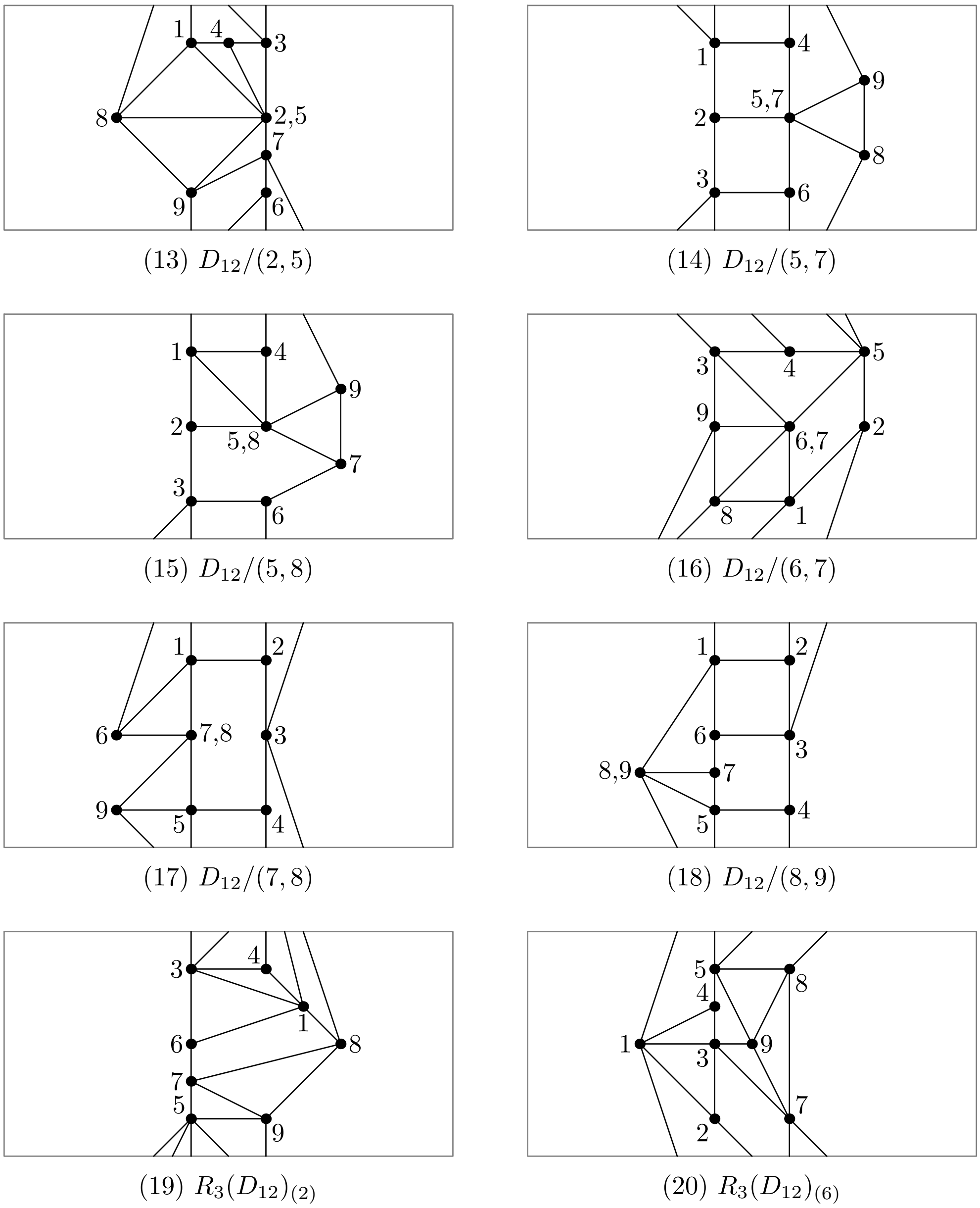
(10) $D_{12}/(1, 2)$



(11) $D_{12}/(1, 6)$



(12) $D_{12}/(1, 8)$



(13) $D_{12}/(2,5)$

(14) $D_{12}/(5,7)$

(15) $D_{12}/(5,8)$

(16) $D_{12}/(6,7)$

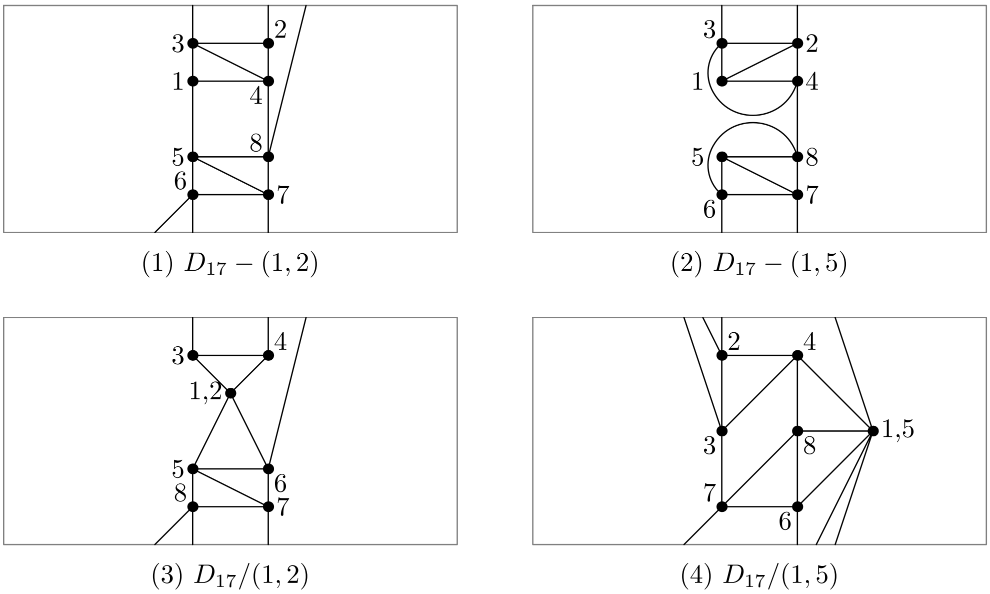
(17) $D_{12}/(7,8)$

(18) $D_{12}/(8,9)$

(19) $R_3(D_{12})_{(2)}$

(20) $R_3(D_{12})_{(6)}$

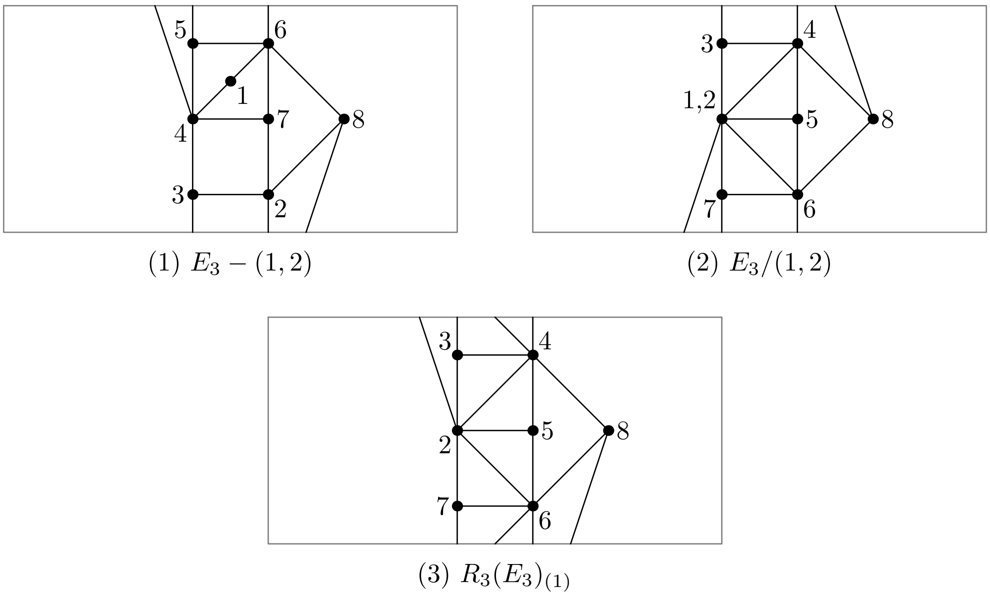
Figure 2.9: Embeddings of $R_i(D_{12})$, $i \in \{0, 2, 3\}$, into the Klein surface



(1) $D_{17} - (1,2)$ (2) $D_{17} - (1,5)$

(3) $D_{17}/(1,2)$ (4) $D_{17}/(1,5)$

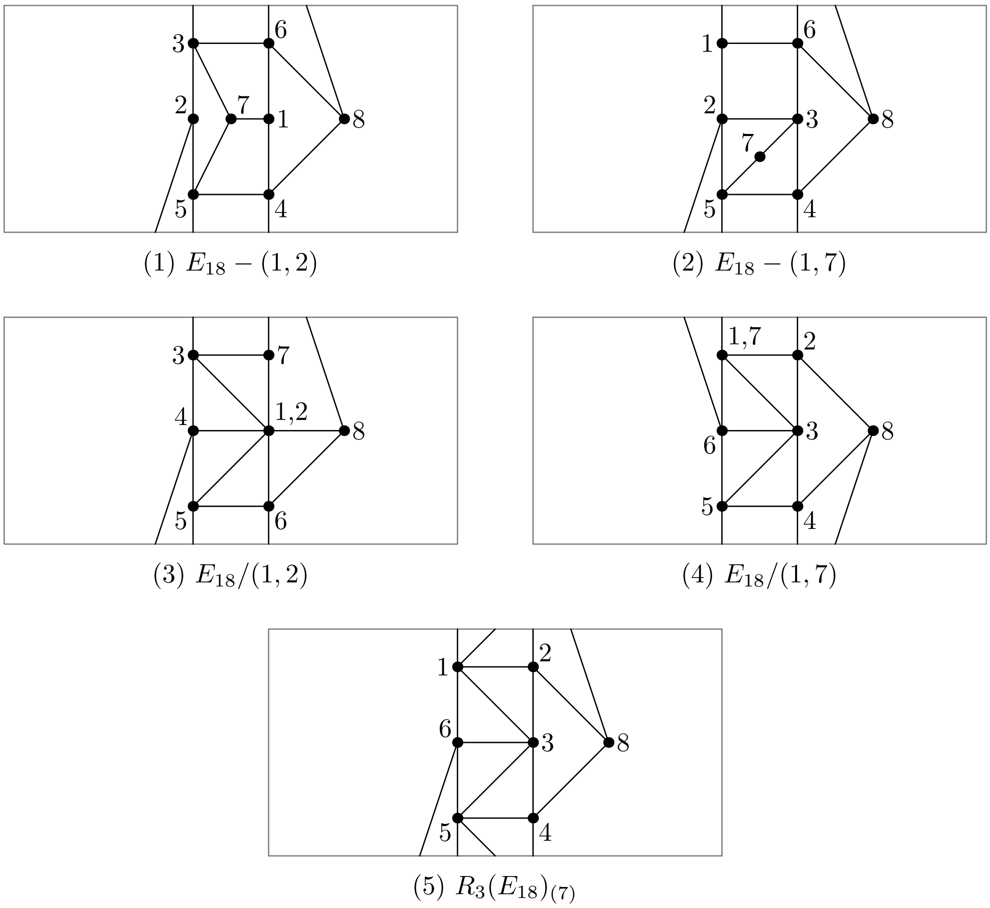
Figure 2.10: Embeddings of $R_i(D_{17})$, $i \in \{0,2\}$, into the Klein surface



(1) $E_3 - (1,2)$ (2) $E_3/(1,2)$

(3) $R_3(E_3)_{(1)}$

Figure 2.11: Embeddings of $R_i(E_3)$, $i \in \{0,2,3\}$, into the Klein surface

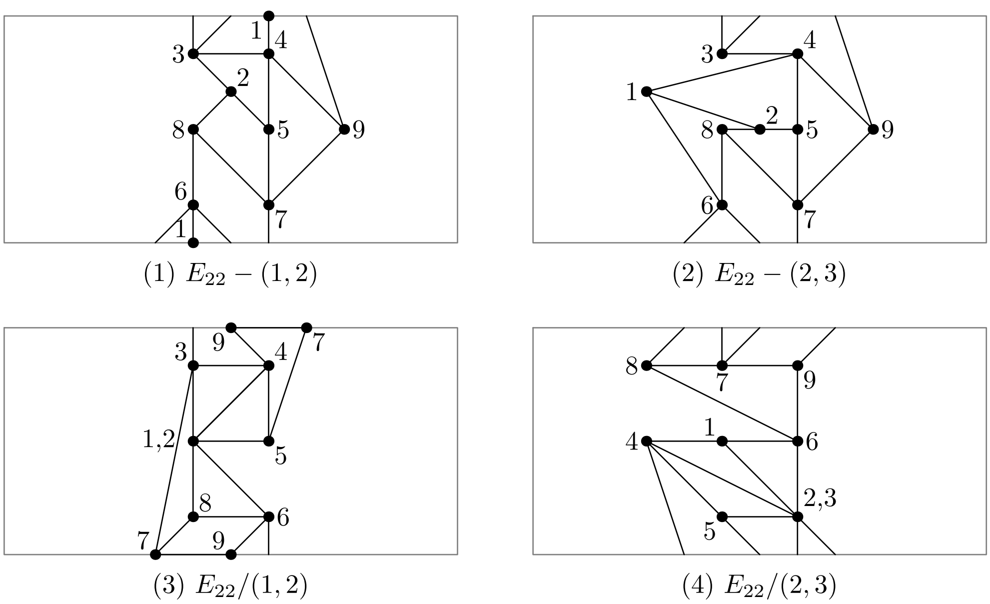


(1) $E_{18} - (1,2)$ (2) $E_{18} - (1,7)$

(3) $E_{18}/(1,2)$ (4) $E_{18}/(1,7)$

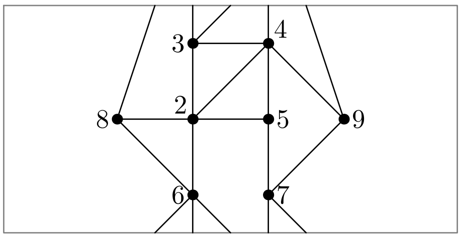
(5) $R_3(E_{18})_{(7)}$

Figure 2.12: Embeddings of $R_i(E_{18})$, $i \in \{0,2,3\}$, into the Klein surface



(1) $E_{22} - (1,2)$ (2) $E_{22} - (2,3)$

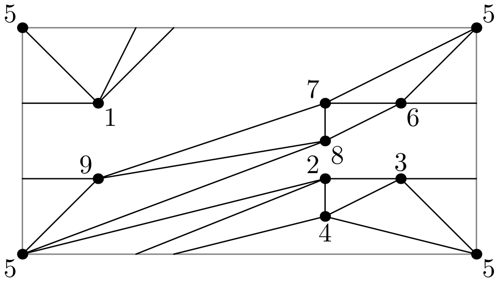
(3) $E_{22}/(1,2)$ (4) $E_{22}/(2,3)$



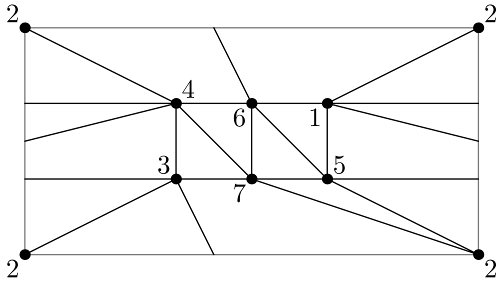
(5) $R_3(E_{22})_{(1)}$

Figure 2.13: Embeddings of $R_i(E_{22})$, $i \in \{0, 2, 3\}$, into the Klein surface

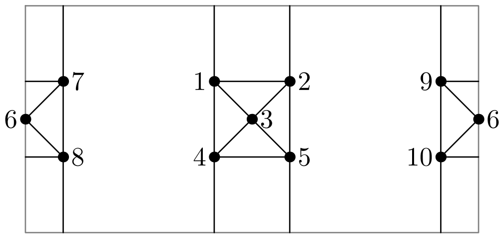
In addition, the irreducible graphs of the projective plane themselves can easily be embedded into the Klein surface:



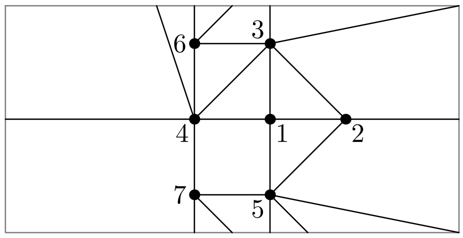
(1) A_1



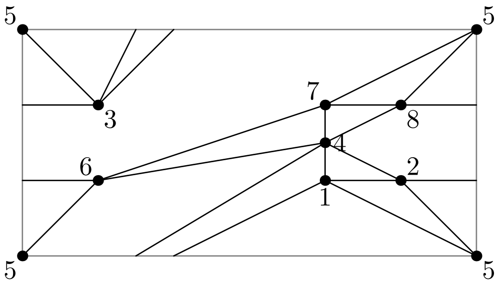
(2) A_2



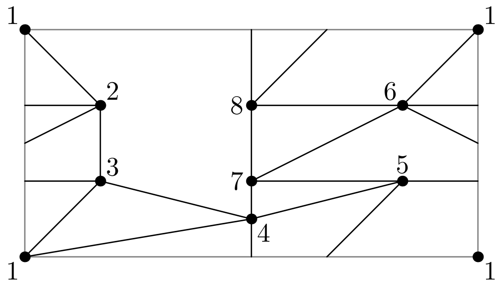
(3) A_5



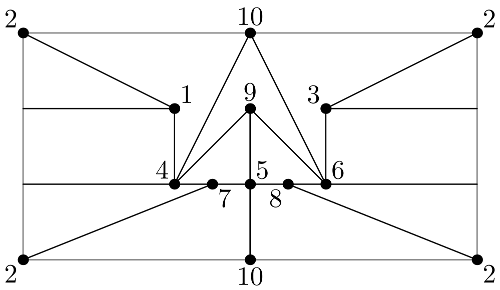
(4) B_1



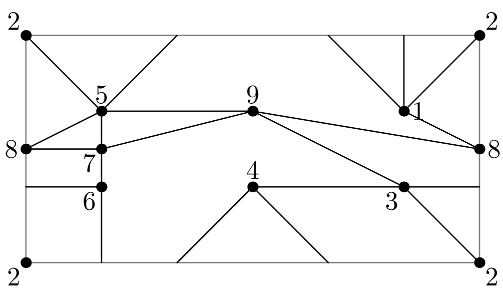
(5) B_3



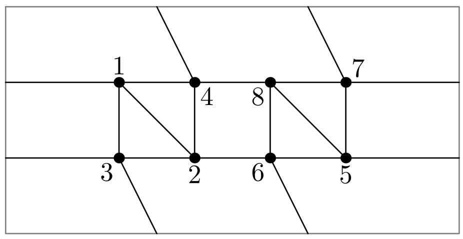
(6) C_7



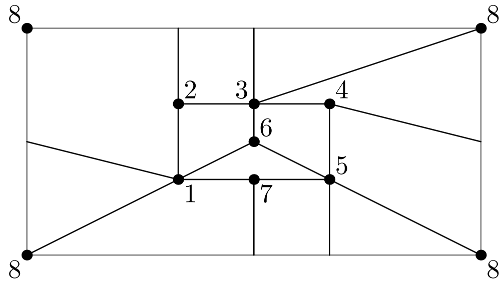
(7) D_9



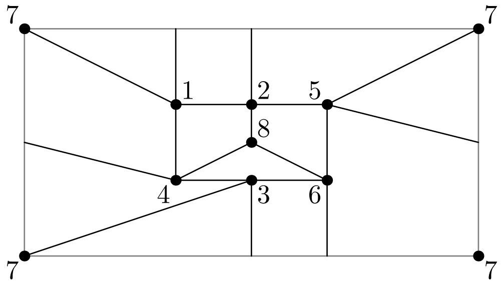
(8) D_{12}



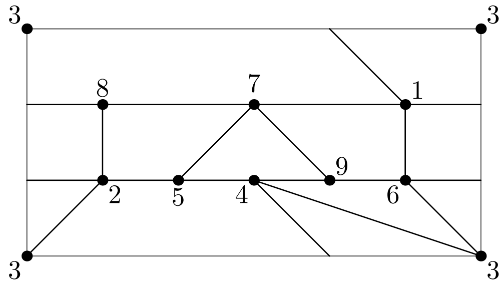
(9) D_{17}



(10) E_3



(11) E_{18}



(12) E_{22}

Figure 2.14: Embeddings of graphs in $M_4(\mathcal{S}_1)$ into the Klein surface

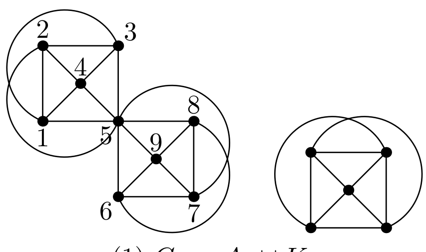
The knowledge about these embeddings will be used in the next chapters, as these are the basic considerations for the construction of graphs in $M_4(\mathcal{S}_2)$.

2.2 Disconnected graphs

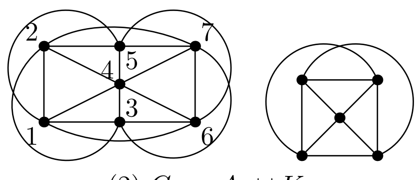
In order to find disconnected irreducible graphs regarding the Klein surface, we take the irreducible graph for the plane, the K_5, and graphs of $M_4(\mathcal{S}_1)$ of the projektive plane and combine these. By doing this, we get 12 disconnected irreducible graphs for the Klein surface.

With the knowledge we have about the graphs in $M_4(\mathcal{S}_1)$ and the structure about the Klein surface, it is obvious that the following proposition holds:

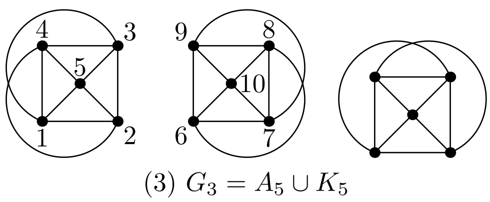
Proposition 2.1
The graphs $G_1, \ldots, G_{12}$ are elements of the minimal basis $M_4(\mathcal{S}_2)$.



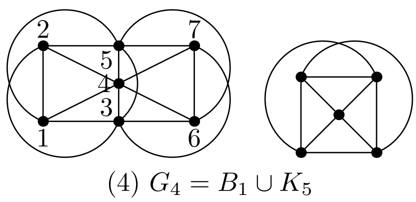
(1) $G_1 = A_1 \cup K_5$



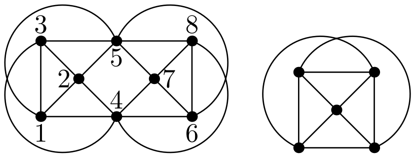
(2) $G_2 = A_2 \cup K_5$



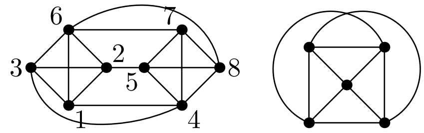
(3) $G_3 = A_5 \cup K_5$



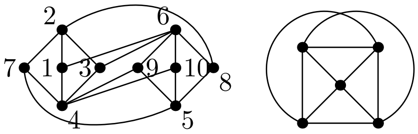
(4) $G_4 = B_1 \cup K_5$



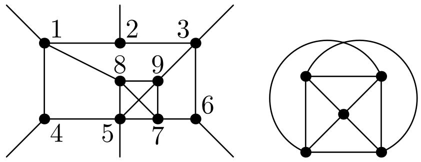
(5) $G_5 = B_3 \cup K_5$



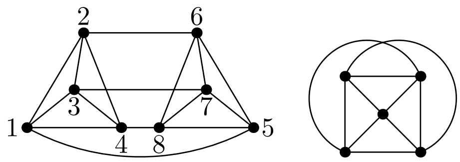
(6) $G_6 = C_7 \cup K_5$



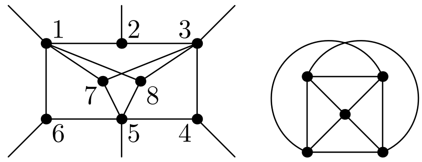
(7) $G_7 = D_9 \cup K_5$



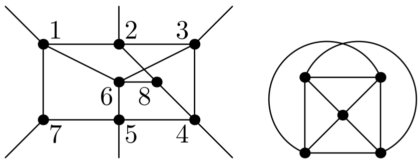
(8) $G_8 = D_{12} \cup K_5$



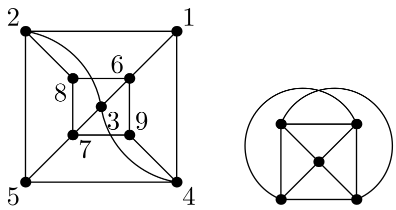
(9) $G_9 = D_{17} \cup K_5$



(10) $G_{10} = E_3 \cup K_5$



(11) $G_{11} = E_{18} \cup K_5$



(12) $G_{12} = E_{22} \cup K_5$

Figure 2.15: The graphs $G_1, \ldots, G_{12}$ in $M_4(\mathcal{S}_2)$

Proof
Each of the graphs G_i, $i \in I$, is a disjoint union of one of the graphs in $M_4(\mathcal{S}_1)$ and the K_5. We already know that the graphs $G_1, \ldots, G_{12}$ are not embeddable into the Klein surface.

Now we still have to show that the graphs $G_1, \ldots, G_{12}$ are irreducible for the Klein surface. Thus we have to show that every graph is embeddable if the relations $R_0, \ldots, R_4$ are applied.

As already shown in Section 2.1, the graphs in $M_4(\mathcal{S}_1)$ can be embedded into the Möbius strip, if one of the relations R_i, $i \in \{0, 1, 2, 3, 4\}$, is applied. The K_5 can easily be embedded into the remaining space of the Klein surface, if the Möbius strip in the middle is already used for the $M_4(\mathcal{S}_1)$-component of the graphs:

Figure 2.16: Embedding of K_5 into the Klein surface

Thus we have twelve disconnected graphs, $G_1, \dots, G_{12}$, which are irreducible for the Klein surface. □

Theorem 2.2
Besides the irreducible Graphs $G_1, \dots, G_{12}$, there are no further graphs with connectivity $\kappa = 0$ in $M_4(\mathcal{S}_2)$.

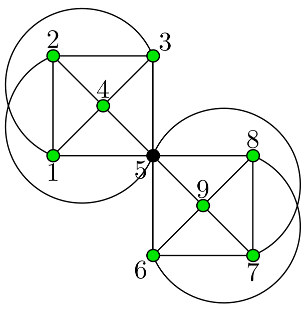
Proof
Let H be an arbitrary graph in $M_4(\mathcal{S}_2)$ with connectivity $\kappa = 0$. Then we have two subgraphs H' and H'' of H, with the properties $H' \cup H'' = H$ and $H' \cap H'' = \emptyset$, so that the graphs are non-planar and non-projective-planar respectively (w.l.o.g. we let H' be the irreducible planar and H'' be the irreducible projective-planar graph), as otherwise H would already be embeddable into the Klein surface. Thus we have $H' \succcurlyeq_j K_5$ or $H' \succcurlyeq_j K_{3,3}$ and $H'' \succcurlyeq_j H$, with $j \in \{0, \dots, 4\}$ and $H \in M_4(\mathcal{S}_1)$. We already know that $K_{3,3} \succcurlyeq_4 K_5$ holds, thus also $H' \succcurlyeq_4 K_5$. As H has to be irreducible, $H \in \{G_1, \dots, G_{12}\}$ has to suffice. □

2.3 Graphs including one base-point

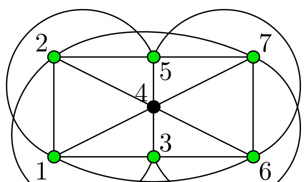
The same way we found the disconnected graphs, we can find graphs with $\kappa = 1$, which are irreducible for the Klein surface.

Proposition 2.3
The graphs $G_{13}, \dots, G_{41}$ are elements of the minimal basis $M_4(\mathcal{S}_2)$. These graphs are constructed by attaching a K_5 to one representative of each vertex-orbit of the irreducible projective-planar graphs $A_1, A_2, B_1, B_3, C_7, D_9, D_{12}, D_{17}, E_3, E_{18}$ or E_{22} and for each attachment to one vertex we get a new graph with connectivity $\kappa = 1$.

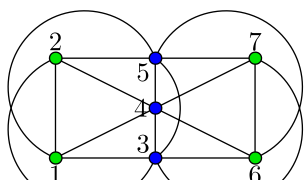
We do not attach the K_5 to A_5, as the resulting graph would be isomorphic to G_1, which we already considered in Section 2.2.



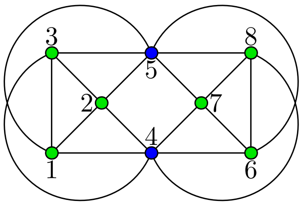
(1) A_1 and its vertex-orbits



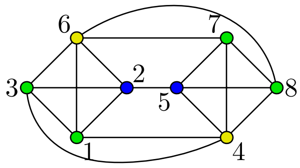
(2) A_2 and its vertex-orbits



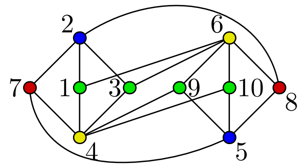
(3) B_1 and its vertex-orbits



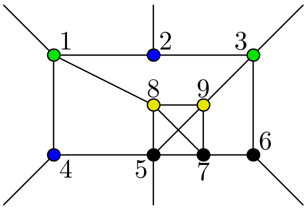
(4) B_3 and its vertex-orbits



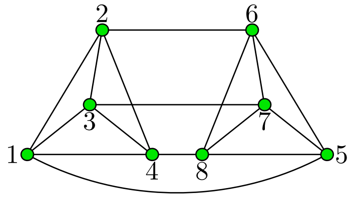
(5) C_7 and its vertex-orbits



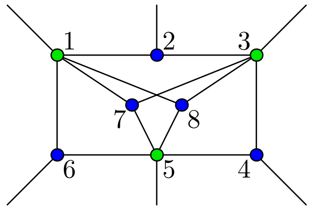
(6) D_9 and its vertex-orbits



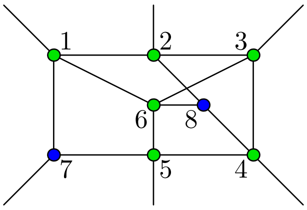
(7) D_{12} and its vertex-orbits



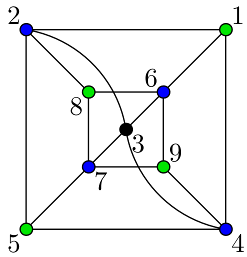
(8) D_{17} and its vertex-orbits



(9) E_3 and its vertex-orbits



(10) E_{18} and its vertex-orbits



(11) E_{22} and its vertex-orbits

Figure 2.17: The graphs in $M_4(\mathcal{S}_1)$ and their vertex-orbits. Vertices in one orbit are of the same color, black vertices indicate individual orbits.

Proof

We already know, that the graphs $G_{13}, \ldots, G_{41}$ cannot be embedded into the Klein surface. We still have to proof that these graphs can be embedded, if one of the relations $R_0, \ldots, R_4$ is applied. Again we consider the embeddability of $R_j(G)$, $G \in M_4(\mathcal{S}_1)$ and $j = 0, \ldots, 4$. As already shown, these graphs can be embedded into the Möbius strip. Additionally, we can find embeddings of these graphs into the Möbius strip, so that a representative of each vertex-orbit can be drawn in the outer face of the Möbius strip regarding one of the possible embeddings. Thus the K_5 can be attached to a representative of each vertex-orbit of the graphs $G_2, \ldots, G_{12}$. We still have to consider those cases, where an edge of the K_5 is deleted or contracted. These cases obviously also work, as $K_5 - e$ or K_5/e can easily be embedded in each of the faces and attached to a representative of each vertex-orbit, the graphs in $M_4(\mathcal{S}_1)$, leave when embedded into the Klein surface as shown in Section 2.1 □

2.4 Graphs including two base-points

In this section, we will construct irreducible graphs by identification of two vertices of a K_5 with two vertices of our graphs in $M_4(\mathcal{S}_1)$. This case of construction is more complex then the previous ones, as not every graph constructed this way, really is an element of $M_4(\mathcal{S}_2)$.

Proposition 2.4
The graphs $G_{42}, \ldots, G_{59}$ are the only graphs in $M_4(\mathcal{S}_2)$, which are constructed by attaching vertices v_1 and v_2 of a $K_5 - (v_1, v_2)$ to a pair of vertices in graphs of $M_4(\mathcal{S}_1)$.

We already know that these graphs are not embeddable into the Klein surface. The minimality of these graphs will be shown in the lemmata of this section. We will also show that none of the other graphs, constructed the same way, can be an element of $M_4(\mathcal{S}_2)$.

As in the cases before, we can embed the irreducible projective-planar graphs into the Möbius strip, if we apply one of the relations $R_0, \ldots, R_4$. In order to be able to attach the K_5 to two arbitrary vertices v_1 and v_2 of $H \in M_4(\mathcal{S}_1)$ we have to be able to embed H into the Möbius strip with v_1 and v_2 on the boundary of one face. This obviously does not work for every combination of two vertices. We will do this for each graph in $M_4(\mathcal{S}_1)$ and its possible attachments individually.

The graph A_1

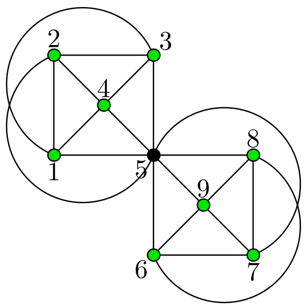


Figure 2.18: A_1 and its vertex-orbits

The only orbit of pairs of vertices we have to look at for the graph A_1 is:

$$\begin{aligned} A = \Big\{ &\{1,6\},\{1,7\},\{1,8\},\{1,9\},\{2,6\},\{2,7\},\{2,8\},\{2,9\}, \\ &\{3,6\},\{3,7\},\{3,8\},\{3,9\},\{4,6\},\{4,7\},\{4,8\},\{4,9\} \Big\} \end{aligned}$$

Lemma 2.5
There is no irreducible graph constructed by identification of the vertices v_1 and v_2 in $K_5 - (v_1, v_2)$ with a pair of vertices of orbit A of A_1.

Proof

We consider the possible embeddings of $A_1 - (1, 2)$, $A_1 - (2, 5)$ and $A_1/(1, 2)$. We have to show that each pair of vertices in A can be drawn on the boundary of one face when embedding these graphs into the Möbius strip. As we cannot find a planar embedding of $K_5 - (v_1, v_2)$, where the vertices v_1 and v_2 lie on the boundary of one face, we also cannot find an embedding of $A_1 - (2, 5)$ into the Möbius strip where vertex 2 and one of the vertices 6, 7, 8 or 9 respectively lie on the boundary of one face, as vertex 5 has to be on the outside to be attached to the $K_5 - (v_1, v_2)$. Consequently the $K_5 - (v_1, v_2)$ cannot be attached to any representative of orbit A to form an irreducible graph of the Klein surface. $\square$

The graph A_2

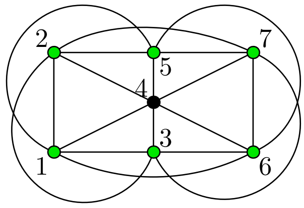


Figure 2.19: A_2 and its vertex-orbits

The only orbit of pairs of vertices we have to consider for A_2 is:

$$A = \Big\{\{1,7\}, \{2,6\}, \{3,5\}\Big\}$$

Lemma 2.6
Attachment of vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to a pair of vertices of orbit A of A_2 does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
As the pair of vertices $\{3,5\}$ lies in the same orbit as $\{3,4\}$ for $A_2 - (3,4)$, vertices 3 and 5 cannot be drawn on the boundary of one face of the Möbius strip, when embedding $A_2 - (3,4)$. Consequently the graph constructed by attaching vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to vertices 3 and 5 in $A_2 - (3,4)$ is not embeddable into the Klein surface. Thus attaching vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to one of the pairs of vertices in orbit A does not result in an irreducible graph for the Klein surface. □

The graph A_5

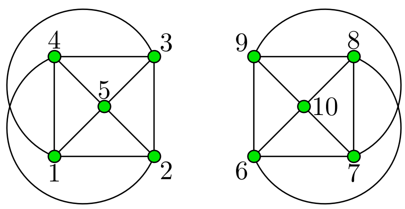


Figure 2.20: A_5 and its vertex-orbits

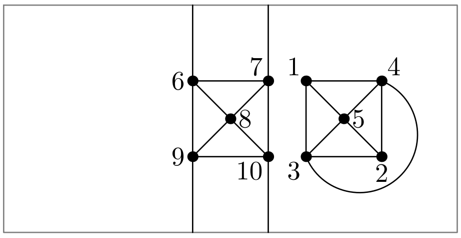
The only orbit of pairs of vertices we have to look at for the graph A_5 is:

$$\begin{aligned} A = \Big\{&\{1,6\}, \{1,7\}, \{1,8\}, \{1,9\}, \{1,10\}, \{2,6\}, \{2,7\}, \{2,8\}, \{2,9\}, \{2,10\}, \\ &\{3,6\}, \{3,7\}, \{3,8\}, \{3,9\}, \{3,10\}, \{4,6\}, \{4,7\}, \{4,8\}, \{4,9\}, \{4,10\}, \\ &\{5,6\}, \{5,7\}, \{5,8\}, \{5,9\}, \{5,10\}\Big\} \end{aligned}$$

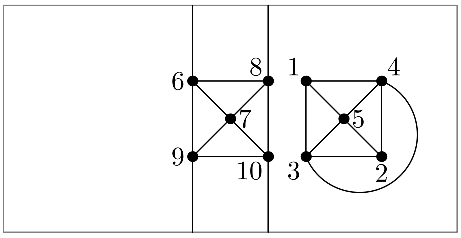
Lemma 2.7
The graph G_{42}, which is obtained by identification of two vertices v_1 and v_2 of a K_5 with one of the pairs of vertices of orbit A and deleting the edge (v_1, v_2), is an element of $M_4(\mathcal{S}_2)$.

Proof

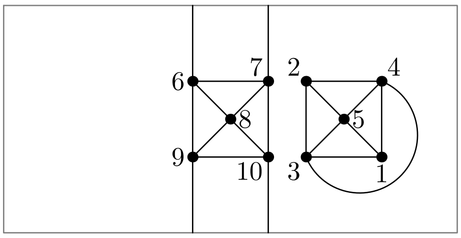
The Graph G_{42} is embeddable into the Klein surface after one of the relations $R_0, \dots, R_4$ is applied. To show that this is correct, we use possible embeddings of the graphs $A_5 - (1,2)$ and $A_5/(1,2)$. As we can find embeddings where each of the pairs of vertices in orbit A, can be drawn on the boundary of one face on the Möbius strip, the $K_5 - (v_1, v_2)$ can be attached to each of the pairs of vertices.



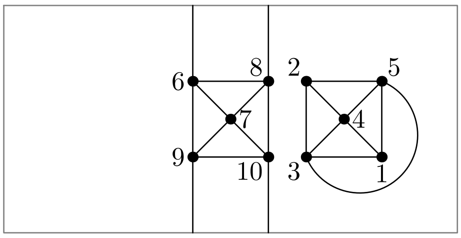
(1) Embedding of $A_5 - (1,2)$, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{1,9\}$, $\{1,10\}$, $\{3,6\}$, $\{3,7\}$, $\{3,9\}$, $\{3,10\}$, $\{4,6\}$, $\{4,7\}$, $\{4,9\}$, $\{4,10\}$ on the boundary of a face.



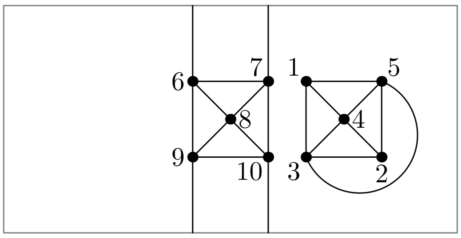
(2) Embedding of $A_5 - (1,2)$, vertex-pairs $\{1,8\}$, $\{3,8\}$, $\{4,8\}$ on the boundary of a face.



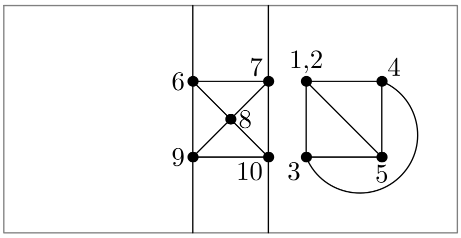
(3) Embedding of $A_5 - (1,2)$, vertex-pairs $\{2,6\}$, $\{2,7\}$, $\{2,9\}$, $\{2,10\}$ on the boundary of a face.



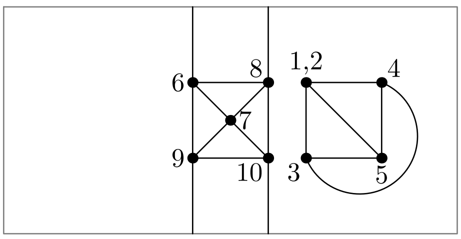
(4) Embedding of $A_5 - (1,2)$, vertex-pairs $\{2,8\}$,$\{5,8\}$ on the boundary of a face.



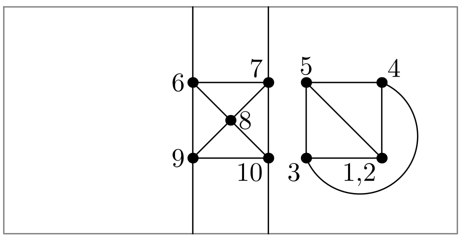
(5) Embedding of $A_5 - (1,2)$, vertex-pairs $\{5,6\}$, $\{5,7\}$, $\{5,9\}$, $\{5,10\}$ on the boundary of a face.



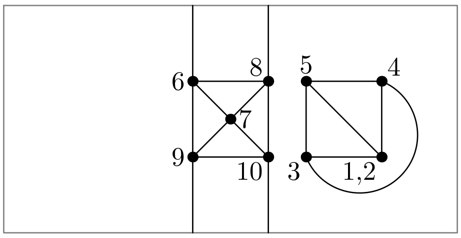
(6) Embedding of $A_5/(1,2)$, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{1,9\}$, $\{1,10\}$, $\{2,6\}$, $\{2,7\}$, $\{2,9\}$, $\{2,10\}$, $\{3,6\}$, $\{3,7\}$, $\{3,9\}$, $\{3,10\}$, $\{4,6\}$, $\{4,7\}$, $\{4,9\}$, $\{4,10\}$ on the boundary of a face.



(7) Embedding of $A_5/(1,2)$, vertex-pairs $\{1,8\}$, $\{2,8\}$, $\{3,8\}$, $\{4,8\}$ on the boundary of a face.



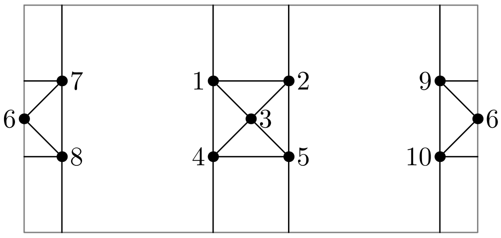
(8) Embedding of $A_5/(1,2)$, vertex-pairs $\{5,6\}$, $\{5,7\}$, $\{5,9\}$, $\{5,10\}$ on the boundary of a face.



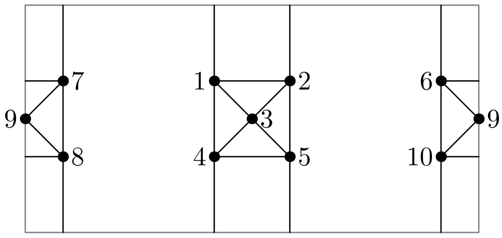
(9) Embedding of $A_5/(1,2)$, vertex-pair $\{5,8\}$ on the boundary of a face.

Figure 2.21: Embeddings of $R_i(A_5)$, $i \in \{0,2\}$, into the Möbius strip, vertex-pairs of orbit A on the boundary of a face.

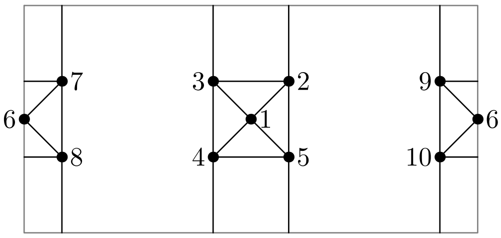
We can also find embeddings of A_5, where each pair of vertices from orbit A can be drawn on the boundary of one face. Thus the $K_5 - (v_1, v_2) - e$ and the $(K_5 - (v_1, v_2))/e$ can be drawn within this face.



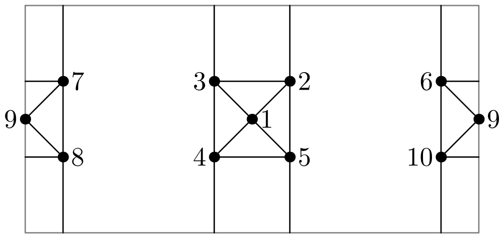
(1) Embedding of A_5, vertex-pairs $\{1,7\}$, $\{1,8\}$, $\{1,9\}$, $\{1,10\}$, $\{2,7\}$, $\{2,8\}$, $\{2,9\}$, $\{2,10\}$, $\{4,7\}$, $\{4,8\}$, $\{4,9\}$, $\{4,10\}$, $\{5,7\}$, $\{5,8\}$, $\{5,9\}$, $\{5,10\}$ on the boundary of a face.



(2) Embedding of A_5, vertex-pairs $\{1,6\}$, $\{2,6\}$, $\{4,6\}$, $\{5,6\}$ on the boundary of a face.



(3) Embedding of A_5, vertex-pairs $\{3,7\}$, $\{3,8\}$, $\{3,9\}$, $\{3,10\}$ on the boundary of a face.



(4) Embedding of A_5, vertex-pair $\{3,6\}$ on the boundary of a face.

Figure 2.22: Embedding of A_5 into the Klein surface, vertex-pairs of orbit A on the boundary of a face.

As this applies to each element in orbit A, G_{42} is an element of $M_4(\mathcal{S}_2)$. $\square$

The graph B_1

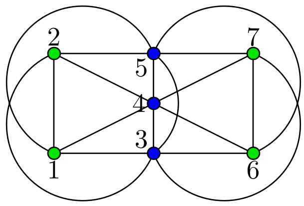


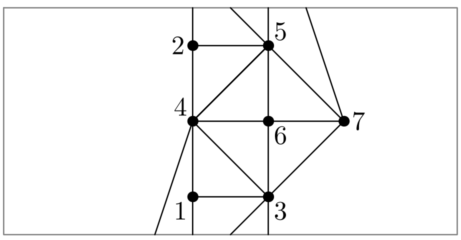
Figure 2.23: B_1 and its vertex-orbits

The orbit of pairs of vertices we have to consider for B_1 is:

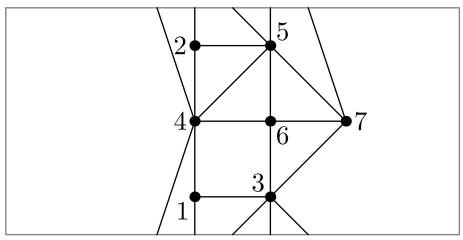
$$A = \Big\{\{1,6\},\{1,7\},\{2,6\},\{2,7\}\Big\}$$

Lemma 2.8
The graph G_{43}, which is constructed by identification of the vertices v_1 and v_2 of $K_5-(v_1,v_2)$ with vertices in orbit A of B_1, is an element of $M_4(\mathcal{S}_2)$.

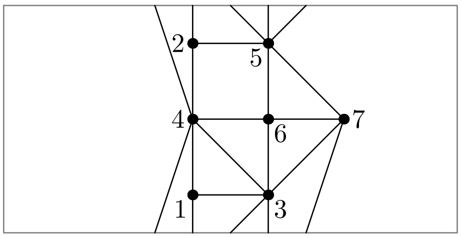
Proof
By considering the Möbius strip embeddings of the graphs $B_1 - e$ and B_1/e, $e \in \{(1,2),(1,3),(3,4)\}$ one can see that each pair of vertices from orbit A can be drawn on the boundary of one face, so that the $K_5-(v_1,v_2)$ can be attached to each of the vertices in orbit A.



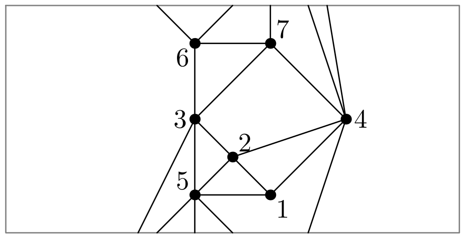
(1) Embedding of $B_1-(1,2)$, vertex-pairs $\{1,\}$, $\{2,7\}$ on the boundary of a face.



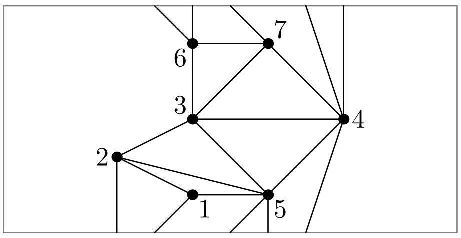
(2) Embedding of $B_1-(1,2)$, vertex-pair $\{1,6\}$ on the boundary of a face.



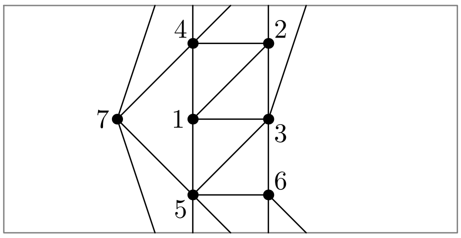
(3) Embedding of $B_1-(1,2)$, vertex-pair $\{2,6\}$ on the boundary of a face.



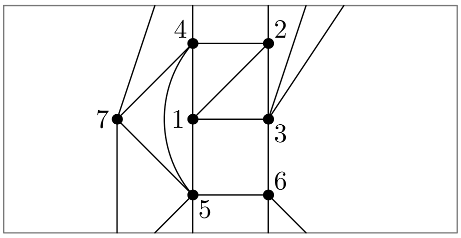
(4) Embedding of $B_1-(1,3)$, vertex-pairs $\{1,6\}$, $\{2,7\}$ on the boundary of a face.



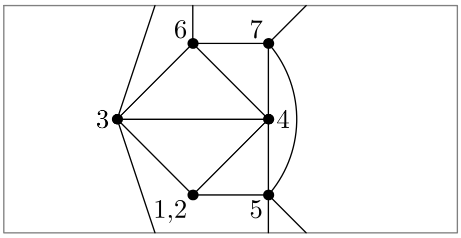
(5) Embedding of $B_1-(1,3)$, vertex-pairs $\{1,7\}$, $\{2,6\}$ on the boundary of a face.



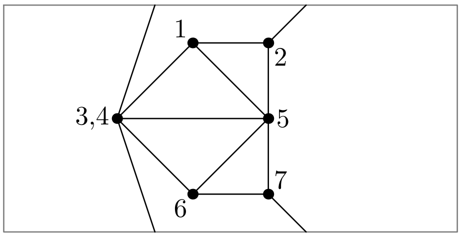
(6) Embedding of $B_1-(3,4)$, vertex-pairs $\{1,7\}$, $\{2,7\}$ on the boundary of a face



(7) Embedding of $B_1-(3,4)$, vertex-pairs $\{1,6\}$, $\{2,6\}$ on the boundary of a face.



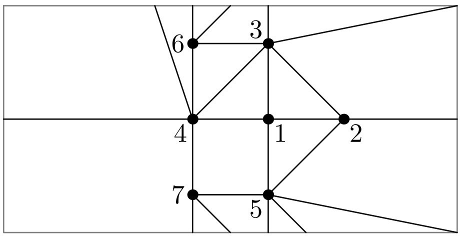
(8) Embedding of $B_1/(1,2)$, all vertex-pairs of orbit A on the boundary of a face.



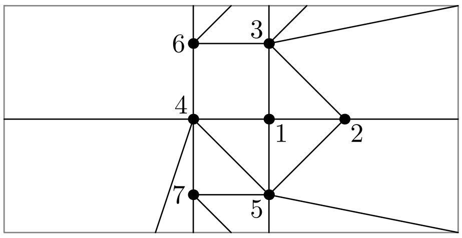
(9) Embedding of $B_1/(3,4)$, all vertex-pairs of orbit A on the boundary of a face.

Figure 2.24: Embeddings of $R_i(B_1)$, $i \in \{0,2\}$, into the Möbius strip.

These pairs of vertices can also be drawn on the boundary of one face of the embedding of B_1 into the Klein surface, so that $K_5-(v_1,v_2)-e$ and $(K_5-(v_1,v_2))/e$ can be attached the same way.



(1) Embedding of B_1, vertex-pairs $\{1,7\}$, $\{2,7\}$ on the boundary of a face.



(2) Embedding of B_1, vertex-pairs $\{1,6\}$, $\{2,6\}$ on the boundary of a face.

Figure 2.25: Embeddings of B_1 into the Klein surface

Consequently G_{43} is an element of $M_4(\mathcal{S}_2)$. $\square$

The graph B_3

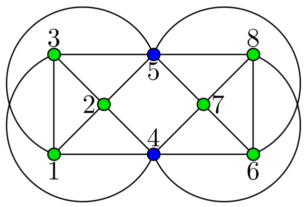


Figure 2.26: B_3 and its vertex-orbits

For B_3 we have to consider the two remaining orbits of vertex-pairs:

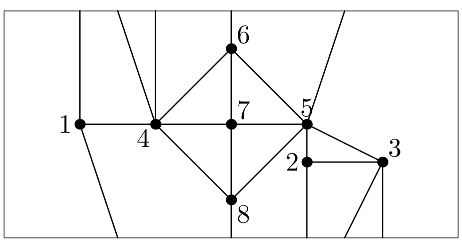
$$A = \Big\{\{1,6\},\{1,7\},\{1,8\},\{2,6\},\{2,7\},\{2,8\},\{3,6\},\{3,7\},\{3,8\}\Big\},$$
$$B = \Big\{\{4,5\}\Big\}.$$

Lemma 2.9
Attaching the $K_5 - (v_1, v_2)$ to a pair of vertices in orbit A does not result in an element of $M_4(\mathcal{S}_2)$.

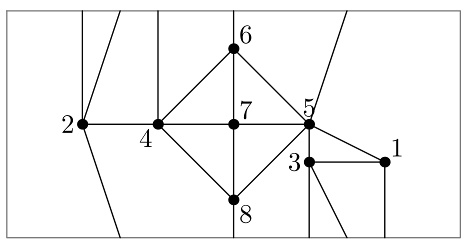
Proof
The graph B_3 consists of two K_5's, which are attached at two vertices (in this case they are called 4 and 5) and the edge $(4,5)$ is deleted. The $K_5 - (4,5) - (1,2)$ cannot be embedded into the plane with vertices $3,4$ and 5 on the boundary of one face. Hence vertex 3 together with one of the vertices $6,7$ or 8 respectively, can never be drawn on the boundary of one face of the Möbius strip, when embedding $B_3 - (1,2)$. Consequently, attaching the $K_5 - (v_1, v_2)$ to a pair of vertices in orbit A does not result in an irreducible graph for the Klein surface. □

Lemma 2.10
The graph G_{44}, which is constructed by attaching vertices v_1 and v_2 of K_5 to vertices 4 and 5 in B_3 and deleting the edge (v_1, v_2) is an element of $M_4(\mathcal{S}_2)$.

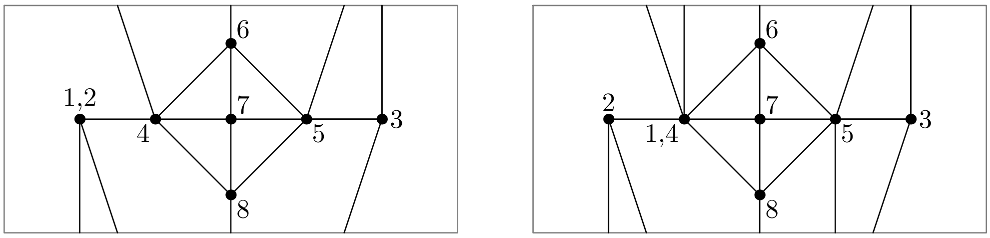
Proof
We consider the embeddings of $B_3 - e$ and B_3/e, $e \in \{(1,2),(1,4)\}$. For each of these graphs we can find an embedding into the Möbius strip, where vertices 4 and 5 lie on the boundary of one face.



(1) Embedding of $B_3 - (1,2)$, vertex-pair $\{4,5\}$ on the boundary of a face.



(2) Embedding of $B_3 - (1,4)$, vertex-pair $\{4,5\}$ on the boundary of a face.



(3) Embedding of $B_3/(1,2)$, vertex-pair $\{4,5\}$ on the boundary of a face.

(4) Embedding of $B_3/(1,4)$, vertex-pair $\{4,5\}$ on the boundary of a face.

Figure 2.27: Embeddings of $R_i(B_3)$, $i \in \{0,2\}$, into the Möbius strip.

Additionally we have an embedding of B_3 into the Klein surface, with vertices 4 and 5 on the boundary of one face:

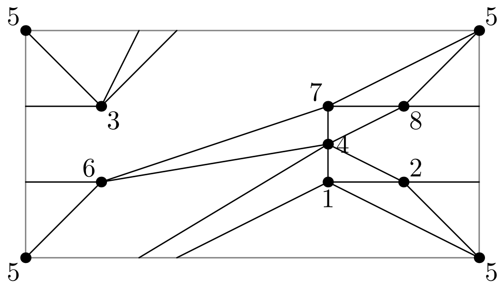


Figure 2.28: Embedding of B_3 into the Klein surface, vertex-pair $\{4,5\}$ on the boundary of a face.

Consequently, for each of those graphs, we can attach the $K_5 - (v_1, v_2)$ to vertices 4 and 5 of B_3, and this is an irreducible graph for the Klein surface and thus an element of $M_4(\mathcal{S}_2)$. $\square$

The graph C_7

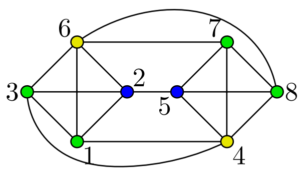


Figure 2.29: C_7 and its vertex-orbits

The orbits of pairs of vertices we have to consider for attachments of $K_5 - (v_1, v_2)$ to C_7 are:

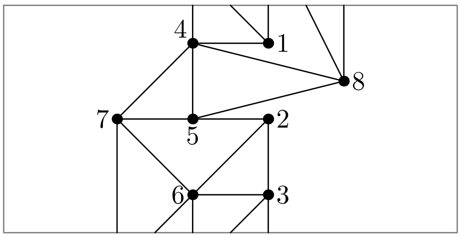
$$A = \Big\{\{1,5\},\{2,7\},\{2,8\},\{3,5\}\Big\}, \qquad B = \Big\{\{1,7\},\{1,8\},\{3,7\},\{3,8\}\Big\},$$
$$C = \Big\{\{2,4\},\{5,6\}\Big\}, \qquad D = \Big\{\{4,6\}\Big\}$$

Lemma 2.11
Construction of a graph where the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ are attached to a pair of vertices of orbit A in C_7 does not result in an element of $M_4(\mathcal{S}_2)$.

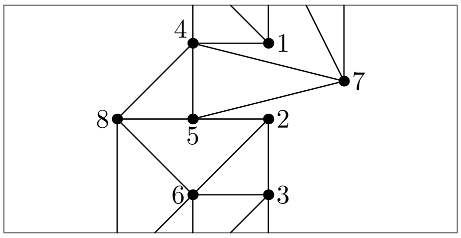
Proof
As the pair of vertices $\{1,5\}$ lies in the same orbit as the pair of vertices $\{1,4\}$ for $C_7-(1,4)$, vertices 1 and 5 cannot be drawn on the boundary of one face of the Möbius strip, when embedding $A_2-(1,4)$. Consequently the graph constructed by attaching vertices v_1 and v_2 of $K_5-(v_1,v_2)$ to vertices 1 and 5 in $C_7-(1,4)$ is not embeddable into the Klein surface and thus attaching vertices v_1 and v_2 of $K_5-(v_1,v_2)$ to one of the pairs of vertices in orbit A does not result in an irreducible graph for the Klein surface. □

Lemma 2.12
The graph G_{45}, which is constructed by attaching vertices v_1 and v_2 of the $K_5-(v_1,v_2)$ to a pair of vertices in orbit B, is an element of $M_4(\mathcal{S}_2)$.

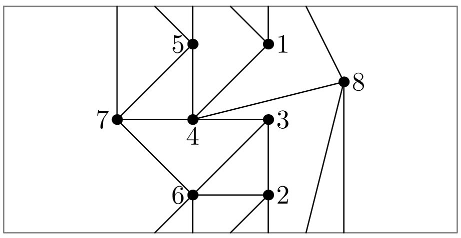
Proof
By considering the Möbius strip embeddings of the graphs C_7-e and C_7/e, $e\in\{(1,2),(1,3),(1,4),(1,6),(2,5),(2,6)\}$ as well as the embedding of C_7 itself into the Klein surface, we can see that each of the pairs of vertices from orbit B can be drawn on the boundary of one face, so that vertices v_1 and v_2 of $K_5-(v_1,v_2)$ can be attached to each of the vertices in orbit B.



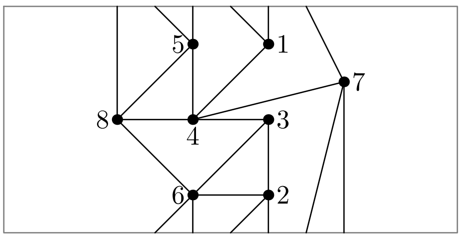
(1) Embedding of $C_7-(1,2)$, vertex pairs $\{1,8\}$, $\{3,7\}$, $\{3,8\}$ on the boundary of a face.



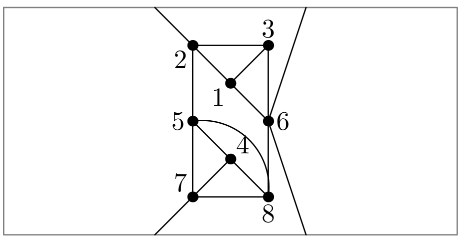
(2) Embedding of $C_7-(1,2)$, vertex-pair $\{1,7\}$ on the boundary of a face.



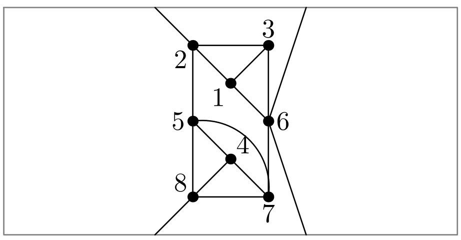
(3) Embedding of $C_7-(1,3)$, vertex-pairs $\{1,8\}$, $\{3,7\}$, $\{3,8\}$ on the boundary of a face.



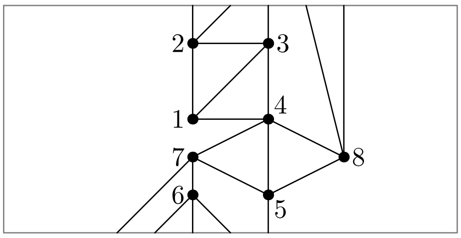
(4) Embedding of $C_7-(1,3)$, vertex pair $\{1,7\}$ on the boundary of a face.



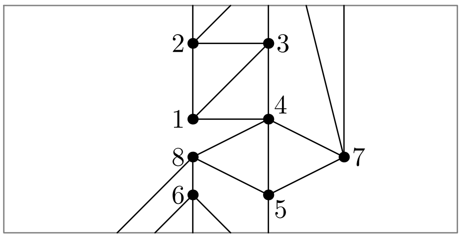
(5) Embedding of $C_7-(1,4)$, vertex-pairs $\{1,8\}$, $\{3,7\}$, $\{3,8\}$ on the boundary of a face.



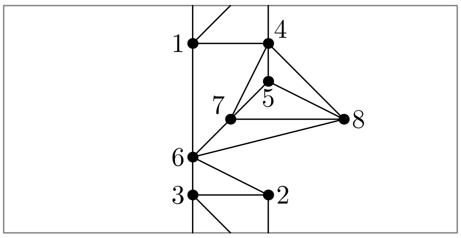
(6) Embedding of $C_7-(1,4)$, vertex-pair $\{1,7\}$ on the boundary of a face.



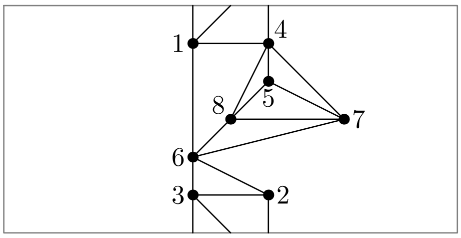
(7) Embedding of $C_7-(1,6)$, vertex-pairs $\{1,7\}$, $\{1,8\}$, $\{3,8\}$ on the boundary of a face.



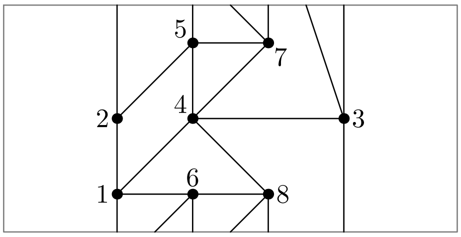
(8) Embedding of $C_7-(1,6)$, vertex-pair $\{3,7\}$ on the boundary of a face.



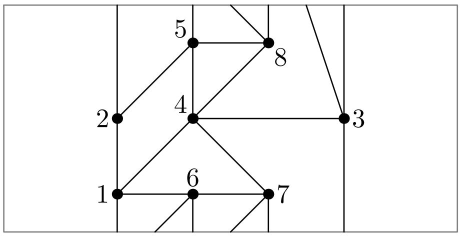
(9) Embedding of $C_7-(2,5)$, vertex-pairs $\{1,7\}$, $\{1,8\}$, $\{3,8\}$ on the boundary of a face.



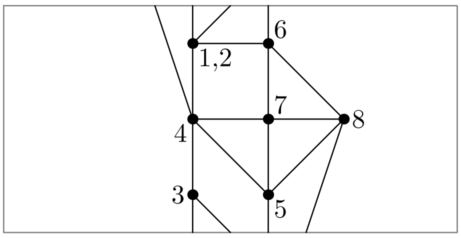
(10) Embedding of $C_7-(2,5)$, vertex-pair $\{3,8\}$ on the boundary of a face.



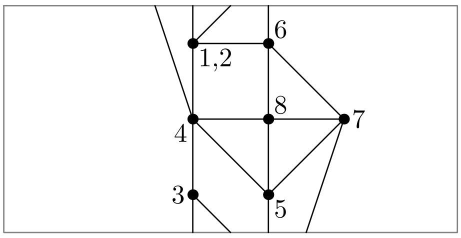
(11) Embedding of $C_7-(2,6)$, vertex-pairs $\{1,8\}$, $\{3,7\}$, $\{3,8\}$ on the boundary of a face.



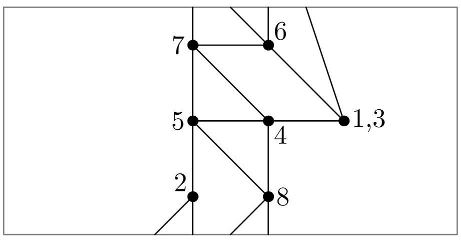
(12) Embedding of $C_7-(2,6)$, vertex-pair $\{1,7\}$ on the boundary of a face.



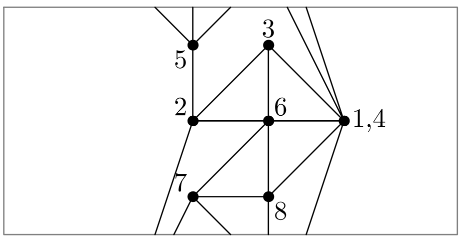
(13) Embedding of $C_7/(1,2)$, vertex-pairs $\{1,7\}$, $\{1,8\}$, $\{3,8\}$ on the boundary of a face.



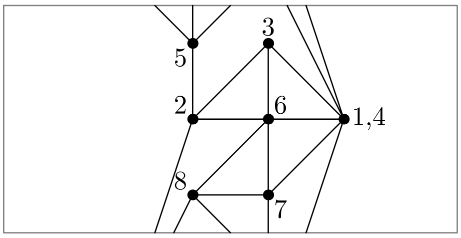
(14) Embedding of $C_7/(1,2)$, vertex-pair $\{3,7\}$ on the boundary of a face.



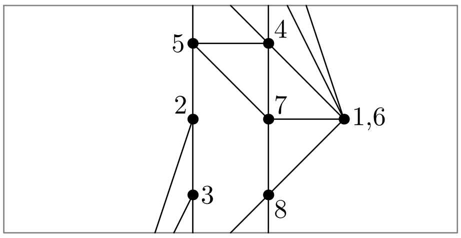
(15) Embedding of $C_7/(1,3)$, each vertex-pair of orbit B on the boundary of a face.



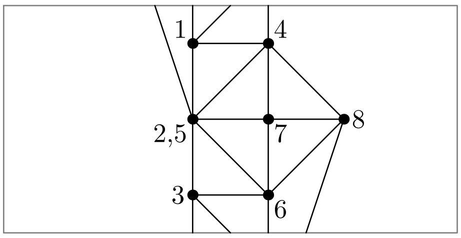
(16) Embedding of $C_7/(1,4)$, vertex-pairs $\{1,7\}$, $\{1,8\}$, $\{3,7\}$ on the boundary of a face.



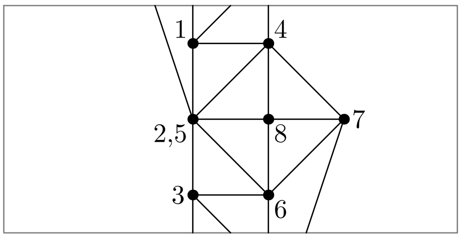
(17) Embedding of $C_7/(1,4)$, vertex-pair $\{3,8\}$ on the boundary of a face.



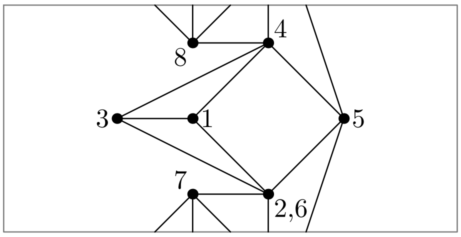
(18) Embedding of $C_7/(1,6)$, each vertex-pair of orbit B on the boundary of a face.



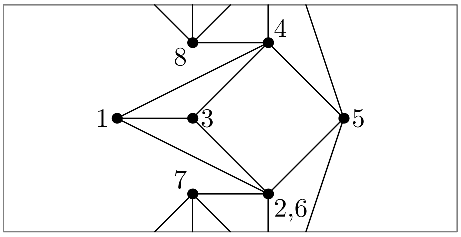
(19) Embedding of $C_7/(2,5)$, vertex-pairs $\{1,8\}$, $\{3,8\}$ on the boundary of a face.



(20) Embedding of $C_7/(2,5)$, vertex-pairs $\{1,7\}$, $\{3,7\}$ on the boundary of a face.



(21) Embedding of $C_7/(2,6)$, vertex-pairs $\{3,7\}$, $\{3,8\}$ on the boundary of a face.



(22) Embedding of $C_7/(2,6)$, vertex-pairs $\{1,7\}$, $\{1,8\}$ on the boundary of a face.

Figure 2.30: Embeddings of $R_i(C_7)$, $i \in \{0,2\}$, into the Möbius strip.

And the embedding of C_7 into the Klein surface:

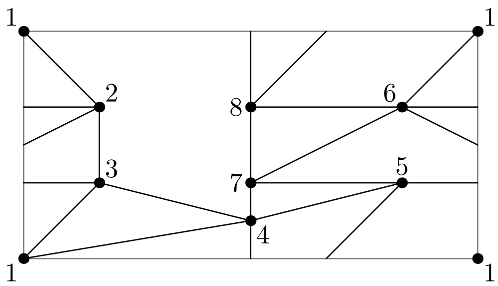


Figure 2.31: Embedding of C_7 into the Klein surface, each vertex-pair of orbit B on the boundary of a face.

□

Lemma 2.13
A graph constructed by attachment of vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to a pair of vertices of orbit C in C_7 cannot be an element of $M_4(\mathcal{S}_2)$.

Proof

Using the Euler characteristic for the graph $C_7 - (2,6)$ we see that the embedding of this graph has to form nine faces on the Möbius strip. As there are only eight cycles of length three in the graph, and as four of these are in a K_4, there can at most be six faces with three vertices on their boundary, each. As three of the cycles of length three, which do not lie in the K_4 include the edge $(1,3)$, only two of these could be embedded. Consequently it is clear that the maximum size of a face, embedded into the Möbius strip, is five. The only cycles of length five, which include vertices 5 and 6, are:

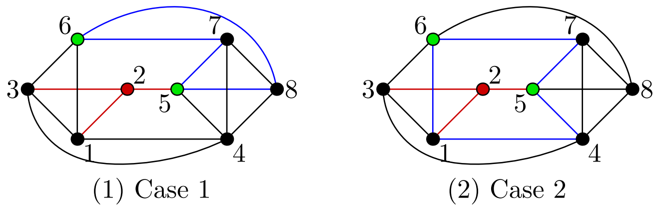
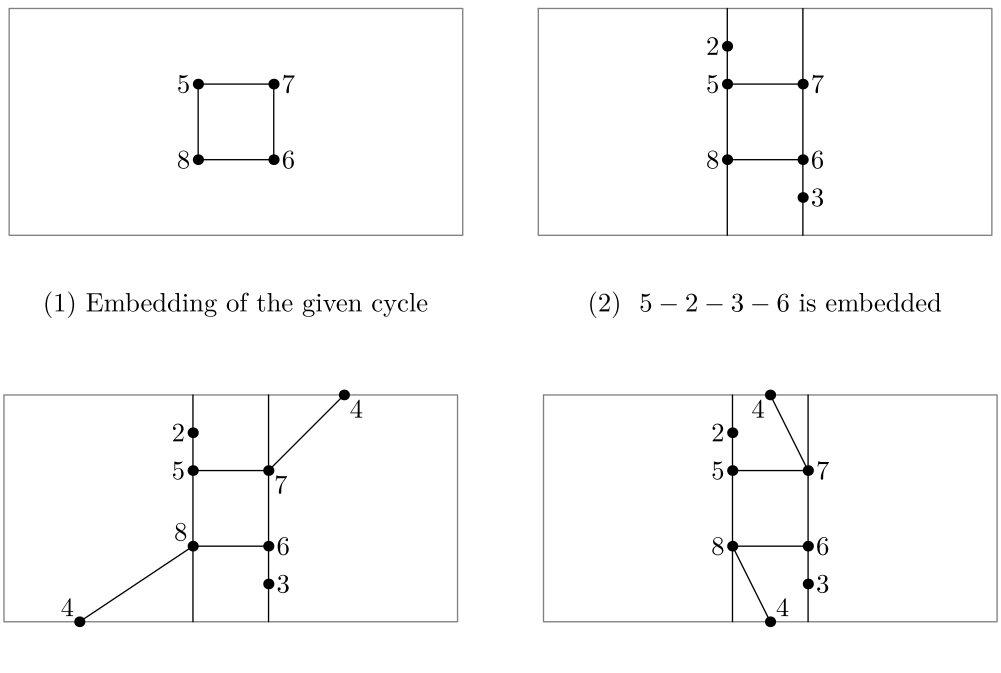


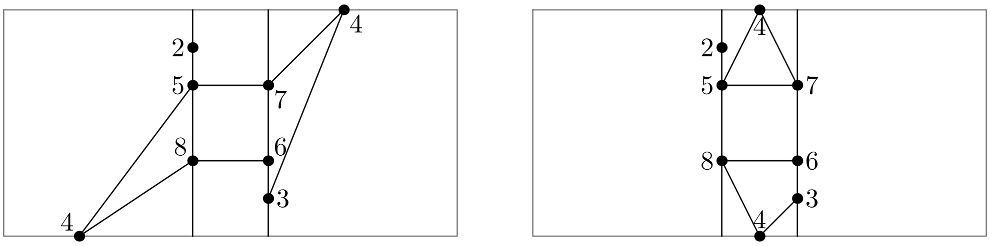
Figure 2.32: Cycles in $C_7 - (2,6)$, including vertices 5 and 6

It is easy to see that an embedding into the Möbius strip, including these cycles as boundaries of one face, is not possible:

Embedding of Case 1:



(3) $7-4-8$ is embedded

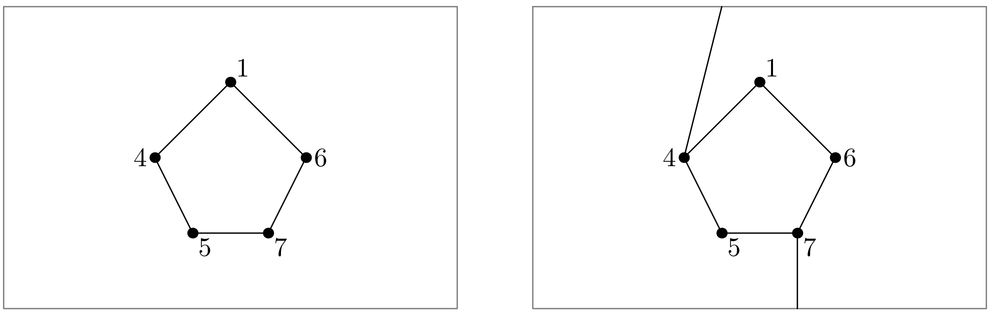


(4) $3-4-5$ is embedded

Figure 2.33: Embedding of Case 1

Figure 2.33 shows that an embedding, with the given restrictions, is not possible, as vertex 1 cannot be embedded.

Embedding of Case 2:

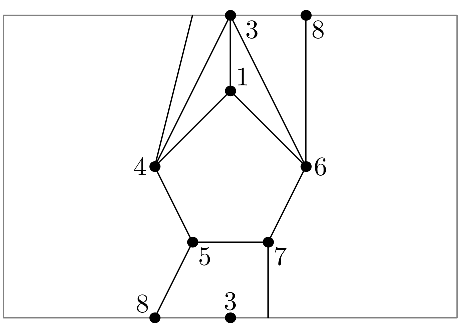


(1) Embedding of the given cycle

(2) $4-7$ is embedded

(3) $5-8-6$ is embedded

(4) $1-3-6$ is embedded



(5) $3-4$ is embedded

Figure 2.34: Embedding of Case 2

Figure 2.34 shows that an embedding, with the given restrictions, is not possible, as vertex 2 cannot be embedded. □

Lemma 2.14
A graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to vertices 4 and 6 in C_7 is no element of $M_4(\mathcal{S}_2)$.

Proof
The graph $C_7/(2,5)$ is isomorphic to $B_1 - (3,5)$, for which we have already shown that the vertices 3 and 5 cannot be drawn on the boundary of one face, when embedding the graph into the Möbius strip. As the graphs $B_1 - (3,5)$ and $C_7/(2,5)$ are isomorphic, we can also say that the pair of vertices $\{3,5\}$ in $B_1 - (3,5)$ is mapped to the pair of vertices $\{4,6\}$ in $C_7/(2,5)$. Consequently vertices 4 and 6 can also not be drawn on the boundary of one face, when embedding $C_7/(2,5)$ into the Möbius strip. Consequently attaching the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to vertices 4 and 6 does not deliver an irreducible graph for the Klein surface. □

The graph D_9

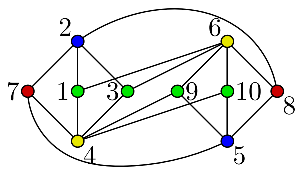


Figure 2.35: D_9 and its vertex-orbits

The orbits of pairs of vertices we have to consider for D_9 are:

$$
\begin{aligned}
A &= \Big\{\{1,3\},\{9,10\}\Big\},\\
B &= \Big\{\{1,5\},\{2,9\},\{2,10\},\{3,5\}\Big\},\\
C &= \Big\{\{1,7\},\{1,8\},\{3,7\},\{3,8\},\{7,9\},\{7,10\},\{8,9\},\{8,10\}\Big\},\\
D &= \Big\{\{1,9\},\{1,10\},\{3,9\},\{3,10\}\Big\},\\
E &= \Big\{\{2,4\},\{2,6\},\{4,5\},\{5,6\}\Big\},\\
F &= \Big\{\{2,5\}\Big\},\\
G &= \Big\{\{4,6\}\Big\},\\
H &= \Big\{\{4,8\},\{6,7\}\Big\},\\
I &= \Big\{\{7,8\}\Big\}.
\end{aligned}
$$

Lemma 2.15
The construction of a graph where the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ are attached to a pair of vertices of orbit A in D_9 does not result in an element of $M_4(\mathcal{S}_2)$.

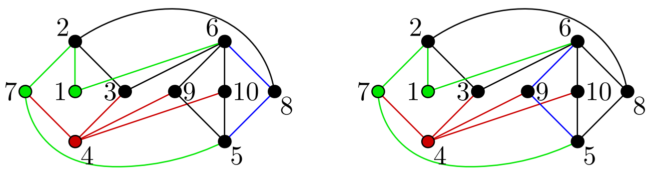
Proof
The graph $R_3(D_9)_{(8)}$ without vertices 9 and 10 (and adjacent edges) contains a $K_{3,3}$ with vertices $1, 2, 3, 4, 6$ and 7, where vertex 5 lies on the edge joining vertices 6 and 7. The embedding of this $K_{3,3}$ and the additional edges adjacent to its vertices already uses the Möbius strip characteristics and only leaves cellular faces. Vertices 9 and 10 consequently cannot be embedded on the boundary of one face. □

Lemma 2.16
Construction of a graph where the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ are attached to a pair of vertices of orbit B in D_9 does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
As the pair of vertices $\{1, 5\}$ lies in the same orbit as the pair $\{1, 4\}$ for $D_9 - (1, 4)$, vertices 1 and 5 cannot be drawn on the boundary of one face of the Möbius strip, when embedding $D_{12} - (1, 4)$. Consequently the graph constructed by attaching vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to vertices 1 and 5 in $D_{12} - (1, 4)$ is not embeddable into the Klein surface and thus attaching the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to one of the pairs of vertices in B does not deliver an irreducible graph for the Klein surface. □

Lemma 2.17
A graph with the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of orbit C in D_9, is no element of $M_4(\mathcal{S}_2)$.

Proof
The graph $D_9 - (1, 4)$ has ten vertices and 17 edges and with the Euler characteristic the number of faces of its embedding into the Möbius strip must be eight. As all faces have an even number of vertices on their boundary, the embedding of $D_9 - (1, 4)$ has one face with six vertices and the remaining faces with only four vertices on their boundaries. The only remaining non-isomorphic cycles in $D_9 - (1, 4)$ including vertices 1 and 7 are:

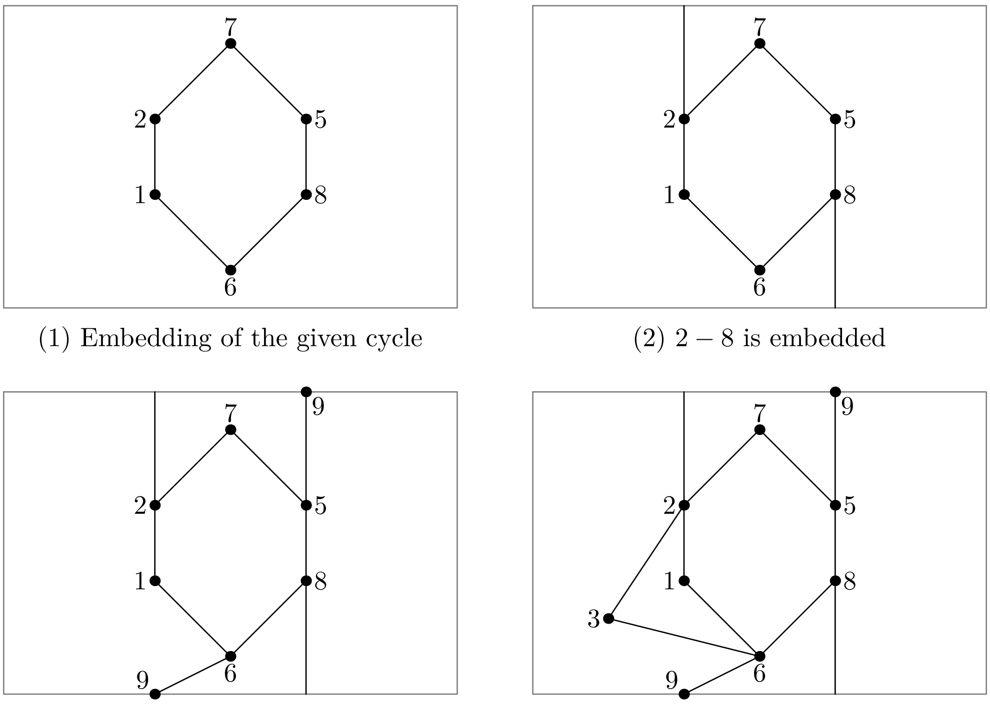


(1) Case 1 (2) Case 2

Figure 2.36: Cycles in $D_9 - (1, 4)$, including vertices 1 and 7

For the embeddings of $D_9 - (1, 4)$ with these cycles as boundaries of a face, we do the following case-differentiation:

Embedding of Case 1:



(1) Embedding of the given cycle

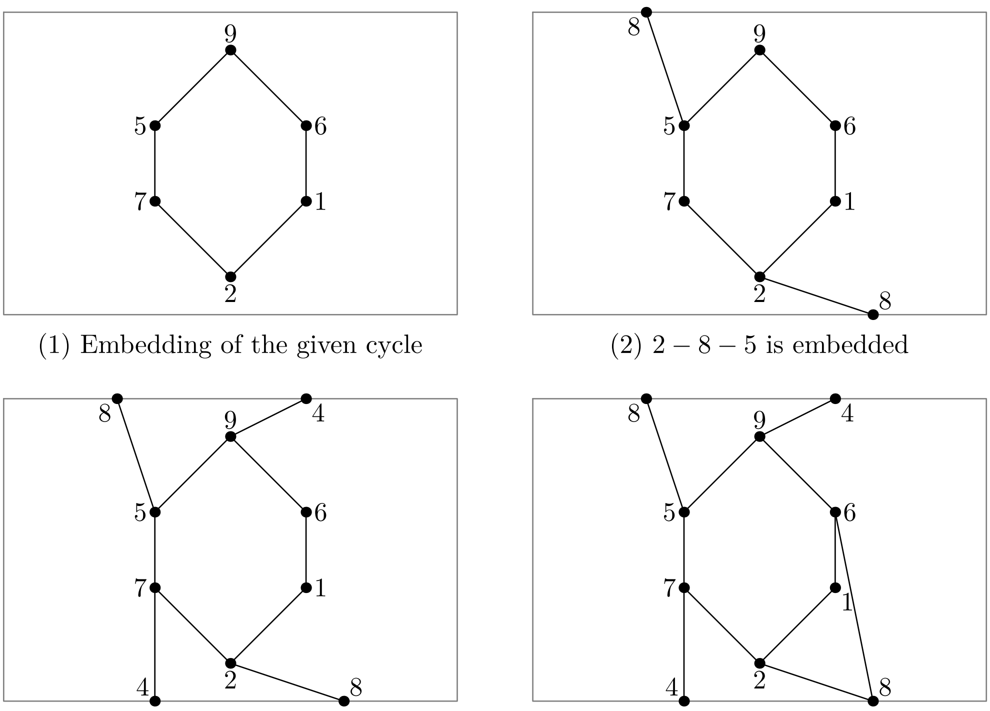
(2) $2-8$ is embedded

(3) $6-9-5$ is embedded

(4) $2-3-6$ is embedded

Figure 2.37: Embedding of Case 1

Figure 2.37 shows that an embedding, with the given restrictions, is not possible, as the path $3-4-7$ cannot be embedded. Embedding of Case 2:



(1) Embedding of the given cycle

(2) $2-8-5$ is embedded

(3) $7-4-9$ is embedded

(4) $6-8$ is embedded

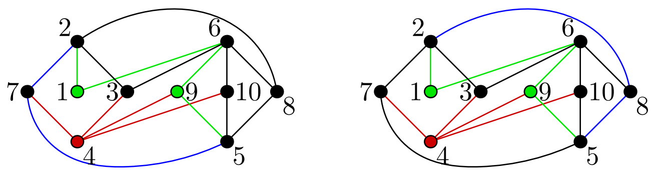
Figure 2.38: Embedding of Case 2

Figure 2.38 shows that an embedding, with the given restrictions, is not possible, as vertex 3 cannot be embedded.

Consequently it is shown, that the given graph is not an element of $M_4(\mathcal{S}_2)$. □

Lemma 2.18
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of the orbit D in D_9, does not result in an element of $M_4(\mathcal{S}_2)$.

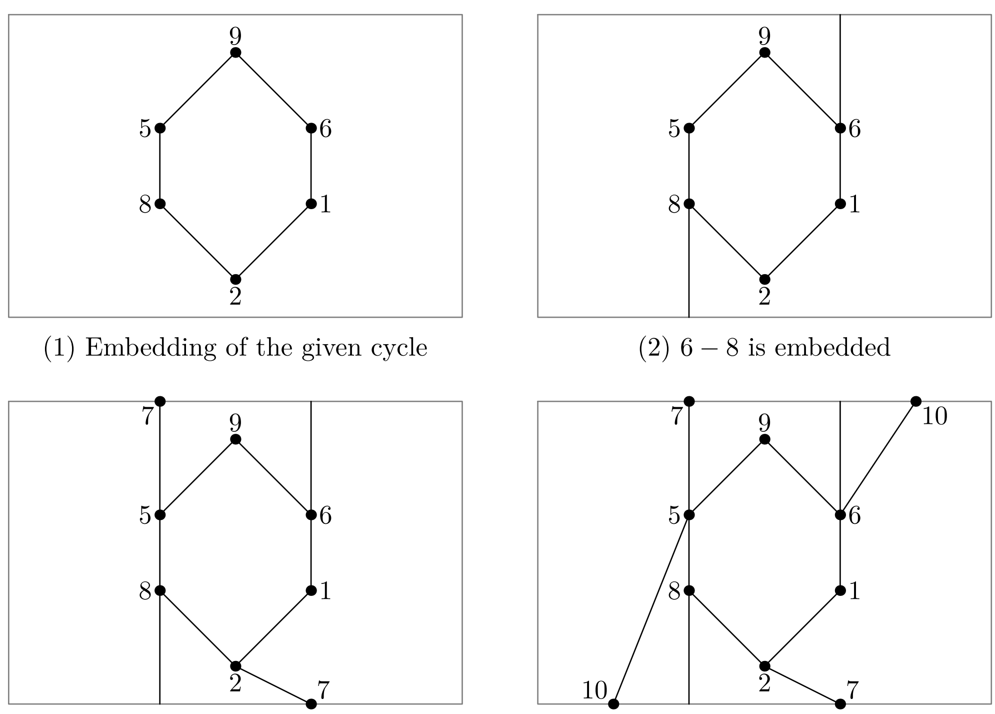
Proof
As in Lemma 2.17 the embedding of $D_9 - (1, 4)$ into the Möbius strip consists of one face with six vertices on its boundary and the remaining faces with four vertices on their boundary. The only remaining non-isomorphic cycles including vertices 1 and 9 are:



(1) Case 1 (2) Case 2

Figure 2.39: Cycles in $D_9 - (1, 4)$, including vertices 1 and 9

The cycle in Case 1 is the same as the one in Case 2 of Lemma 2.17 and thus already shown. For the cycle in Case 2, we again try to embed the graph starting with the given cycle as the boundary of one face:



(1) Embedding of the given cycle (2) $6 - 8$ is embedded

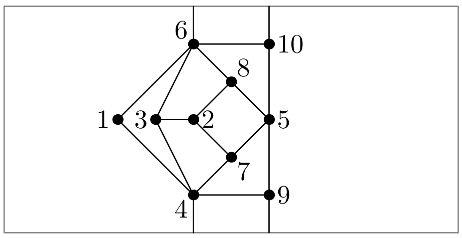
(3) $5 - 7 - 2$ is embedded (4) $5 - 10 - 6$ is embedded

Figure 2.40: Embedding of Case 2

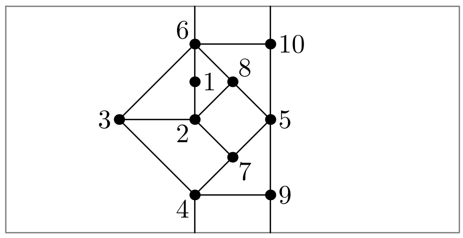
Figure 2.40 shows that an embedding, with the given restrictions, is not possible, as the path $9 - 4 - 10$ cannot be embedded. □

Lemma 2.19
The graphs G_{46}, G_{47} and G_{48}, which are obtained by identifying two vertices v_1 and v_2 of a K_5 with one of the pairs of vertices of orbits E, F or G, and deleting the edge (v_1, v_2), are elements of $M_4(\mathcal{S}_2)$.

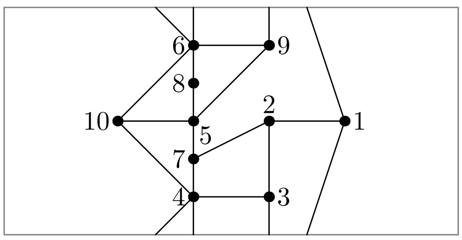
Proof
The graphs G_{46}, G_{47} and G_{48} are embeddable into the Klein surface as soon as one of the relations $R_0, \ldots, R_4$ is applied. To show that this is correct, we use possible embeddings of the graphs $D_9 - e$ and D_9/e, $e \in \{(1,2), (1,4), (2,8), (4,7)\}$ as well as $R_3(D_9)_{(1)}$ and $R_3(D_9)_{(8)}$ into the Möbius strip. As we can find embeddings where each of the pairs of vertices, which lie in orbits E, F and G, can be drawn on the boundary of one face on the Möbius strip, the $K_5 - (v_1, v_2)$ can be attached to each of these pairs of vertices.



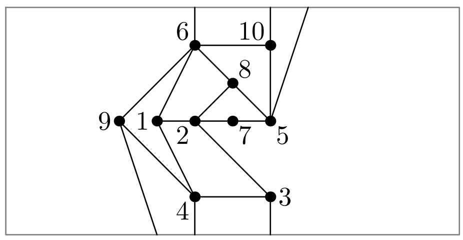
(1) Embedding of $D_9 - (1,2)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



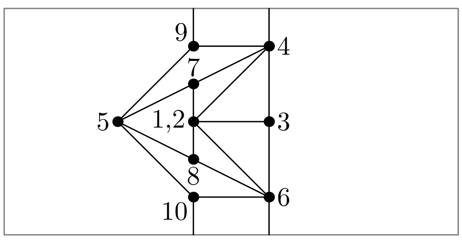
(2) Embedding of $D_9 - (1,4)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



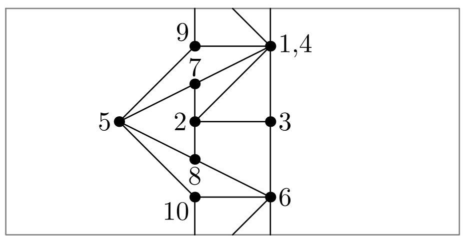
(3) Embedding of $D_9 - (2,8)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



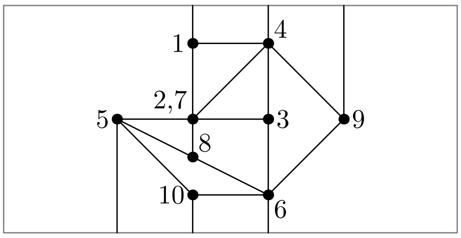
(4) Embedding of $D_9 - (4,7)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



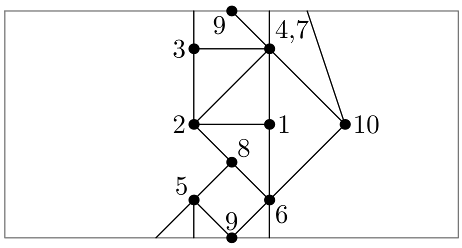
(5) Embedding of $D_9/(1,2)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



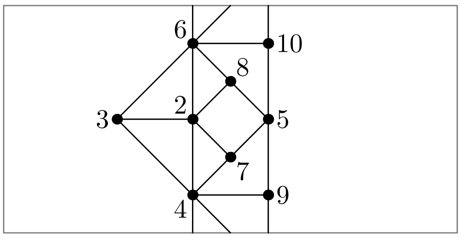
(6) Embedding of $D_9/(1,4)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



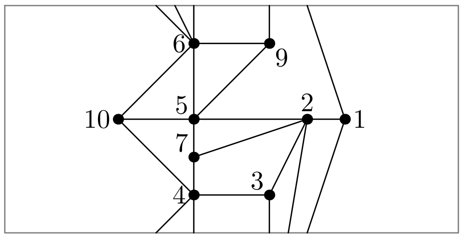
(7) Embedding of $D_9/(2,7)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



(8) Embedding of $D_9/(4,7)$, each vertex-pair of orbits E, F, and G on the boundary of a face.



(9) Embedding of $R_3(D_9)_{(1)}$, each vertex-pair of orbits E, F, and G on the boundary of a face.



(10) Embedding of $R_3(D_9)_{(8)}$, each vertex-pair of orbits E, F, and G on the boundary of a face.

Figure 2.41: Embeddings of $R_i(D_9)$, $i \in \{0, 2, 3\}$, into the Möbius strip.

We can also find an embedding of D_9, where each pair of vertices in orbits E, F and G can be drawn on the boundary of one face. Thus the $K_5 - (v_1, v_2) - e$ and the $(K_5 - (v_1, v_2))/e$ can be drawn within this face.

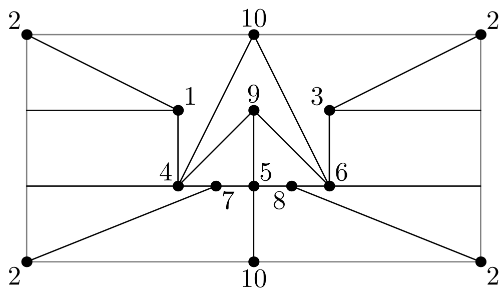
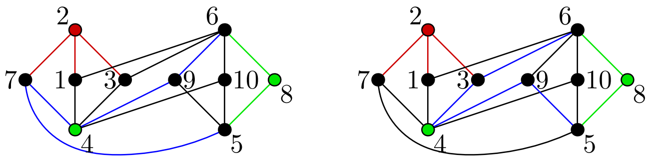


Figure 2.42: Embedding of D_9 into the Klein surface, each vertex-pair of orbits E, F, and G on the boundary of a face.

As this applies to each element in E, F and G, G_{46}, G_{47} and G_{48} are elements of $M_4(\mathcal{S}_2)$. $\square$

Lemma 2.20
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of orbit H in D_9, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
As in Lemma 2.17 the embedding of $D_9 - (2, 8)$ consists of one face with six vertices on its boundary and the remaining faces with four vertices on their boundary. The only remaining non-isomorphic cycles including vertices 4 and 8 are:



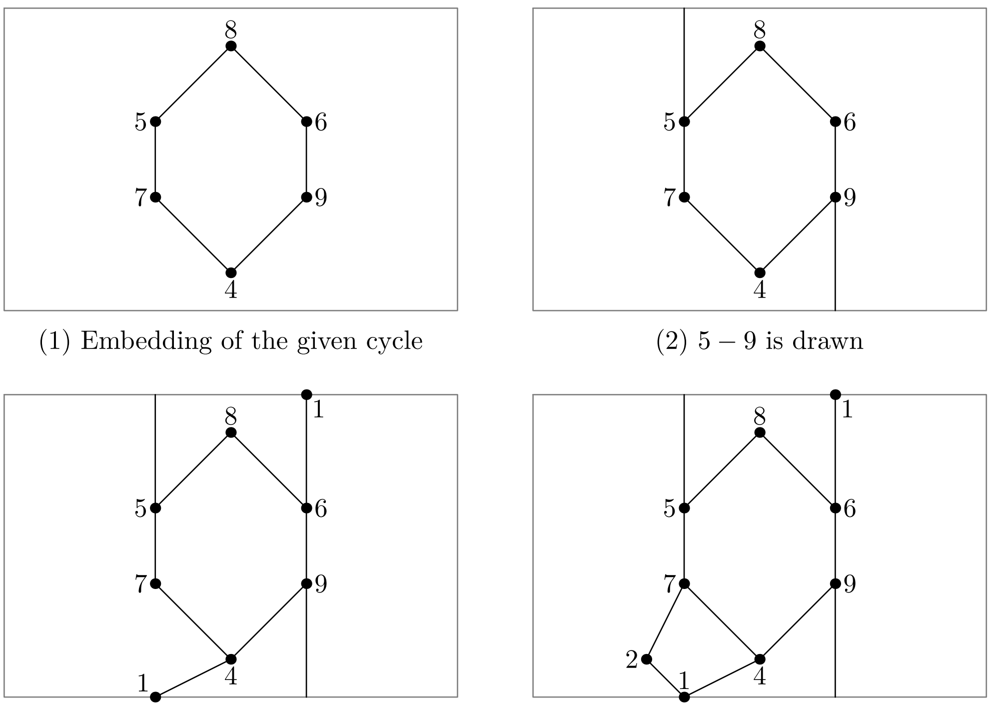
(1) Case 1 (2) Case 2

(3) Case 3 (4) Case 4

Figure 2.43: Cycles in $D_9 - (2, 8)$, including vertices 4 and 8

For these four cycles we again do a case-distinction to show that these cycles cannot exist as boundaries of one face, when embedding $D_9 - (2, 8)$ into the Möbius strip.

Embedding of Case 1:



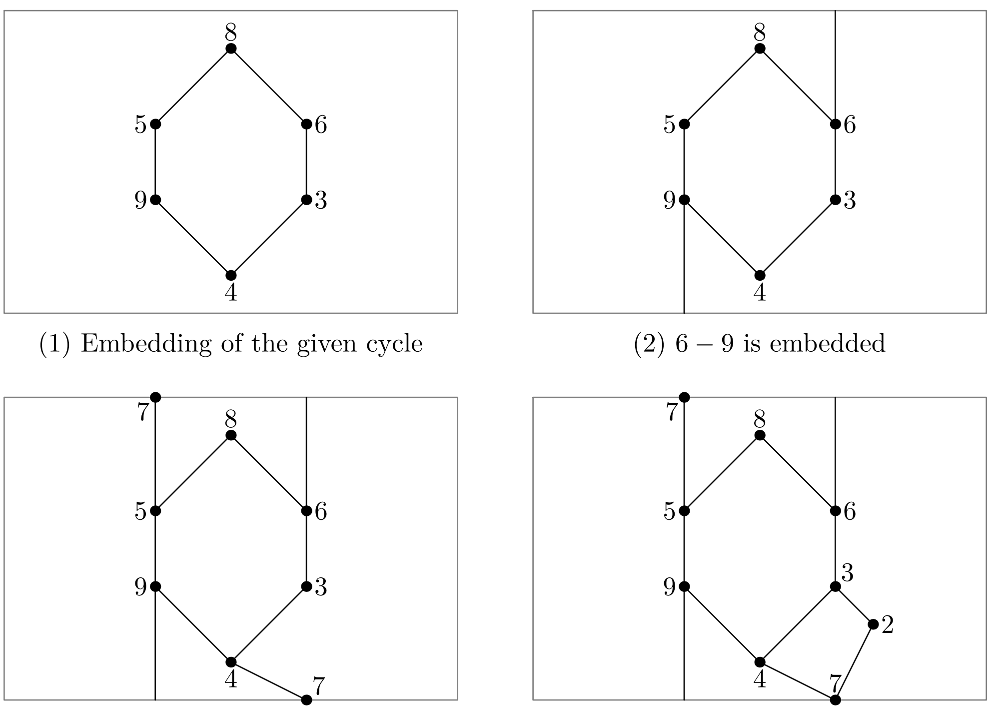
(1) Embedding of the given cycle (2) $5-9$ is drawn

(3) $4-1-6$ is embedded (4) $1-2-7$ is embedded

Figure 2.44: Embedding of Case 1

Figure 2.44 shows that an embedding, with the given restrictions, is not possible, as vertex 3 cannot be embedded.

Embedding of Case 2:



(1) Embedding of the given cycle

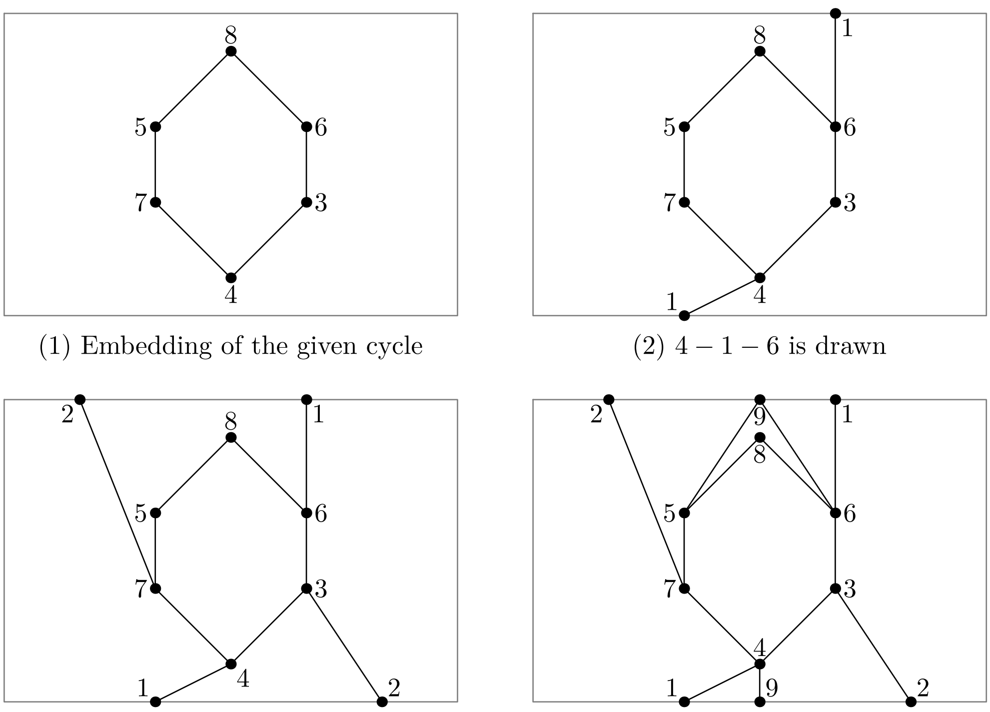
(2) $6-9$ is embedded

(3) $4-7-5$ is embedded

(4) $3-2-7$ is embedded

Figure 2.45: Embedding of Case 2

Figure 2.45 shows that an embedding, with the given restrictions, is not possible, as vertex 1 cannot be embedded. Embedding of Case 3:



(1) Embedding of the given cycle

(2) $4-1-6$ is drawn

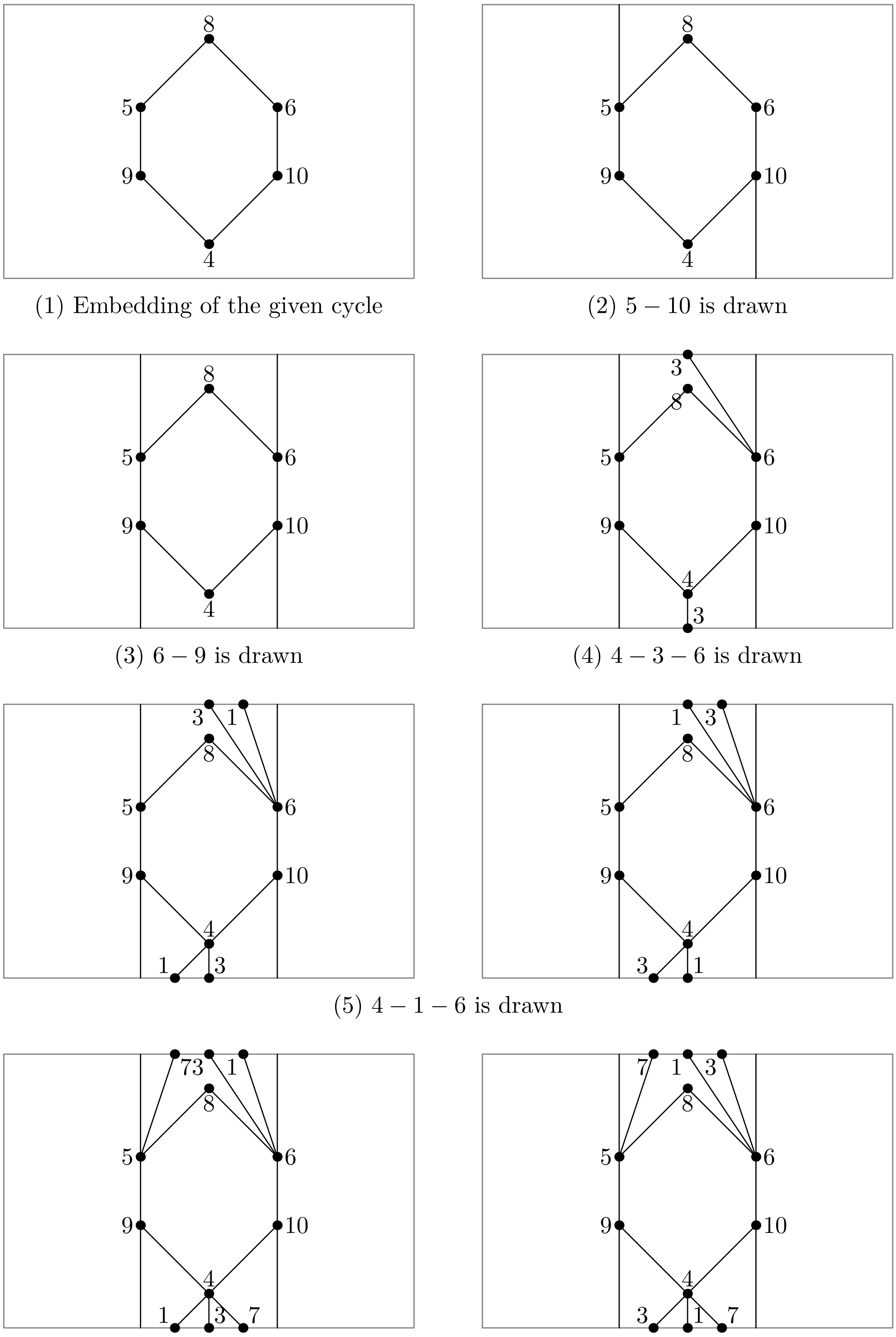
(3) $3-2-7$ is drawn

(4) 9 is embedded

Figure 2.46: Embedding of Case 3

Figure 2.46 shows that an embedding, with the given restrictions, is not possible, as vertex 10 cannot be embedded.

Embedding of Case 4:



(1) Embedding of the given cycle

(2) $5 - 10$ is drawn

(3) $6 - 9$ is drawn

(4) $4 - 3 - 6$ is drawn

(5) $4 - 1 - 6$ is drawn

(6) $4 - 7 - 5$ is drawn

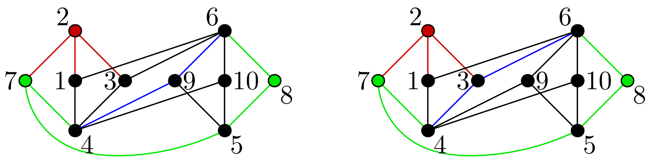
Figure 2.47: Embedding of Case 4

Figure 2.47 shows that an embedding, with the given restrictions, is not possible, as vertex 2 cannot be embedded.

Consequently non of the given cycles can be found as boundaries of one face in an embedding of $D_9 - (2,8)$. □

Lemma 2.21
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of orbit I in D_9, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The same conditions as in Lemma 2.20 apply for this lemma. Consequently the only cycles we have to look at in $D_9 - (2,8)$ are:



(1) Case 1 (2) Case 2

Figure 2.48: Cycles in $D_9 - (2,8)$, including vertices 7 and 8

As for these cycles we already found a contradiction in the proof of Lemma 2.20, nothing else has to be shown. □

The graph D_{12}

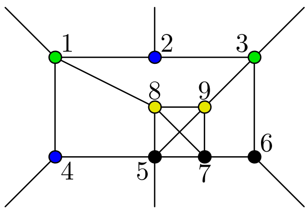


Figure 2.49: D_{12} and its vertex-orbits

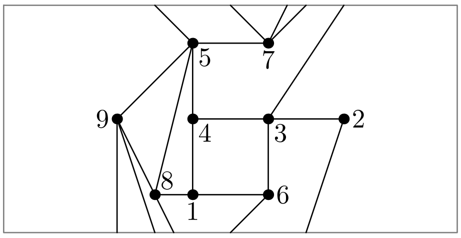
For the graph D_{12} we have to consider the following orbits of pairs of vertices:

$$\begin{aligned} A &= \big\{\{1,3\}\big\}, & B &= \big\{\{1,5\},\{3,5\}\big\}, \\ C &= \big\{\{1,7\},\{3,7\}\big\}, & D &= \big\{\{1,9\},\{3,8\}\big\}, \\ E &= \big\{\{2,4\}\big\}, & F &= \big\{\{2,6\},\{4,6\}\big\}, \\ G &= \big\{\{2,7\},\{4,7\}\big\}, & H &= \big\{\{2,8\},\{2,9\},\{4,8\},\{4,9\}\big\}, \\ I &= \big\{\{5,6\}\big\}, & J &= \big\{\{6,8\},\{6,9\}\big\}. \end{aligned}$$

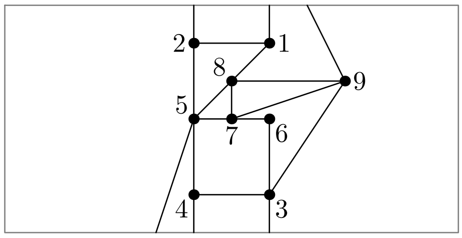
Lemma 2.22
The graphs G_{49}, G_{50} and G_{51} which are obtained by identifying two vertices v_1 and v_2 of a K_5 with pairs of vertices in orbit A, B or I and deleting the edge (v_1, v_2), are elements of $M_4(\mathcal{S}_2)$.

Proof

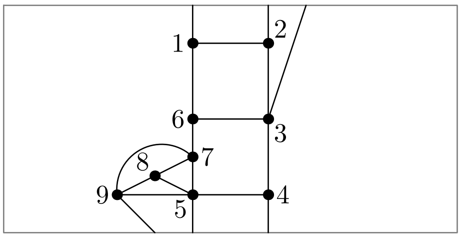
We consider embeddings of $G_4 - e$ and G_4/e for $e \in \{(1,2),\ (1,6),\ (1,8),\ (2,5),\ (5,7),\ (5,8),\ (6,7),\ (7,8),\ (8,9)\}$ as well as $R_3(D_{12})_{(2)}$ and $R_3(D_{12})_{(6)}$ into the Möbius strip.



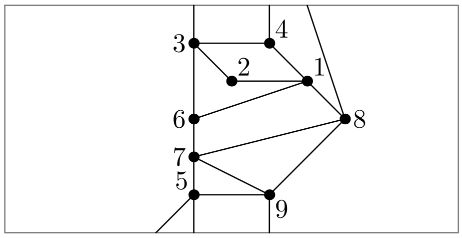
(1) Embedding of $D_{12} - (1,2)$, each vertex-pair of orbits A, B and I on the boundary of one face.



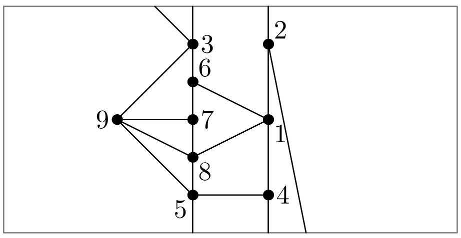
(2) Embedding of $D_{12} - (1,6)$, each vertex-pair of orbits A, B and I on the boundary of one face.



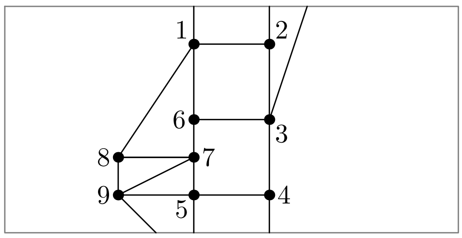
(3) Embedding of $D_{12} - (1,8)$, each vertex-pair of orbits A, B and I on the boundary of one face.



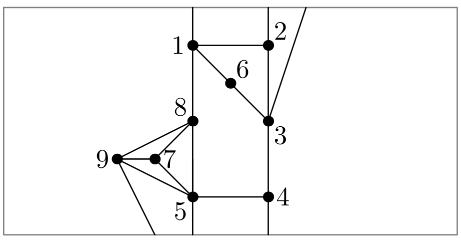
(4) Embedding of $D_{12} - (2,5)$, each vertex-pair of orbits A, B and I on the boundary of one face.



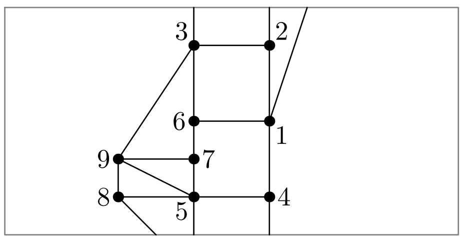
(5) Embedding of $D_{12} - (5,7)$, each vertex-pair of orbits A, B and I on the boundary of one face.



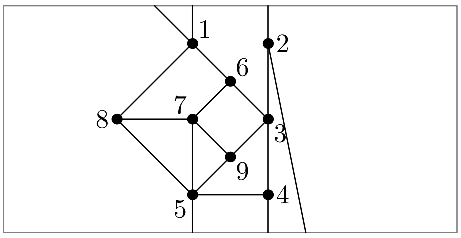
(6) Embedding of $D_{12} - (5,8)$, each vertex-pair of orbits A, B and I on the boundary of one face.



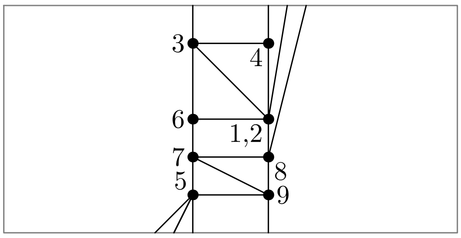
(7) Embedding of $D_{12} - (6,7)$, each vertex-pair of orbits A, B and I on the boundary of one face.



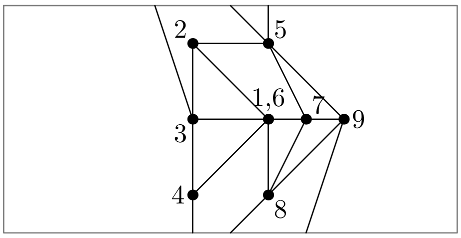
(8) Embedding of $D_{12} - (7,8)$, each vertex-pair of orbits A, B and I on the boundary of one face.



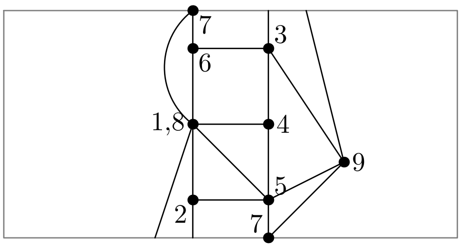
(9) Embedding of $D_{12} - (8,9)$, each vertex-pair of orbits A, B and I on the boundary of one face.



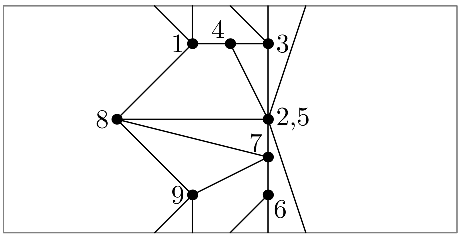
(10) Embedding of $D_{12}/(1,2)$, each vertex-pair of orbits A, B and I on the boundary of one face.



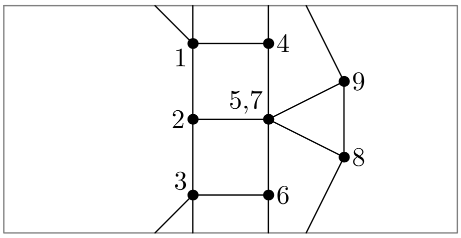
(11) Embedding of $D_{12}/(1,6)$, each vertex-pair of orbits A, B and I on the boundary of one face.



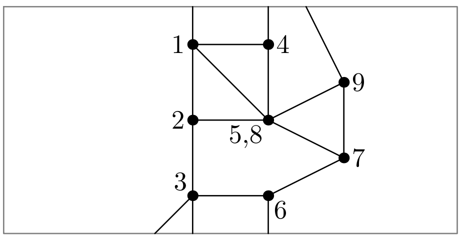
(12) Embedding of $D_{12}/(1,8)$, each vertex-pair of orbits A, B and I on the boundary of one face.



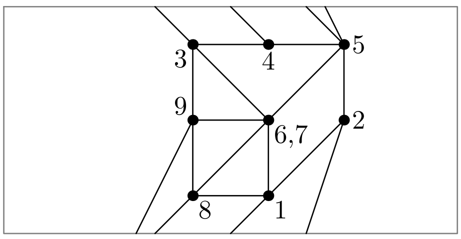
(13) Embedding of $D_{12}/(2,5)$, each vertex-pair of orbits A, B and I on the boundary of one face.



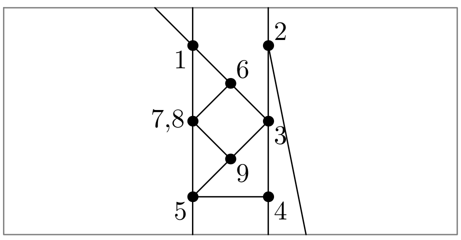
(14) Embedding of $D_{12}/(5,7)$, each vertex-pair of orbits A, B and I on the boundary of one face.



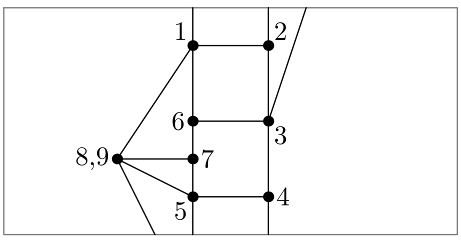
(15) Embedding of $D_{12}/(5,8)$, each vertex-pair of orbits A, B and I on the boundary of one face.



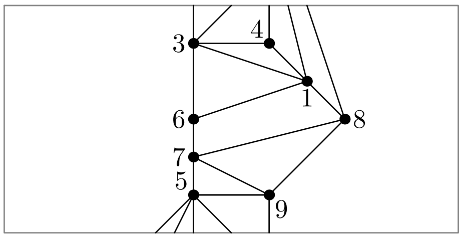
(16) Embedding of $D_{12}/(6,7)$, each vertex-pair of orbits A, B and I on the boundary of one face.



(17) Embedding of $D_{12}/(7,8)$, each vertex-pair of orbits A, B and I on the boundary of one face.



(18) Embedding of $D_{12}/(8,9)$, each vertex-pair of orbits A, B and I on the boundary of one face.



(19) Embedding of $R_3(D_{12})_{(2)}$, each vertex-pair of orbits A, B and I on the boundary of one face.

Figure 2.50: Embeddings of $R_i(D_{12})$, $i \in \{0, 2, 3\}$, into the Möbius strip

We can also find embeddings of D_{12} into the Klein surface, where the pairs of vertices in orbits A, B and I lie on the boundary of one face:

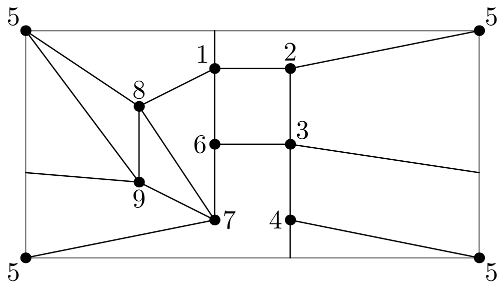


Figure 2.51: Embedding of D_{12} into the Klein surface, each vertex-pair of orbits A, B and I on the boundary of one face.

Consequently G_{49}, G_{50} and G_{51} are elements of $M_4(\mathcal{S}_2)$. □

Lemma 2.23
Attaching the $K_5 - (v_1, v_2)$ to a pair of vertices from orbit C of D_{12} does not result in an element of $M_4(\mathcal{S}_2)$.

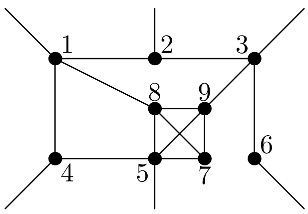


Figure 2.52: $D_{12} - (6, 7)$

Proof
The graph, which we have to consider to show that the constructed graph is not embeddable, if we delete one random edge, is $D_{12} - (6, 7)$ with v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to one of the pairs of vertices in orbit C. Vertices $5, 8$ and 9 have to be connected to vertices in the $K_{3,3} - (5, 6)$ (vertex 5 is already in $K_{3,3} - (5, 6)$). As vertices $5, 8$ and 9 are adjacent to each other, the vertices $1, 3$ and 5 of the $K_{3,3}$ have to lie on the boundary of one face. Thus the Möbius characteristics have to be used. Consequently the K_4 with vertices $5, 7, 8$ and 9 has to be embedded into one face and vertex 7 consequently has to be embedded

into the triangular face with vertices $5, 8$ and 9. Thus vertex 7 cannot lie in the same face as vertices 1 or 3. □

Lemma 2.24
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of orbits D, H or J in D_{12}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The $K_{3,3}$ with the vertices $1, 2, 3, 4, 5$ and 6 uses the Möbius characteristics. Therefore the attached K_4 with the vertices $5, 7, 8$ and 9 has to be drawn within one face. The vertices $5, 7$ and 9 have to lie on the boundary of one face, so they can each be connected to or identified with a vertex of the $K_{3,3}$. The vertices $5, 7$ and 9 thus form a triangle, in which vertex 8 has to be embedded. Consequently vertex 8 cannot lie in the same face as vertices $2, 3, 4$ or 6. □

Lemma 2.25
A graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of orbit G in D_{12} cannot be an element of $M_4(\mathcal{S}_2)$.

Proof
As the pair of vertices {2, 7} lies in the same orbit as {2, 5} for $D_{12} - (2, 5)$, vertices 2 and 7 cannot be drawn on the boundary of one face of the Möbius strip, when embedding $D_{12} - (2, 5)$. Consequently the graph constructed by attaching vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to vertices 2 and 7 in $D_{12} - (2, 5)$ is not embeddable into the Klein surface and thus attaching vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to one of the pairs of vertices in F does not deliver an irreducible graph for the Klein surface. □

For the orbits E and F we take the possible embeddings of the K_4 into the Möbius strip into abbount. The only embedding, where all four vertices of the K_4 lie on the boundary of one face, is:

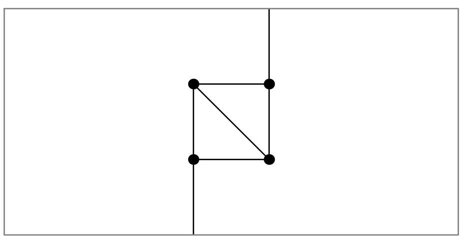


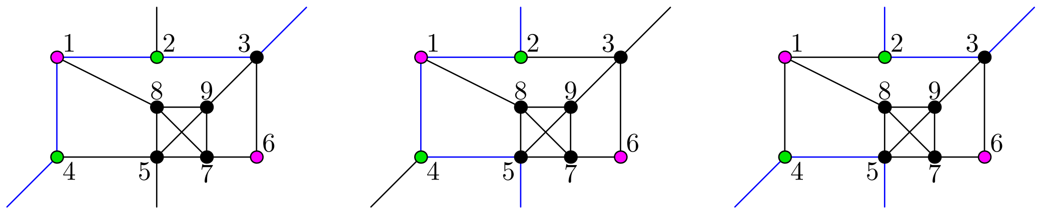
Figure 2.53: The K_4 embedded into the Möbius strip

This embedding of the K_4 has only two triangular faces. Although we can find four cycles of length three in the K_4, this is the maximal number of triangular faces under the given conditions.

Using this, the next two lemmata can be proven.

Lemma 2.26
The construction of a graph whith vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of orbit E in D_{12}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
In $D_{12} - (1, 6)$ the shortest paths between vertices 2 and 4 are $2 - 1 - 4$, $2 - 3 - 4$ and $2 - 5 - 4$, additionally vertices $1, 3$ and 5 are the only vertices 2 and 4 are adjacent to. Using Lemma 1.11, the cycles of length four in Figure 2.54are the only ones we have to consider for possible embeddings of $D_{12} - (1, 6)$ with vertices 2 and 4 on the boundary of one face.

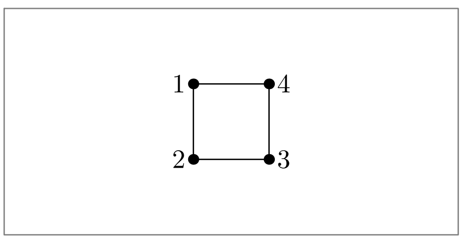


(1) Case 1 (2) Case 2 (3) Case 3

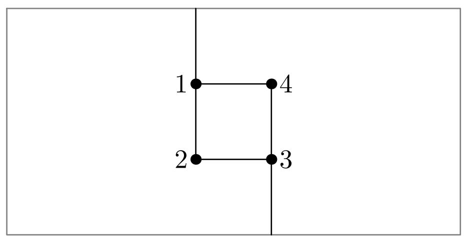
Figure 2.54: Cycles in $D_{12} - (1, 6)$, including vertices 2 and 4; vertices 1 and 6 cannot lie on the same cycle.

For these cycles, we try to find embeddings with these cycles as boundaries of one face:

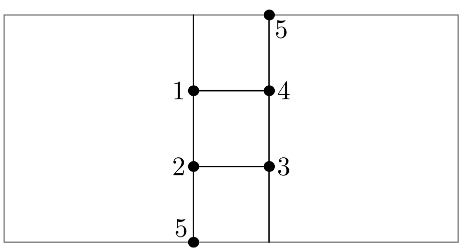
Embedding of Case 1:



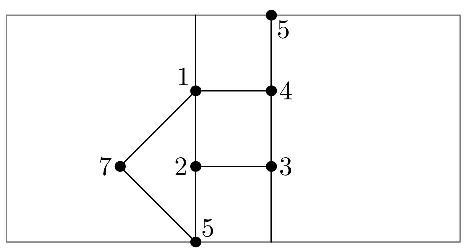
(1) Embedding of the given cycle



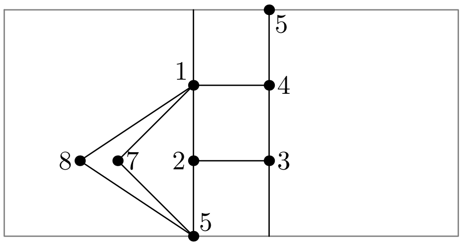
(2) $1 - 3$ is embedded



(3) $2 - 5 - 4$ is embedded



(4) $1 - 7 - 5$ is embedded

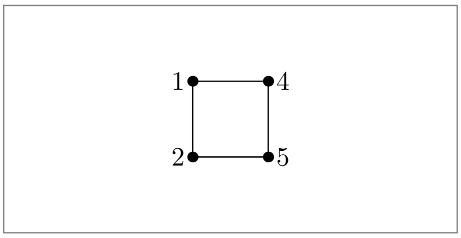


(5) $1-8-5$ is embedded

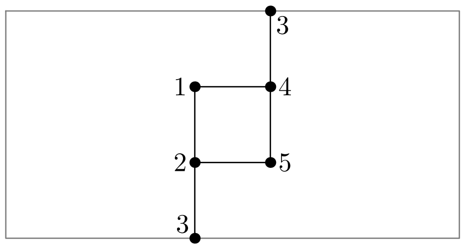
Figure 2.55: Embedding of Case 1

Figure 2.55 shows that an embedding, with the given restrictions, is not possible, as vertex 9 cannot be embedded.

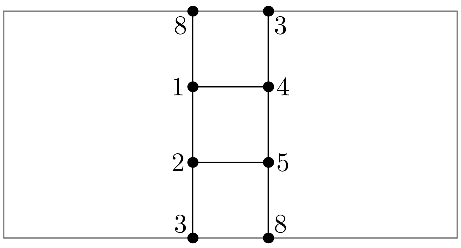
Embedding of Case 2:



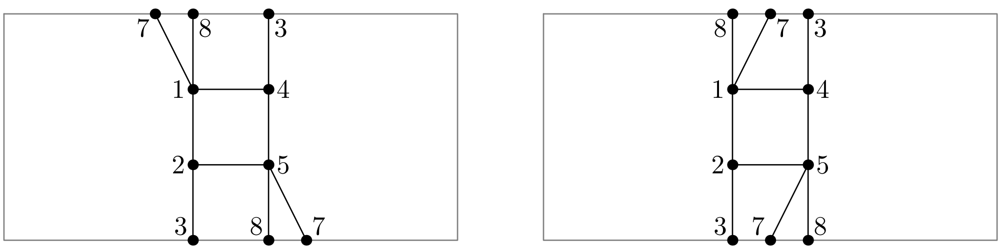
(1) Embedding of the given cycle



(2) $2-3-4$ is embedded



(3) $1-8-5$ is embedded

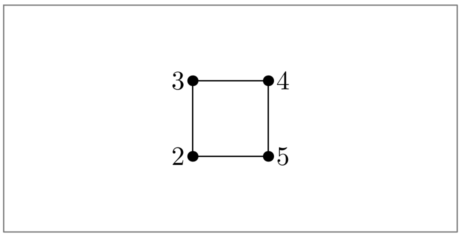


(4) $1-7-5$ is embedded

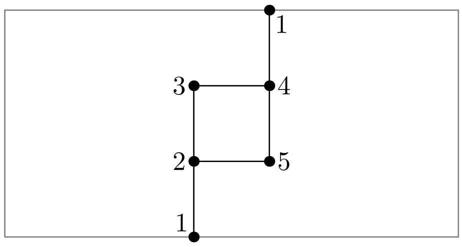
Figure 2.56: Embedding of Case 2

Figure 2.56 shows that an embedding, with the given restrictions, is not possible, as vertex 9 cannot be embedded.

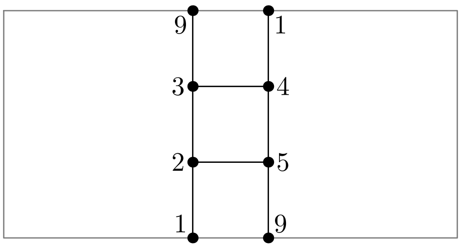
Embedding of Case 3:



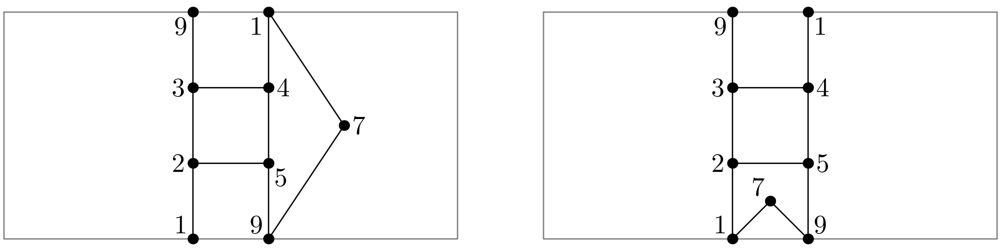
(1) Embedding of the given cycle



(2) $2-1-4$ is embedded



(3) $5-9-3$ is embedded



(4) $1-7-9$ is embedded

Figure 2.57: Embedding of Case 3

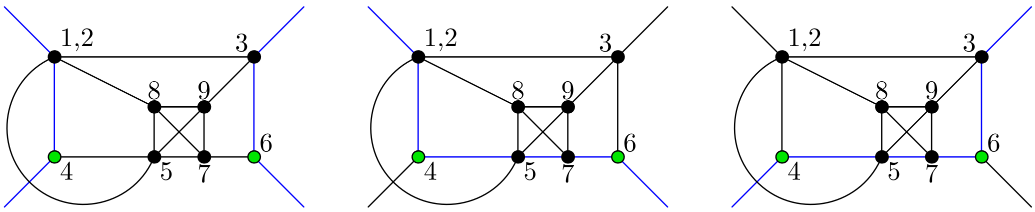
Figure 2.57 shows that an embedding, with the given restrictions, is not possible, as vertex 8 cannot be embedded.

Consequently none of the given cycles in $D_{12}-(1,6)$ can be embedded as the boundary of one face. □

Lemma 2.27
The graph with vertices v_1 and v_2 of $K_5-(v_1,v_2)$ attached to a pair of vertices of orbit F in D_{12}, cannot be an element of $M_4(\mathcal{S}_2)$.

Proof
The graph $D_{12}/(1,2)$ has eight vertices and 16 edges and with the Euler characteristic

the number of faces of its embedding into the Möbius strip must be nine. As there are only eight cycles of length three in the graph, where four of these are within a K_4, the embedding of $D_{12}/(1,2)$ can have at most six triangular faces and consequently faces cannot have more than six vertices on their boundary. After application of Lemma 1.11 the only remaining non-isomorphic cycles including vertices 4 and 6 are:

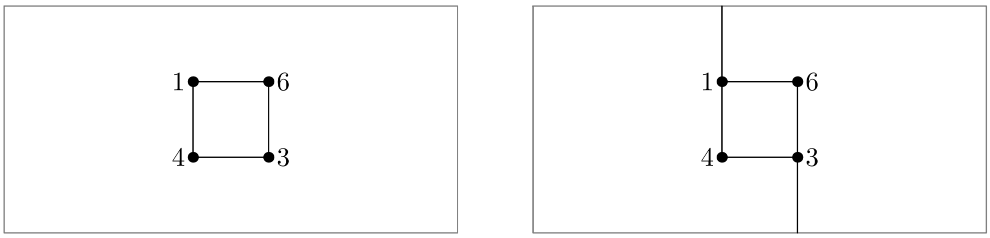


(1) Case 1 (2) Case 2 (3) Case 3

Figure 2.58: Cycles in $D_{12}/(1,2)$, including vertices 4 and 6.

For embeddings with vertices 4 and 6 on the boundary of one face, we have to do a case-distinction:

Embedding of Case 1:



(1) Embedding of the given cycle (2) $1-3$ is embedded

Figure 2.59: Embedding of Case 1

Figure 2.59 shows that an embedding, with the given restrictions, is not possible, as the remaining K_4 with vertices $5,7,8$ and 9 cannot be embedded.

Embedding of Case 2:

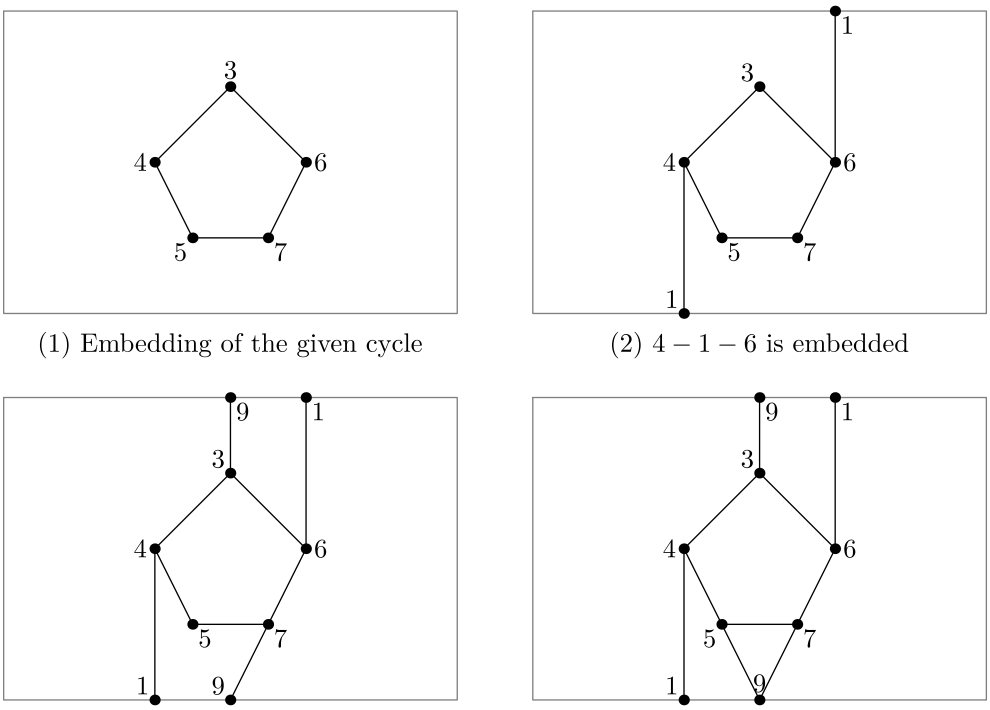
(1) Embedding of the given cycle (2) $1-5$ is embedded

(3) $4-3-6$ is embedded (4) 9 is embedded

Figure 2.60: Embedding of Case 2

Figure 2.60 shows that an embedding, with the given restrictions, is not possible, as vertex 8 cannot be embedded.

Embedding of Case 3:



(1) Embedding of the given cycle (2) $4-1-6$ is embedded

(3) $7-9-3$ is embedded (4) $5-9$ is embedded

Figure 2.61: Embedding of Case 3

Figure 2.61 shows that an embedding, with the given restrictions, is not possible, as vertex 8 cannot be embedded. Consequently none of the graphs can be an element of $M_4(\mathcal{S}_2)$. $\square$

The graph D_{17}

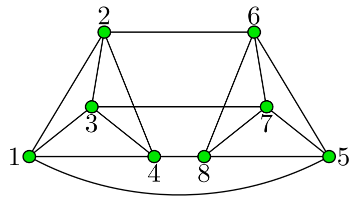


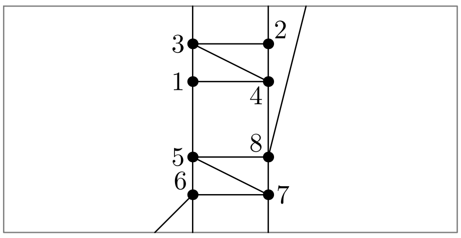
Figure 2.62: D_{17} and its vertex-orbits

The only orbit of pairs of vertices we have to look at for the graph D_{17} is:

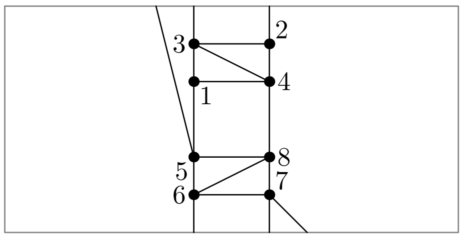
$$A = \Big\{\{1,6\},\{1,7\},\{1,8\},\{2,5\},\{2,7\},\{2,8\},\{3,5\},\{3,6\},\{3,8\},\{4,5\},\{4,6\},\{4,7\}\Big\}.$$

Lemma 2.28
The graph G_{52}, which is constructed by attaching the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to a pair of vertices of orbit A, is an element of $M_4(\mathcal{S}_2)$.

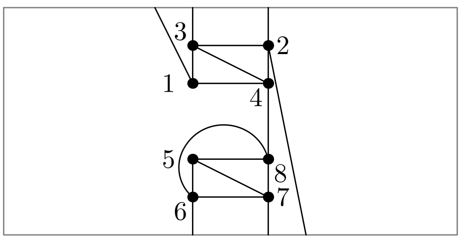
Proof
Looking at possible embeddings of $D_{17} - e$ and D_{17}/e, $e \in \{(1,2), (1,5)\}$ into the Möbius strip, we find embeddings where each pair of vertices in orbit A can be drawn on the boundary of one face of the Möbius strip, so that vertices v_1 and v_2 from $K_5 - (v_1, v_2)$ can be drawn in the outer face and thus attached to each representative of orbit A.



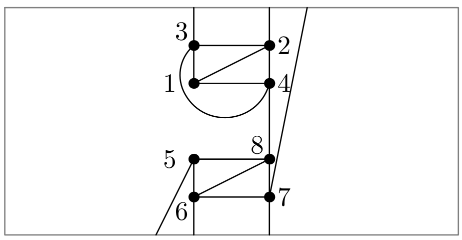
(1) Embedding of $D_{17} - (1,2)$, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{1,8\}$, $\{2,7\}$, $\{2,8\}$, $\{3,5\}$, $\{3,6\}$, $\{3,8\}$, $\{4,5\}$, $\{4,6\}$ on the boundary of one face.



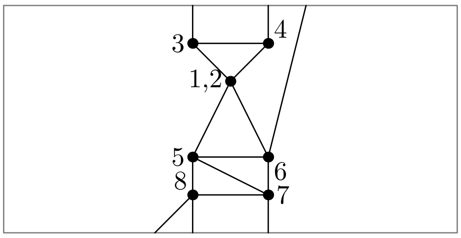
(2) Embedding of $D_{17} - (1,2)$, vertex-pairs $\{2,5\}$, $\{4,7\}$ on the boundary of one face.



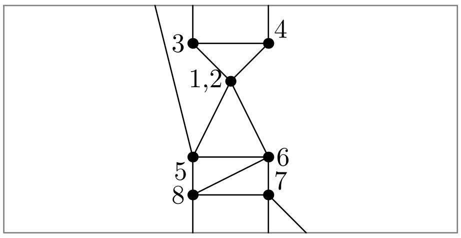
(3) Embedding of $D_{17} - (1,5)$, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{1,8\}$, $\{2,7\}$, $\{2,8\}$, $\{3,6\}$, $\{3,8\}$, $\{4,6\}$, $\{4,7\}$ on the boundary of one face.



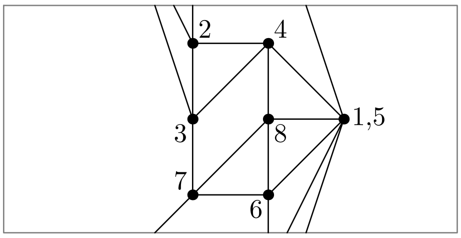
(4) Embedding of $D_{17} - (1,5)$, vertex-pairs $\{2,5\}$, $\{3,5\}$, $\{4,5\}$ on the boundary of one face.



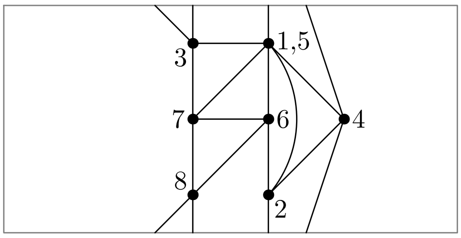
(5) Embedding of $D_{17}/(1,2)$, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{1,8\}$, $\{2,5\}$, $\{2,7\}$, $\{2,8\}$, $\{3,5\}$, $\{3,6\}$, $\{3,8\}$, $\{4,6\}$, $\{4,7\}$ on the boundary of one face.



(6) Embedding of $D_{17}/(1,2)$, vertex-pair $\{4,5\}$ on the boundary of one face.



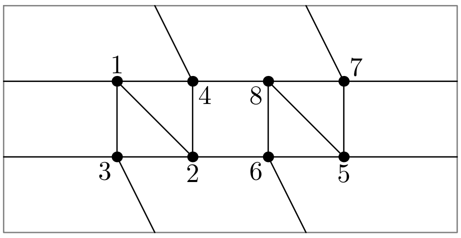
(7) Embedding of $D_{17}/(1,5)$, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{1,8\}$, $\{2,5\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{4,5\}$, $\{4,6\}$, $\{4,7\}$ on the boundary of one face.



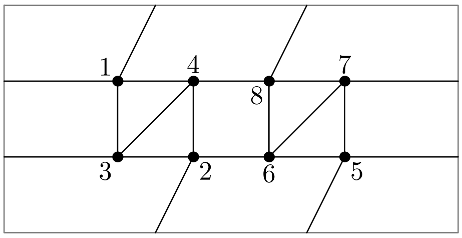
(8) Embedding of $D_{17}/(1,5)$, vertex-pairs $\{2,8\}$, $\{3,6\}$ on the boundary of one face.

Figure 2.63: Embeddings of $R_i(D_{17})$, $i \in \{0,2\}$, into the Möbius strip.

Also the embeddings of D_{17} itself into the Klein surface, allow that each pair of vertices in orbit A lies on the boundary of one face, so that vertices v_1 and v_2 of $K_5 - (v_1, v_2) - e$ and $(K_5/(v_1, v_2))/e$ can be attached to each representative.



(1) Embedding of D_{17}, vertex-pairs $\{1,6\}$, $\{1,7\}$, $\{2,7\}$, $\{2,8\}$, $\{3,5\}$, $\{3,6\}$, $\{4,5\}$, $\{4,6\}$, $\{4,7\}$ on the boundary of one face.



(2) Embedding of D_{17}, vertex-pairs $\{1,8\}$, $\{2,5\}$, $\{3,8\}$ on the boundary of one face.

Figure 2.64: Embedding of D_{17} into the Klein surface

Consequently G_{52} is an element of $M_4(\mathcal{S}_2)$. □

The graph E_3

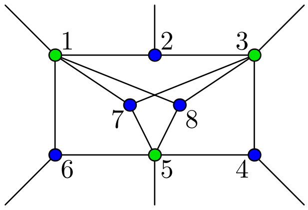


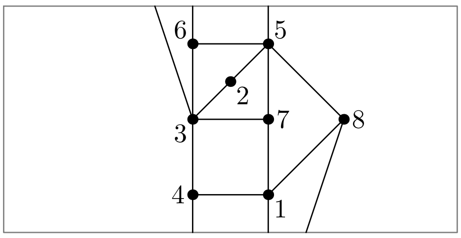
Figure 2.65: E_3 and its vertex-orbits

The only orbits of pairs of vertices we have to look at for the graph E_3 are:

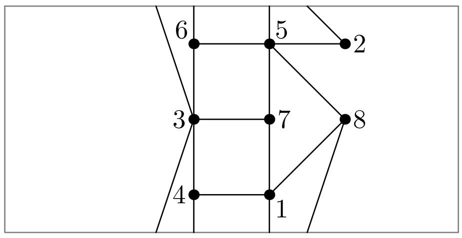
$$A = \Big\{\{1,3\},\{1,5\},\{3,5\}\Big\},$$
$$B = \Big\{\{2,4\},\{2,6\},\{2,7\},\{2,8\},\{4,6\},\{4,7\},\{4,8\},\{6,7\},\{6,8\},\{7,8\}\Big\}.$$
$$A = \Big\{\{1,3\},\{1,5\},\{1,7\},\{1,8\},\{3,5\},\{3,7\},\{3,8\},\{5,7\},\{5,8\},\{7,8\}\Big\},$$
$$B = \Big\{\{2,4\},\{2,6\},\{4,6\}\Big\}.$$

Lemma 2.29
The graphs G_{53} and G_{54}, which are constructed by attaching the $K_5 - (v_1, v_2)$ to a pair of vertices from orbits A and B, are elements of $M_4(\mathcal{S}_2)$.

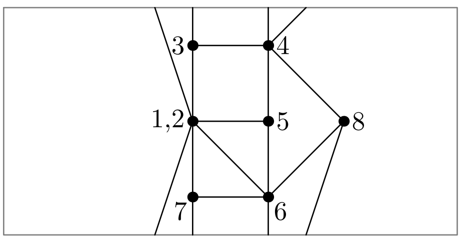
Proof
For both graphs, we have to check if the elements of A and B can be drawn on the boundary of one face of the embeddings of $E_3 - (1,2)$, $E_3/(1,2)$ and $R_3(E_3)_{(1)}$ into the Möbius strip, and all these embeddings are possible:



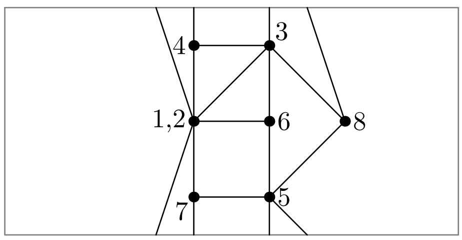
(1) Embedding of $E_3-(1,2)$, vertex-pairs $\{2,6\}$, $\{2,7\}$, $\{1,3\}$, $\{1,5\}$, $\{4,6\}$, $\{4,7\}$, $\{4,8\}$, $\{3,5\}$, $\{6,8\}$, $\{7,8\}$ on the boundary of a face.



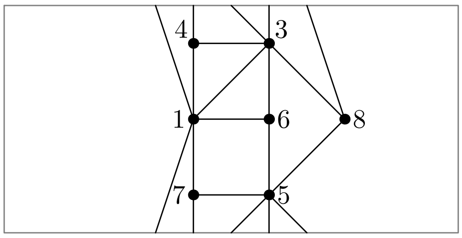
(2) Embedding of $E_3-(1,2)$, vertex-pairs $\{2,4\}$, $\{2,8\}$, $\{6,7\}$ on the boundary of a face.



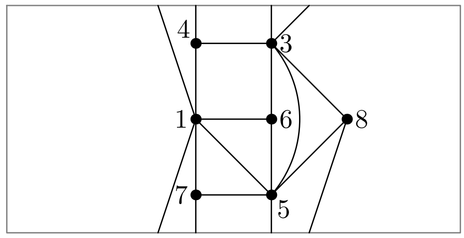
(3) Embedding of $E_3/(1,2)$, vertex-pairs $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{2,8\}$, $\{1,3\}$, $\{1,5\}$, $\{4,6\}$, $\{4,7\}$, $\{4,8\}$, $\{3,5\}$, $\{6,8\}$ on the boundary of a face.



(4) Embedding of $E_3/(1,2)$, vertex-pairs $\{6,7\}$, $\{7,8\}$ on the boundary of a face.



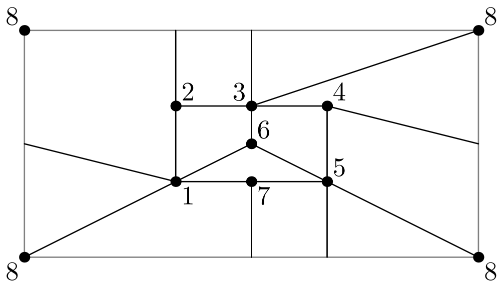
(5) Embedding of $R_3(E_3)_{(2)}$, vertex-pairs $\{1,3\}$, $\{1,5\}$, $\{3,5\}$, $\{6,7\}$, $\{6,8\}$, $\{7,8\}$ on the boundary of a face.



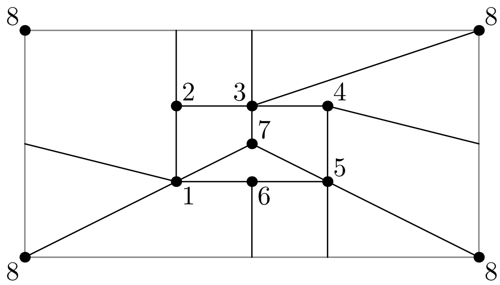
(6) Embedding of $R_3(E_3)_{(2)}$, vertex-pairs $\{4,6\}$, $\{4,7\}$, $\{4,8\}$ on the boundary of a face.

Figure 2.66: Embeddings of $R_i(E_3)$, $i \in \{0,2,3\}$, into the Möbius strip.

Also every pair of vertices of orbits A and B can be drawn on the boundary of one face, when embedding E_3 into the Klein surface. Thus vertices v_1 and v_2 of $K_5 - (v_1, v_2) - e$ and $(K_5 - (v_1, v_2))/e$ can be attached to each of the pairs of vertices in orbits A and B.



(1) Embedding of E_3, vertex-pairs $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{2,8\}$, $\{1,3\}$, $\{1,5\}$, $\{4,6\}$, $\{4,8\}$, $\{3,5\}$, $\{6,7\}$, $\{7,8\}$ on the boundary of a face.



(2) Embedding of E_3, vertex-pairs $\{4,7\}$, $\{6,8\}$ on the boundary of a face.

Figure 2.67: Embedding of E_3 into the Klein surface

Consequently G_{53} and G_{54} are elements of $M_4(\mathcal{S}_2)$. □

The graph E_{18}

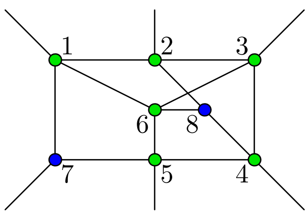


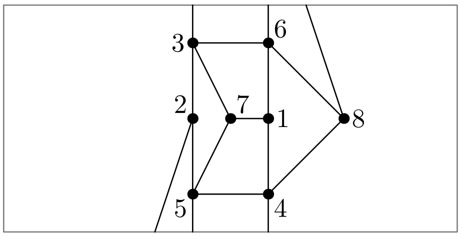
Figure 2.68: E_{18} and its vertex-orbits

The remaining orbits of pairs of two vertices, we have to consider for E_{18} are:

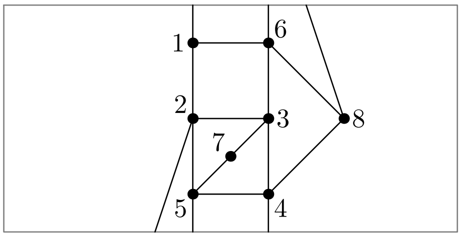
$$A = \Big\{\{1,3\},\{1,5\},\{2,4\},\{2,6\},\{3,5\},\{4,6\}\Big\},$$
$$B = \Big\{\{1,8\},\{2,7\},\{3,8\},\{4,7\},\{5,8\},\{6,7\}\Big\},$$
$$C = \Big\{\{7,8\}\Big\}.$$

Lemma 2.30
The graphs G_{55} and G_{56}, are graphs constructed by attaching the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ to two vertices of E_{18}, which lie in orbits A and B. Both graphs are elements of $M_4(\mathcal{S}_2)$.

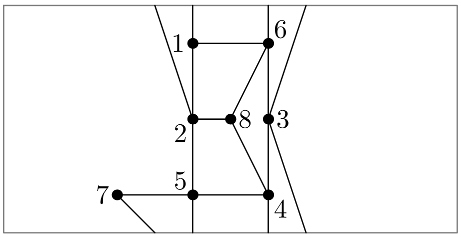
Proof
For each pair of vertices from orbits A and B we can find an embedding of $E_{18} - e$ and E_{18}/e, $e \in \{(1,2), (1,7)\}$, as well as $R_3(E_{18})_{(7)}$ into the Möbius strip, where they lie on the boundary of one face, so that we can attach the $K_5 - (v_1, v_2)$.



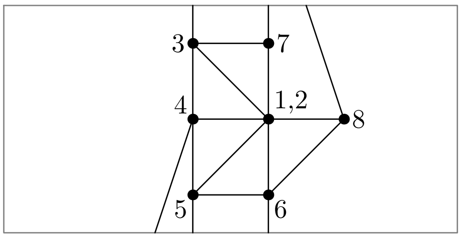
(1) Embedding of $E_{18}-(1,2)$, each vertex-pair of orbits A and B on the boundary of a face.



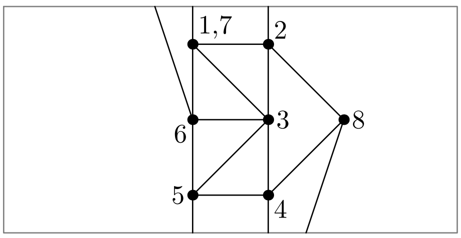
(2) Embedding of $E_{18} - (1,7)$, vertex-pairs $\{1,3\}$, $\{1,5\}$, $\{1,8\}$, $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{4,6\}$, $\{4,7\}$, $\{5,8\}$ on the boundary of a face.



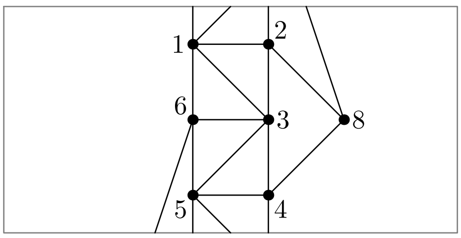
(3) Embedding of $E_{18}-(1,7)$, vertex-pair $\{6,7\}$ on the boundary of a face.



(4) Embedding of $E_{18}/(1,2)$, each vertex-pair of orbits A and B on the boundary of a face.



(5) Embedding of $E_{18}/(1,7)$, each vertex-pair of orbits A and B on the boundary of a face.



(6) Embedding of $R_3(E_{18})_{(7)}$, vertex-pairs $\{1,3\}$, $\{1,5\}$, $\{1,8\}$, $\{2,4\}$, $\{2,6\}$, $\{3,5\}$, $\{3,8\}$, $\{4,6\}$, $\{5,8\}$ on the boundary of a face.

Figure 2.69: Embeddings of $R_i(E_{18})$, $i \in \{0,2,3\}$ into the Möbius strip.

We also find embeddings of E_{18} itself into the Klein surface, so that each pair of vertices from orbits A and B lies on the boundary of one face and the $K_5 - (v_1, v_2) - e$ and $(K_5 - (v_1, v_2))/e$ can be attached.

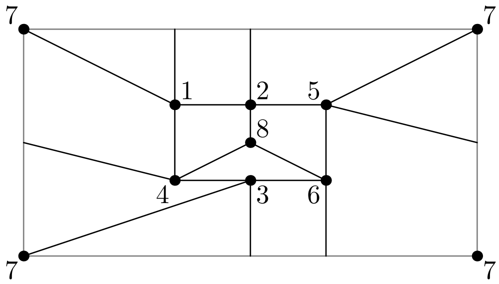


Figure 2.70: Embedding of E_{18} into the Klein surface, each vertex-pair of orbits A and B on the boundary of a face.

Consequently G_{55} and G_{56} are elements of $M_4(\mathcal{S}_2)$. □

Lemma 2.31
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to vertices 7 and 8 in E_{18}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
As the pairs of vertices {1, 2} and {7, 8} lie in the same orbit for $E_{18} - (1,2)$, when embedding this graph into the Möbius strip, vertices 7 and 8 can never be drawn on the boundary of one face, as this would be a contradiction to the irreducibility of E_{18} for the projective plane. □

The graph E_{22}

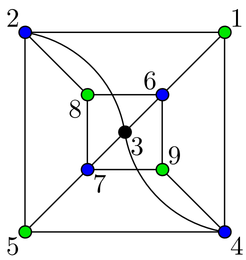


Figure 2.71: E_{22} and its vertex-orbits

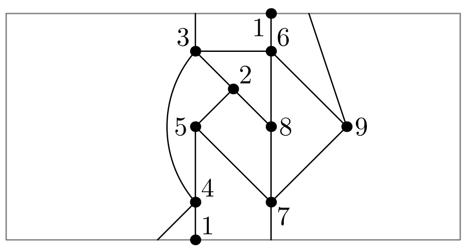
The orbits of pairs of vertices for E_{22}, which we have to consider, are:

$$\begin{aligned}
A &= \Big\{\{1,3\},\{3,5\},\{3,8\},\{3,9\}\Big\},\\
B &= \Big\{\{1,5\},\{1,8\},\{1,9\},\{5,8\},\{5,9\},\{8,9\}\Big\},\\
C &= \Big\{\{1,7\},\{2,9\},\{4,8\},\{5,6\}\Big\},\\
D &= \Big\{\{2,4\},\{2,6\},\{2,7\},\{4,6\},\{4,7\},\{6,7\}\Big\}.
\end{aligned}$$

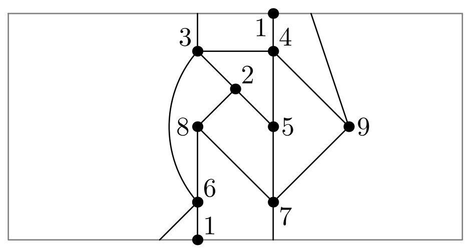
Lemma 2.32
Attaching vertices v_1 and v_2 to a pair of vertices of orbits A, B or D, is the construction of the irreducible graphs G_{57}, G_{58} and G_{59}, which are elements of $M_4(\mathcal{S}_2)$.

Proof

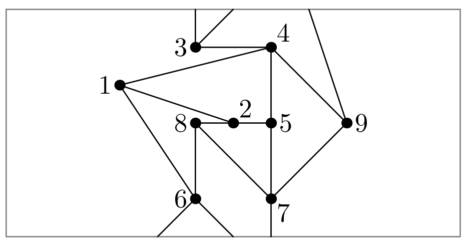
Considering the embeddings of $E_{22} - e$, E_{22}/e, $e \in \{(1,2), (2,3)\}$, and $R_3(E_{22})_{(1)}$ into the Möbius strip, we can see that we can find embeddings of these graphs, so that the pairs of vertices from A, B and D can be drawn on the boundary of one face and vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ can be attached to each representative of A, B and D.



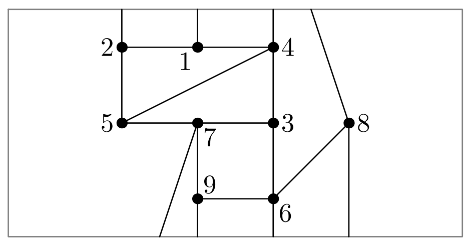
(1) Embedding of $E_{22} - (1,2)$, vertex pairs $\{1,3\}$, $\{1,5\}$, $\{1,9\}$, $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{3,9\}$, $\{4,6\}$, $\{4,7\}$, $\{5,8\}$, $\{6,7\}$, $\{8,9\}$ on the boundary of a face.



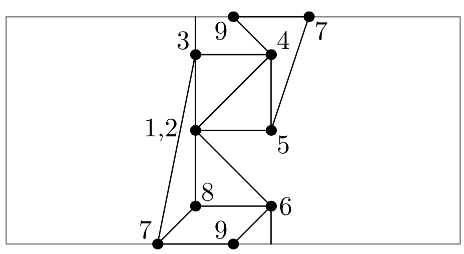
(2) Embedding of $E_{22} - (1,2)$, vertex pairs $\{1,8\}$, $\{5,9\}$ on the boundary of a face.



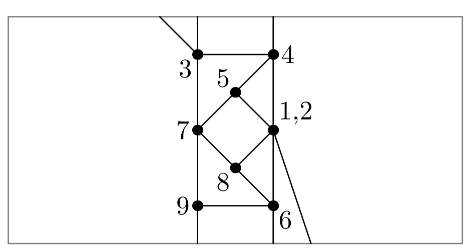
(3) Embedding of $E_{22} - (2,3)$, vertex pairs $\{1,3\}$, $\{1,5\}$, $\{1,8\}$, $\{1,9\}$, $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,8\}$, $\{3,9\}$, $\{4,6\}$, $\{4,7\}$, $\{5,8\}$, $\{5,9\}$, $\{6,7\}$, on the boundary of a face.



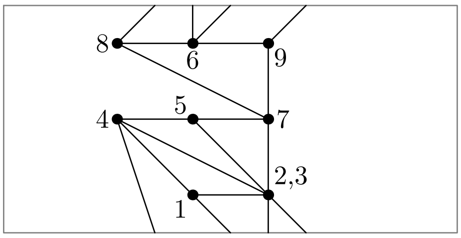
(4) Embedding of $E_{22} - (2,3)$, vertex pairs $\{3,5\}$, $\{8,9\}$ on the boundary of a face.



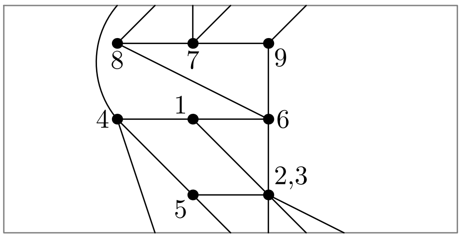
(5) Embedding of $E_{22}/(1,2)$, vertex pairs $\{1,3\}$, $\{1,5\}$, $\{1,8\}$, $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{3,9\}$, $\{4,7\}$, $\{5,9\}$, $\{6,7\}$, $\{8,9\}$ on the boundary of a face.



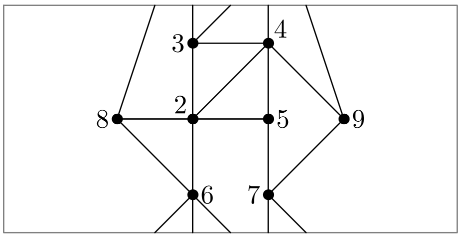
(6) Embedding of $E_{22}/(1,2)$, vertex pairs $\{1,9\}$, $\{4,6\}$, $\{5,8\}$ on the boundary of a face.



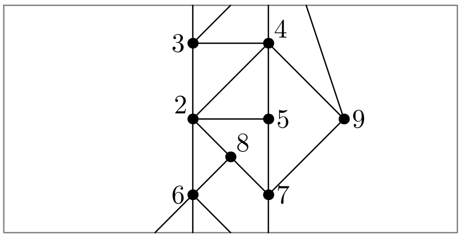
(7) Embedding of $E_{22}/(2,3)$, vertex pairs $\{1,3\}$, $\{1,9\}$, $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{3,9\}$, $\{4,6\}$, $\{4,7\}$, $\{5,8\}$, $\{5,9\}$, $\{6,7\}$, $\{8,9\}$ on the boundary of a face.



(8) Embedding of $E_{22}/(2,3)$, vertex pairs $\{1,5\}$, $\{1,8\}$ on the boundary of a face.



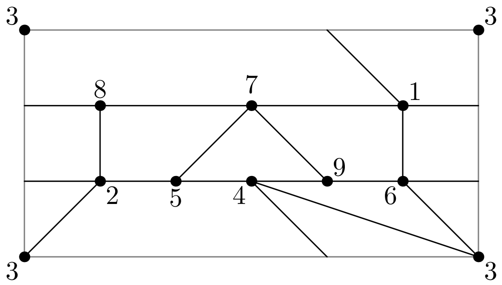
(9) Embedding of $R_3(E_{22})_{(1)}$, vertex-pairs $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{4,6\}$, $\{4,7\}$, $\{5,9\}$, $\{6,7\}$, $\{8,9\}$ on the boundary of a face.



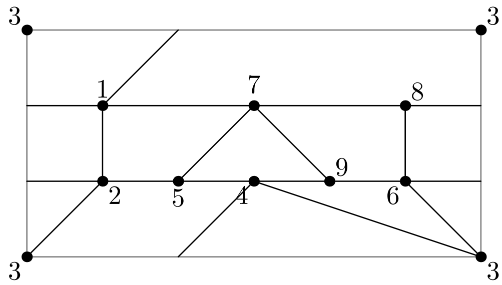
(10) Embedding of $R_3(E_{22})_{(1)}$, vertex-pairs $\{3,9\}$, $\{5,8\}$ on the boundary of a face.

Figure 2.72: Embeddings of $R_i(E_{22})$, $i \in \{0,2,3\}$, into the Möbius strip.

The embedding of E_{22} into the Klein surface, also has each pair of vertices in orbits A, B and D on the boundary of one face, so that the vertices v_1 and v_2 from $K_5 - (v_1, v_2) - e$ as well as $(K_5 - (v_1, v_2))/e$ can be attached to these vertices.



(1) Embedding of E_{22}, vertex-pairs $\{1,3\}$, $\{1,8\}$, $\{1,9\}$, $\{2,4\}$, $\{2,6\}$, $\{2,7\}$, $\{3,5\}$, $\{3,8\}$, $\{3,9\}$, $\{4,6\}$, $\{4,7\}$, $\{5,8\}$, $\{5,9\}$, $\{6,7\}$ on the boundary of a face.



(2) Embedding of E_{22}, vertex-pairs $\{1,5\}$, $\{8,9\}$ on the boundary of a face.

Figure 2.73: Embedding of E_{22} into the Klein surface

Hence G_{57}, G_{58} and G_{59} are elements of $M_4(\mathcal{S}_2)$. □

Lemma 2.33
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ attached to a pair of vertices of C in E_{22}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
As the pair of vertices $\{2,9\}$ lies in the same orbit as $\{2,3\}$ for $E_{22}-(2,3)$, vertices 2 and 9 cannot be drawn on the boundary of one face, when embedding $E_{22}-(2,3)$ into the Möbius strip: Consequently the graph constructed by attaching vertices v_1 and v_2 of $K_5-(v_1,v_2)$ to vertices 2 and 9 in $E_{22}-(2,3)$, is not embeddable into the Klein surface. Thus attaching vertices v_1 and v_2 of $K_5-(v_1,v_2)$ to one of the pairs of vertices in C, does not deliver an irreducible graph for the Klein surface. $\square$

2.5 Graphs including three base-points

In this section, we will look at the possible combinations of the graphs in $M_4(\mathcal{S}_1)$ and a subgraph of the K_5 with the property that these graphs have three shared vertices.

Proposition 2.34
None of the graphs, which are constructed by attaching the vertices v_1, v_2 and v_3 of a $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ to three pairwise different vertices of any graph in $M_4(\mathcal{S}_1)$is an element of the minimal basis $M_4(\mathcal{S}_2)$.

Considering the results of Proposition 2.4, most of the orbits of triples do not have to be considered, as they already consist of pairs of vertices, for which is already shown that an attachment of a $K_5-(v_1,v_2)$ does not result in an element of $M_4(\mathcal{S}_2)$.

The graph D_9

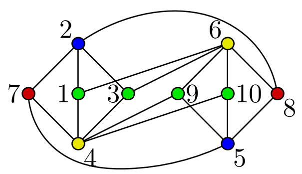


Figure 2.74: D_9 and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph D_9 are:

$$A=\Big\{\{2,4,5\},\{2,5,6\}\Big\}, \qquad B=\Big\{\{2,4,6\},\{4,5,6\}\Big\}.$$

Lemma 2.35
The graph, where vertices v_1, v_2 and v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ are identified with triples of vertices from orbit A of D_9, is not an element of $M_4(\mathcal{S}_2)$.

Proof
The embedding of $D_9-(4,7)$ consists of one face of size six and seven faces of size four (the graph has 17 edges and the embedding consists of eight faces). The only cycle, which includes vertices $2,4$ and 5, is:

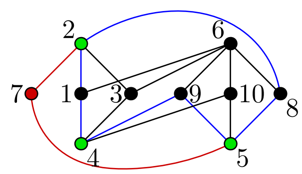


Figure 2.75: Cycle in $D_9 - (4,7)$, including vertices $2, 4$ and 5

Trying to embed $D_9 - (4,7)$ containing this cycle as the boundary of one face, delivers the following result:

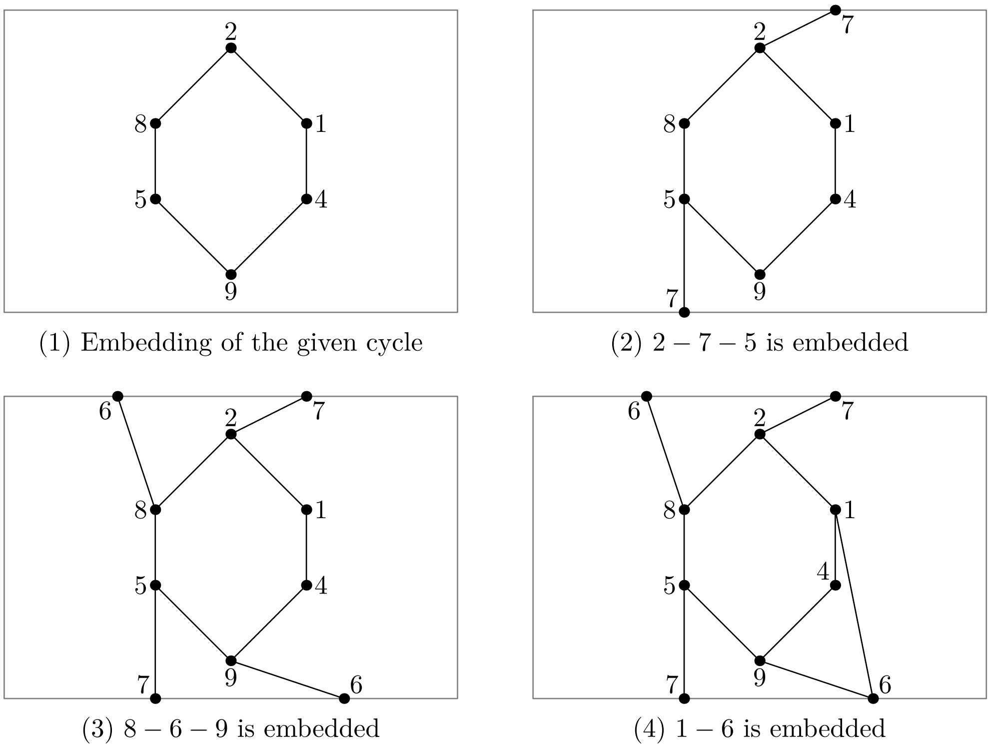


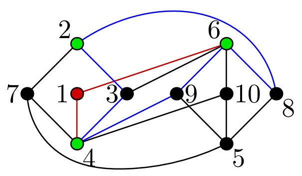
Figure 2.76: Embedding of $D_9 - (4,7)$

Figure 2.76 shows that an embedding, with the given restrictions, is not possible, as vertex 10 cannot be embedded.

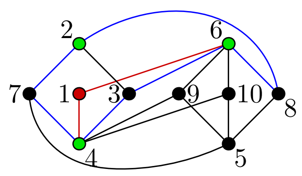
Consequently vertices $2, 4$ and 5 can never lie on the boundary of one face, when embedding the graph $D_9 - (4,7)$ into the Möbius strip. □

Lemma 2.36
The graph, with vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ identified with a triple of vertices from orbit B of D_9, is not an element of $M_4(\mathcal{S}_2)$.

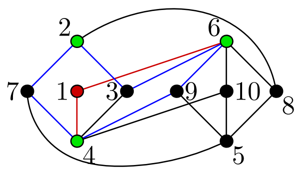
Proof
The embedding of $D_9 - (1,2)$ consists of one face of size six and seven faces of size four (the graph has 17 edges and the embedding consists of eight faces). The four cycles including vertices $2, 4$ and 6 are:



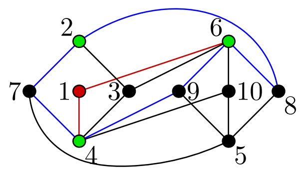
(1) Case 1



(2) Case 2



(3) Case 3

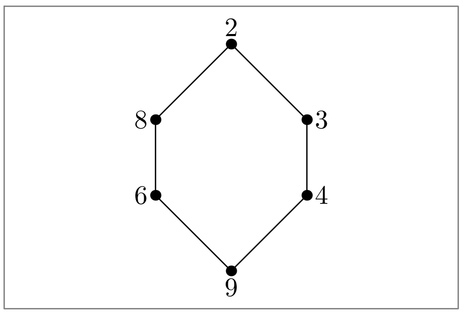


(4) Case 4

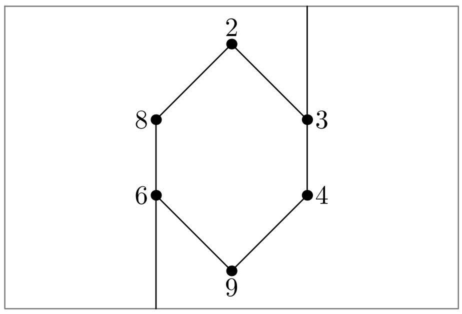
Figure 2.77: Cycles in $D_9 - (1, 2)$, including vertices $2, 4$ and 6

Trying to embed $D_9 - (4, 7)$ containing these cycles as boundaries of a face, delivers the following results:

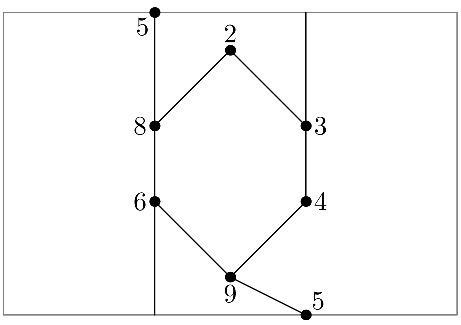
Embedding of Case 1:



(1) Embedding of the given cycle



(2) $3 - 6$ is drawn

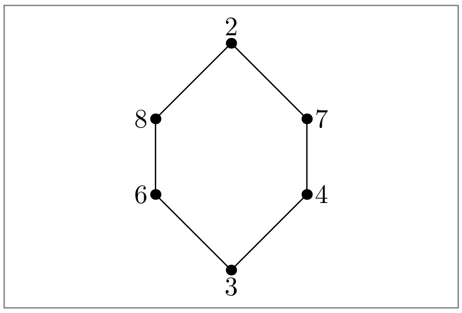


(3) $9 - 5 - 8$ is embedded

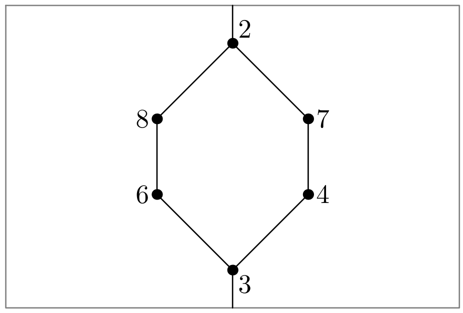
Figure 2.78: Embedding of Case 1

Figure 2.78 shows that an embedding, with the given restrictions, is not possible, as the path $2 - 4 - 5$ cannot be embedded.

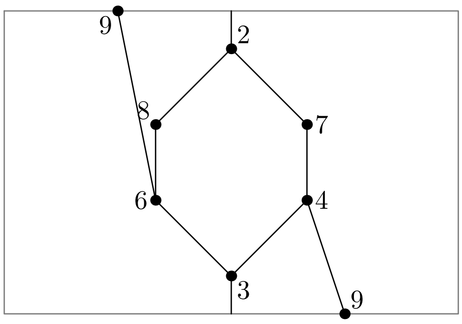
Embedding of Case 2:



(1) Embedding of the given cycle



(2) $2 - 3$ is drawn

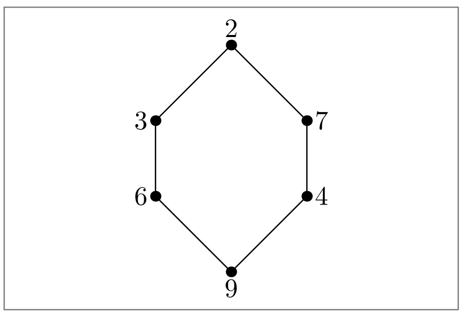


(3) $4 - 9 - 6$ is embedded

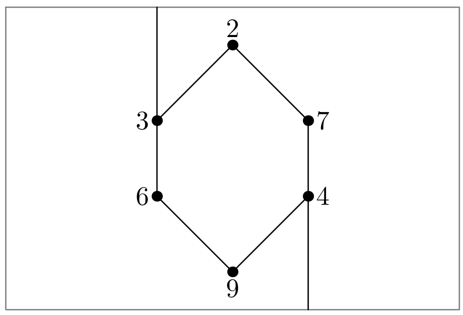
Figure 2.79: Embedding of Case 2

Figure 2.79 shows that an embedding, with the given restrictions, is not possible, as the path $7 - 5 - 8$ cannot be embedded.

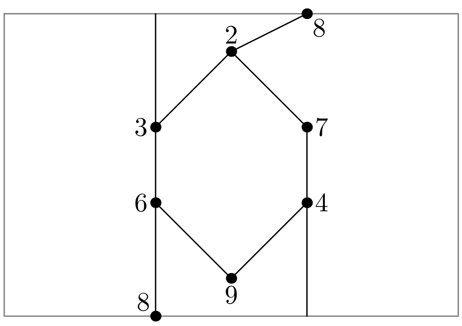
Embedding of Case 3:



(1) Embedding of the given cycle



(2) $3 - 4$ is embedded

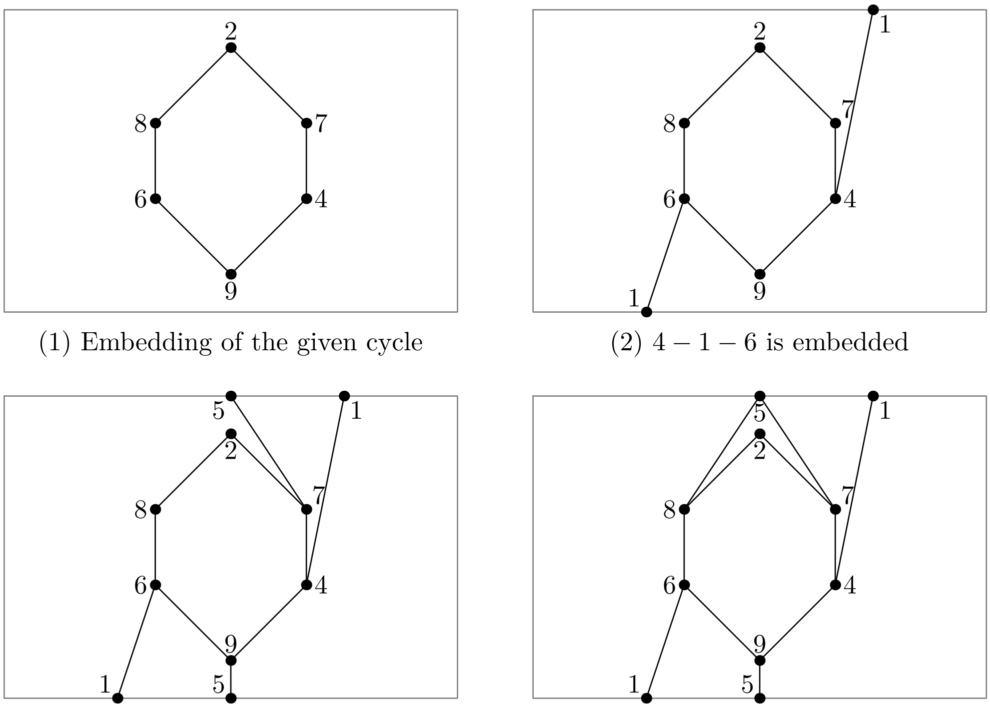


(3) $2 - 8 - 6$ is embedded

Figure 2.80: Embedding of Case 3

Figure 2.80 shows that an embedding, with the given restrictions, is not possible, as the path $7-5-9$ cannot be embedded.

Embedding of Case 4:



(1) Embedding of the given cycle

(2) $4-1-6$ is embedded

(3) $7-5-9$ is embedded

(4) $5-8$ is embedded

Figure 2.81: Embedding of Case 4

Figure 2.81 shows that an embedding, with the given restrictions, is not possible, as the path $2-3-6$ cannot be embedded.

Consequently no embedding of $D_9-(4,7)$ into the Möbius strip, with vertices $2, 4$ and 6 on the boundary of one face, exists. $\square$

The graph D_{12}

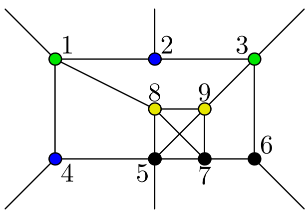


Figure 2.82: D_{12} and its vertex-orbits

The only triple of vertices we have to consider for D_{12} is $\{1, 3, 5\}$.

Lemma 2.37
The graph, where vertices v_1, v_2 and v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ are identified with vertices $1, 3$ and 5 of the graph D_{12}, cannot be an element of $M_4(\mathcal{S}_2)$.

Proof
The vertices 1, 3 and 5 cannot lie on the boundary of one face for $D_{12} - (1, 2)$. This is the case because the K_4 with vertices $5, 7, 8$ and 9 needs the Möbius strip characteristics, so that they can be connected with other vertices in the graph. The also included $K_{3,3} - (1, 2)$ with the vertices $1, 2, 3, 4, 5$ and 7 consequently cannot use the characteristics of the Möbius strip and thus the vertices $1, 3$ and 5, which all lie within the same set of vertices regarding the $K_{3,3}$, cannot all be drawn on the boundary of the same face. □

The graph E_3

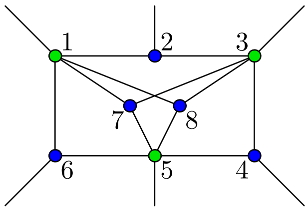


Figure 2.83: E_3 and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph E_3 are:

$$\begin{aligned} A = \Big\{ &\{2,4,6\}, \{2,4,7\}, \{2,4,8\}, \{2,6,7\}, \{2,6,8\}, \{2,7,8\}, \{4,6,7\}, \\ &\{4,6,8\}, \{4,7,8\}, \{6,7,8\} \Big\}, \\ B = \Big\{ &\{1,3,5\} \Big\} . \end{aligned}$$

Lemma 2.38
The construction of graphs, with vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ identified with triples of vertices from orbit A or B of the graph E_3 does not result in elements of $M_4(\mathcal{S}_2)$.

Proof
As the Euler characteristic for a graph G embedded into the Möbius strip is $\chi(G) = v - e + f = 1$, and $E_3 - (1, 2)$ has eight vertices and 14 edges, the embedding of $E_3 - (1, 2)$ has seven faces. Because the graph is bipartite, a face of the embedding has to have at least four vertices on its boundary. As the graph only has 14 edges for seven faces and each edge can only be on the boundary of two faces, each face of the embedding of the graph has size four. Consequently three vertices which lie within the same set of vertices in the bipartite graph, cannot lie on the boundary of the same face. □

The graph E_{18}

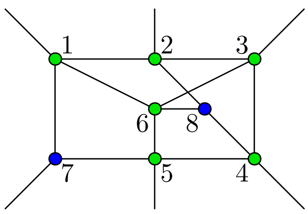


Figure 2.84: E_{18} and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph E_{18} are:

$$A = \Big\{\{1,3,5\},\{2,4,6\}\Big\},$$
$$B = \Big\{\{1,3,8\},\{1,5,8\},\{2,4,7\},\{2,6,7\},\{3,5,8\},\{4,6,7\}\Big\}.$$

Lemma 2.39
The construction of graphs, with vertices v_1, v_2 and v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ identified with triples of vertices from orbit A or B of the graph E_{18}, does not result in elements of $M_4(\mathcal{S}_2)$.

Proof
For the graph $E_{18}-(1,2)$ the same arguments as for the graph $E_3-(1,2)$ in Lemma 2.38 apply. □

The graph E_{22}

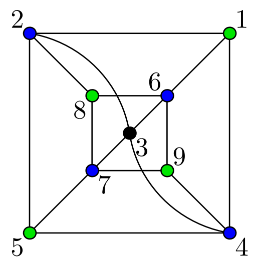


Figure 2.85: E_{22} and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph E_{22} are:

$$A = \Big\{\{1,3,5\},\{1,3,8\},\{1,3,9\},\{3,5,8\},\{3,5,9\},\{3,8,9\}\Big\},$$
$$B = \Big\{\{1,5,8\},\{1,5,9\},\{1,8,9\},\{5,8,9\}\Big\},$$
$$C = \Big\{\{2,4,6\},\{2,4,7\},\{2,6,7\},\{4,6,7\}\Big\}.$$

Lemma 2.40
The construction of a graph, with vertices v_1, v_2 and v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ identified with triples of vertices from orbit A of the graph E_{22} does not result in an element $M_4(\mathcal{S}_2)$.

Proof
The graph $E_{22}-(1,2)$ is bipartite and with the Euler characteristic, we know that the embedding of the graph consists of seven faces, one of size six and the others of size four. The only cycle including vertices $1,3$ and 9 is:

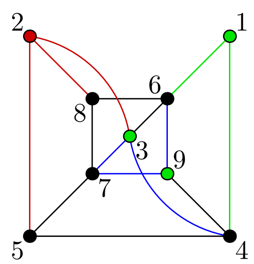
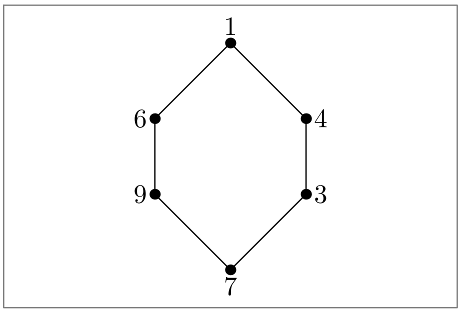
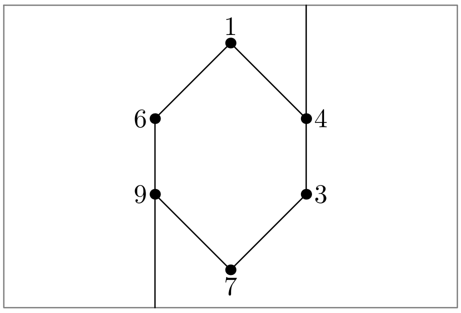


Figure 2.86: Cycle in $E_{22}-(1,2)$, including vertices $1,3$ and 9

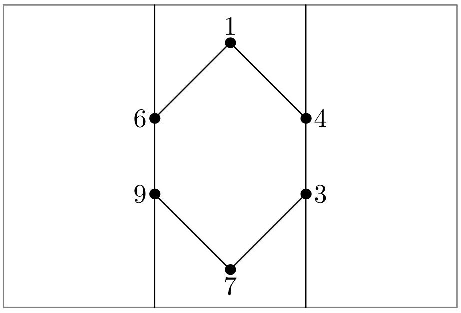
By trying to embed the graph $E_{22} - (1,2)$, having this cycle as the boundary of one face, we get:



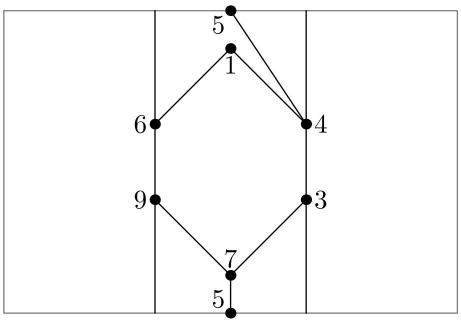
(1) Embedding of the given cycle



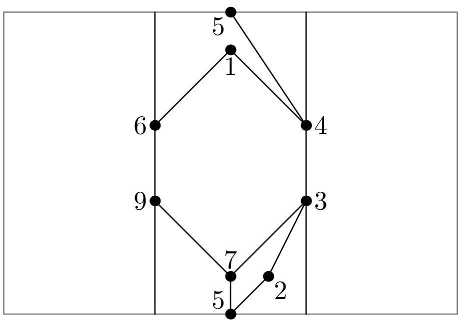
(2) $4-9$ is embedded



(3) $3-6$ is embedded



(4) $4-5-7$ is embedded



(5) $3-2-5$ is embedded

Figure 2.87: Embedding of $E_{22} - (1,2)$

Figure 2.87 shows that an embedding, with the given restrictions, is not possible, as the path $6-8-7$ cannot be embedded.

Consequently vertices $1,3$ and 5 cannot lie on the boundary of the same face, when embedding $E_{22} - (1,2)$ into the Möbius strip. $\square$

Lemma 2.41
The graph, with vertices v_1, v_2 and v_3 of $K_5 - (v_1,v_2) - (v_1,v_3) - (v_2,v_3)$ identified with triples of vertices from orbit B of the graph E_{22}, is not an element of $M_4(\mathcal{S}_2)$.

Proof
In this Lemma the same considerations about face sizes as in Lemma 2.40 apply. The only remaining cycle including vertices $1,5$ and 8 is:

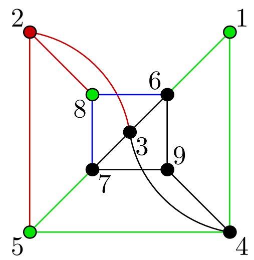
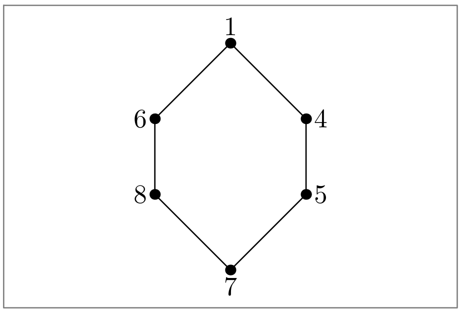
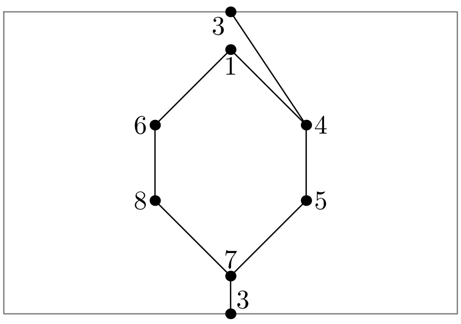


Figure 2.88: Cycle in $E_{22} - (1, 2)$, including vertices $1, 5$ and 8

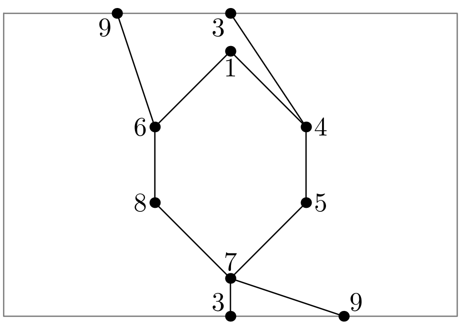
As before, embedding of $E_{22} - (1, 2)$ including this cycle as the boundary of one face, is done:



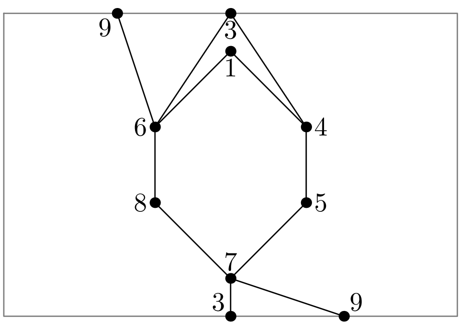
(1) Embedding of the given cycle



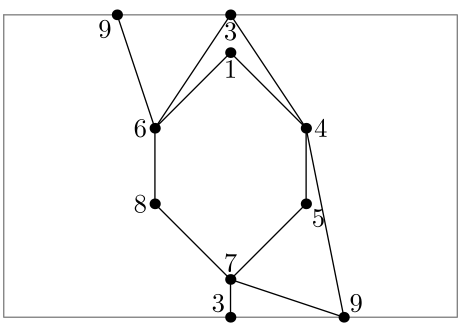
(2) $4 - 3 - 7$ is embedded



(3) $6 - 9 - 7$ is embedded



(4) $3 - 6$ is embedded



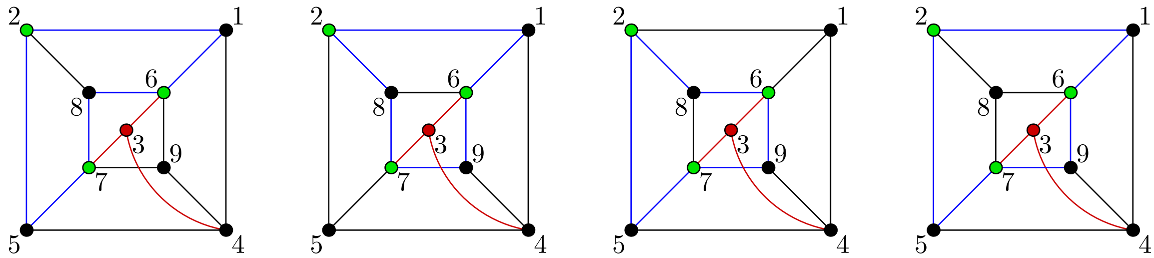
(5) $4 - 9$ is embedded

Figure 2.89: Embedding of $E_{22} - (1, 2)$

Figure 2.89 shows that an embedding, with the given restrictions, is not possible, as the path $5 - 2 - 8$ cannot be embedded. □

Lemma 2.42
The construction of a graph, with vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ identified with triples of vertices from orbit C of the graph E_{22} does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The maximal size of a face in $E_{22} - (2, 3)$ is six, as already shown above. Thus the only cycles including vertices $2, 6$, and 7, which have to be considered, are:

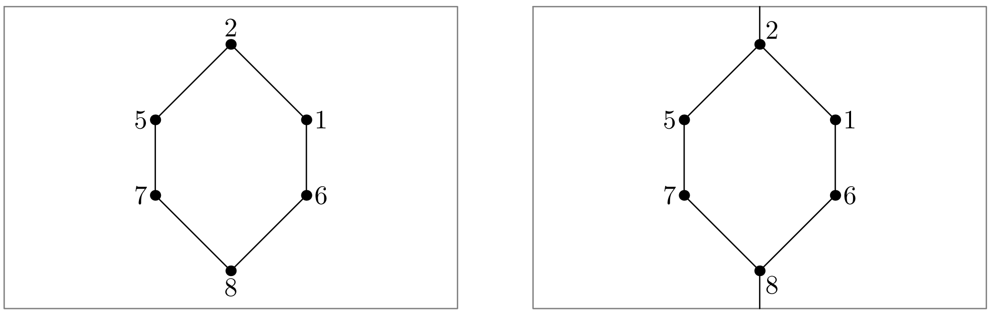


(1) Case 1 (2) Case 2 (3) Case 3 (4) Case 4

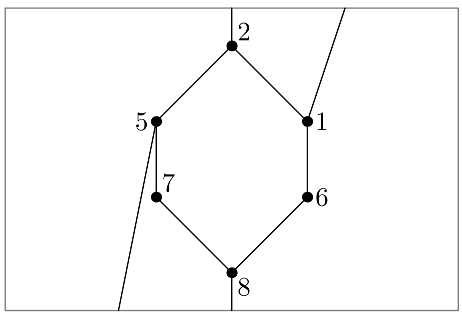
Figure 2.90: Cycles in $E_{22} - (2, 3)$, including vertices $2, 6$ and 7

Again we try to embed the graph $E_{22} - (2, 3)$ with these cycles as boundaries of one face, and we get:

Embedding of Case 1:



(1) Embedding of the given cycle (2) $2 - 8$ is embedded

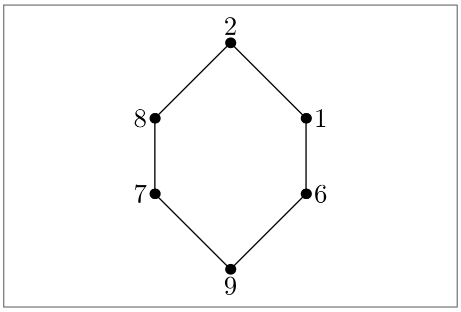


(3) $1 - 4 - 5$ is embedded

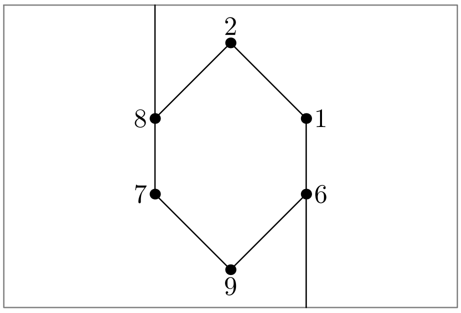
Figure 2.91: Embedding of Case 1

Figure 2.91 shows that an embedding, with the given restrictions, is not possible, as the path $6-9-7$ cannot be embedded.

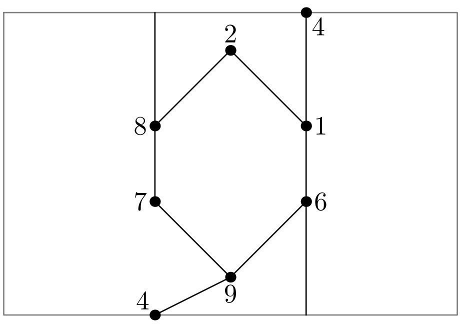
Embedding of Case 2:



(1) Embedding of the given cycle



(2) $6-8$ is embedded

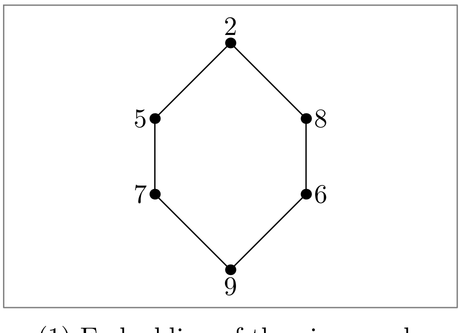


(3) $1-4-9$ is embedded

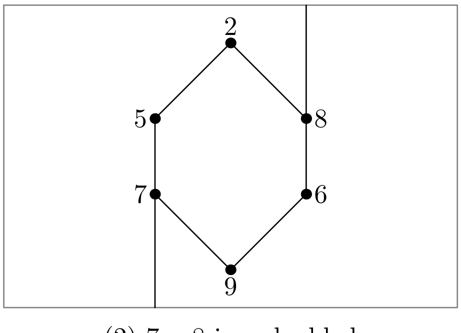
Figure 2.92: Embedding of Case 2

Figure 2.92 shows that an embedding, with the given restrictions, is not possible, as the path $2-5-7$ cannot be embedded.

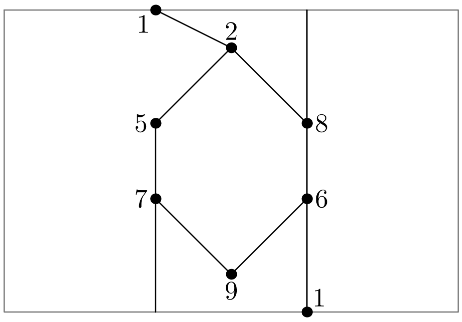
Embedding of Case 3:



(1) Embedding of the given cycle



(2) $7-8$ is embedded

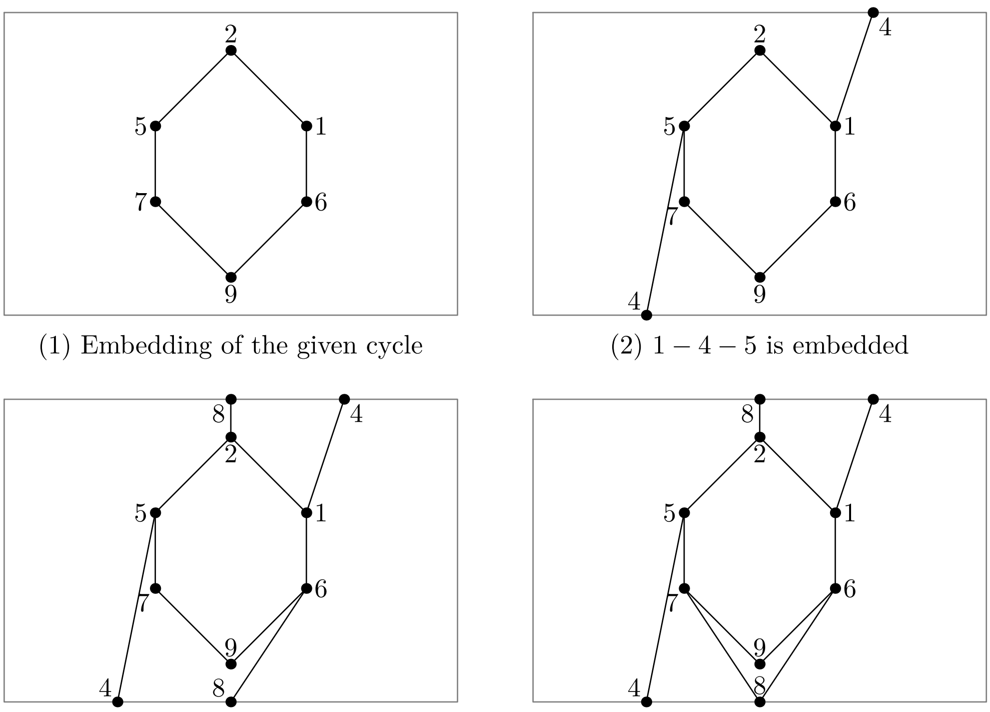


(3) $2-1-6$ is embedded

Figure 2.93: Embedding of Case 3

Figure 2.93 shows that an embedding, with the given restrictions, is not possible, as vertex 4 cannot be embedded.

Embedding of Case 4:



(1) Embedding of the given cycle

(2) $1-4-5$ is embedded

(3) $2-8-6$ is embedded

(4) $7-8$ is embedded

Figure 2.94: Embedding of Case 4

Figure 2.94 shows that an embedding, with the given restrictions, is not possible, as the path $5-4-9$ cannot be embedded.

Consequently an embedding of $E_{22}-(2,3)$, with vertices $2,6$ and 7 on the boundary of one face, is not possible. $\square$

3 $M_3(\mathcal{S}_1)$ to $M_4(\mathcal{S}_2)$

In this chapter we will study the graphs of the minimal basis $M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$ of the projective plane. We only have to consider an attachment of a subgraph of the K_5 to a graph G from $M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$ for combinations of vertices which eliminate the application of R_4 and thus transforms the graph into another one from $M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$ or $M_4(\mathcal{S}_1)$. Figure 3.1 shows the possible graphs and the relations we have to foreclose:

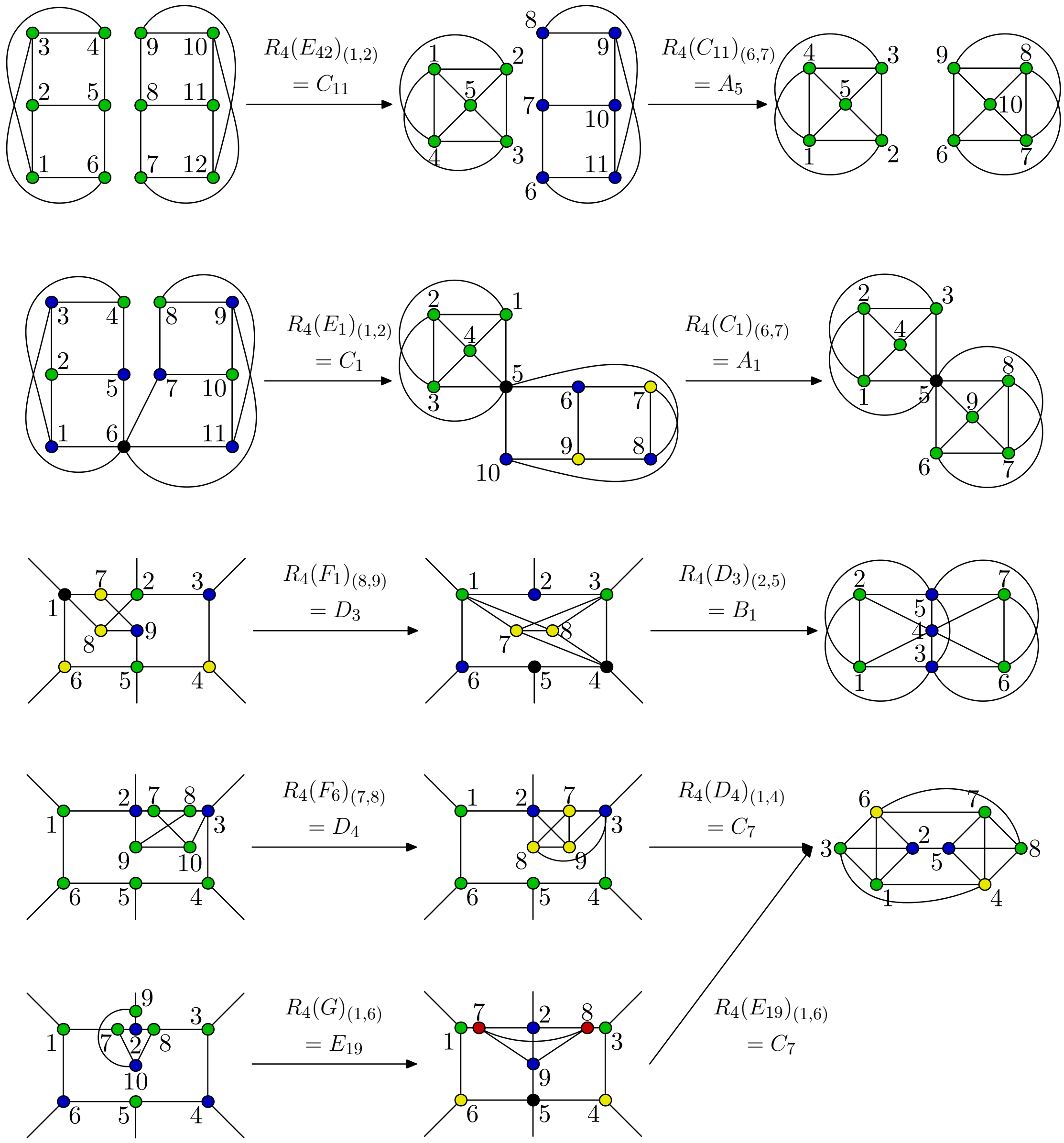


Figure 3.1: Genealogy of graphs in $M_3(\mathcal{S}_1)$

3.1 A graph including one base-point

The only graph in $M_4(\mathcal{S}_1) - M_3(\mathcal{S}_1)$, where an attachment of the K_5 to one vertex eliminates all possibilities of applying the relation R_4, is the graph D_3.

Proposition 3.1
The graph G_{60} which is obtained by attachment of a K_5 to vertex 5 of D_3 is an element of $M_4(\mathcal{S}_2)$.

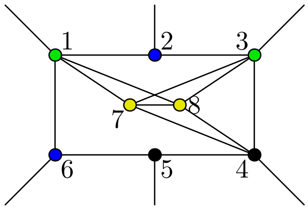


Figure 3.2: D_3 and its vertex-orbits

Proof
As the graph D_3 is in $M_3(\mathcal{S}_1)$, it is embeddable on the Möbius strip as soon as one of the relations R_0, R_1, R_2 or R_3 is applied. Consequently an attachment of a K_5 to vertex 5, which forecloses the application of R_4 on D_3, results in an element of $M_4(\mathcal{S}_2)$. $\square$

3.2 Graphs including two base-points

In this section we deal with the attachment of the $K_5 - (v_1, v_2)$ to two vertices v_i, v_j, $i \neq j$, v_i in $G \in M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$. We again do this one by one and show whether or not the attachment of the K_5 results in a graph which is an element of $M_4(\mathcal{S}_2)$.

Proposition 3.2
The graphs $G_{61}, \ldots, G_{66}$ are the only elements in $M_4(\mathcal{S}_2)$ constructed by combination of a graph in $M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$ and a $K_5 - (v_1, v_2)$ as a relative component with v_1 and v_2 base-points.

The graph C_1

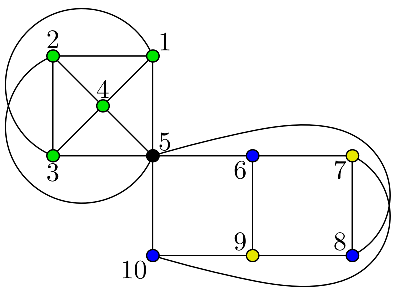
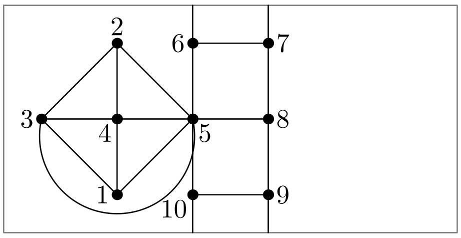


Figure 3.3: C_1 and its vertex-orbits

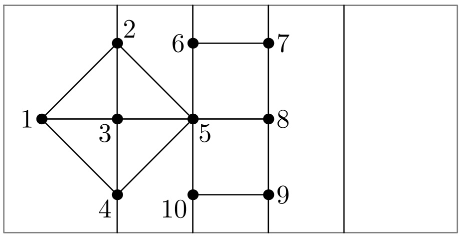
The only pair of vertices we have to consider for the graph C_1 is: $\{7, 9\}$.

Lemma 3.3
The graph G_{61}, which is obtained by attaching the $K_5 - (v_1, v_2)$ to C_1 by identification of vertices v_1 and v_2 of the K_5 and vertices 7 and 9 in C_1, is an element of $M_4(\mathcal{S}_2)$.

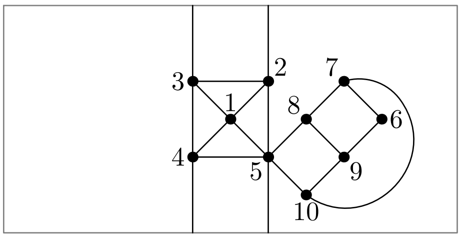
Proof
It is sufficient to show that the graphs $C_1 - e$ and C_1/e with $e \in \{(1, 2), (1, 5), (5, 6), (6, 7)\}$ and $R_3(C_1)_{(6)}$ can be embedded into the Möbius strip, and C_1 itself into the Klein surface, with vertices 7 and 9 on the boundary of one face:



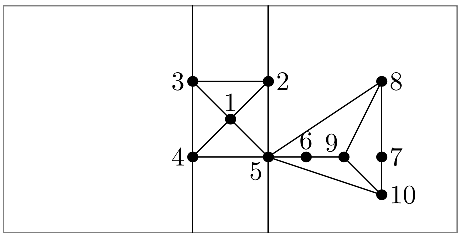
(1) Embedding of $C_1 - (1, 2)$, vertices 7 and 9 on the boundary of a face.



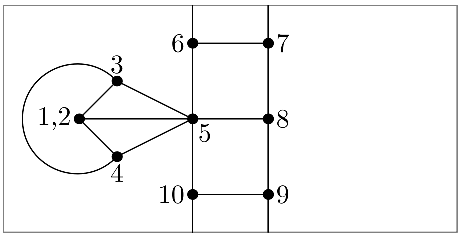
(2) Embedding of $C_1 - (1, 5)$, vertices 7 and 9 on the boundary of a face.



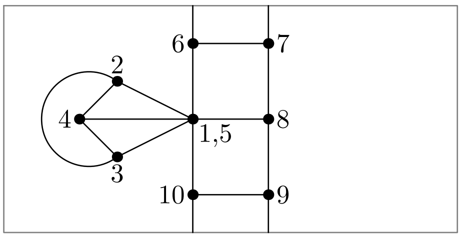
(3) Embedding of $C_1 - (5, 6)$, vertices 7 and 9 on the boundary of a face.



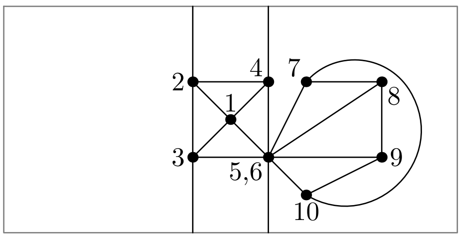
(4) Embedding of $C_1 - (6, 7)$, vertices 7 and 9 on the boundary of a face.



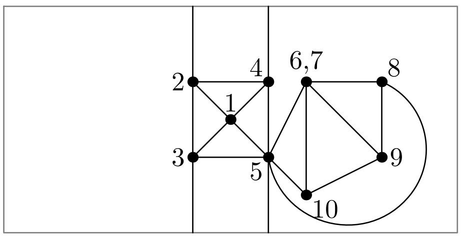
(5) Embedding of $C1/(1, 2)$, vertices 7 and 9 on the boundary of a face.



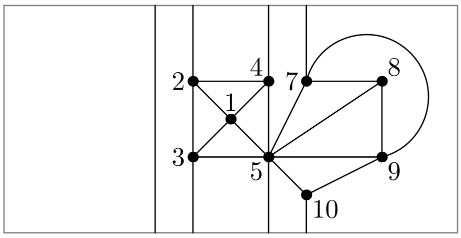
(6) Embedding of $C_1/(1, 5)$, vertices 7 and 9 on the boundary of a face.



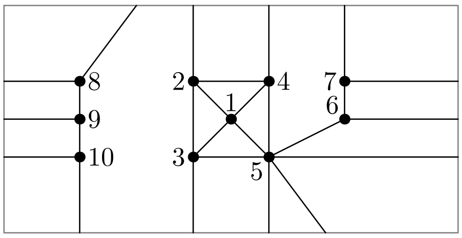
(7) Embedding of $C_1/(5, 6)$, vertices 7 and 9 on the boundary of a face.



(8) Embedding of $C_1/(6, 7)$, vertices 7 and 9 on the boundary of a face.



(9) Embedding of $R_3(C_1)_{(6)}$, vertices 7 and 9 on the boundary of a face.



(10) Embedding of C_1, vertices 7 and 9 on the boundary of a face.

Figure 3.4: (1)-(9): Embeddings of $R_i(C_1)$, $i \in (0, 2, 3)$, into the Möbius strip, (10): Embedding of C_1 into the Klein surface

Consequently G_{61} is an element of $M_4(\mathcal{S}_2)$. □

The graph D_3

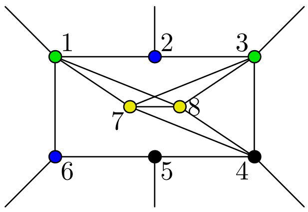


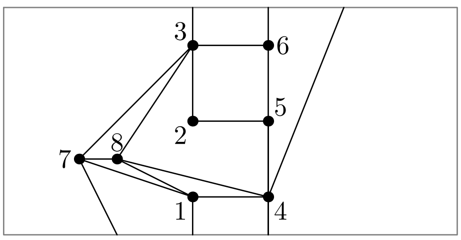
Figure 3.5: D_3 and its vertex-orbits

The orbits of pairs of vertices we have to consider for the graph D_3 are:

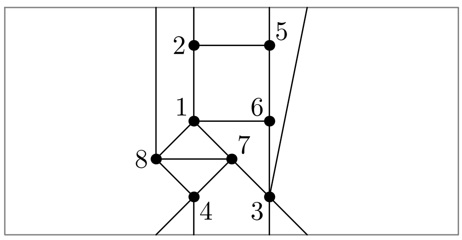
$$A = \Big\{\{1,5\},\{3,5\}\Big\}, \qquad B = \Big\{\{2,6\}\Big\},$$
$$C = \Big\{\{5,7\},\{5,8\}\Big\}.$$

Lemma 3.4
The graphs G_{62} and G_{63} which can be obtained by identification of the vertices v_1 and v_2 in $K_5 - (v_1, v_2)$ with a pair of vertices from orbits A or B of D_3, are elements of $M_4(\mathcal{S}_2)$.

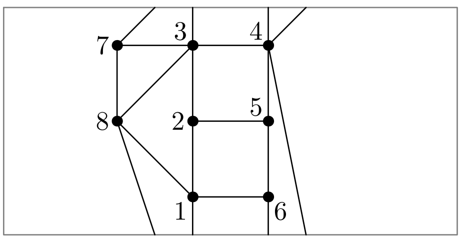
Proof
As can be seen in Figure 3.6, the graphs $D_3 - e$, D_3/e, with $e \in \{(1,2), (1,4), (1,7), (2,5), (4,5), (7,8)\}$ and $R_3(D_3)_{(2)}$ and $R_3(D_3)_{(5)}$ can be embedded into the Möbius strip as well as D_3 itself into the Klein surface, in a way that the pairs of vertices from orbits A and B each lie on the boundary of one face:



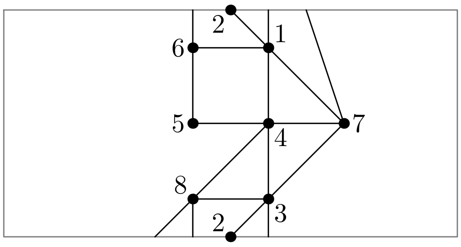
(1) Embedding of $D_3-(1,2)$, each vertex-pair of orbits A and B on the boundary of a face.



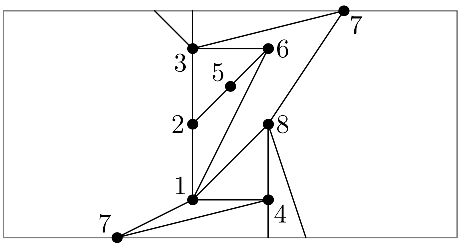
(2) Embedding of $D_3-(1,4)$, each vertex-pair of orbits A and B on the boundary of a face.



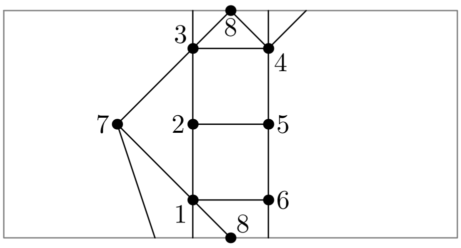
(3) Embedding of $D_3-(1,7)$, each vertex-pair of orbits A and B on the boundary of a face.



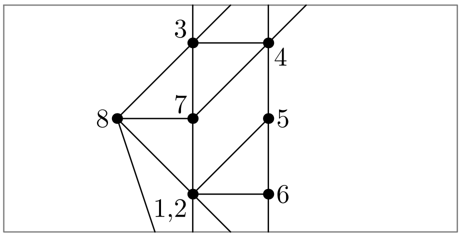
(4) Embedding of $D_3-(2,5)$, each vertex-pair of orbits A and B on the boundary of a face.



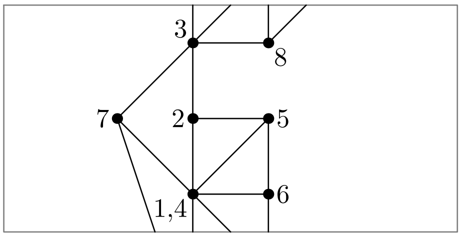
(5) Embedding of $D_3-(4,5)$, each vertex-pair of orbits A and B on the boundary of a face.



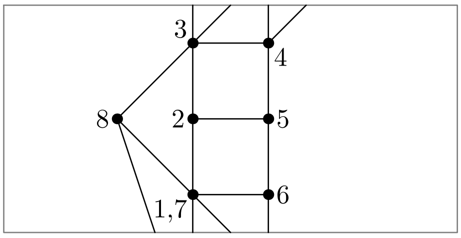
(6) Embedding of $D_3-(7,8)$, each vertex-pair of orbits A and B on the boundary of a face.



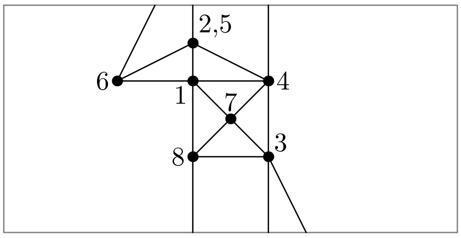
(7) Embedding of $D_3/(1,2)$, each vertex-pair of orbits A and B on the boundary of a face.



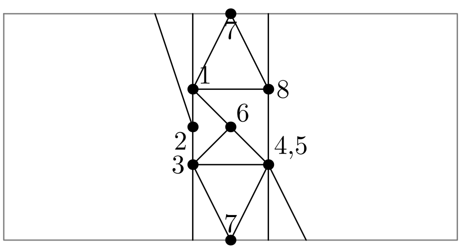
(8) Embedding of $D_3/(1,4)$, each vertex-pair of orbits A and B on the boundary of a face.



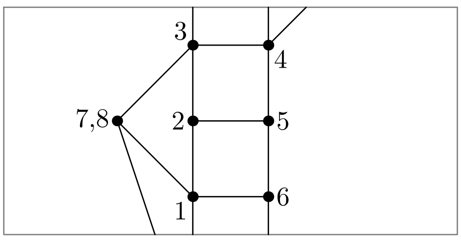
(9) Embedding of $D_3/(1,7)$, each vertex-pair of orbits A and B on the boundary of a face.



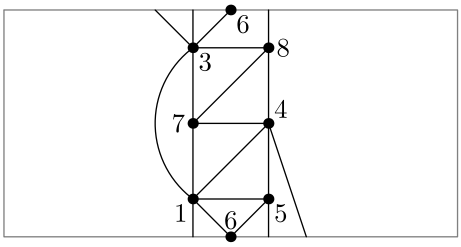
(10) Embedding of $D_3/(2,5)$, each vertex-pair of orbits A and B on the boundary of a face.



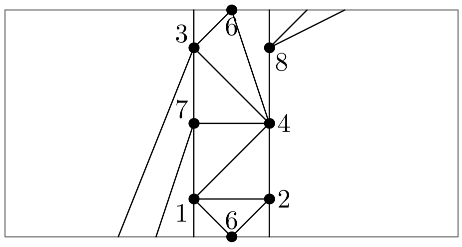
(11) Embedding of $D_3/(4,5)$, each vertex-pair of orbits A and B on the boundary of a face.



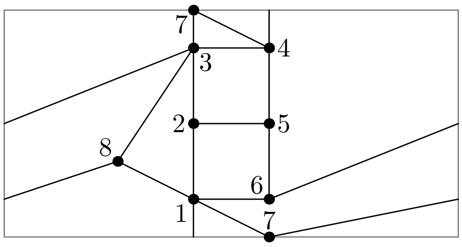
(12) Embedding of $D_3/(7,8)$, each vertex-pair of orbits A and B on the boundary of a face.



(13) Embedding of $R_3(D_3)_{(2)}$, vertex-pairs $\{1,5\}$, $\{3,5\}$ on the boundary of a face.



(14) Embedding of $R_3(D_3)_{(5)}$, vertex-pair $\{2,6\}$ on the boundary of a face.



(15) Embedding of D_3, each vertex-pair of orbits A and B on the boundary of a face.

Figure 3.6: (1)-(14): Embeddings of $R_i(D_3)$, $i \in (1,2,3)$, into the Möbius strip, (15): Embedding of D_3 into the Klein surface

Consequently G_{62} and G_{63} are elements of $M_4(\mathcal{S}_2)$. $\square$

Lemma 3.5
The construction of a graph, with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with a pair of vertices from orbit C of the graph D_3, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The graph $D_3 - (2,5)$ is not embeddable into the Möbius strip, in a way that vertices 5 and 7 lie on the boundary of one face. The only cycle we have to consider for this graph is:

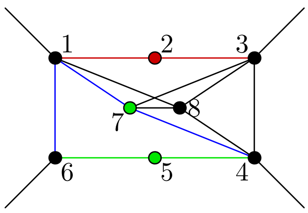
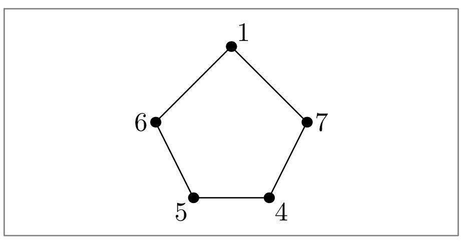
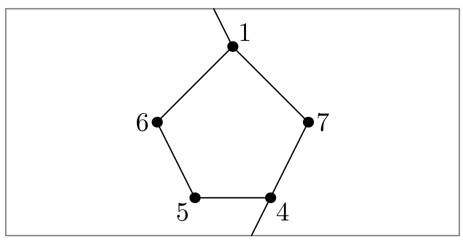


Figure 3.7: Cycle in $D_3 - (2,5)$, including vertices 5 and 7

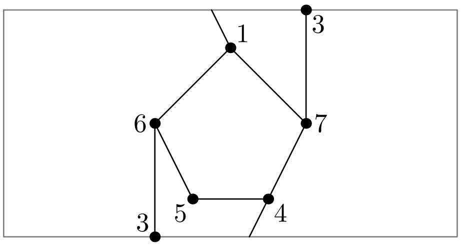
Embedding the graph starting with this cycle as the boundary of one face is not possible:



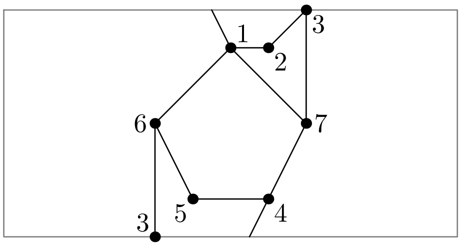
(1) Embedding of the given cycle



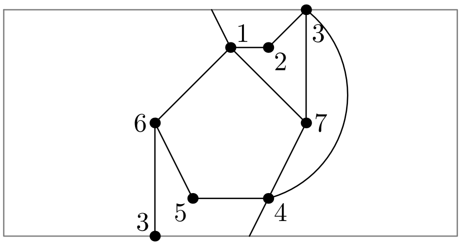
(2) $1-4$ is embedded.



(3) $6 - 3 - 7$ is embedded



(4) $1 - 2 - 3$ is embedded



(5) $3 - 4$ is embedded

Figure 3.8: Embedding of $D_3 - (4, 5)$

Figure 3.8 shows that an embedding, with the given restrictions, is not possible, as vertex 8 cannot be embedded. □

The graph D_4

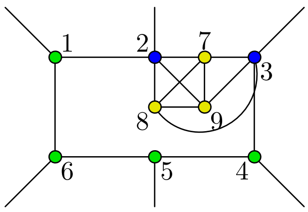


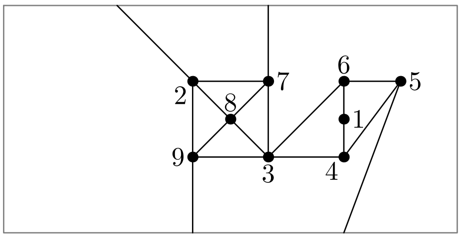
Figure 3.9: D_4 and its vertex-orbits

The only orbit of pairs of vertices we have to consider for the graph D_4 is:

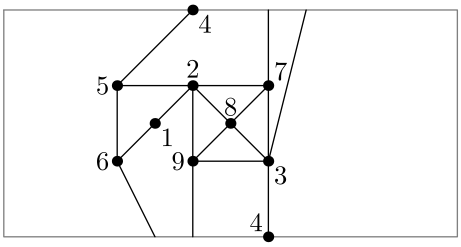
$$A = \Big\{\{1, 5\}, \{4, 6\}\Big\}.$$

Lemma 3.6
The graph G_{64}, which is obtained by identification of vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ and a pair of vertices from orbit A of D_4, is an element of $M_4(\mathcal{S}_2)$.

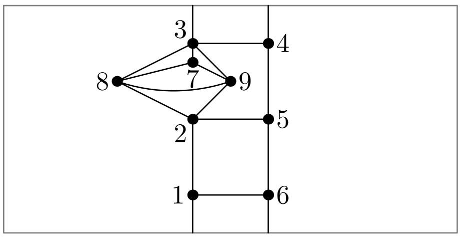
Proof
As can be seen in Figure 3.10, each graph $D_4 - e$, D_4/e, with $e \in \{(1, 2), (1, 4), (2, 7), (7, 8)\}$, and $R_3(D_4)_{(1)}$ can be embedded into the Möbius strip in a way that the pairs of vertices $\{1, 5\}$ and $\{4, 6\}$ each lie on the boundary of one face of these embeddings:



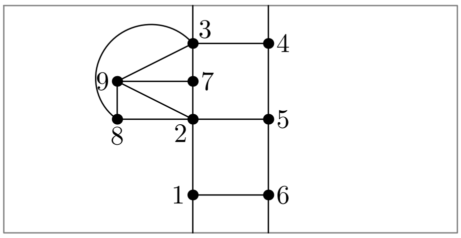
(1) Embedding of $D_4-(1,2)$, each vertex-pair of orbit A on the boundary of a face.



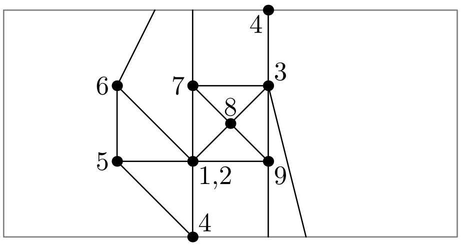
(2) Embedding of $D_4-(1,4)$, each vertex-pair of orbit A on the boundary of a face.



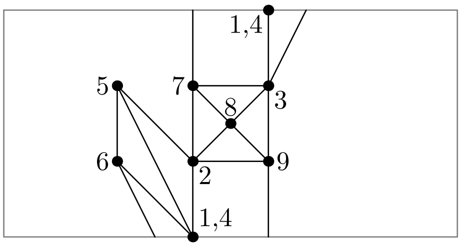
(3) Embedding of $D_4-(2,7)$, each vertex-pair of orbit A on the boundary of a face.



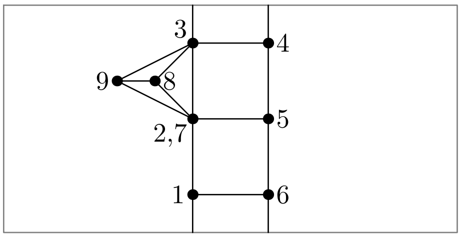
(4) Embedding of $D_4-(7,8)$, each vertex-pair of orbit A on the boundary of a face.



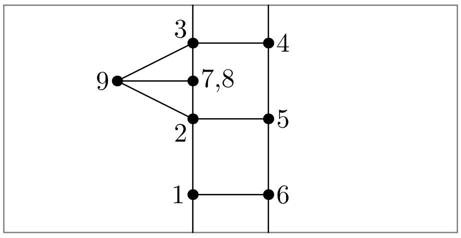
(5) Embedding of $D_4/(1,2)$, each vertex-pair of orbit A on the boundary of a face.



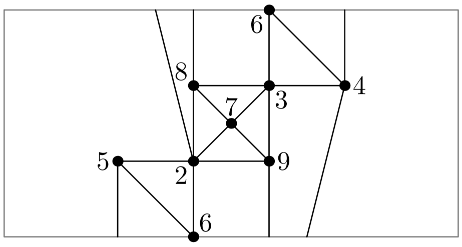
(6) Embedding of $D_4/(1,4)$, each vertex-pair of orbit A on the boundary of a face.



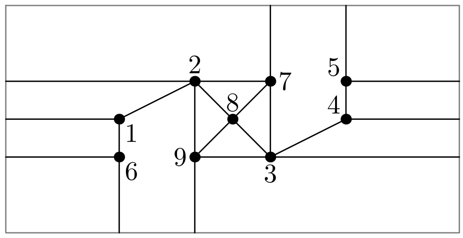
(7) Embedding of $D_4/(2,7)$, each vertex-pair of orbit A on the boundary of a face.



(8) Embedding of $D_4/(7,8)$, each vertex-pair of orbit A on the boundary of a face.



(9) Embedding of $R_3(D_4)_{(1)}$, vertices 4 and 6 on the boundary of a face.



(10) Embedding of D_4, each vertex-pair of orbit A on the boundary of a face.

Figure 3.10: (1)-(9): Embeddings of $R_i(D_4)$, $i \in (0, 2, 3)$, into the Möbius strip, (10): Embedding of D_4 into the Klein surface

□

The graph E_{19}

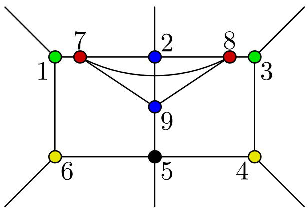


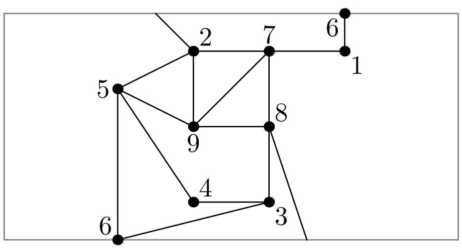
Figure 3.11: E_{19} and its vertex-orbits

The only pairs of vertices we have to consider for the graph E_{19} are:

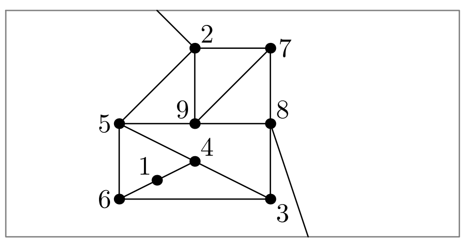
$$A = \Big\{\{1, 3\}\Big\}, \qquad B = \Big\{\{4, 6\}\Big\}.$$

Lemma 3.7
The graphs G_{65} and G_{66}, which are elements of $M_4(\mathcal{S}_2)$, can be obtained by identification of vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ and the pairs of vertices $\{1, 3\}$ and $\{4, 6\}$ of E_{19}.

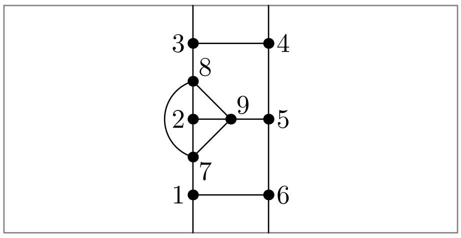
Proof
Looking at the embeddings of $E_{19} - e$ and E_{19}/e with $e \in \{(1, 4), (1, 7), (2, 5), (2, 7), (2, 9), (4, 5), (7, 8)\}$, $R_3(E_{19})_{(1)}$ and $R_3(E_{19})_{(4)}$ into the Möbius strip as well as E_{19} into the Klein surface, we can find embeddings, such that the pairs of vertices $\{1, 3\}$ as well as $\{4, 6\}$ each lie on the boundary of one face:



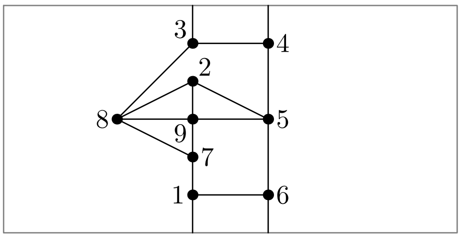
(1) Embedding of $E_{19} - (1, 4)$, vertex-pairs $\{1, 3\}$ and $\{4, 6\}$ on the boundary of a face.



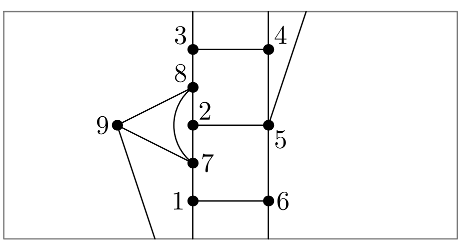
(2) Embedding of $E_{19} - (1, 7)$, vertex-pairs $\{1, 3\}$ and $\{4, 6\}$ on the boundary of a face.



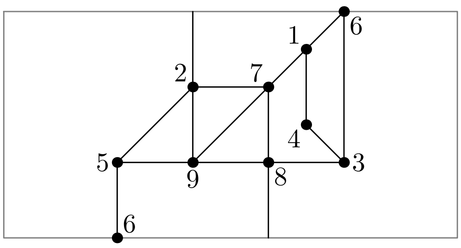
(3) Embedding of $E_{19} - (2,5)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



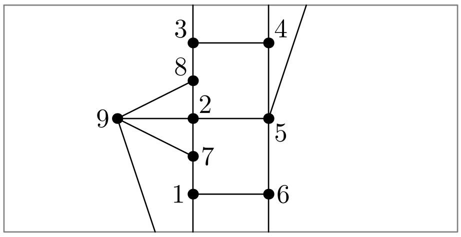
(4) Embedding of $E_{19} - (2,7)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



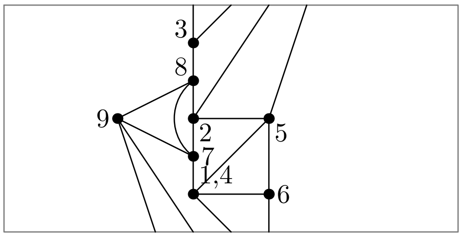
(5) Embedding of $E_{19} - (2,9)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



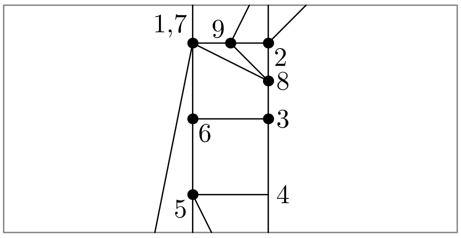
(6) Embedding of $E_{19} - (4,5)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



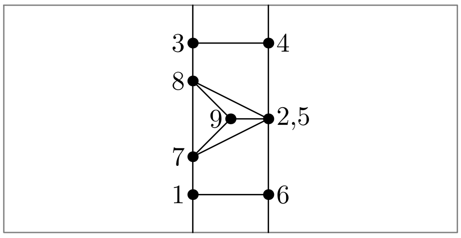
(7) Embedding of $E_{19} - (7,8)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



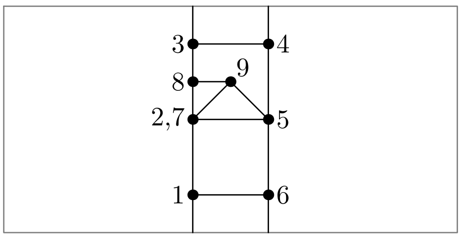
(8) Embedding of $E_{19}/(1,4)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



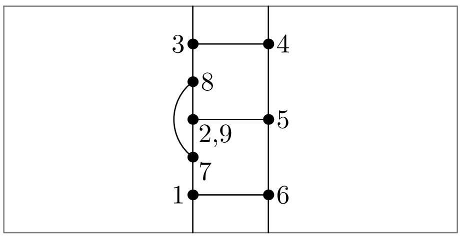
(9) Embedding of $E_{19}/(1,7)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



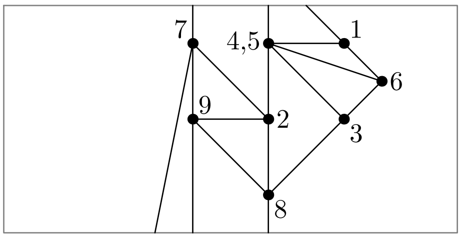
(10) Embedding of $E_{19}/(2,5)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



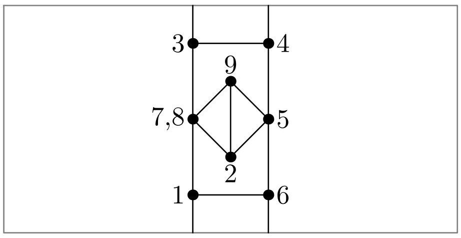
(11) Embedding of $E_{19}/(2,7)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



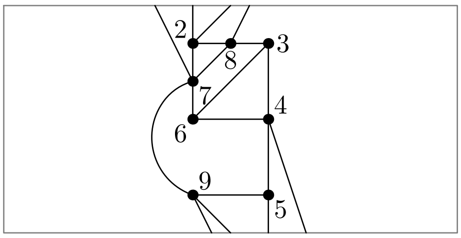
(12) Embedding of $E_{19}/(2,9)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



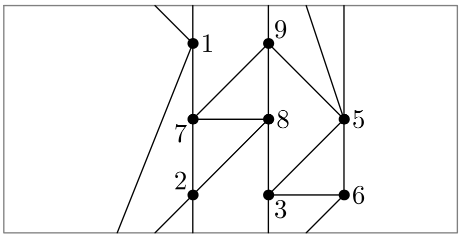
(13) Embedding of $E_{19}/(4,5)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



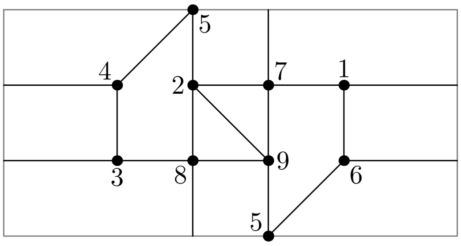
(14) Embedding of $E_{19}/(7,8)$, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.



(15) Embedding of $R_3(E_{19})_{(1)}$, vertex-pair $\{4,6\}$ on the boundary of a face.



(16) Embedding of $R_3(E_{19})_{(4)}$, vertex-pair $\{1,3\}$ on the boundary of a face.



(17) Embedding of E_{19}, vertex-pairs $\{1,3\}$ and $\{4,6\}$ on the boundary of a face.

Figure 3.12: (1)-(16): Embeddings of $R_i(E_{19})$, $i \in (0,2,3)$, into the Möbius strip, (17): Embedding of E_{19} into the Klein surface

□

The graph F_1

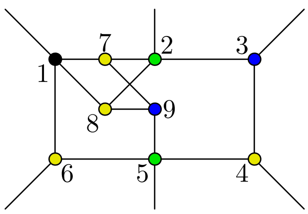


Figure 3.13: F_1 and its vertex-orbits

The only pair of vertices we have to consider for the graph F_1 is: $\{3, 9\}$

Lemma 3.8
The graph constructed by identification of vertices 3 and 9 of F_1 and vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ is not an element of $M_4(\mathcal{S}_2)$.

Proof
It is sufficient to show that the graph $F_1 - (2,3)$ cannot be embedded into the Möbius strip with vertices 3 and 9 on the boundary of the same face. The only cycle, which we have to consider, is:

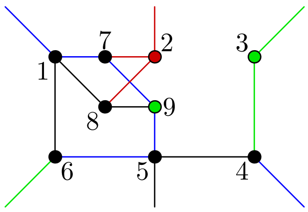
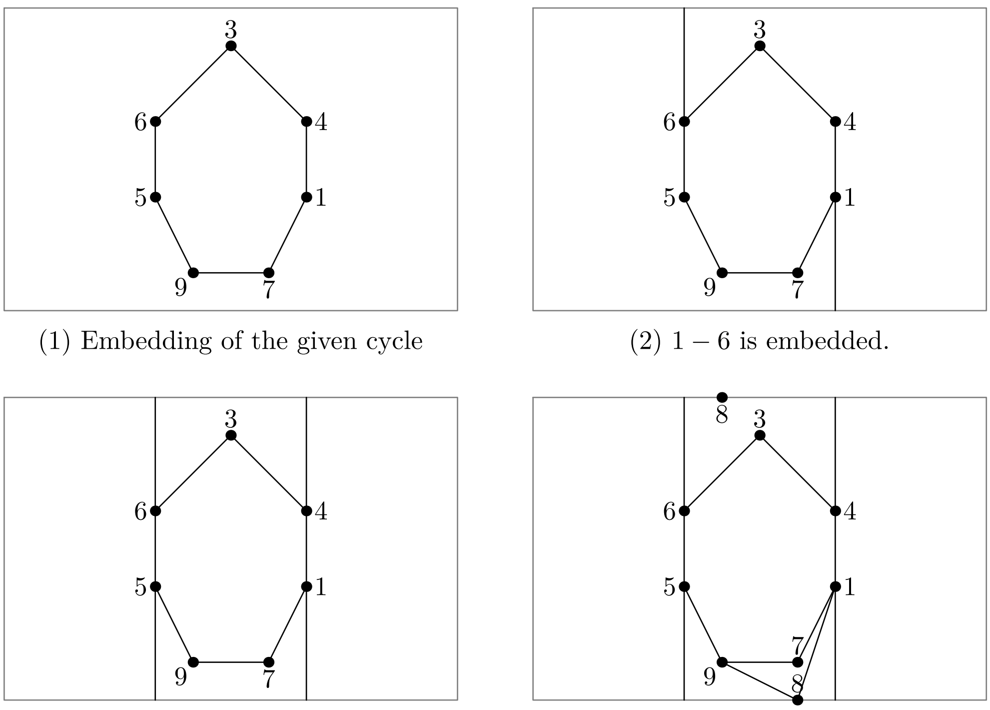


Figure 3.14: Cycle in $F_1 - (2,3)$, including vertices 3 and 9

We cannot find an embedding of $F_1 - (2,3)$ with this cycle as the boundary of one face:



(1) Embedding of the given cycle

(2) $1-6$ is embedded.

(3) $4-5$ is embedded

(4) $1-8-9$ is embedded

Figure 3.15: Embedding of $F_1 - (2,3)$

Figure 3.15 shows that an embedding, with the given restrictions, is not possible, as the path $6-7$ cannot be embedded. □

3.3 Graphs including three base-points

In this section, we will proof that none of the graphs constructed by identification of vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ with three vertices of the graphs in $M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$ is an element of $M_4(\mathcal{S}_2)$.

Proposition 3.9
None of the graphs, which are constructed by identification of three vertices in one of the graphs in $M_3(\mathcal{S}_1) - M_4(\mathcal{S}_1)$ with three vertices in K_5 (w.l.o.g. v_1, v_2, v_3) and deleting the edges (v_1, v_2), (v_1, v_3) and (v_2, v_3), is an element of $M_4(\mathcal{S}_2)$.

The graph C_1

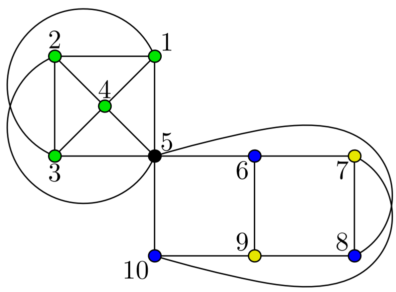
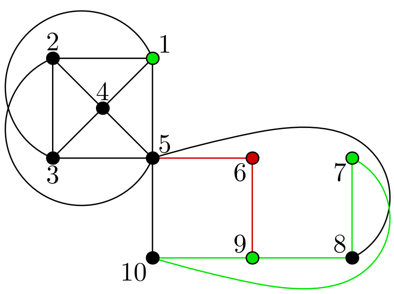


Figure 3.16: C_1 and its vertex-orbits

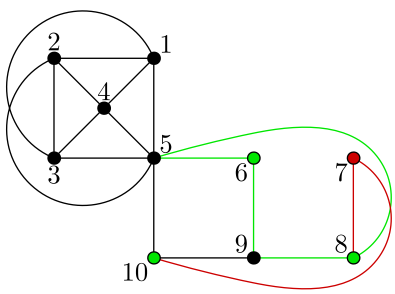
The only orbits of triples of vertices we have to consider for the graph C_1 are:

$$A = \Big\{\{1,7,9\}, \{2,7,9\}, \{3,7,9\}, \{4,7,9\}\Big\}, \qquad\qquad B = \Big\{\{6,8,10\}\Big\}.$$

Lemma 3.10
The construction of graphs by identification of vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ with a triple of vertices from orbit A or B of C_1 does not result in elements of $M_4(\mathcal{S}_2)$.



(1) $C_1 - (6, 7)$, vertices $1, 7$ and 9 cannot lie in one cylce



(2) $C_1 - (6, 7)$, vertices $6, 8$ and 10 cannot lie in one cycle

Figure 3.17: Cycles, including the vertex-sets $\{1, 7, 9\}$ or $\{6, 8, 10\}$, cannot be found in $C_1 - (6, 7)$.

Proof
As can be seen in Figure 3.17, it is not possible to find a cycle in $C_1 - (6,7)$ where the vertex-triples $\{1,7,9\} \in A$ or $\{6,8,10\} \in B$ are included, as already two vertices of these triples lie in a smaller cycle (cf. Lemma 1.11). $\square$

The graph C_{11}

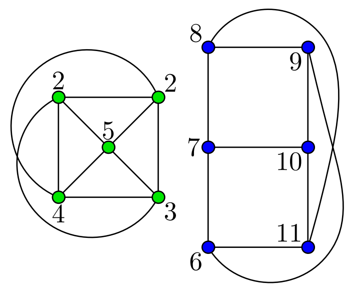


Figure 3.18: C_{11} and its vertex-orbits

The only orbit of triples of vertices we have to consider for the graph C_{11} is:

$$A = \Big\{\{6,8,10\},\{7,9,11\}\Big\}$$

Lemma 3.11
The construction of a graph by identification of vertices v_1, v_2, v_3 of $K_5 - (v_1,v_2) - (v_1,v_3) - (v_2,v_3)$ and a triple of vertices in orbit A of C_{11}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
As can be seen in Figure 3.19, vertices $7,9$ and 11 cannot lie on one cycle in $C_{11} - (6,7)$, as vertices 7 and 11 already lie on a smaller cycle (cf. Lemma 1.11).

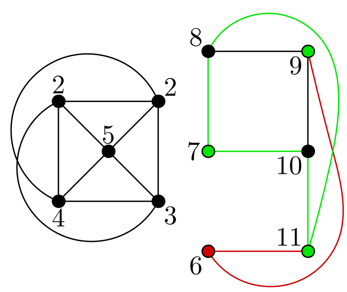


Figure 3.19: Vertices $7,9$ and 11 cannot lie on one cycle in $C_{11} - (6,7)$

Consequently no embedding of $C_{11} - (6,7)$ with vertices from orbit A on the boundary of one face is possible. $\square$

The graph D_3

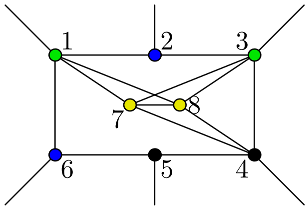


Figure 3.20: D_3 and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph D_3 are:

$$A = \Big\{\{1,3,5\}\Big\}, \qquad B = \Big\{\{2,6,7\},\{2,6,8\}\Big\}.$$

Lemma 3.12
The construction of a graph by identification of vertices v_1, v_2, v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ with vertices $1, 3$ and 5 of D_3, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The only cycle, including vertices $1, 3$ and 5, we have to look at for $D_3 - (4,5)$ is:

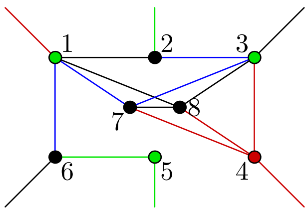
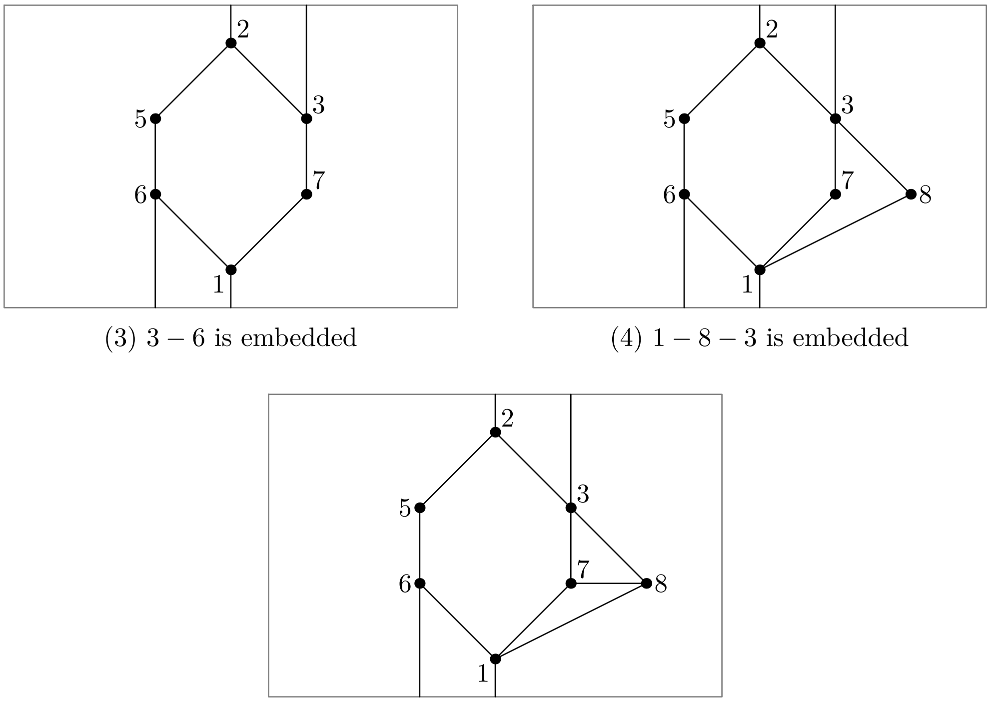


Figure 3.21: Cycle in $D_3 - (4,5)$, including vertices $1, 3$ and 5

This is because the adjacent vertices of vertex 5 are given, from vertex 2 we can either go to vertex 1 or 3, as these are isomorphic, w.l.o.g. we choose vertex 3. From vertex 3 we can only choose between vertices 7 and 8, but again, these are isomorphic, so w.l.o.g. we choose vertex 7. From here we can only choose vertex 1 as the next vertex, then we close the cycle. This is the shortest possible cycle, which is isomorphic to all other cycles of length six including the given three vertices. Every other possible cycle already includes this (or an isomorphic) cycle and thus (Lemma 1.11) does not have to be considered. Trying to embed $D_3 - (4,5)$ including this cycle as the boundary of one face delivers:

(1) Embedding of the given cycle

(2) $1 - 2$ is embedded



(3) $3-6$ is embedded

(4) $1-8-3$ is embedded

(5) $7-8$ is embedded

Figure 3.22: Embedding of $D_3 - (4,5)$

Figure 3.22 shows that an embedding, with the given restrictions, is not possible, as vertex 4 cannot be embedded. $\square$

Lemma 3.13
The graph constructed by identification of vertices v_1, v_2, v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ with a triple of vertices in orbit B of D_3 cannot be an element of $M_4(\mathcal{S}_2)$.

Proof
Regarding the graph $D_3-(1,2)$ and trying to find a cycle including vertices $2,6$ and 7 is not successful, as vertices 2 and 6 already lie on a cycle which does not include vertex 7:

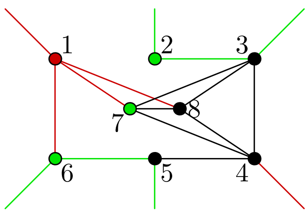


Figure 3.23: A cycle, including vertices 2, 6 and 7, cannot be found in $D_3-(1,2)$

Hence this graph cannot be an element of $M_4(\mathcal{S}_2)$. $\square$

The graph D_4

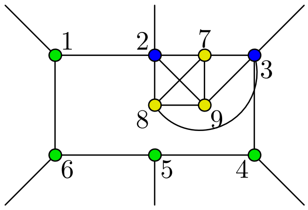


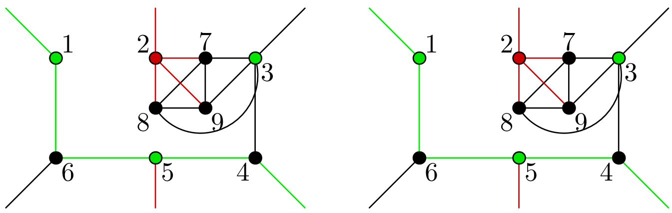
Figure 3.24: D_4 and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph D_4 are:

$$A = \Big\{\{1,3,5\},\{2,4,6\}\Big\},$$
$$B = \Big\{\{1,5,7\},\{1,5,8\},\{1,5,9\},\{4,6,7\},\{4,6,8\},\{4,6,9\}\Big\}.$$

Lemma 3.14
The graphs constructed by identification of vertices v_1, v_2, v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ with a triple of vertices in orbit A or B of D_4 cannot be elements of $M_4(\mathcal{S}_2)$.

Proof
Looking at the graph $D_4 - (1,2)$ and trying to find a cycle including vertices $1, 3$ and 5 or $1, 5$ and 7, we can see that vertices 1 and 5 already have to lie on a cycle of length 4. The third vertex cannot be included in this cycle.



(1) Cycle in $D_4 - (1,2)$, including vertices 1 and 5

(2) Cycle in $D_4 - (1,2)$, including vertices 1 and 5

Figure 3.25: Cycles in $D_4 - (1,2)$ cannot include the vertex-triples $\{1,3,5\}$ or $\{1,5,7\}$.

□

The graph E_{19}

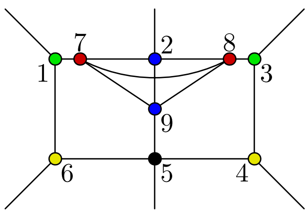


Figure 3.26: E_{19} and its vertex-orbits

The orbits of triples of vertices we have to consider for E_{19} are:

$$A = \Big\{\{1,2,3\},\{1,3,9\}\Big\}, \qquad B = \Big\{\{1,3,5\}\Big\},$$
$$C = \Big\{\{2,4,6\},\{4,6,9\}\Big\}, \qquad D = \Big\{\{4,6,7\},\{4,6,8\}\Big\}.$$

Lemma 3.15
The graph obtained by identification of vertices v_1, v_2, v_3 of $K_5 - (v_1,v_2) - (v_1,v_3) - (v_2,v_3)$ with a triple of vertices in orbit A of E_{19} is not an element of $M_4(\mathcal{S}_2)$.

Proof
The only cycle, we have to look at, is:

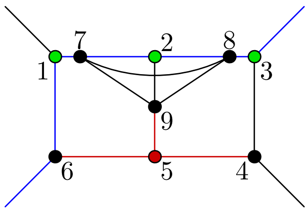
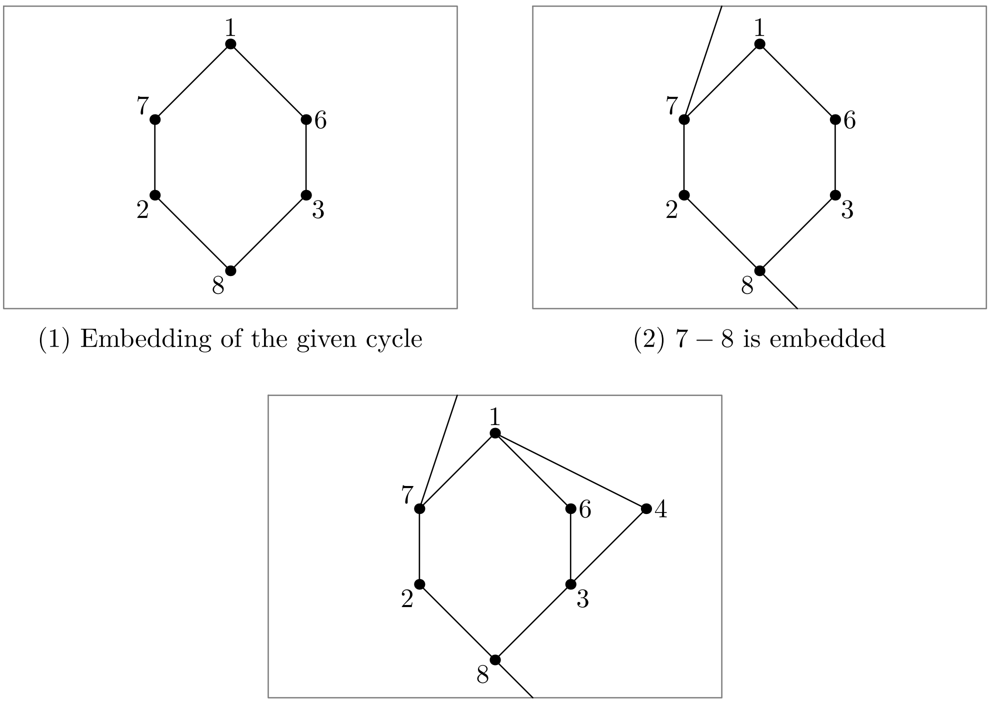


Figure 3.27: Cycle in $E_{19} - (2,5)$, including vertices 1, 2 and 3

All other possible cycles are either isomorphic to this one, or they already include this cycle using Lemma 1.11. Trying to embed the graph, having this cycle as the boundary of one face, delivers:



(1) Embedding of the given cycle

(2) $7-8$ is embedded

(3) $3-4-1$ is embedded

Figure 3.28: Embedding of $E_{19} - (2,5)$

Figure 3.28 shows that an embedding, with the given restrictions, is not possible, as the path $6-5-9-2$ cannot be embedded. $\square$

Lemma 3.16
The construction of a graph by identification of vertices v_1, v_2, v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ and a triple of vertices of orbit B of E_{19} does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The only cycle we have to look at, due to isomorphisms is:

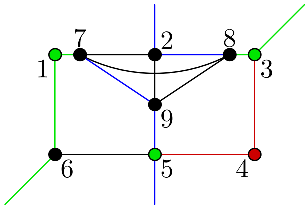
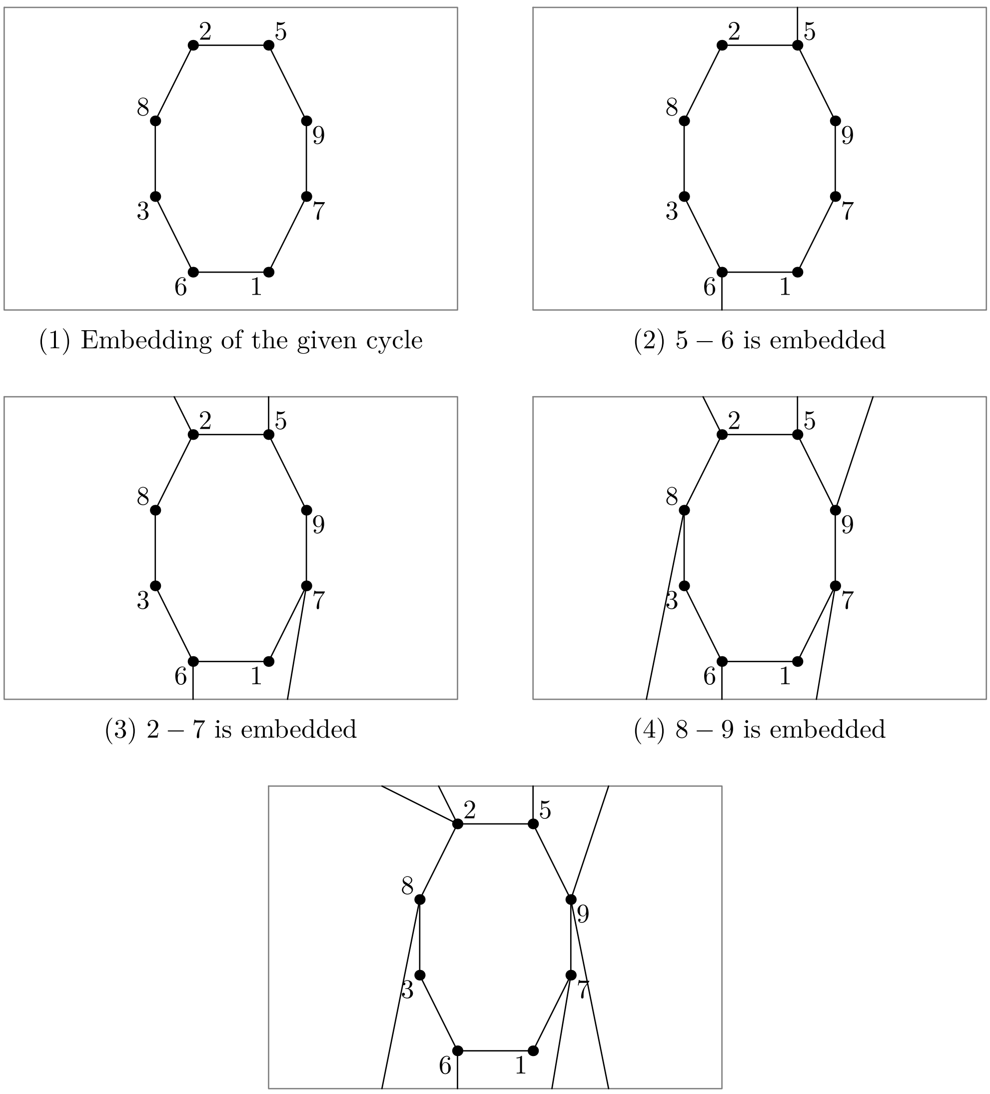


Figure 3.29: Cycle in $E_{19}-(1,4)$, including vertices $1,3$ and 5

Trying to embed E_{19} into the Möbius strip including this cycle as the boundary of one face leads to a contradiction:



(1) Embedding of the given cycle

(2) $5-6$ is embedded

(3) $2-7$ is embedded

(4) $8-9$ is embedded

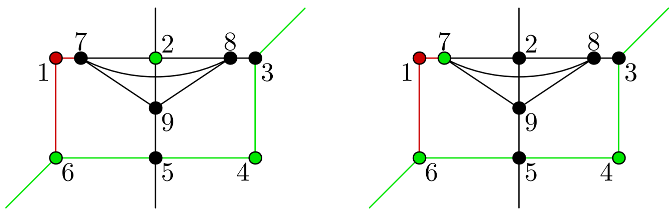
(5) $2-9$ is embedded

Figure 3.30: Embedding of $E_{19}-(1,4)$

Figure 3.30 shows that an embedding, with the given restrictions, is not possible, as the path $7 - 8$ cannot be embedded. □

Lemma 3.17
The graphs obtained by identification of vertices v_1, v_2, v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ with a triple of vertices in orbit C or D of E_{19} cannot be elements of $M_4(\mathcal{S}_2)$.

Proof
The graph $E_{19} - (1, 4)$ only leaves one possible cycle including vertices 4 and 6, this cycle is of length four and does not include the third vertex of the desired triples.



(1) Cycle in E_{19}, including vertices $2, 4$ and 6

(2) Cycle in E_{19}, including vertices $4, 6$ and 7

Figure 3.31: Cycles including vertex-triples $\{2, 4, 6\}$ or $\{4, 6, 7\}$ cannot be found in $E_{19} - (1, 4)$

Consequently these graphs cannot be elements of $M_4(\mathcal{S}_2)$. □

The graph F_1

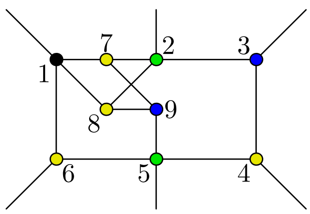


Figure 3.32: F_1 and its vertex-orbits

The only orbit of triples of vertices we have to consider for the graph F_1 is:

$$A = \Big\{\{3, 7, 8\}, \{4, 6, 9\}\Big\}.$$

Lemma 3.18
The construction of a graph by identification of vertices v_1, v_2, v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ with a triple of vertices in orbit A of F_1 does not result in elements of $M_4(\mathcal{S}_2)$.

Proof
Regarding the graph $F_1 - (2, 7)$ and trying to find a cycle including the vertex-triple $\{3, 7, 8\}$ from orbit A is not possible, as vertices 7 and 8 already lie on a cycle of length 4, which cannot be enlarged so that it also includes vertex 3.

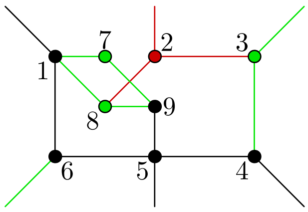
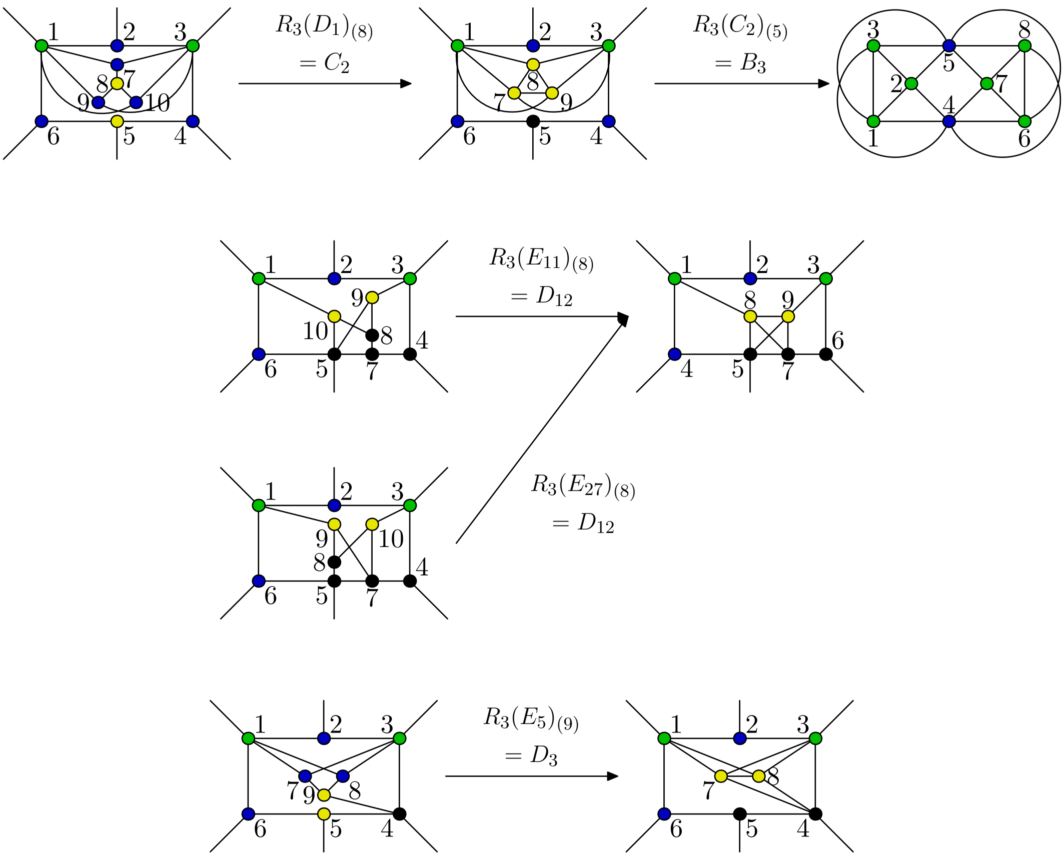


Figure 3.33: In $F_1 - (2,7)$, vertices $3, 7$ and 8 cannot be included in one cycle.

□

4 $M_2(\mathcal{S}_1)$ to $M_4(\mathcal{S}_2)$

In this chapter we look at the graphs of the minimal basis $M_2(\mathcal{S}_1) - M_3(\mathcal{S}_1)$ of the projective plane. We only have to consider an attachment of the K_5 to a graph from $M_2(\mathcal{S}_1) - M_3(\mathcal{S}_1)$ for combinations of vertices which eliminate the application of R_3, which transforms the graph into another one from $M_2(\mathcal{S}_1) - M_3(\mathcal{S}_1)$ or $M_3(\mathcal{S}_1)$ and thus makes the graph with attachment a graph of $M_4(\mathcal{S}_2)$ of the Klein surface. Figure 4.1 shows the respective graphs and the relations between these:



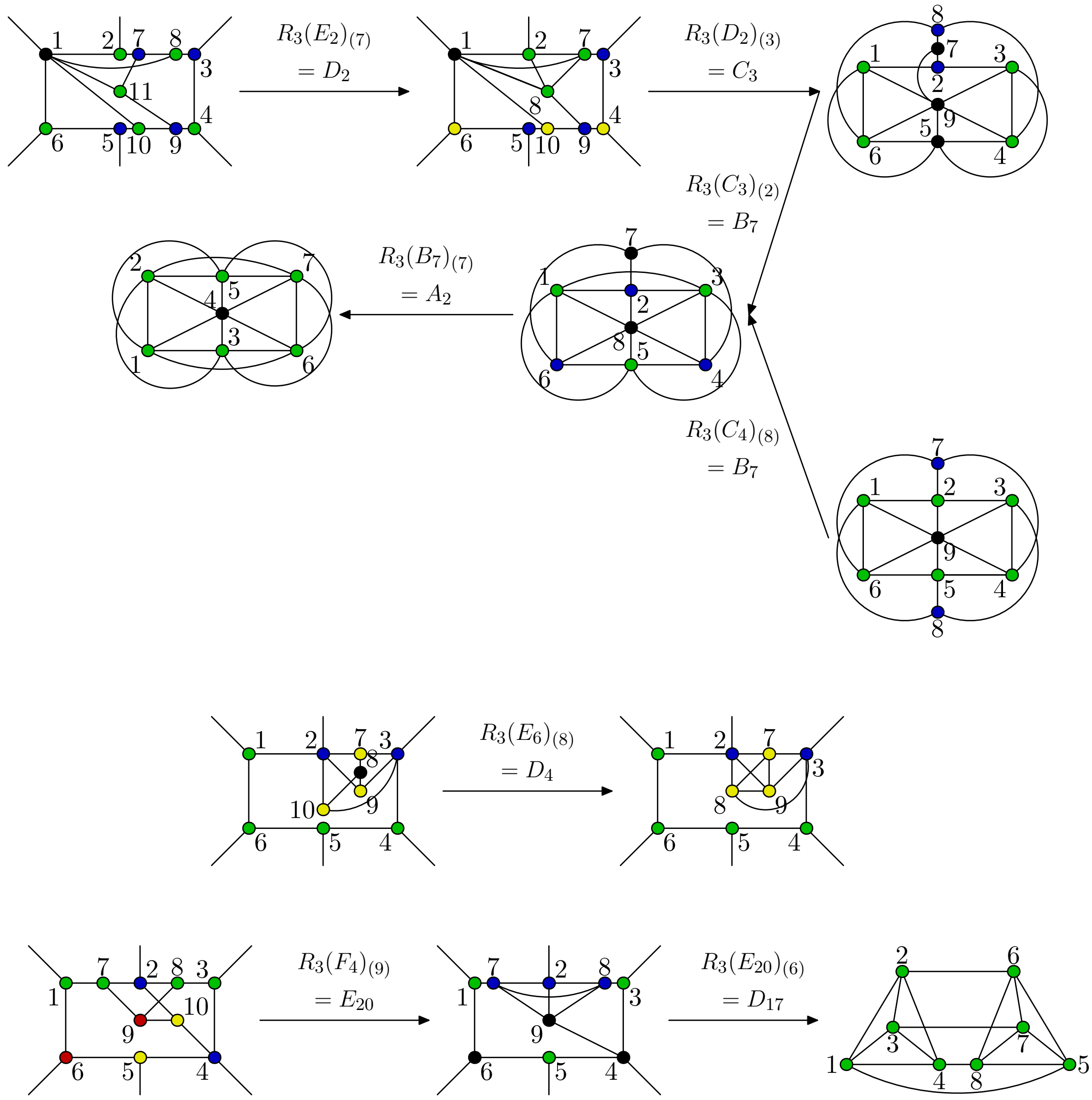


Figure 4.1: Genealogy of graphs in $M_2(\mathcal{S}_1)$

4.1 Graphs including one base-point

Proposition 4.1
The graphs $G_{67}, G_{68}, G_{69}, G_{70}$ and G_{71} which are obtained by attachment of a K_5 to vertex 7 of B_7, vertex 5 of C_2, vertex 8 of E_6, vertex 8 of E_{11}, vertex 6 of E_{20} and vertex 8 of E_{27} are elements of $M_4(\mathcal{S}_2)$.

Proof
As all graphs, the K_5 is attached to, are in $M_2(\mathcal{S}_1)$, they are embeddable on the Möbius strip as soon as one of the relations R_0, R_1 and R_2 is applied. Consequently an attachment of a K_5 to a vertex of these graphs, which forecloses the application of R_3 on these vertices is possible and results in graphs which are elements of $M_4(\mathcal{S}_2)$. $\square$

4.2 Graphs including two base-points

Proposition 4.2
The graphs $G_{72}, \ldots, G_{83}$ are the only graphs in $M_4(\mathcal{S}_2)$, which are constructed by attaching the vertices v_1 and v_2 of a $K_5 - (v_1, v_2)$ to a pair of vertices in graphs of $M_2(\mathcal{S}_1) - M_3(\mathcal{S}_1)$.

We already know that these graphs are not embeddable into the Klein surface. The minimality of these graphs will be shown in the lemmata of this section. We will also show that none of the other graphs constructed the same way, is an element of $M_4(\mathcal{S}_2)$.

The graph B_7

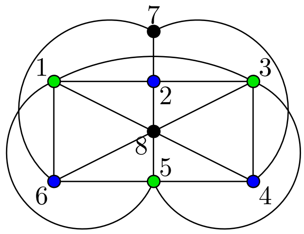


Figure 4.2: B_7 and its vertex-orbits

The orbits of pairs of vertices we have to consider for the graph B_7 are:

$$A = \Big\{\{1,7\},\{3,7\},\{5,7\}\Big\}, \qquad\qquad B = \Big\{\{7,8\}\Big\}.$$

Lemma 4.3
The graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with a pair of vertices from orbit A of B_7 is no element of $M_4(\mathcal{S}_2)$.

Proof
Using Lemma 1.15 we know that the embedding of $B_7/(2,7)$ on the Möbius strip has 11 faces.

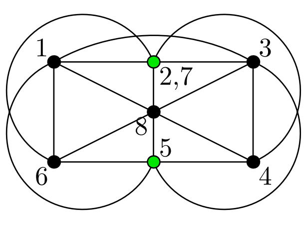
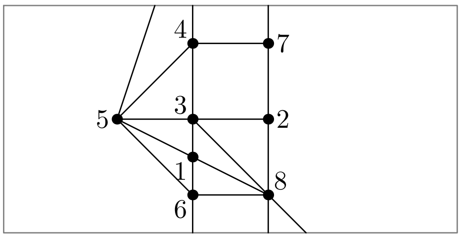


Figure 4.3: $B_7/(2,7)$, vertices 5 and 7 supposed to lie in one cycle.

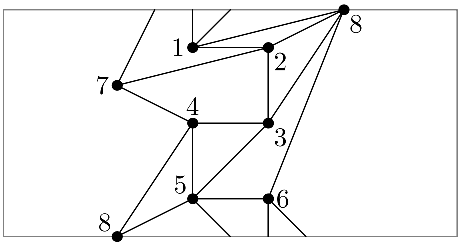
The graph $B_7/(2,7)$ has 17 edges which means that a possible embedding has to consist of ten triangular and one quadrangular face. As vertices 1 and 7 as well as vertices 3 and 7 are adjacent in $B_7/(2,7)$, we have to show that vertices 5 and 7 cannot lie on the boundary of one face, when embedding this graph. We can find 17 cycles of length three in $B_7/(2,7)$ but obviously none of these includes both vertices 5 and 7. Thus the only face of size four in the embedding has to include these vertices. Only three non-isomorphic cycles of length four in $B_7/(2,7)$ include both of the vertices. These cycles are: $1-2-3-5-1$, $1-2-4-5-1$ and $1-2-8-5-1$. If $B_7/(2,7)$ was embeddable with one of these cycles as the boundary of a face, we would have to find ten additional cycles of length three, which include each of the edges in the cycle of length four exactly once and each other edge exactly twice. As we cannot find a combination of ten cycles of length three with these properties, $B_7/(2,7)$ cannot be embedded with vertices 5 and 7 on the boundary of one face. Thus the construction of a graph with an attached $K_5-(v_1,v_2)$ and base points 5 and 7 is no element of $M_4(\mathcal{S}_2)$. $\square$

Lemma 4.4
The graph G_{72}, which is obtained by attaching the $K_5-(v_1,v_2)$ to B_7 by identification of vertices v_1 and v_2 of the K_5 and vertices 7 and 8 in B_7, is an element of $M_4(\mathcal{S}_2)$.

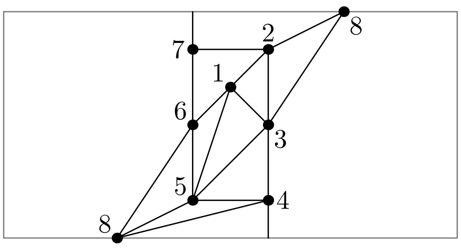
Proof
It is sufficient to show that the graphs B_7-e and B_7/e with $e\in\{(1,2),\ (1,3),\ (1,8),\ (2,7),\ (2,8)\}$ can be drawn on the Möbius strip, and B_7 itself on the Klein surface, with vertices 7 and 8 on the boundary of one face:



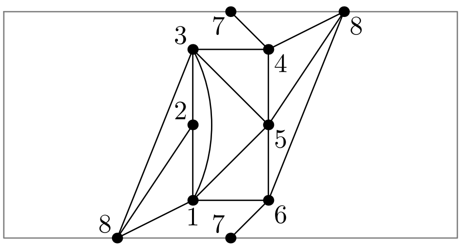
(1) Embedding of $B_7-(1,2)$, vertices 7 and 8 on the boundary of one face.



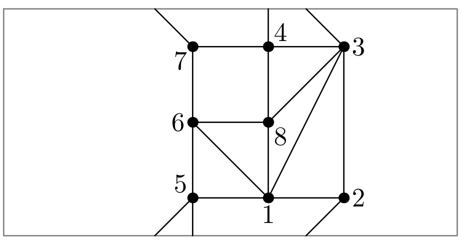
(2) Embedding of $B_7-(1,3)$, vertices 7 and 8 on the boundary of one face.



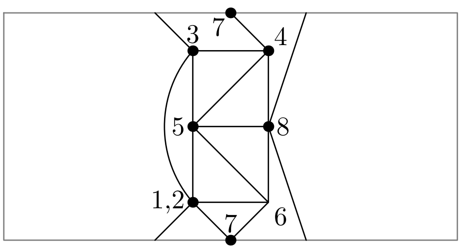
(3) Embedding of $B_7-(1,8)$, vertices 7 and 8 on the boundary of one face.



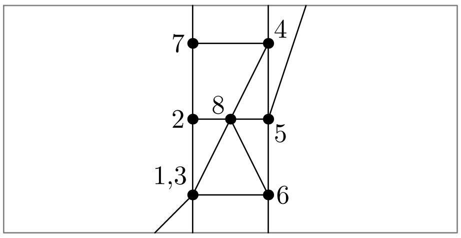
(4) Embedding of $B_7-(2,7)$, vertices 7 and 8 on the boundary of one face.



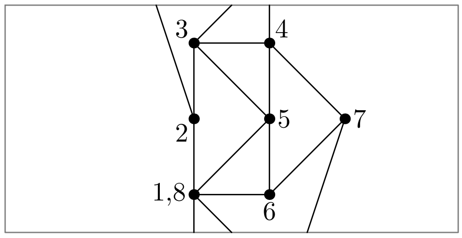
(5) Embedding of $B_7-(2,8)$, vertices 7 and 8 on the boundary of one face.



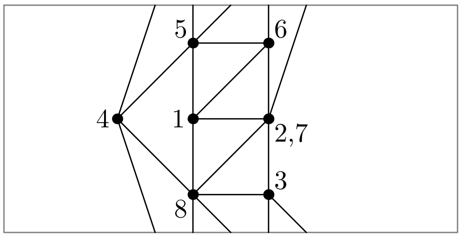
(6) Embedding of $B_7/(1,2)$, vertices 7 and 8 on the boundary of one face.



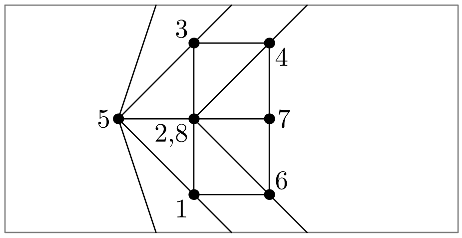
(7) Embedding of $B_7/(1,3)$, vertices 7 and 8 on the boundary of one face.



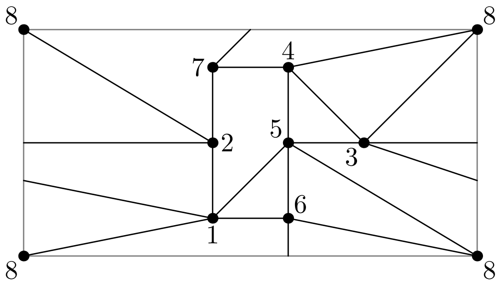
(8) Embedding of $B_7/(1,8)$, vertices 7 and 8 on the boundary of one face.



(9) Embedding of $B_7/(2,7)$, vertices 7 and 8 on the boundary of one face.



(10) Embedding of $B_7/(2,8)$, vertices 7 and 8 on the boundary of one face.



(11) Embedding of B_7, vertices 7 and 8 on the boundary of one face.

Figure 4.4: (1)-(10): Embeddings of $R_i(C_1)$, $i \in (0,2)$, on the Möbius strip, (11): Embedding of B_7 on the Klein surface

Consequently G_{72} is an element of $M_4(\mathcal{S}_2)$. $\square$

The graph C_2

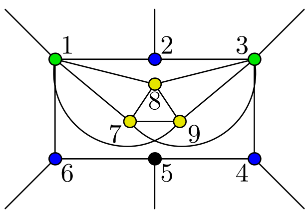


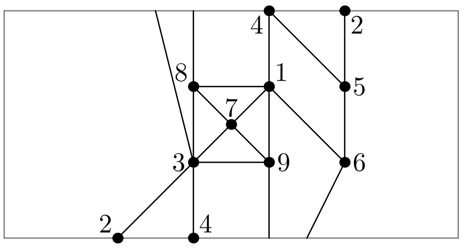
Figure 4.5: C_2 and its vertex-orbits

The orbits of pairs of vertices we have to consider for the graph C_2 are:

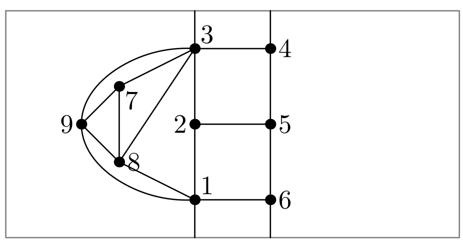
$$A = \Big\{\{1,5\},\{3,5\}\Big\}, \qquad B = \Big\{\{5,7\},\{5,8\},\{5,9\}\Big\}.$$

Lemma 4.5
The graph G_{73}, which is obtained by attaching the $K_5 - (v_1, v_2)$ to C_2 by identification of vertices v_1 and v_2 of the K_5 and a pair of vertices from orbit A in B_7, is an element of $M_4(\mathcal{S}_2)$.

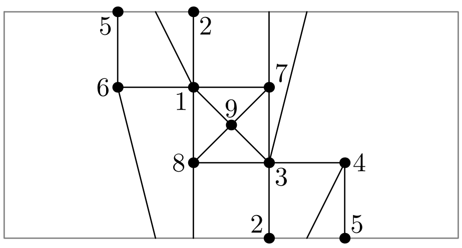
Proof
It is sufficient to show that the graphs $C_2 - e$, C_2/e with $e \in \{(1,2), (1,7), (2,5), (7,8)\}$ and $R_3(C_2)_{(2)}$ can be drawn on the Möbius strip, with each pair of vertices in orbit A on the boundary of one face:



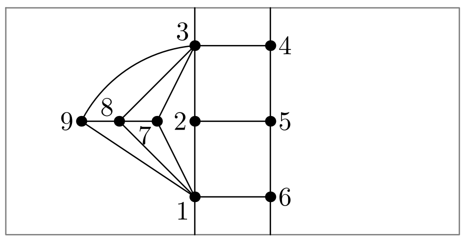
(1) Embedding of $C_2 - (1,2)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



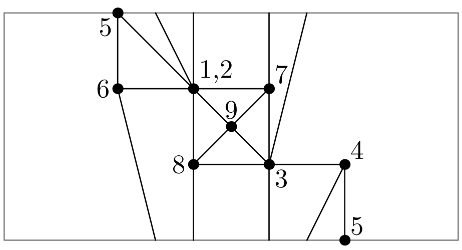
(2) Embedding of $C_2 - (1,7)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



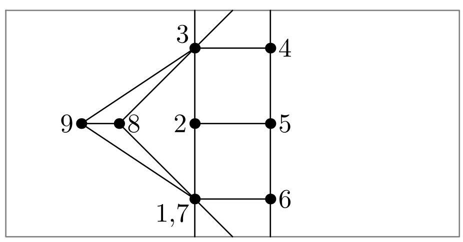
(3) Embedding of $C_2 - (2,5)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



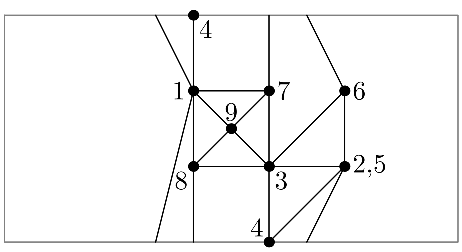
(4) Embedding of $C_2 - (7,8)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



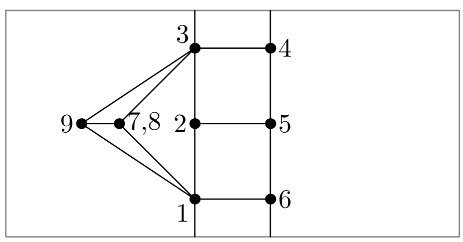
(5) Embedding of $C_2/(1,2)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



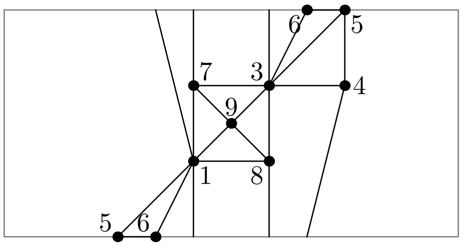
(6) Embedding of $C_2/(1,7)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



(7) Embedding of $C_2/(2,5)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



(8) Embedding of $C_2/(7,8)$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.



(9) Embedding of $R_3(C_2)_{(2)}$, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.

Figure 4.6: Embeddings of $R_i(C_2)$, $i \in \{0,2,3\}$, on the Möbius strip.

We can also find an embedding of C_2 into the Klein surface with each pair of vertices in orbit A on the boundary of one face:

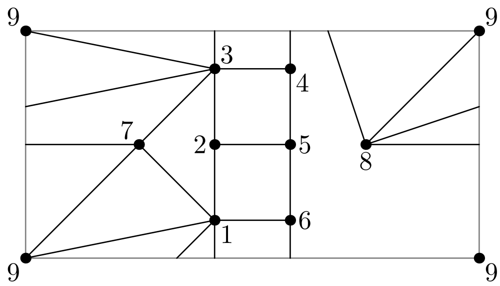


Figure 4.7: Embedding of C_2 into the Klein surface, vertex-pairs $\{1,5\}$ and $\{3,5\}$ on the boundary of a face.

Consequently G_{73} is an element of $M_4(\mathcal{S}_2)$. □

Lemma 4.6
The graph, with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with pairs of vertices from orbit B of the graph C_2, cannot be an element of $M_4(\mathcal{S}_2)$.

Proof
We take the graph $C_2 - (1,7)$ into consideration to show that vertices 5 and 7 cannot lie on the boundary of one face, when embedding this graph on the Möbius strip.

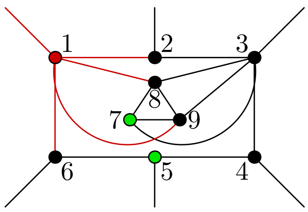


Figure 4.8: $C_2 - (1,7)$

As can be seen, we cannot find a cycle including vertices 5 and 7, as vertex 3 would always have to be passed twice. □

The graph C_3

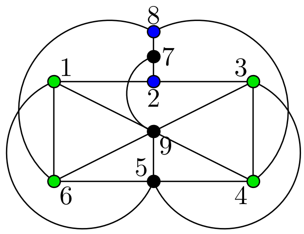
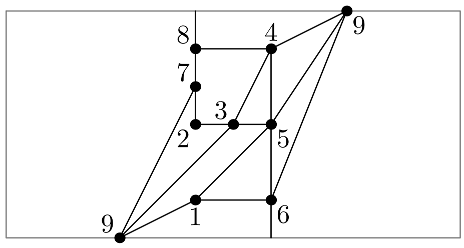


Figure 4.9: C_3 and its vertex-orbits

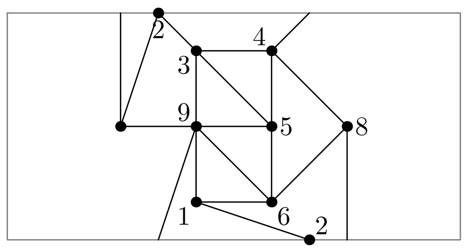
The only pair of vertices we have to consider for the graph C_3 is $\{2, 8\}$.

Lemma 4.7
The graph G_{74} which is obtained by attaching the $K_5 - (v_1, v_2)$ to C_3 by identification of vertices v_1 and v_2 of the K_5 and vertices 2 and 8 in C_3 is an element of $M_4(\mathcal{S}_2)$.

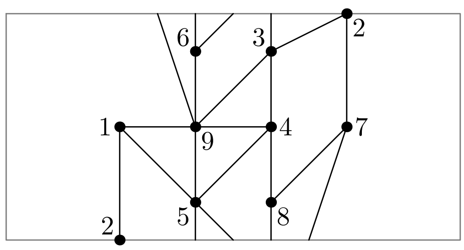
Proof
It is sufficient to show that the graphs $C_3 - e$, C_3/e with $e \in \{(1,2), (1,5), (1,6), (1,9), (2,7), (5,9), (7,9)\}$ and $R_3(C_3)_{(7)}$ can be embedded into the Möbius strip with vertices 2 and 8 on the boundary of one face:



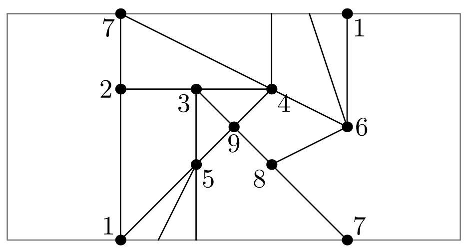
(1) Embedding of $C_3 - (1,2)$, vertices 2 and 8 on the boundary of one face.



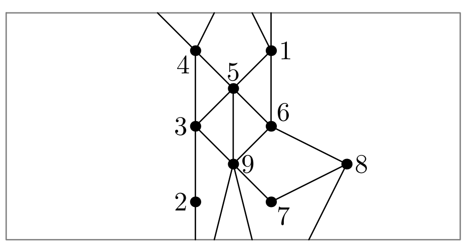
(2) Embedding of $C_3 - (1,5)$, vertices 2 and 8 on the boundary of one face.



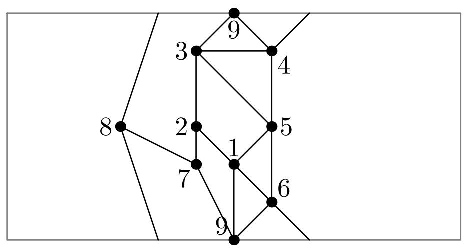
(3) Embedding of $C_3 - (1,6)$, vertices 2 and 8 on the boundary of one face.



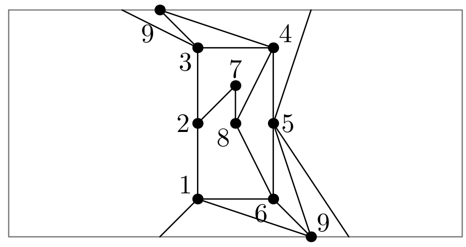
(4) Embedding of $C_3 - (1,9)$, vertices 2 and 8 on the boundary of one face.



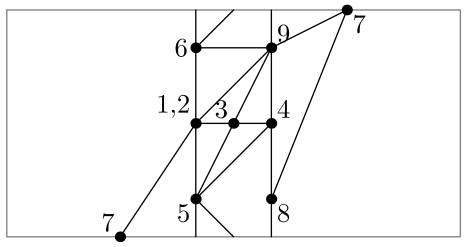
(5) Embedding of $C_3 - (2,7)$, vertices 2 and 8 on the boundary of one face.



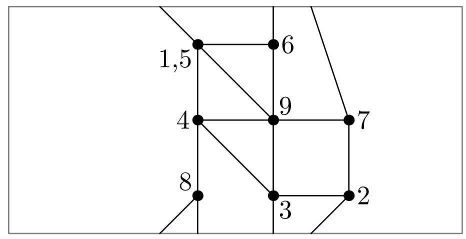
(6) Embedding of $C_3 - (5,9)$, vertices 2 and 8 on the boundary of one face.



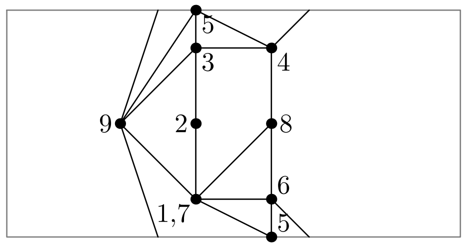
(7) Embedding of $C_3 - (7, 9)$, vertices 2 and 8 on the boundary of one face.



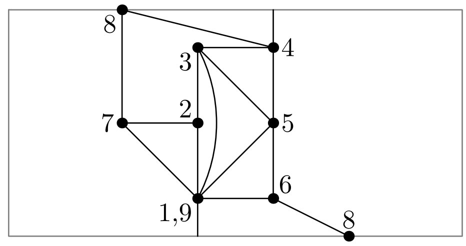
(8) Embedding of $C_3/(1, 2)$, vertices 2 and 8 on the boundary of one face.



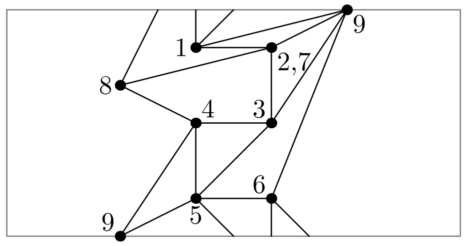
(9) Embedding of $C_3/(1, 5)$, vertices 2 and 8 on the boundary of one face.



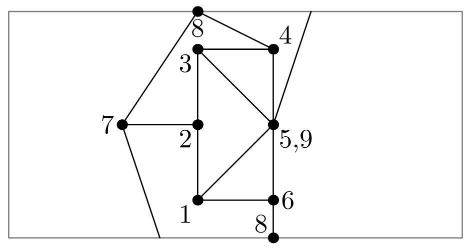
(10) Embedding of $C_3/(1, 7)$, vertices 2 and 8 on the boundary of one face.



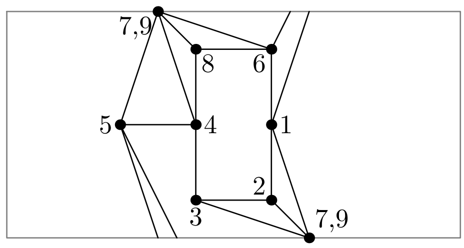
(11) Embedding of $C_3/(1, 9)$, vertices 2 and 8 on the boundary of one face.



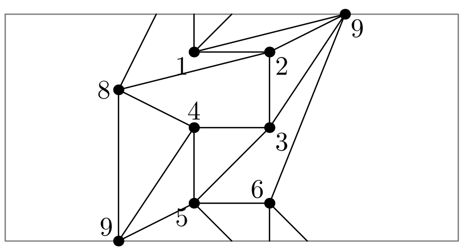
(12) Embedding of $C_3/(2, 7)$, vertices 2 and 8 on the boundary of one face.



(13) Embedding of $C_3/(5, 9)$, vertices 2 and 8 on the boundary of one face.



(14) Embedding of $C_3/(7, 9)$, vertices 2 and 8 on the boundary of one face.



(15) Embedding of $R_3(C_3)_{(7)}$, vertices 2 and 8 on the boundary of one face.

Figure 4.10: Embeddings of $R_i(C_3)$, $i \in \{0, 2, 3\}$, on the Möbius strip.

We can also find an embedding of C_3 into the Klein surface with vertices 2 and 8 on the boundary of one face:

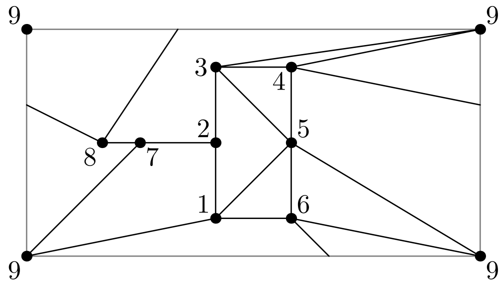


Figure 4.11: Embedding of C_3 into the Klein surface, vertices 2 and 8 on the boundary of one face.

Consequently G_{74} is an element of $M_4(\mathcal{S}_2)$. $\square$

The graph C_4

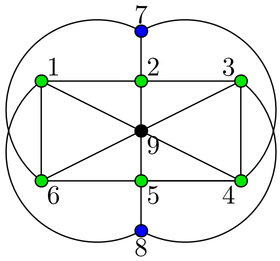
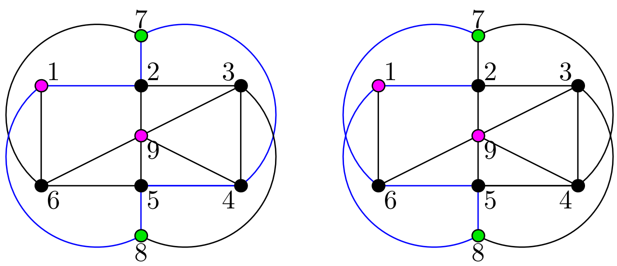


Figure 4.12: C_4 and its vertex-orbits

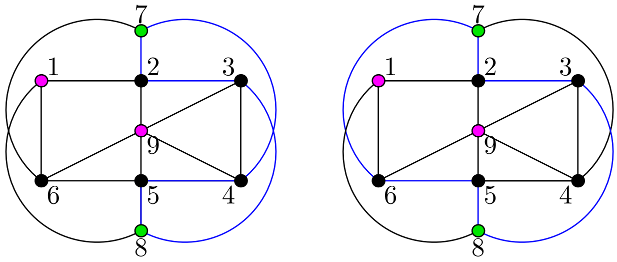
The only pair of vertices we have to consider for the graph C_4 is $\{7, 8\}$.

Lemma 4.8
The graph, with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with vertices 7 and 8 of C_4 is not an element of $M_4(\mathcal{S}_2)$.

Proof
For $C_4 - (1, 9)$ we have to consider four cycles including vertices 7 and 8 as the boundary of a face on the Möbius strip. Vertices 1 and 9 cannot lie in the same cycle.



(1) Case 1 (2) Case 2

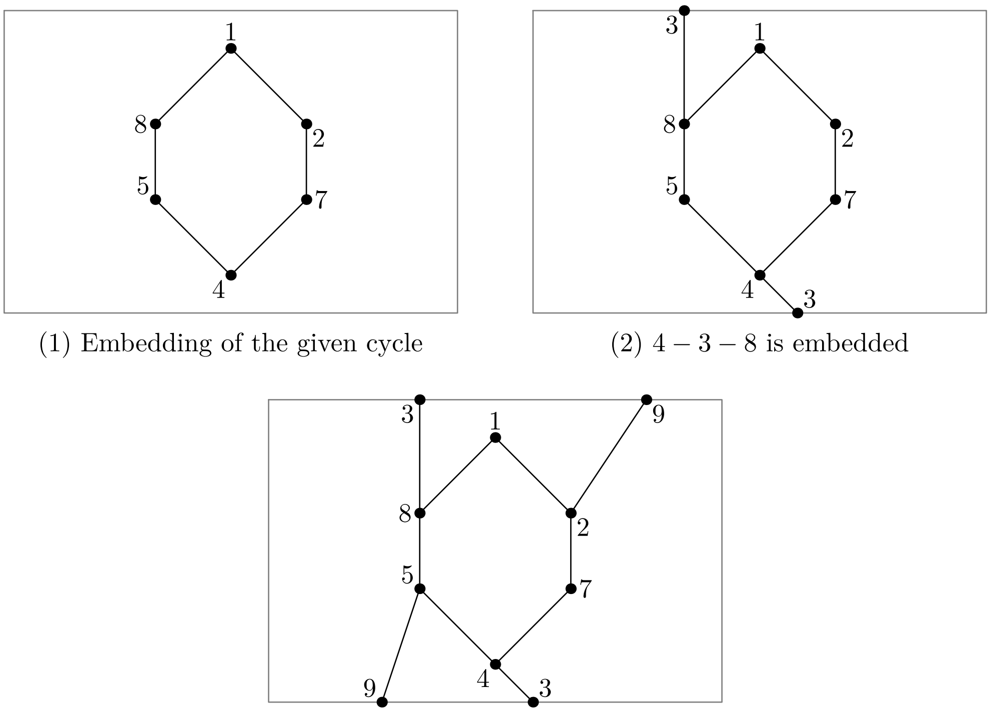


(3) Case 3 (4) Case 4

Figure 4.13: Cycles in $C_4 - (1, 9)$, including vertices 7 and 8

All other cycles do not have to be considered following Lemma 1.11. Trying to embed the graph $C_4 - (1, 9)$ with each one of these four cycles as the boundary of one face results in:

Embedding of Case 1:



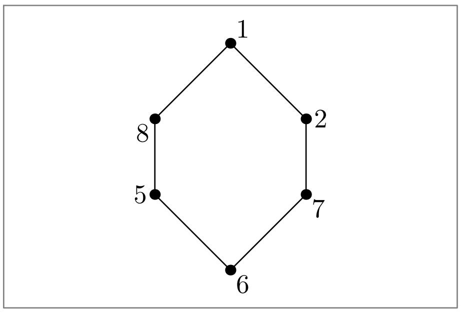
(1) Embedding of the given cycle (2) $4 - 3 - 8$ is embedded

(3) $2 - 9 - 5$ is embedded

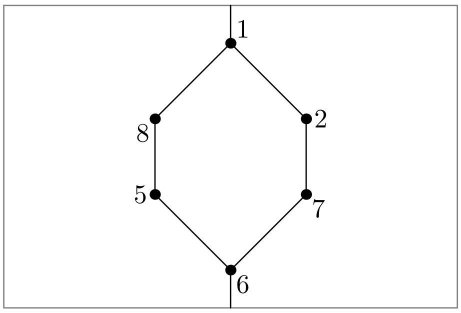
Figure 4.14: Embedding of Case 1

Figure 4.14 shows that an embedding, with the given restrictions, is not possible, as the path $1 - 6 - 7$ cannot be embedded.

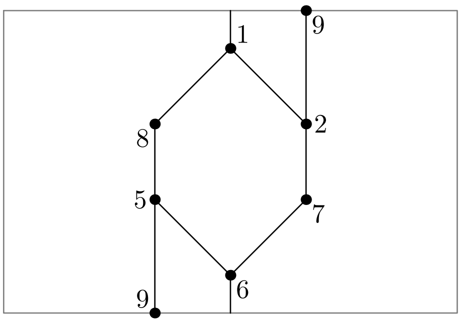
Embedding of Case 2:



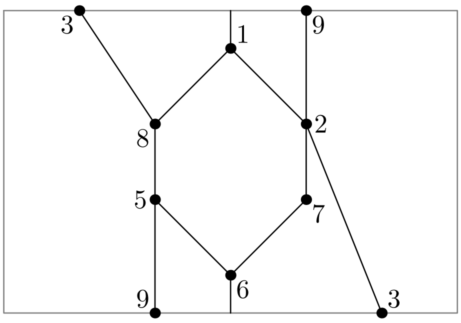
(1) Embedding of the given cycle



(2) $1-6$ is embedded



(3) $2-9-5$ is embedded

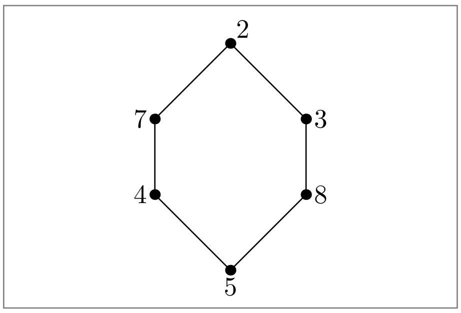


(4) $2-3-8$ is embedded

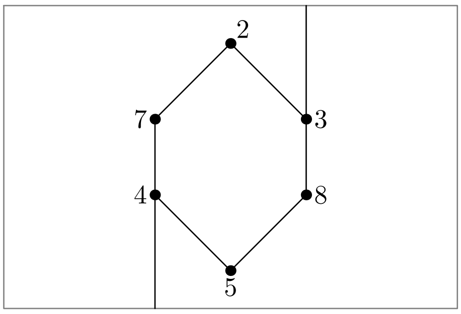
Figure 4.15: Embedding of Case 2

Figure 4.15 shows that an embedding, with the given restrictions, is not possible, as the path $5-4-7$ cannot be embedded.

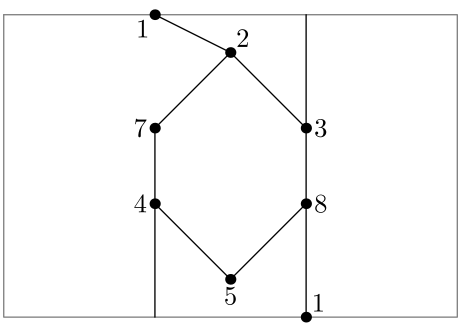
Embedding of Case 3:



(1) Embedding of the given cycle



(2) $3-4$ is embedded

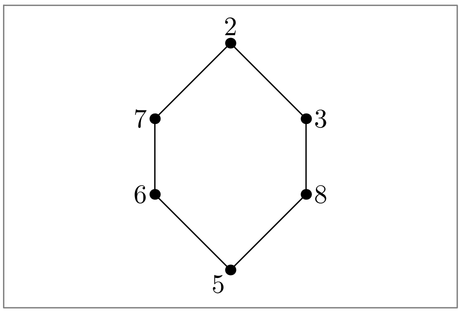


(3) $2-1-8$ is embedded

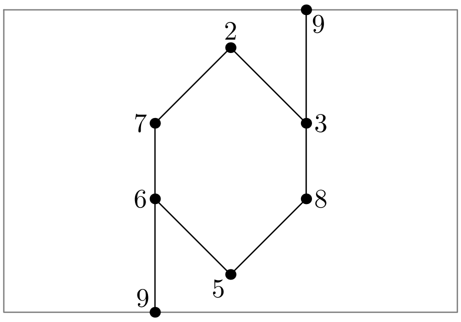
Figure 4.16: Embedding of Case 3

Figure 4.16 shows that an embedding, with the given restrictions, is not possible, as the path $5-6-7$ cannot be embedded.

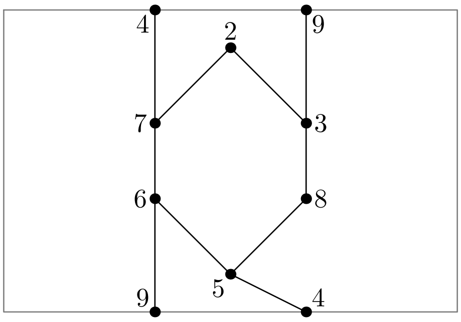
Embedding of Case 4:



(1) Embedding of the given cycle



(2) $3-9-6$ is embedded



(3) $5-4-7$ is embedded

Figure 4.17: Embedding of Case 4

Figure 4.17 shows that an embedding, with the given restrictions, is not possible, as the path $2-1-8$ cannot be embedded.

Consequently $C_4-(1,4)$ cannot be embedded on the Möbius strip with one of the given cycles as boundary of one face. $\square$

The graph D_1

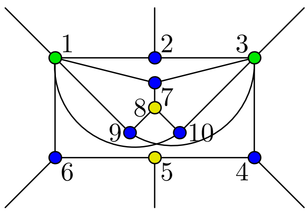


Figure 4.18: D_1 and its vertex-orbits

The only pair of vertices we have to consider for the graph D_2 is $\{5,8\}$.

Lemma 4.9
The construction of a graph with vertices v_1 and v_2 of $K_5-(v_1,v_2)$ identified with vertices 5 and 8 of the graph D_1 does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The shortest cycle in $D_1 - (2,5)$, which includes vertices 5 and 8, is of length eight.

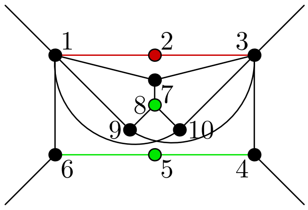


Figure 4.19: $D_1 - (2,5)$, vertices 5 and 8 supposed to be included in one cycle

Using Lemma 1.15 a face of size eight cannot be found in any embedding of $D_1 - (2,5)$ on the Möbius strip. An embedding of $D_1 - (2,5)$ has eight faces and $D_1 - (2,5)$ has 17 edges and each edge lies on the boundary of two faces, lets say we have 34 ‚edge-sides'. Lets assume one face has size eight and thus uses eight edge-sides, now we have 26 edge-sides left for the remaining 7 faces. As we cannot find a cycle of length three in $D_1 - (2,5)$ the smallest cycles have length four. Seven cycles of length four already use 28 edge-sides, consequently a face cannot be of size eight, when embedding $D_1 - (2,5)$. □

The graph E_5

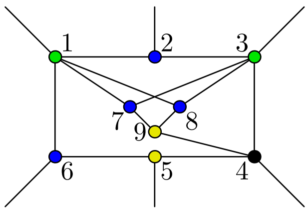


Figure 4.20: E_5 and its vertex-orbits

The only pair or vertices we have to consider for the graph E_5 is $\{5,9\}$.

Lemma 4.10
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with vertices 5 and 9 of the graph C_2 does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
Looking at the graph $E_5 - (4,5)$ and trying to find a cycle including vertices 5 and 9, we already have to include certain edges into the cycle:

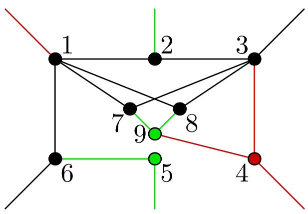


Figure 4.21: Cycle in $E_5 - (4,5)$, including vertices 5 and 9

The vertex-orbits of $E_5 - (4,5)$ are: $\{1,3\}, \{2,6\}, \{4,7,8\}$. Consequently vertices 4 and 7 could be exchanged. An embedding of $E_5 - (4,5)$ with 5 and 9 on the boundary of one face including vertices 7 and 8 consequently means that an embedding of $E_5 - (4,5)$ on the Möbius strip would also be possible with vertices 4 and 5 on the boundary of one face. This is a contradiction to the fact that E_5 is irreducible for the projective plane. □

The graph E_6

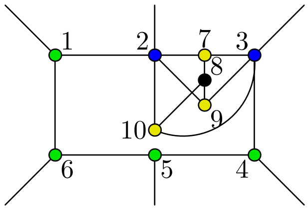


Figure 4.22: E_6 and its vertex-orbits

The only orbits of pairs of vertices we have to consider for the graph E_6 are:

$$A = \Big\{\{1,8\},\{4,8\},\{5,8\},\{6,8\}\Big\}; \qquad B = \Big\{\{2,8\},\{3,8\}\Big\}.$$

Lemma 4.11
The graph, where the vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ are identified with a pair of vertices from orbit A of the graph E_6 does not lie in $M_4(\mathcal{S}_2)$.

Proof
The graph $E_6 - (1,2)$ does not allow an embedding with vertices 1 and 8 on the boundary of one face.

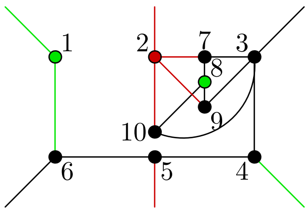
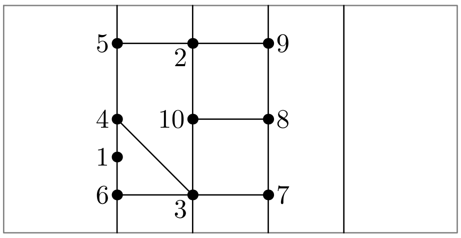


Figure 4.23: $E_6 - (1,2)$

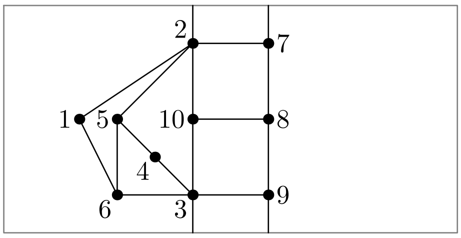
Every possible path from vertex 8 runs through vertex 3, consequently we cannot find a cycle including vertices 1 and 8, which could be the boundary of a face embedded on the Möbius strip. □

Lemma 4.12
The graph G_{75}, which is obtained by attaching the $K_5 - (v_1, v_2)$ to C_2 by identification of vertices v_1 and v_2 of the K_5 and a pair of vertices from orbit B in C_3, is an element of $M_4(\mathcal{S}_2)$.

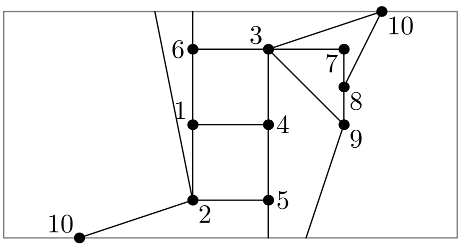
Proof
It is sufficient to show that the graphs $E_6 - e$, E_6/e with $e \in \{(1,2),(1,4),(2,7),(7,8)\}$ $R_3(E_6)_{(1)}$ and $R_3(E_6)_{(7)}$ can be drawn on the Möbius strip, and E_6 itself on the Klein surface with pairs of vertices of orbit B on the boundary of one face:



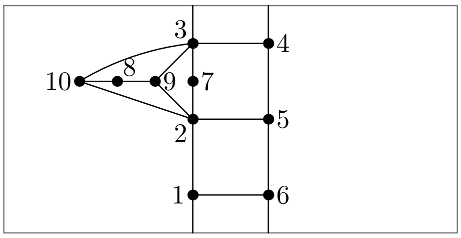
(1) Embedding of $E_6 - (1,2)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



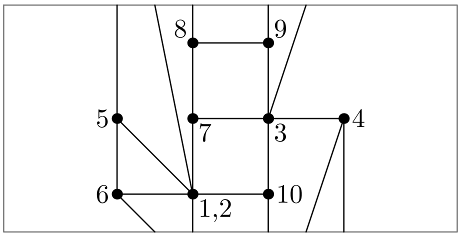
(2) Embedding of $E_6 - (1,4)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



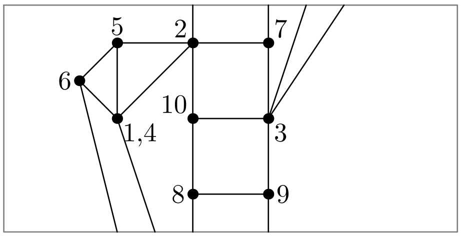
(3) Embedding of $E_6 - (2,7)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



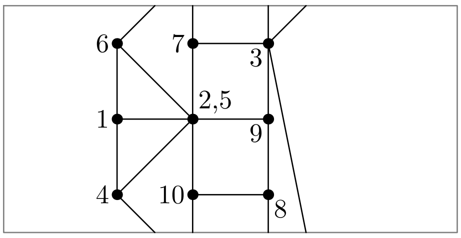
(4) Embedding of $E_6 - (7,8)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



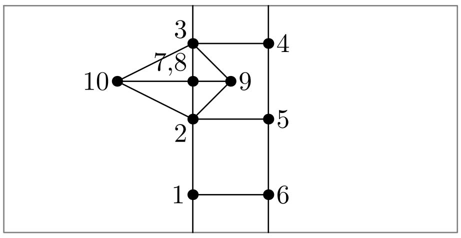
(5) Embedding of $E_6/(1,2)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



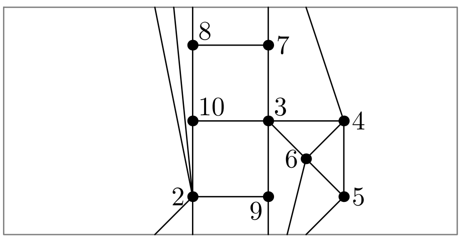
(6) Embedding of $E_6/(1,4)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



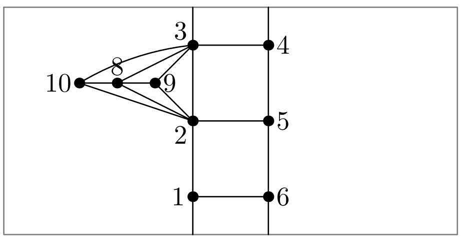
(7) Embedding of $E_6/(2,5)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



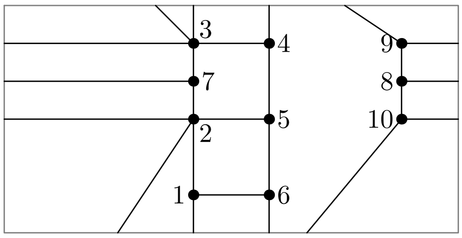
(8) Embedding of $E_6/(7,8)$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



(9) Embedding of $R_3(E_6)_{(1)}$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



(10) Embedding of $R_3(E_6)_{(7)}$, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.



(11) Embedding of E_6, vertex-pairs $\{2,8\}$ and $\{3,8\}$ on the boundary of a face.

Figure 4.24: (1)-(10): Embeddings of $R_i(E_6)$, $i \in \{0,2,3\}$, into the Möbius strip, (11): Embedding of E_6 into the Klein surface

Consequently G_{75} is an element of $M_4(\mathcal{S}_2)$. $\square$

The graph E_{11}

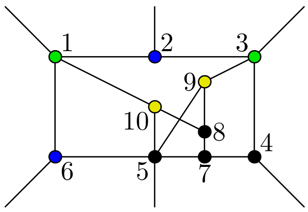


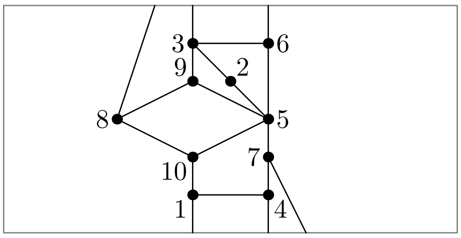
Figure 4.25: E_{11} and its vertex-orbits

The only orbits or pairs of vertices we have to consider for the graph E_{11} are:

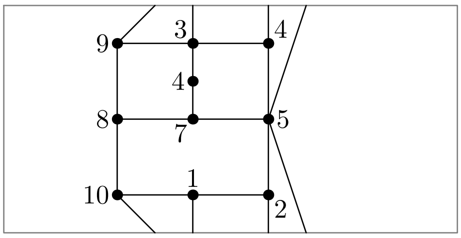
$$A = \Big\{\{1,8\},\{3,8\}\Big\}, \qquad B = \Big\{\{2,8\},\{6,8\}\Big\},$$
$$C = \Big\{\{4,8\}\Big\}, \qquad D = \Big\{\{5,8\}\Big\}.$$

Lemma 4.13
The graphs G_{76}, G_{77} and G_{78}, which can be obtained by identification of vertices v_1 and v_2 in $K_5 - (v_1, v_2)$ with a pair of vertices from orbits A, C or D of E_{11}, are elements of $M_4(\mathcal{S}_2)$.

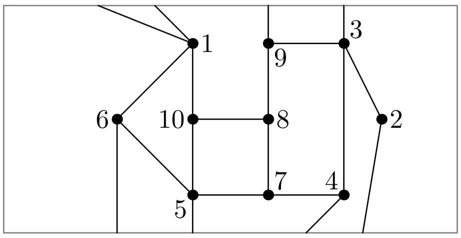
Proof
As can be seen in Figure 4.26, the graphs $E_{11} - e$, E_{11}/e, $e \in \{(1,2), (1,4), (2,5), (3,9), (4,7), (5,7), (7,8), (8,9)\}$, and $R_3(E_{11})_{(j)}$ for $j \in \{2,4,7,9\}$ can be embedded into the Möbius strip as well as E_{11} itself into the Klein surface, in a way that the pairs of vertices from orbits A, C and D each lie on the boundary of one face:



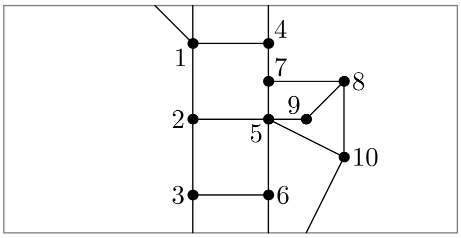
(1) Embedding of $E_{11}-(1,2)$, each vertex-pair from orbits A, C and D on the boundary of a face.



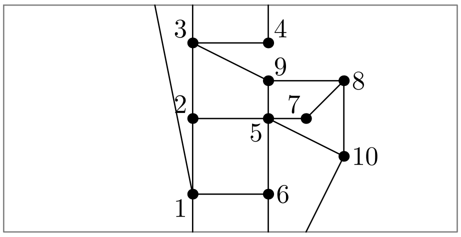
(2) Embedding of $E_{11}-(1,4)$, each vertex-pair from orbits A, C and D on the boundary of a face.



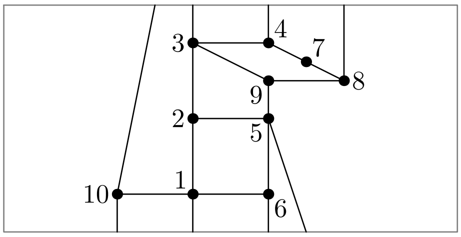
(3) Embedding of $E_{11}-(2,5)$, each vertex-pair from orbits A, C and D on the boundary of a face.



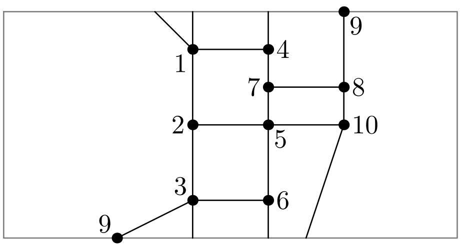
(4) Embedding of $E_{11}-(3,9)$, each vertex-pair from orbits A, C and D on the boundary of a face.



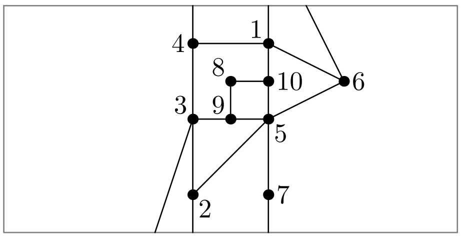
(5) Embedding of $E_{11}-(4,7)$, each vertex-pair from orbits A, C and D on the boundary of a face.



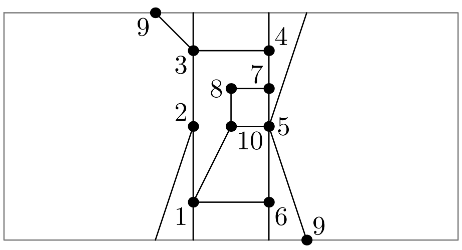
(6) Embedding of $E_{11}-(5,7)$, each vertex-pair from orbits A, C and D on the boundary of a face.



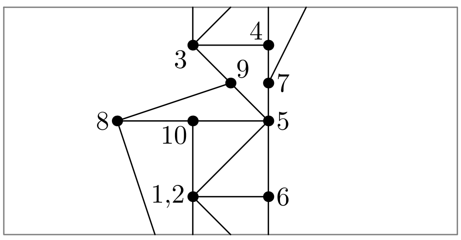
(7) Embedding of $E_{11}-(5,9)$, each vertex-pair from orbits A, C and D on the boundary of a face.



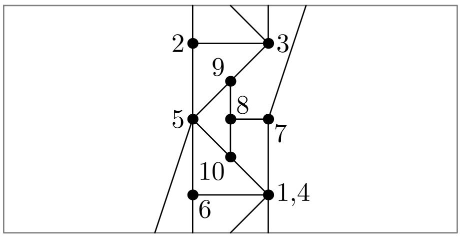
(8) Embedding of $E_{11}-(7,8)$, each vertex-pair from orbits A, C and D on the boundary of a face.



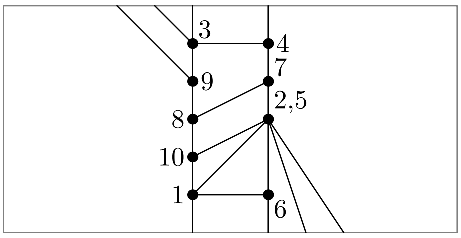
(9) Embedding of $E_{11}-(8,9)$, each vertex-pair from orbits A, C and D on the boundary of a face.



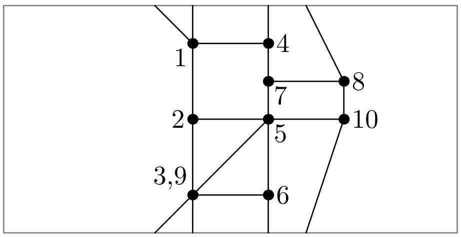
(10) Embedding of $E_{11}/(1,2)$, each vertex-pair from orbits A, C and D on the boundary of a face.



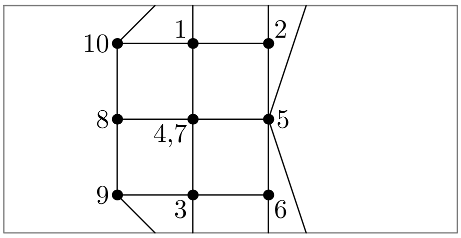
(11) Embedding of $E_{11}/(1,4)$, each vertex-pair from orbits A, C and D on the boundary of a face.



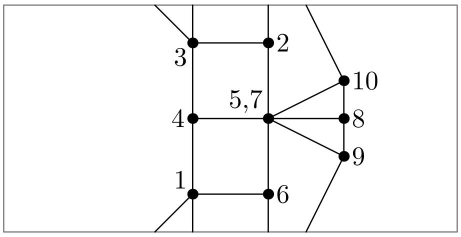
(12) Embedding of $E_{11}/(2,5)$, each vertex-pair from orbits A, C and D on the boundary of a face.



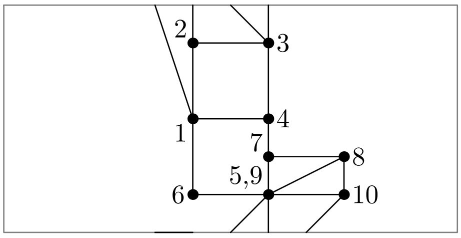
(13) Embedding of $E_{11}/(3,9)$, each vertex-pair from orbits A, C and D on the boundary of a face.



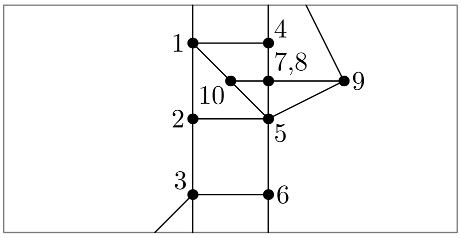
(14) Embedding of $E_{11}/(4,7)$, each vertex-pair from orbits A, C and D on the boundary of a face.



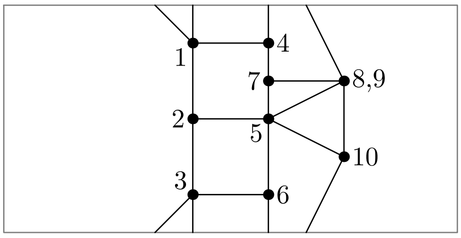
(15) Embedding of $E_{11}/(5,7)$, each vertex-pair from orbits A, C and D on the boundary of a face.



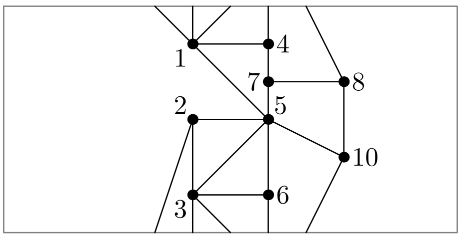
(16) Embedding of $E_{11}/(5,9)$, each vertex-pair from orbits A, C and D on the boundary of a face.



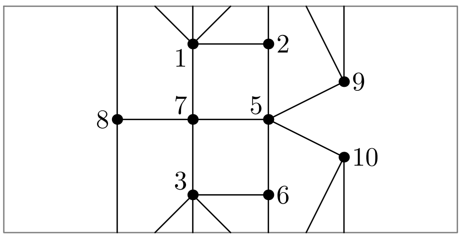
(17) Embedding of $E_{11}/(7,8)$, each vertex-pair from orbits A, C and D on the boundary of a face.



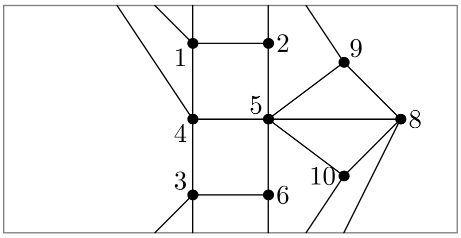
(18) Embedding of $E_{11}/(8,9)$, each vertex-pair from orbits A, C and D on the boundary of a face.



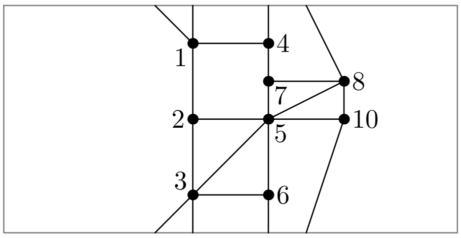
(19) Embedding of $R_3(E_{11})_{(2)}$, each vertex-pair from orbits A, C and D on the boundary of a face.



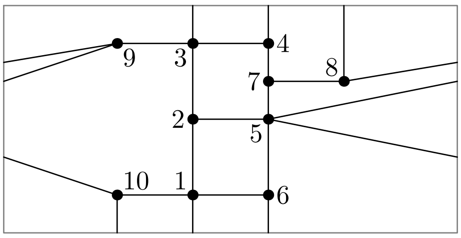
(20) Embedding of $R_3(E_{11})_{(4)}$, each vertex-pair from orbits A, C and D on the boundary of a face.



(21) Embedding of $R_3(E_{11})_{(7)}$ vertex-pairs $\{1,8\}$, $\{3,8\}$, $\{5,8\}$ on the boundary of a face.



(22) Embedding of $R_3(E_{11})_{(9)}$, each vertex-pair from orbits A, C and D on the boundary of a face.



(23) Embedding of E_{11}, each vertex-pair from orbits A, C and D on the boundary of a face.

Figure 4.26: (1)-(22): Embeddings of $R_i(E_{11})$, $i \in (1, 2, 3)$, into the Möbius strip, (23): Embedding of E_{11} into the Klein surface

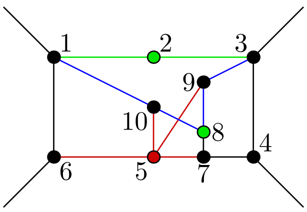
Consequently G_{76}, G_{77} and G_{78} are elements of $M_4(\mathcal{S}_2)$. □

Lemma 4.14

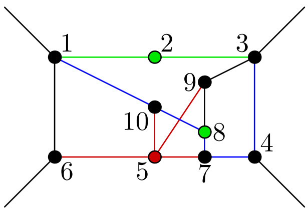
The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with a pair of vertices from orbit B of the graph E_{11} does not result in an element of $M_4(\mathcal{S}_2)$.

Proof

For $E_{11} - (2, 5)$ we have to consider two cycles, including vertices 2 and 8, as the boundary of a face on the Möbius strip:



(1) Case 1

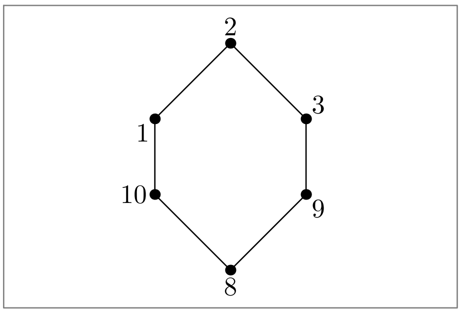


(2) Case 2

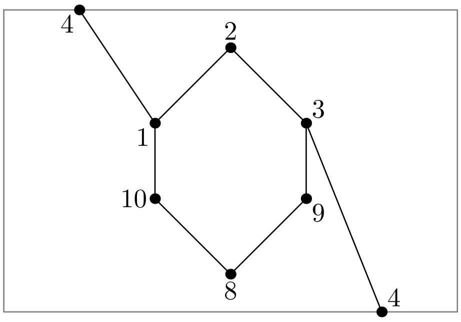
Figure 4.27: Cycles in $E_{11} - (2, 5)$ including vertices 2 and 8

All other cycles do not have to be considered following Lemma 1.11. Trying to embed the graph $E_{11} - (2, 5)$ with one of these two cycles as the boundary of one face delivers:

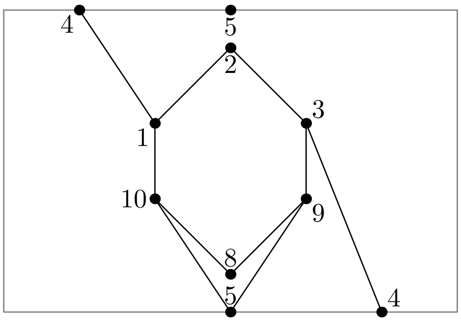
Embedding of Case 1:



(1) Embedding of the given cycle



(2) $1-4-3$ is embedded

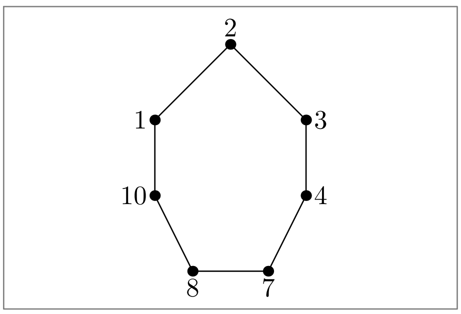


(3) $9-5-10$ is embedded

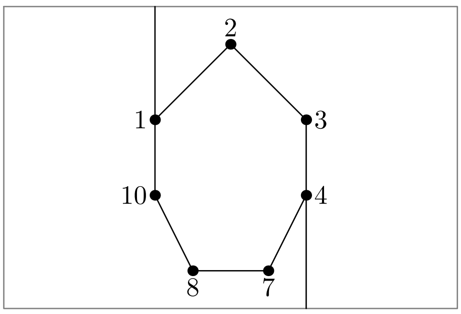
Figure 4.28: Embedding of Case 1

Figure 4.28 shows that an embedding, with the given restrictions, is not possible, as the path $4-7-8$ cannot be embedded.

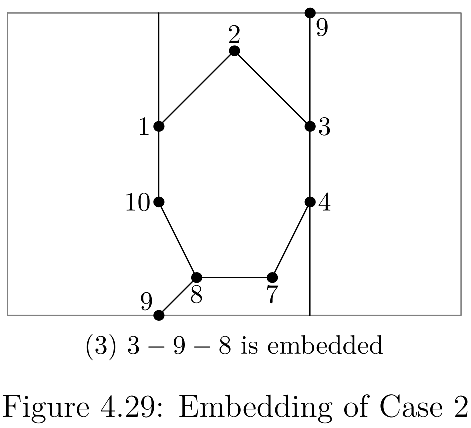
Embedding of Case 2:



(1) Embedding of the given cycle



(2) $1-4$ is embedded



(3) $3-9-8$ is embedded

Figure 4.29: Embedding of Case 2

Figure 4.29 shows that an embedding, with the given restrictions, is not possible, as the path $7-5-10$ cannot be embedded. □

The graph E_{20}

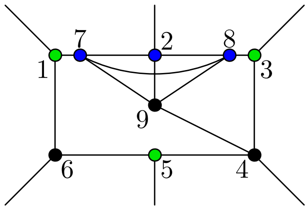


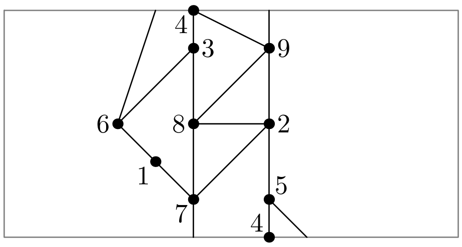
Figure 4.30: E_{20} and its vertex-orbits

The only orbits of pairs of vertices we have to consider for the graph E_{20} are:

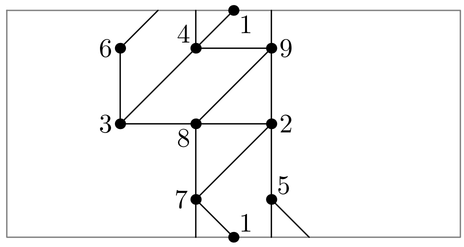
$$A = \Big\{\{2,6\},\{6,7\},\{6,8\}\Big\}, \qquad B = \Big\{\{4,6\}\Big\},$$
$$C = \Big\{\{6,9\}\Big\}.$$

Lemma 4.15
The graphs G_{79} and G_{80}, which can be obtained by identification of vertices v_1 and v_2 in $K_5-(v_1,v_2)$ with a pair of vertices from orbit A or B of E_{20}, are elements of $M_4(\mathcal{S}_2)$.

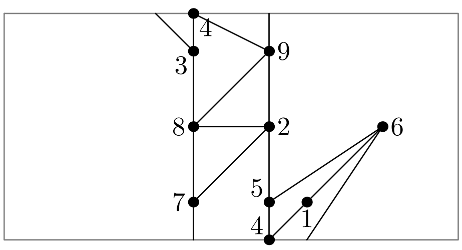
Proof
As can be seen in Figure 4.31, the graphs $E_{20}-e$, E_{20}/e for $e \in \{(1,4),\,(1,6),\,(1,7),\,(2,7),\,(2,9),\,(4,9)\}$ and $R_3(E_{11})_{(1)}$ can be embedded into the Möbius strip as well as E_{20} itself into the Klein surface, in a way that the pairs of vertices from orbits A and B each lie on the boundary of one face:



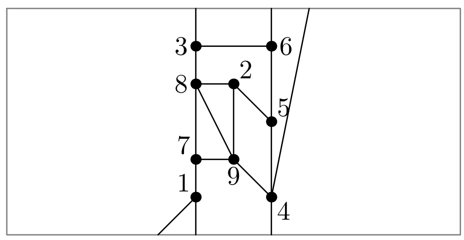
(1) Embedding of $E_{20}-(1,4)$, each vertex-pair of orbits A and B on the boundary of a face.



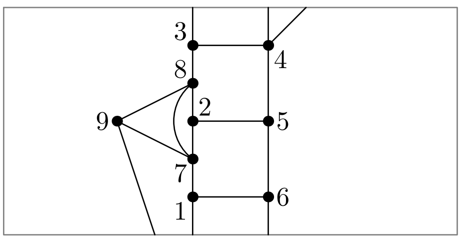
(2) Embedding of $E_{20}-(1,6)$, each vertex-pair of orbits A and B on the boundary of a face.



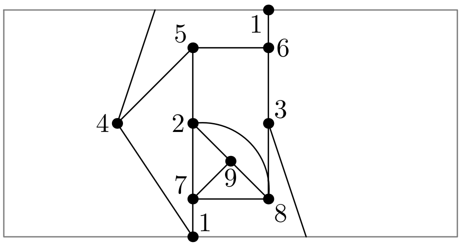
(3) Embedding of $E_{20}-(1,7)$, each vertex-pair of orbits A and B on the boundary of a face.



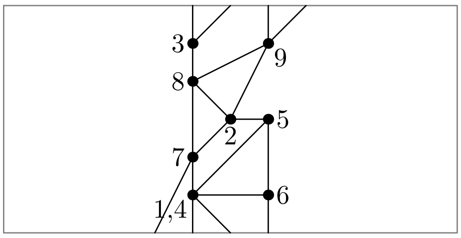
(4) Embedding of $E_{20}-(2,7)$, each vertex-pair of orbits A and B on the boundary of a face.



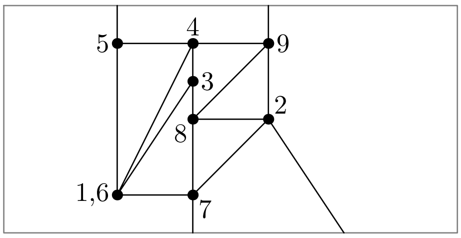
(5) Embedding of $E_{20}-(2,9)$, each vertex-pair of orbits A and B on the boundary of a face.



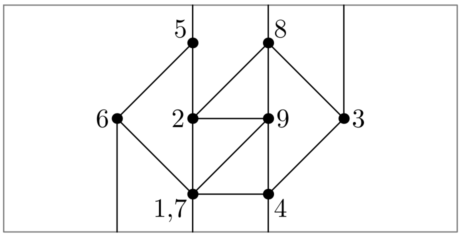
(6) Embedding of $E_{20}-(4,9)$, each vertex-pair of orbits A and B on the boundary of a face.



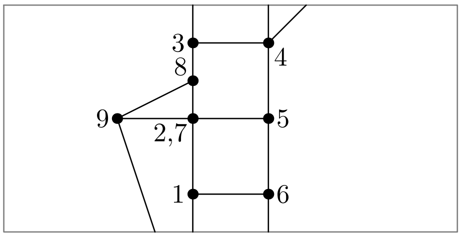
(7) Embedding of $E_{20}/(1,4)$, each vertex-pair of orbits A and B on the boundary of a face.



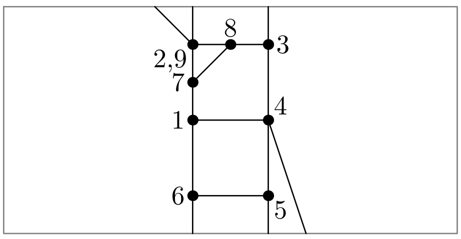
(8) Embedding of $E_{20}/(1,6)$, each vertex-pair of orbits A and B on the boundary of a face.



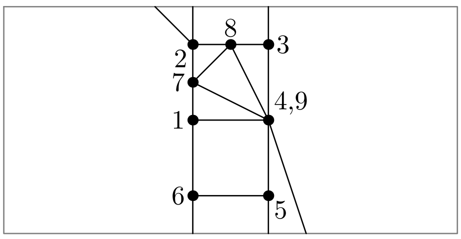
(9) Embedding of $E_{20}/(1,7)$, each vertex-pair of orbits A and B on the boundary of a face.



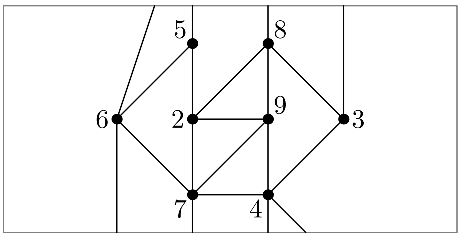
(10) Embedding of $E_{20}/(2,7)$, each vertex-pair of orbits A and B on the boundary of a face.



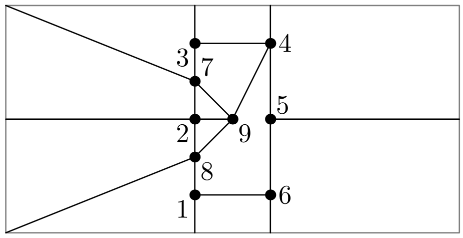
(11) Embedding of $E_{20}/(2,9)$, each vertex-pair of orbits A and B on the boundary of a face.



(12) Embedding of $E_{20}/(4,9)$, each vertex-pair of orbits A and B on the boundary of a face.



(13) Embedding of $R_3(E_{20})_{(1)}$, each vertex-pair of orbits A and B on the boundary of a face.



(14) Embedding of E_{20}

Figure 4.31: (1)-(13): Embeddings of $R_i(E_{20})$, $i \in (1, 2, 3)$, into the Möbius strip, (15): Embedding of E_{20} into the Klein surface

Consequently G_{79} and G_{80} are elements of $M_4(\mathcal{S}_2)$. □

Lemma 4.16
The construction of a graph, with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with pairs of vertices from orbit C of the graph E_{20}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
For the proof, we look at the graph $E_{20} - (4, 9)$ and try to find a cycle including vertices 6 and 9.

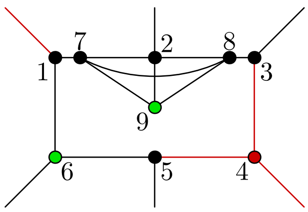


Figure 4.32: $E_{20} - (4, 9)$

The set of vertex-orbits of $E_{20} - (4, 9)$ is: $\big\{\{1,\ 3,\ 5\},\ \{2,\ 7,\ 8\},\ \{4,\ 6\}\big\}$. Consequently vertices 4 and 6 are isomorphic. An embedding of $E_{20} - (4, 9)$ with 6 and 9 on the boundary of one face consequently means that an embedding of $E_{20} - (4, 9)$ into the Möbius strip would also be possible with vertices 4 and 9 on the boundary of one face. This is a contradiction to the irreducibility of E_{20} for the projective plane. □

The graph E_{27}

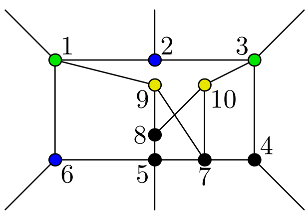


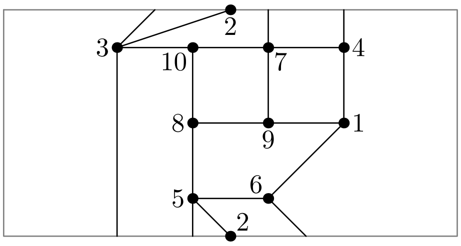
Figure 4.33: E_{27} and its vertex-orbits

The orbits of pairs of vertices we have to consider for the graph E_{27} are:

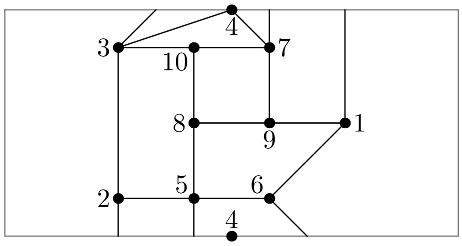
$$A = \Big\{\{1,8\},\{3,8\}\Big\}, \qquad B = \Big\{\{2,8\},\{6,8\}\Big\},$$
$$C = \Big\{\{4,8\}\Big\}, \qquad D = \Big\{\{7,8\}\Big\}.$$

Lemma 4.17
The graphs G_{81} and G_{82}, which can be obtained by identification of vertices v_1 and v_2 in $K_5 - (v_1, v_2)$ with a pair of vertices from orbit A or D of E_{27}, are elements of $M_4(\mathcal{S}_2)$.

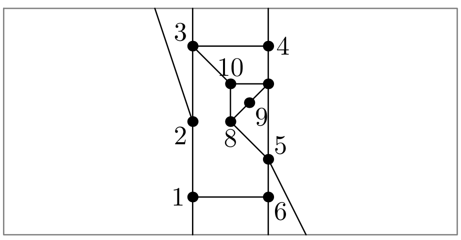
Proof
As can be seen in Figure 4.34, the graphs $E_{27} - e$, E_{27}/e for $e \in \{(1,2), (1,4), (1,9), (2,5), (4,7), (5,7), (5,8), (7,9), (8,9)\}$ and $R_3(E_{27})_{(j)}$ for $j \in \{2,4,9\}$ can be embedded into the Möbius strip as well as E_{27} itself into the Klein surface, in a way that the pairs of vertices from orbits A and D each lie on the boundary of one face:



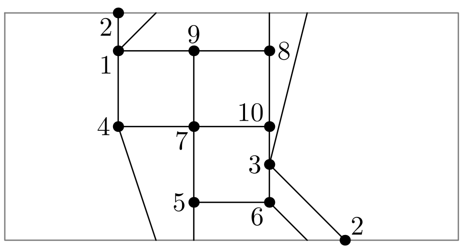
(1) Embedding of $E_{27}-(1,2)$, each vertex-pair of orbits A and D on the boundary of a face.



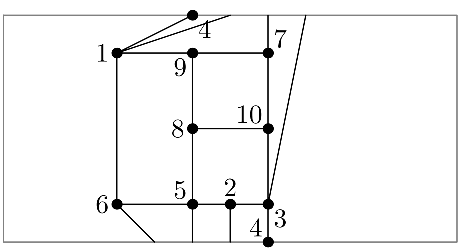
(2) Embedding of $E_{27}-(1,4)$, each vertex-pair of orbits A and D on the boundary of a face.



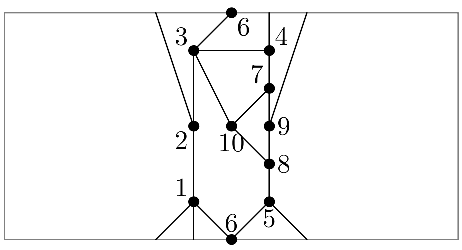
(3) Embedding of $E_{27}-(1,9)$, each vertex-pair of orbits A and D on the boundary of a face.



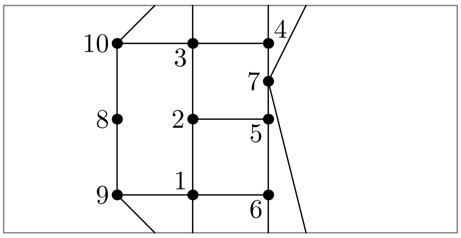
(4) Embedding of $E_{27}-(2.5)$, each vertex-pair of orbits A and D on the boundary of a face.



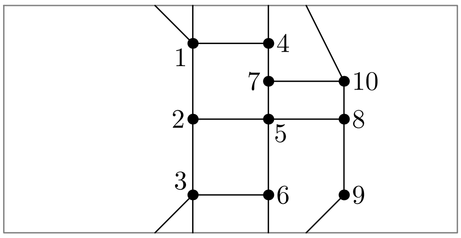
(5) Embedding of $E_{27}-(4,7)$, each vertex-pair of orbits A and D on the boundary of a face.



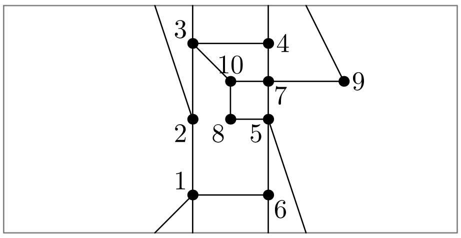
(6) Embedding of $E_{27}-(5,7)$, each vertex-pair of orbits A and D on the boundary of a face.



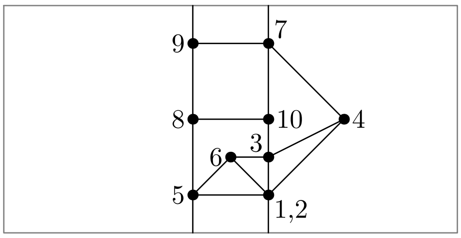
(7) Embedding of $E_{27}-(5,8)$, each vertex-pair of orbits A and D on the boundary of a face.



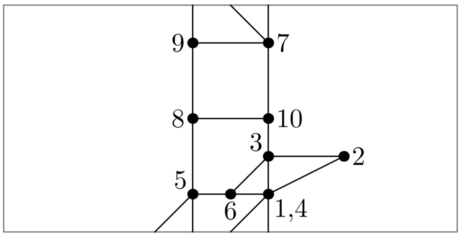
(8) Embedding of $E_{27}-(7,9)$, each vertex-pair of orbits A and D on the boundary of a face.



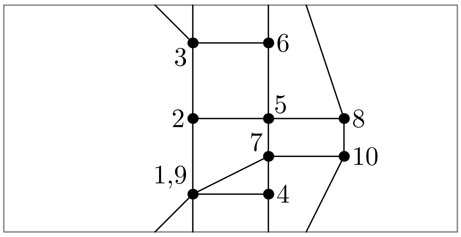
(9) Embedding of $E_{27}-(8,9)$, each vertex-pair of orbits A and D on the boundary of a face.



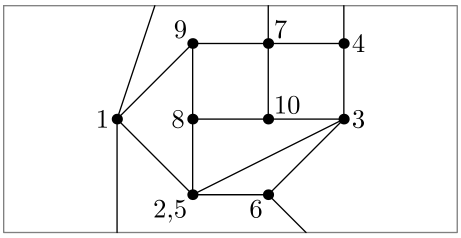
(10) Embedding of $E_{27}/(1,2)$, each vertex-pair of orbits A and D on the boundary of a face.



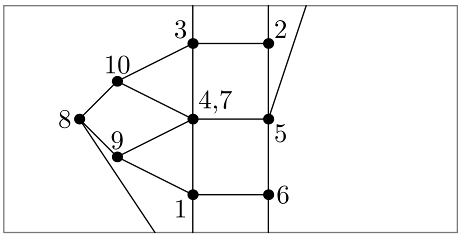
(11) Embedding of $E_{27}/(1,4)$, each vertex-pair of orbits A and D on the boundary of a face.



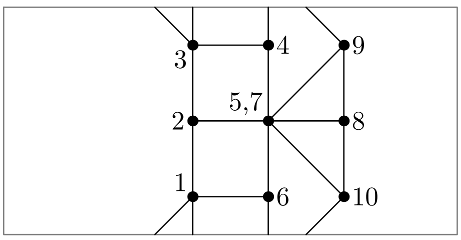
(12) Embedding of $E_{27}/(1,9)$, each vertex-pair of orbits A and D on the boundary of a face.



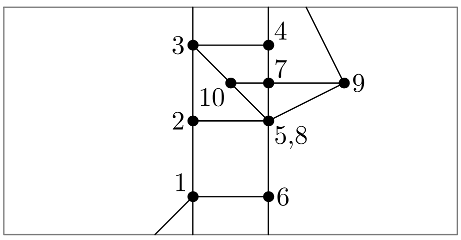
(13) Embedding of $E_{27}/(2,5)$, each vertex-pair of orbits A and D on the boundary of a face.



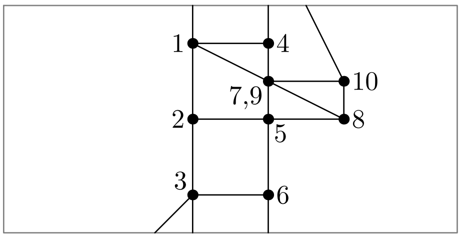
(14) Embedding of $E_{27}/(4,7)$, each vertex-pair of orbits A and D on the boundary of a face.



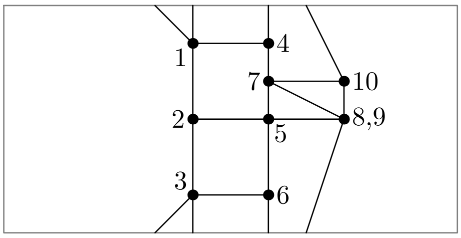
(15) Embedding of $E_{27}/(5,7)$, each vertex-pair of orbits A and D on the boundary of a face.



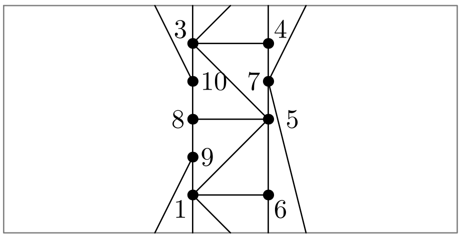
(16) Embedding of $E_{27}/(5,8)$, each vertex-pair of orbits A and D on the boundary of a face.



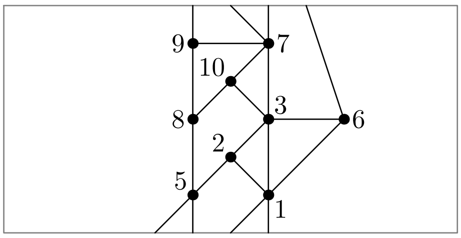
(17) Embedding of $E_{27}/(7,9)$, each vertex-pair of orbits A and D on the boundary of a face.



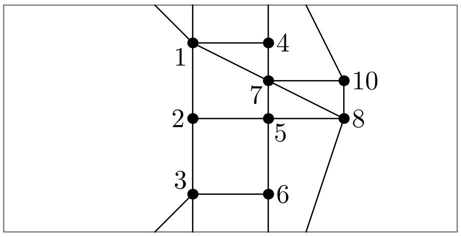
(18) Embedding of $E_{27}/(8,9)$, each vertex-pair of orbits A and D on the boundary of a face.



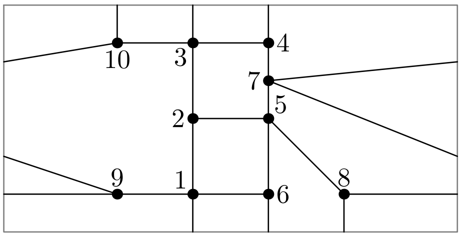
(19) Embedding of $R_3(E_{27})_{(2)}$, each vertex-pair of orbits A and D on the boundary of a face.



(20) Embedding of $R_3(E_{27})_{(4)}$, each vertex-pair of orbits A and D on the boundary of a face.



(21) Embedding of $R_3(E_{27})_{(9)}$, each vertex-pair of orbits A and D on the boundary of a face.



(22) Embedding of E_{27}, each vertex-pair of orbits A and D on the boundary of a face.

Figure 4.34: (1)-(21): Embeddings of $R_i(E_{27})$m $i \in (1,2,3)$, into the Möbius strip, (22): Embedding of E_{27} into the Klein surface

Consequently G_{81} and G_{82} are elements of $M_4(\mathcal{S}_2)$. □

Lemma 4.18

The construction of a graph with vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ identified with pairs of vertices from orbit B of the graph E_{27} does not result in an element of $M_4(\mathcal{S}_2)$.

Proof

Regarding the graph $E_{27} - (2,5)$ the only cycle we have to consider, using Lemma 1.11 and isomorphisms, is:

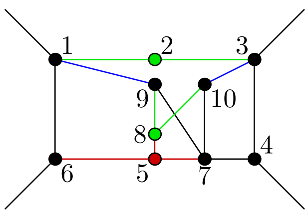
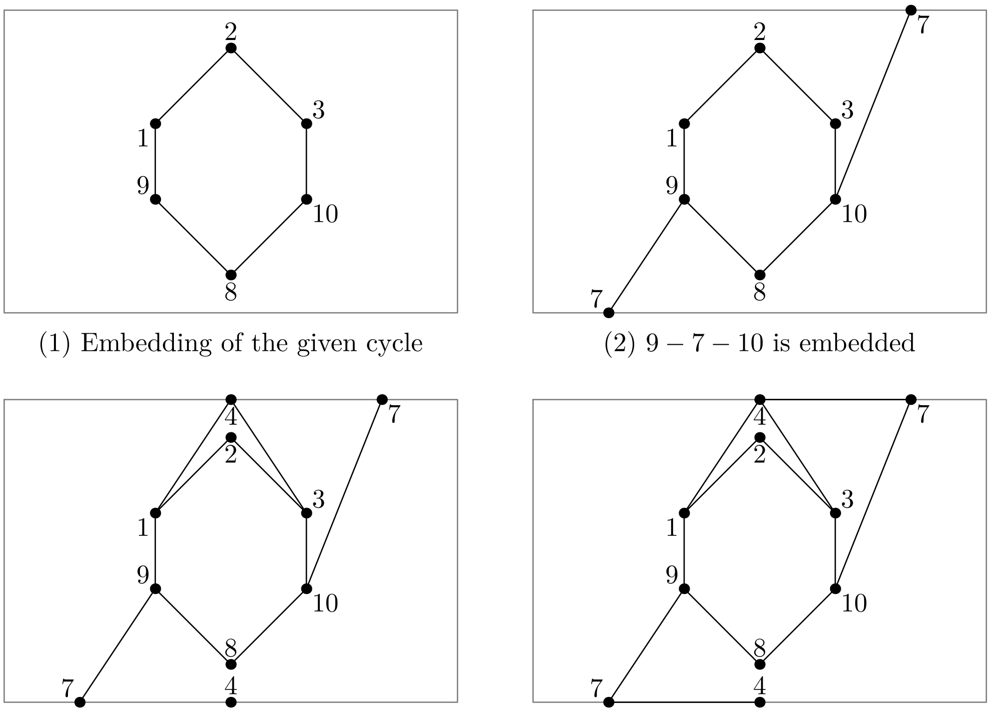


Figure 4.35: Cycle in $E_{27} - (2,5)$, including vertices 2 and 8

Trying to embed $E_{27} - (2,5)$ with this cycle as the boundary of a face on the Möbius strip does not work:



(1) Embedding of the given cycle

(2) $9 - 7 - 10$ is embedded

(3) $1 - 4 - 3$ is embedded

(4) $4 - 7$ is embedded

Figure 4.36: Embedding of $E_{27} - (2,5)$

Figure 4.36 shows that an embedding, with the given restrictions, is not possible, as the path $1 - 6 - 3$ cannot be embedded. □

Lemma 4.19

The construction of a graph, where vertices v_1 and v_2 of $K_5 - (v_1, v_2)$ are identified with pairs of vertices from orbit C of the graph E_{27}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof

Looking at the graph $E_{27} - (4,7)$ and trying to find a cycle including vertices 4 and 8 we already have to include certain edges into the cycle:

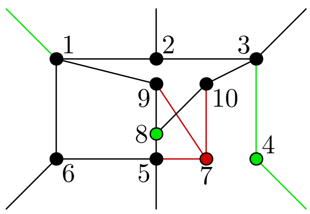


Figure 4.37: $E_{27} - (4,7)$, vertices 4 and 8 supposed to be included in one cycle

The set of vertex-orbits of E_{27} is: $\Big\{\{1, 3\}, \{2, 6\}, \{7, 8\}, \{9,10\}\Big\}$. Consequently vertices 7 and 8 could be exchanged. An embedding of E_{27} with vertices 4 and 8 on the boundary of one face consequently means that an embedding of E_{27} into the Möbius strip would also be possible with vertices 4 and 7 on the boundary of one face. This is a contradiction to the irreducibility of E_{27} for the projective plane. □

The graph F_4

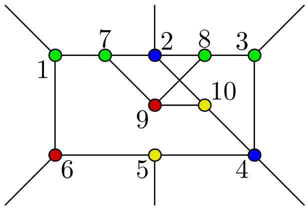
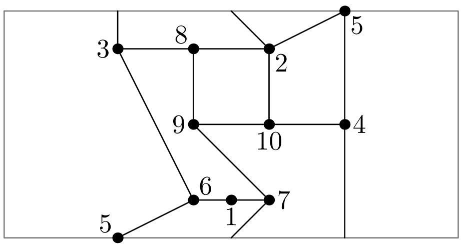


Figure 4.38: F_4 and its vertex-orbits

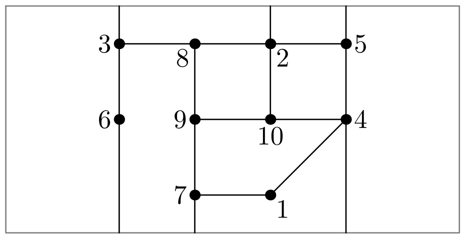
The only pair of vertices we have to consider for the graph F_4 is $\{6, 9\}$.

Lemma 4.20
The graph G_{83}, which is by identification of vertices v_1 and v_2 of the $K_5 - (v_1, v_2)$ and vertices 6 and 9 in F_4, is an element of $M_4(\mathcal{S}_2)$.

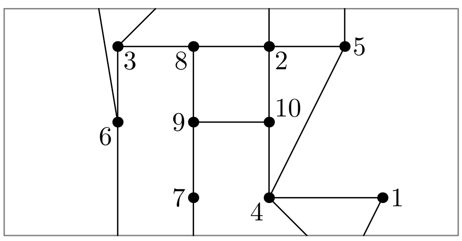
Proof
It is sufficient to show that the graphs $F_4 - e$, F_4/e with $e \in \{(1,4), (1,6), (1,7), (2,5), (2,10), (5,6)\}$, $R_3(F_4)_{(1)}$ and $R_3(F_4)_{(5)}$ can be embedded into the Möbius strip, and F_4 itself into the Klein surface with vertices 6 and 9 on the boundary of one face:



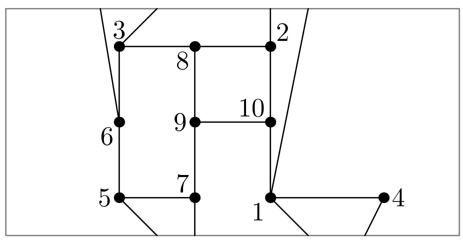
(1) Embedding of $F_4 - (1,4)$, vertices 6 and 9 on the boundary of one face.



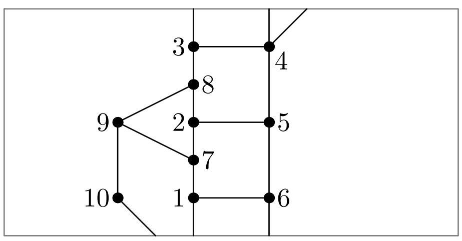
(2) Embedding of $F_4 - (1,6)$, vertices 6 and 9 on the boundary of one face.



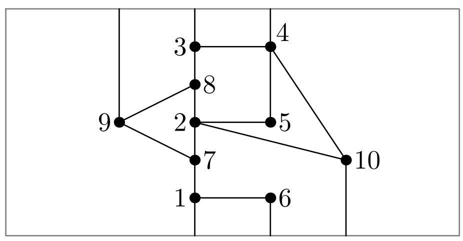
(3) Embedding of $F_4 - (1,7)$, vertices 6 and 9 on the boundary of one face.



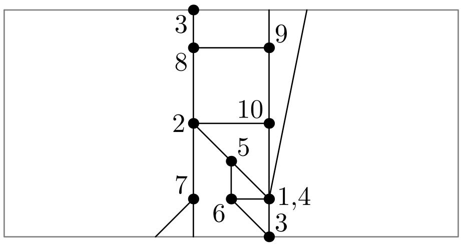
(4) Embedding of $F_4 - (2,5)$, vertices 6 and 9 on the boundary of one face.



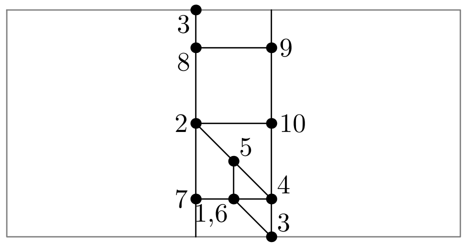
(5) Embedding of $F_4 - (2,10)$, vertices 6 and 9 on the boundary of one face.



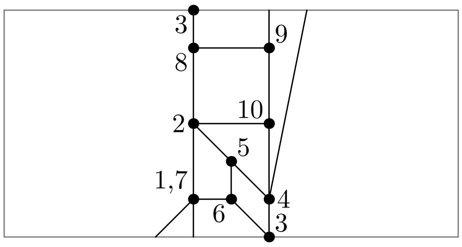
(6) Embedding of $F_4 - (5,6)$, vertices 6 and 9 on the boundary of one face.



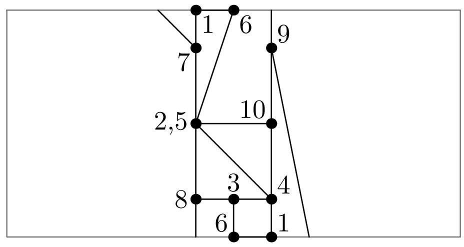
(7) Embedding of $F_4/(1,4)$, vertices 6 and 9 on the boundary of one face.



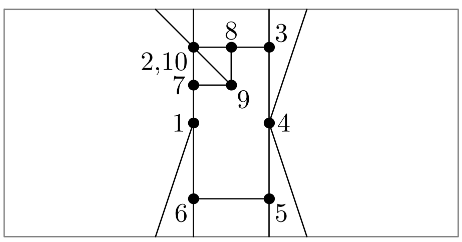
(8) Embedding of $F_4/(1,6)$, vertices 6 and 9 on the boundary of one face.



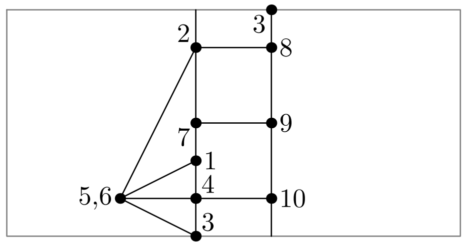
(9) Embedding of $F_4/(1,7)$, vertices 6 and 9 on the boundary of one face.



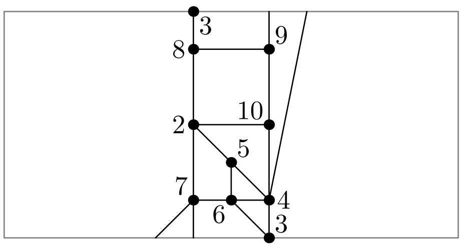
(10) Embedding of $F_4/(2,5)$, vertices 6 and 9 on the boundary of one face.



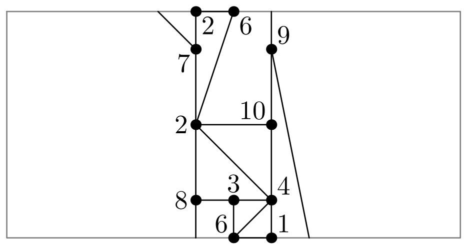
(11) Embedding of $F_4/(2,10)$, vertices 6 and 9 on the boundary of one face.



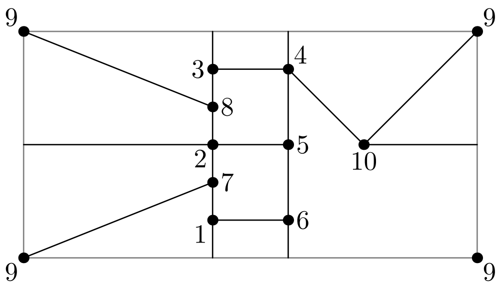
(12) Embedding of $F_4/(5,6)$, vertices 6 and 9 on the boundary of one face.



(13) Embedding of $R_3(F_4)_{(1)}$, vertices 6 and 9 on the boundary of one face.



(14) Embedding of $R_3(F_4)_{(5)}$, vertices 6 and 9 on the boundary of one face.



(15) Embedding of F_4, vertices 6 and 9 on the boundary of one face.

Figure 4.39: (1)-(14): Embeddings of $R_i(F_4)$, $i \in (1,2,3)$, into the Möbius strip, (15): Embedding of F_4 into the Klein surface

Consequently G_{83} is an element of $M_4(\mathcal{S}_2)$. $\square$

4.3 Graphs including three base-points

In this section we will look at the remaining graphs, which still have to be considered for the attachment of the $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ to three of their vertices.

Proposition 4.21
None of the graphs constructed by attachment of a $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ to three vertices in $M_2(\mathcal{S}_1) - M_3(\mathcal{S}_1)$, is in $M_4(\mathcal{S}_2)$.

The graph C_2

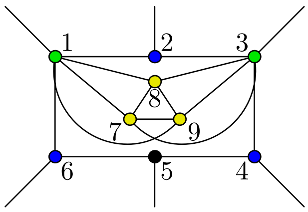


Figure 4.40: C_2 and its vertex-orbits

The only triple of vertices we have to consider for attachments to the graph C_2 is: $\{1, 3, 5\}$.

Lemma 4.22
The construction of a graph, where vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ are identified with vertices $1, 3$ and 5 in B_7, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
The only cycle including vertices $1, 3$ and 5, we have to look at using Lemma 1.11 and isomorphisms is:

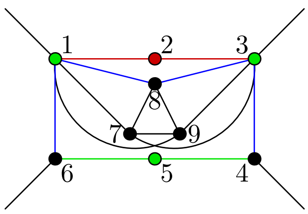
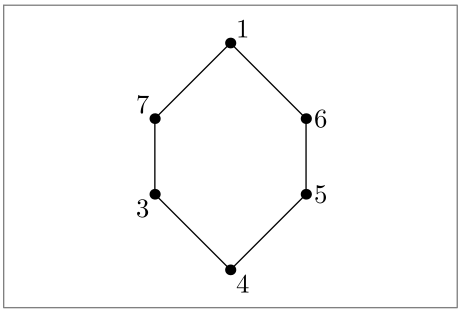
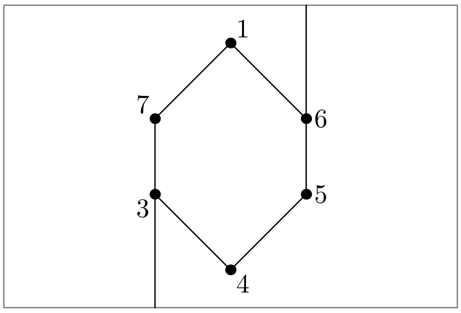


Figure 4.41: Cycle in $C_2 - (2, 5)$, including vertices $1, 3$ and 5

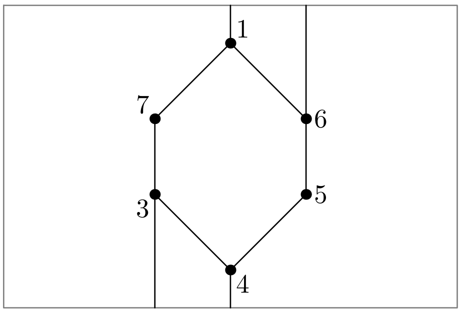
Trying to embed this cycle as the boundary of a face when embedding $C_2 - (2, 5)$ into the Möbius strip is not possible:



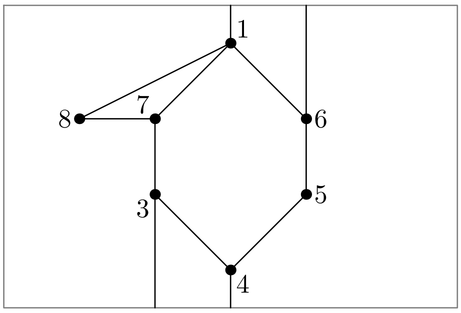
(1) Embedding of the given cycle



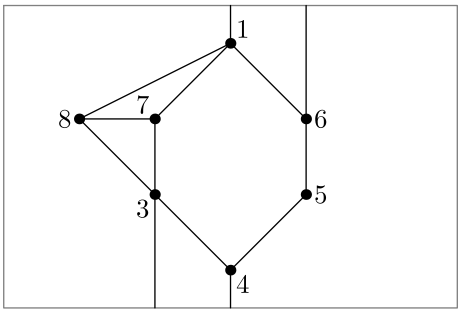
(2) $3 - 6$ is embedded



(3) $1 - 4$ is embedded



(4) $1 - 8 - 7$ is embedded



(5) $3 - 8$ is embedded

Figure 4.42: Embedding of $C_2 - (2, 5)$

Figure 4.42 shows that an embedding, with the given restrictions, is not possible, as vertex 9 cannot be embedded. □

The graph C_3

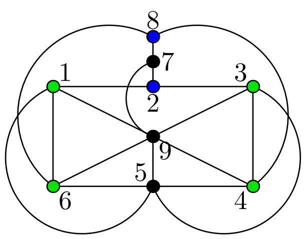


Figure 4.43: C_3 and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph C_3 are:

$$A = \Big\{\{2,5,8\}\Big\}, \qquad\qquad B = \Big\{\{2,8,9\}\Big\}.$$

Lemma 4.23
The graph, where vertices v_1, v_2 and v_3 of $K_5 - (v_1,v_2) - (v_1,v_3) - (v_2,v_3)$ are identified with triples of vertices from orbit A of the graph C_3, cannot be an element of $M_4(\mathcal{S}_2)$.

Proof
We have a look at the graph $C_3 - (2,7)$: The only cycle, which includes vertices $2, 5$ and 8, we have to consider for this case, is:

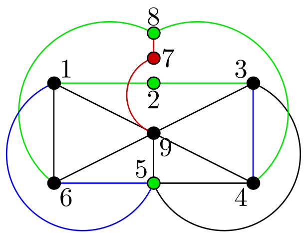
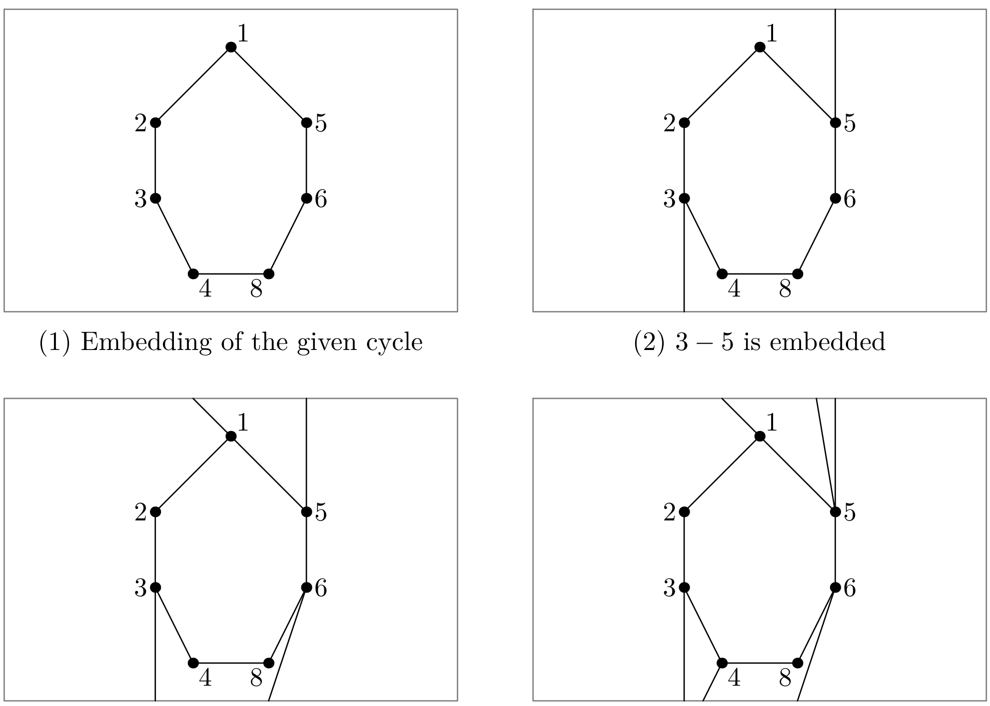


Figure 4.44: Cycle in $C_3 - (2,7)$, including vertices $2, 5$ and 8

Trying to embed this cycle as the boundary of one face of the graph $C_3 - (2,7)$ on the Möbius strip results in:



(1) Embedding of the given cycle

(2) $3-5$ is embedded

(3) $1-6$ is embedded

(4) $4-5$ is embedded

Figure 4.45: Embedding of $C_3 - (2,7)$

Figure 4.45 shows that an embedding, with the given restrictions, is not possible, as vertex 9 cannot be embedded. $\square$

Lemma 4.24
The construction of a graph, where vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ are identified with triples of vertices from orbit B of the graph C_3, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
We consider the graph $C_3 - (7, 9)$. The only cycle we have to try to embed as the boundary of one face is:

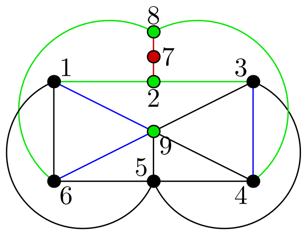
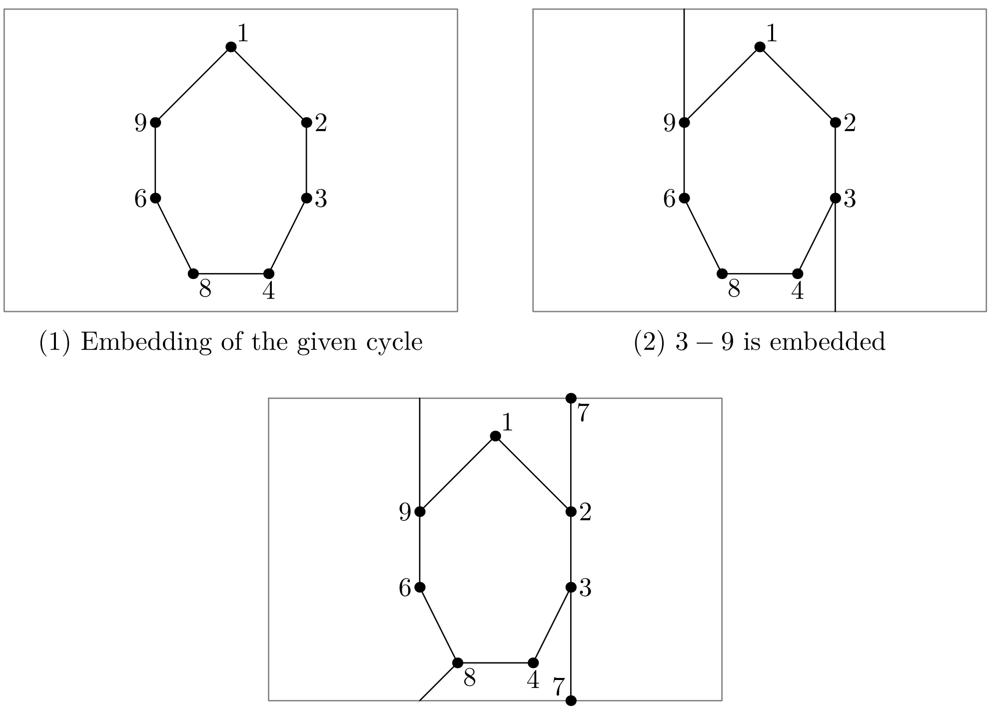


Figure 4.46: Cycle in $C_3 - (7, 9)$, including vertices $2, 8$ and 9

The embedding of this cycle as the boundary of one face does not work:



(1) Embedding of the given cycle

(2) $3 - 9$ is embedded

(3) $2 - 7 - 8$ is embedded

Figure 4.47: Embedding of $C_3 - (7, 9)$

Figure 4.14 shows that an embedding, with the given restrictions, is not possible, as the path $1 - 6$ cannot be embedded. $\square$

The graph D_2

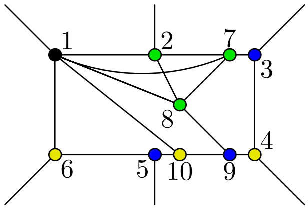


Figure 4.48: D_2 and its vertex-orbits

The only triple of vertices we have to consider for the graph D_2 is: $\{3, 5, 9\}$.

Lemma 4.25
The construction of a graph, where vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ are identified with vertices $3, 5$ and 9 in D_2, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
Regarding the graph $D_2 - (2, 7)$ there is only one cycle including vertices $3, 5$ and 9, which has to be considered:

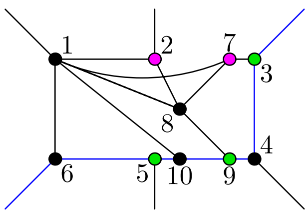
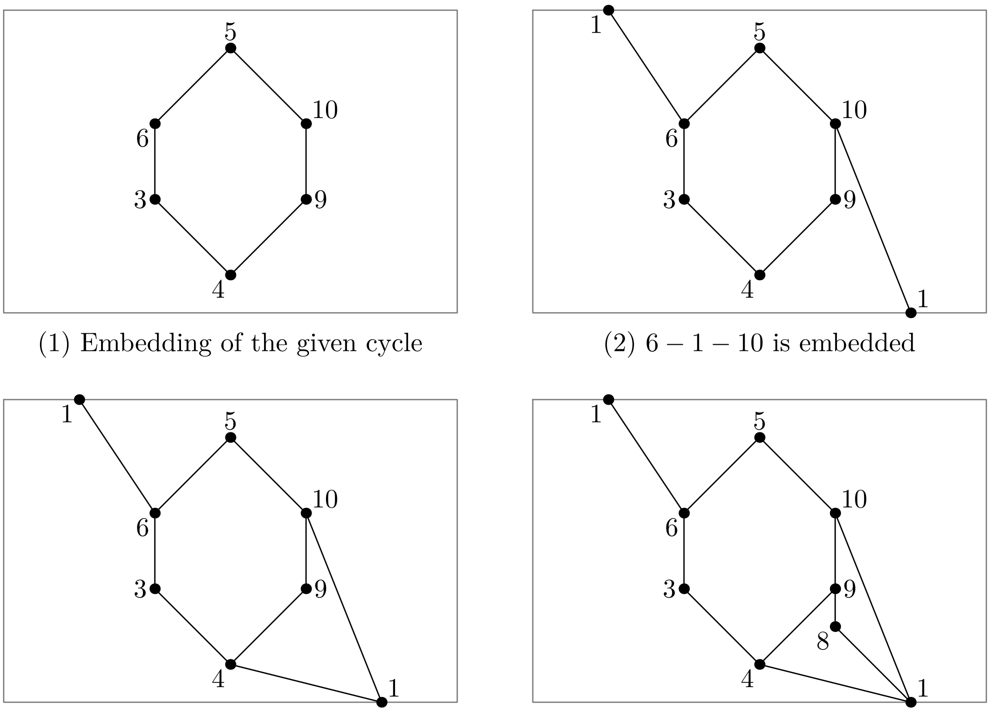


Figure 4.49: Cycle in $D_2 - (2, 7)$, including vertices $3, 5$ and 9

Trying to embed this cycle as the boundary of one face of $D_2 - (2,7)$ on the Möbius strip delivers:



(1) Embedding of the given cycle

(2) $6 - 1 - 10$ is embedded

(3) $1 - 4$ is embedded

(4) $1 - 8 - 9$ is embedded

Figure 4.50: Embedding of $D_2 - (2,7)$

Figure 4.50 shows that an embedding, with the given restrictions, is not possible, as vertex 2 cannot be embedded. □

The graph E_6

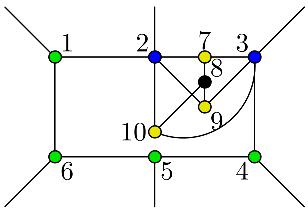


Figure 4.51: E_6 and its vertex-orbits

The only triple of vertices we have to consider for the graph E_6 is: $\{2,3,8\}$.

Lemma 4.26
The construction of a graph with vertices v_1, v_2 and v_3 of $K_5 - (v_1,v_2) - (v_1,v_3) - (v_2,v_3)$ identified with vertices $2,3$ and 8 in E_6 does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
Regarding the graph $E_6 - (2,7)$ there is only one cycle which includes vertices $2,3$ and 8, that has to be considered:

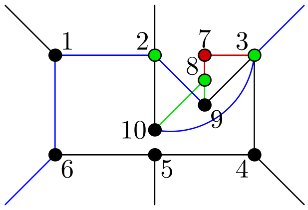
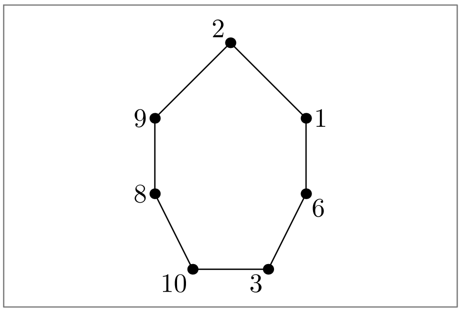
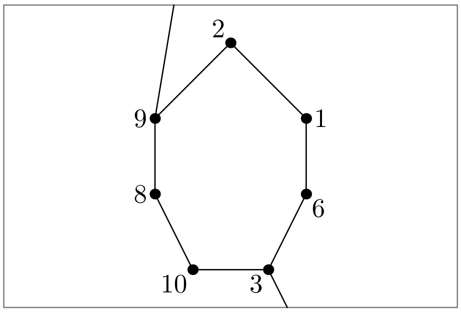


Figure 4.52: Cycle in $E_6 - (2,7)$, including vertices $2, 3$ and 8

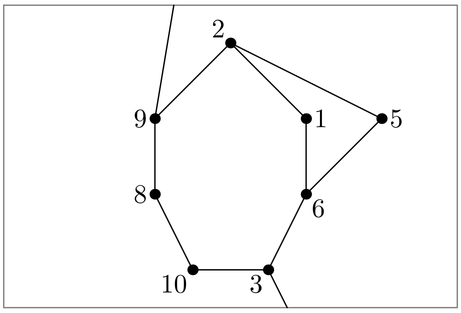
Trying to embed this cycle as the boundary of one face of $E_6 - (2,7)$ on the Möbius strip results in a contradiction.



(1) Embedding of the given cycle



(2) $3 - 9$ is embedded



(3) $2 - 5 - 6$ is embedded

Figure 4.53: Embedding of $E_6 - (2,7)$

Figure 4.53 shows that an embedding, with the given restrictions, is not possible, as the path $1 - 4 - 3$ cannot be embedded. $\square$

The graph E_{11}

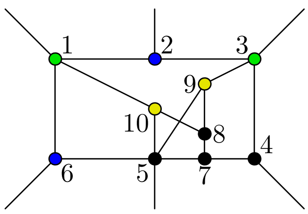


Figure 4.54: E_{11} and its vertex-orbits

The orbits of triples of vertices we have to consider for the graph E_{11} are:

$$A = \Big\{\{1,3,8\}\Big\}, \qquad B = \Big\{\{1,5,8\},\{3,5,8\}\Big\},$$
$$C = \Big\{\{4,5,8\}\Big\}.$$

Lemma 4.27
The construction of a graph with vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ identified with vertices 1, 3 and 8 of the graph E_{11} does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
We are using the graph $E_{11} - (1,4)$ for this case. The smallest cycle which includes vertices 1, 3 and 8, is of length six. We also do not have to consider a cycle of length greater then six because: Let us assume that the cycle has length eight (a cycle of length seven containing all three vertices does not exist). Using the euler-criterion (Lemma 1.15), we know that the embedding consists of seven faces. Consequently we have to find six cycles of length four, which form the boundaries of the remaining six faces of the embedding and which also have to cover all vertices missing in the cycle of length eight. As the cycle of length eight has to include vertex 1 it cannot include vertex 4 (Corollary 1.13). None of the cycles of length four includes vertex 4. Consequently we cannot find six cycles of length four which would complete the embedding of $E_{11} - (1,4)$ when beginning with a cycle of length eight. The only cycle we have to look at for this graph, due to the above mentioned arguments and isomorphisms, is:

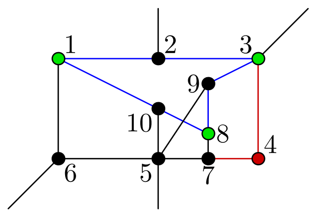
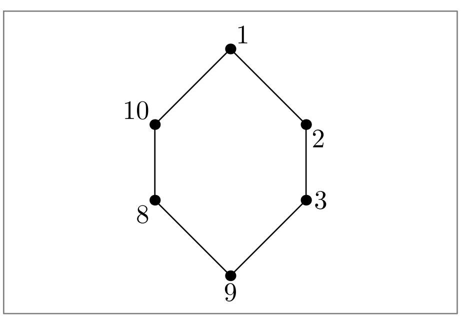
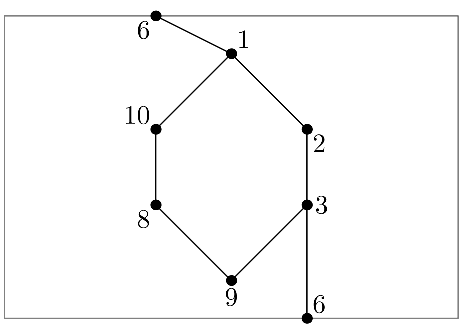


Figure 4.55: Cycle in $E_{11} - (1,4)$, including vertices $1,3$ and 8

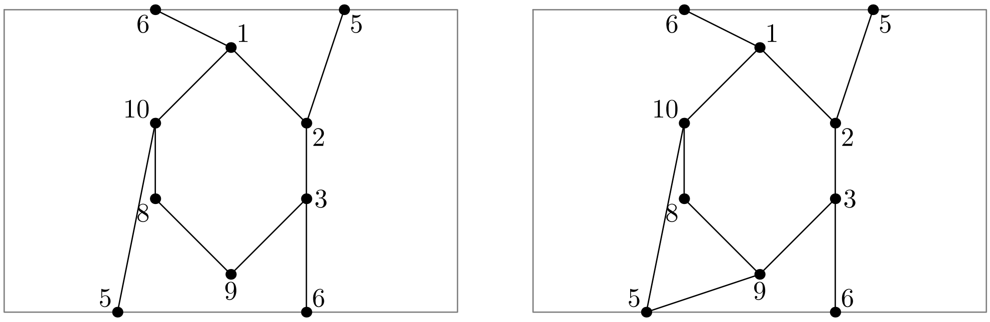
We try to embed the graph starting with this cycle as the boundary of one face:



(1) Embedding of the given cycle



(2) $1-6-3$ is embedded



(3) $2-5-10$ is embedded (4) $5-9$ is embedded

Figure 4.56: Embedding of $E_{11}-(1,4)$

Figure 4.56 shows that an embedding, with the given restrictions, is not possible, as the path $8-7-4-3$ cannot be embedded. □

Lemma 4.28
The construction of a graph, where vertices v_1, v_2 and v_3 of $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ are identified with triples of vertices from orbit B of the graph E_{11}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
In this case we use the graph $E_{11}-(7,8)$ for our considerations. The same arguments as in the proof of Lemma 4.27 apply for the size of the cycles of length eight, only that in this case vertex 7 is the one which is not represented in the cycles of length four. All cycles of length seven include vertex 7 and thus also do not have to be considered. The only cycle of length six we have to consider is:

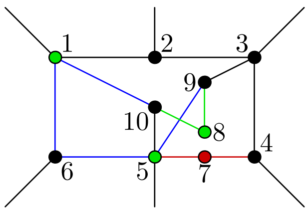
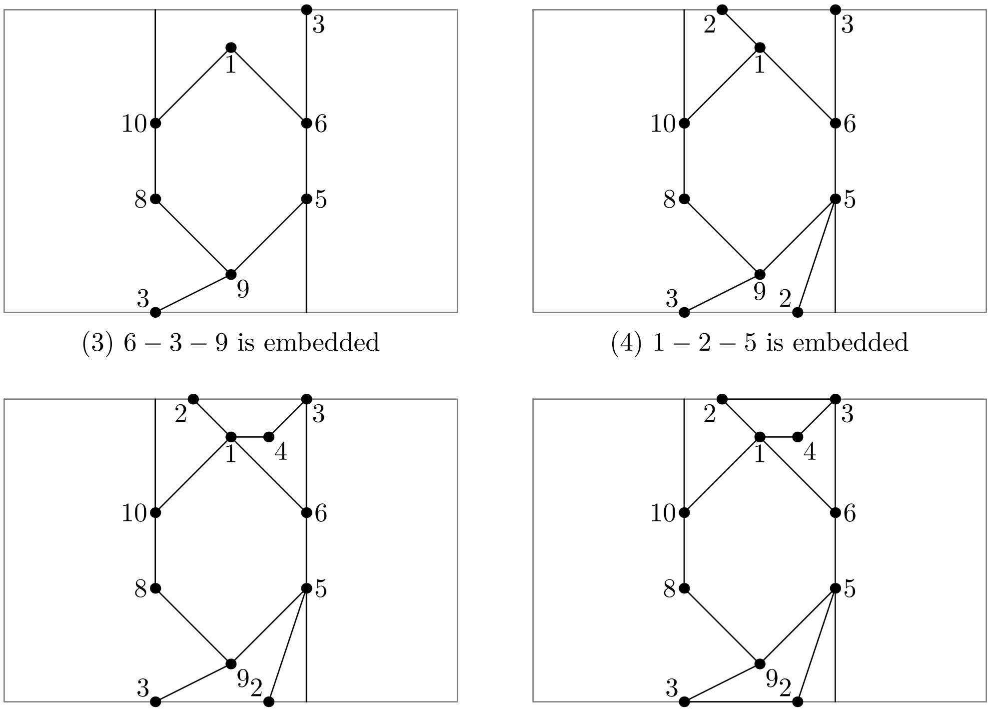


Figure 4.57: Cycle in $E_{11}-(7,8)$, including vertices $1,5$ and 8

Trying to embed the graph starting with this cycle as the boundary of one face delivers:

(1) Embedding of the given cycle (2) $5-10$ is embedded



(3) $6-3-9$ is embedded

(4) $1-2-5$ is embedded

(5) $1-4-3$ is embedded

(6) $2-3$ is embedded

Figure 4.58: Embedding of $E_{11}-(7,8)$

Figure 4.58 shows that an embedding, with the given restrictions, is not possible, as the path $4-7-5$ cannot be embedded. $\square$

Lemma 4.29

The graph, where vertices v_1, v_2 *and* v_3 *of* $K_5-(v_1,v_2)-(v_1,v_3)-(v_2,v_3)$ *are identified with vertices* $4,5$ *and* 8 *in* E_{11}, *is not an element of* $M_4(\mathcal{S}_2)$.

Proof

The only cycle in $E_{11}-(4,7)$, which includes vertices $4,5$ and 8, we have to consider due to isomorphisms is:

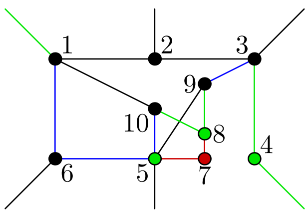
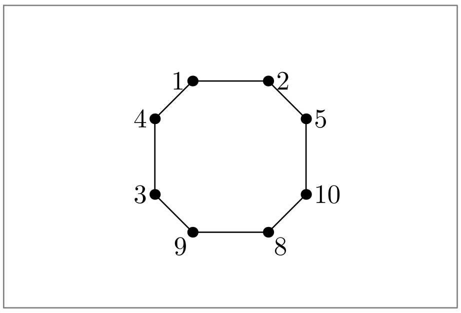
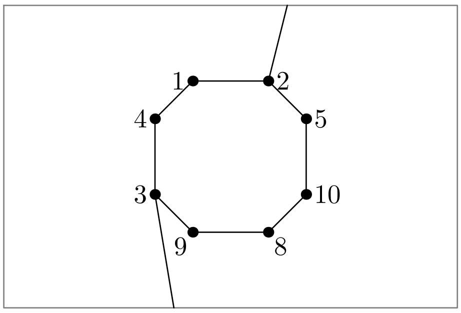


Figure 4.59: Cycle in $E_{11}-(4,7)$, including vertices $4,5$ and 8

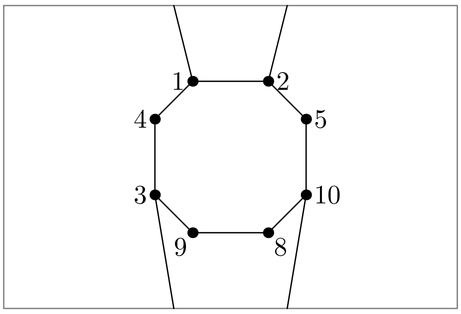
We try to embed the graph starting with this cycle as the boundary of one face:



(1) Embedding of the given cycle



(2) $2 - 3$ is embedded



(3) $1 - 10$ is embedded

Figure 4.60: Embedding of $E_{11} - (4, 7)$

Figure 4.60 shows that an embedding, with the given restrictions, is not possible, as the path $5 - 9$ cannot be embedded. □

The graph E_{20}

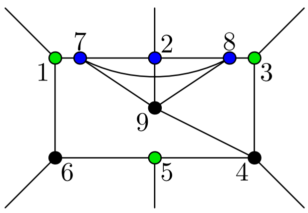


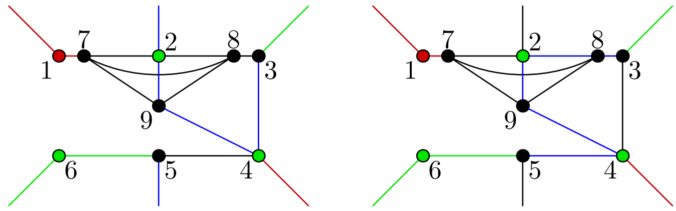
Figure 4.61: E_{20} and its vertex-orbits

The only orbit of triples of vertices we have to consider for the graph E_{11} is:

$$A = \Big\{\{2, 4, 6\}, \{4, 6, 7\}, \{4, 6, 8\}\Big\}.$$

Lemma 4.30
The construction of a graph, where vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ are identified with triples of vertices from orbit A of the graph E_{20}, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof
We only have to consider two cycles in $E_{20} - (1,6)$ as the boundary of a face on the Möbius strip, which contain vertices $2, 4$ and 6:

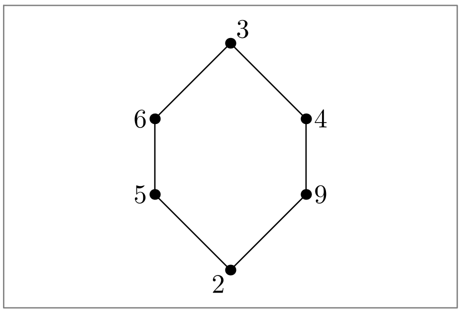


(1) Case 1 (2) Case 2

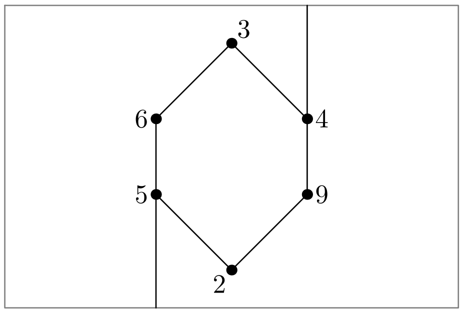
Figure 4.62: Cycles in $E_{20} - (1,6)$ including vertices $2, 4$ and 6

All other cycles do not have to be considered following Lemma 1.11.
Trying to embed the graph $E_{11} - (1,6)$ with these two cycles as the boundary of one face delivers:

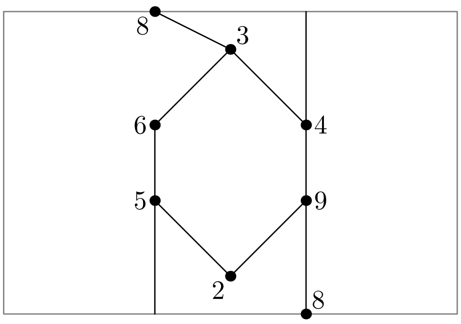
Embedding of Case 1:



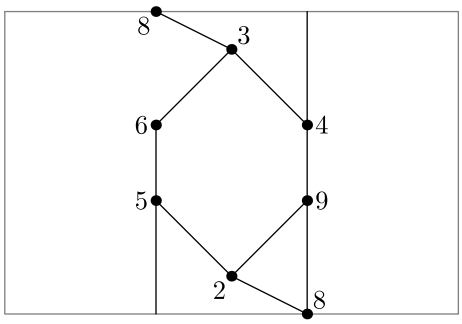
(1) Embedding of the given cycle



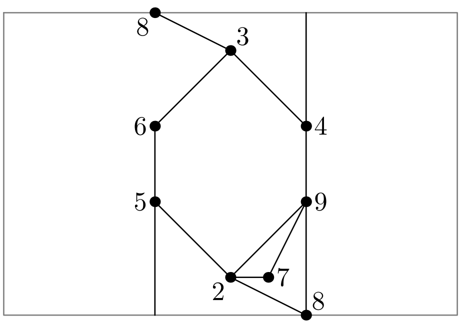
(2) $4 - 5$ is embedded



(3) $3 - 8 - 9$ is embedded



(4) $2 - 8$ is embedded

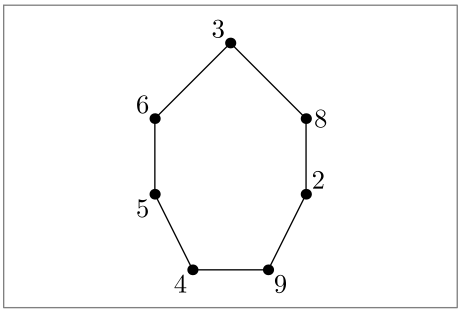


(5) $2 - 7 - 9$ is embedded

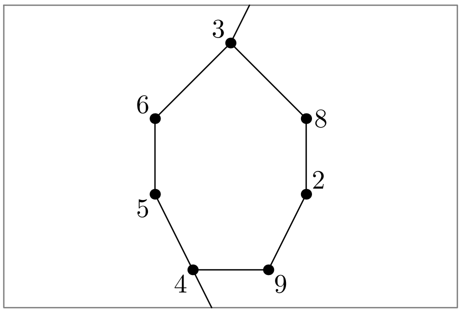
Figure 4.63: Embedding of Case 1

Figure 4.63 shows that an embedding, with the given restrictions, is not possible, as the path $4 - 1 - 7$ cannot be embedded.

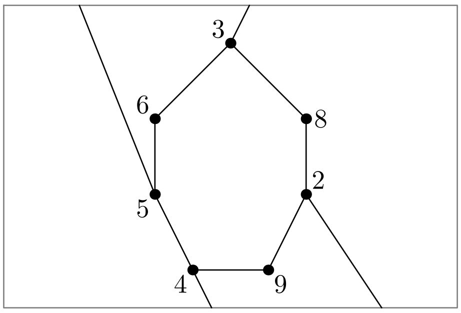
Embedding of Case 2:



(1) Embedding of the given cycle



(2) $3 - 4$ is embedded



(3) $2 - 5$ is embedded

Figure 4.64: Embedding of Case 2

Figure 4.64 shows that an embedding, with the given restrictions, is not possible, as the path $8 - 9$ cannot be embedded. □

The graph E_{27}

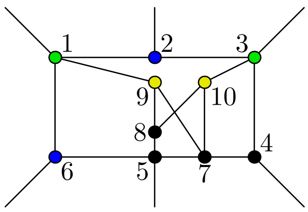


Figure 4.65: E_{27} and its vertex-orbits

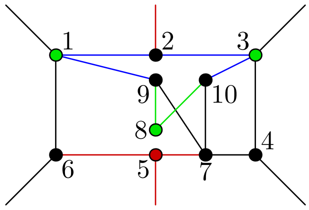
The orbits of triples of vertices we have to consider for the graph E_{27} are:

$$A = \Big\{\{1, 3, 8\}\Big\}, \qquad\qquad B = \Big\{\{1, 7, 8\}, \{3, 7, 8\}\Big\}.$$

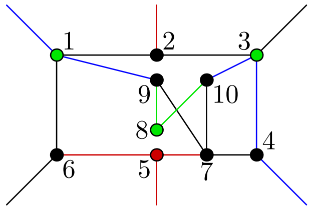
Lemma 4.31
The construction of a graph with vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ identified with vertices $1, 3$ and 8 in E_{27} does not result in an element of $M_4(\mathcal{S}_2)$.

Proof

Regarding the graph $E_{27} - (5, 8)$ the only two cycles including vertices $1, 3$ and 8, we have to consider, using Lemma 1.11 and isomorphisms, are:



(1) Case 1

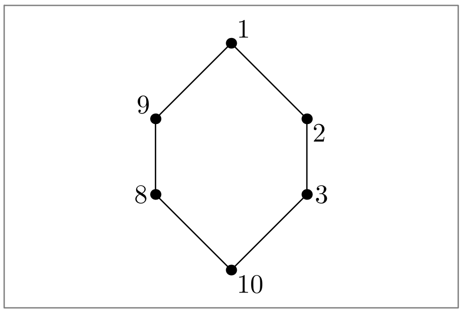


(2) Case 2

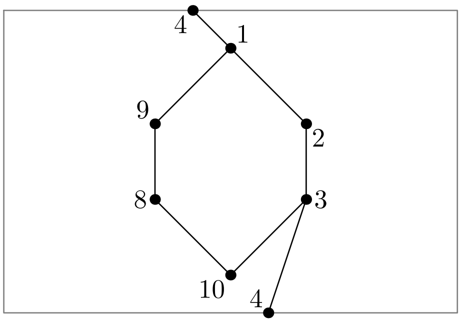
Figure 4.66: Cycles in $E_{27} - (5, 8)$ including vertices $1, 3$ and 8

Trying to embed $E_{27} - (5, 8)$ with these cycles as the boundaries of a face on the Möbius strip does not work, as can be seen below.

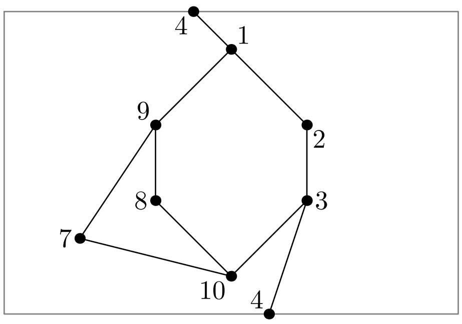
Embedding of Case 1:



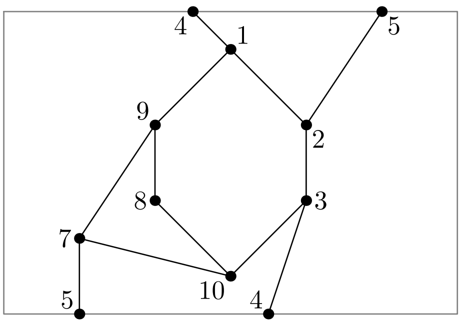
(1) Embedding of the given cycle



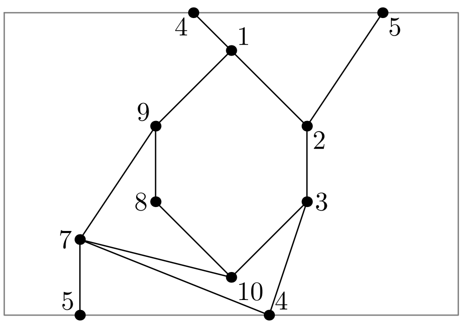
(2) $1 - 4 - 3$ is embedded



(3) $9 - 7 - 10$ is embedded



(4) $2 - 5 - 7$ is embedded

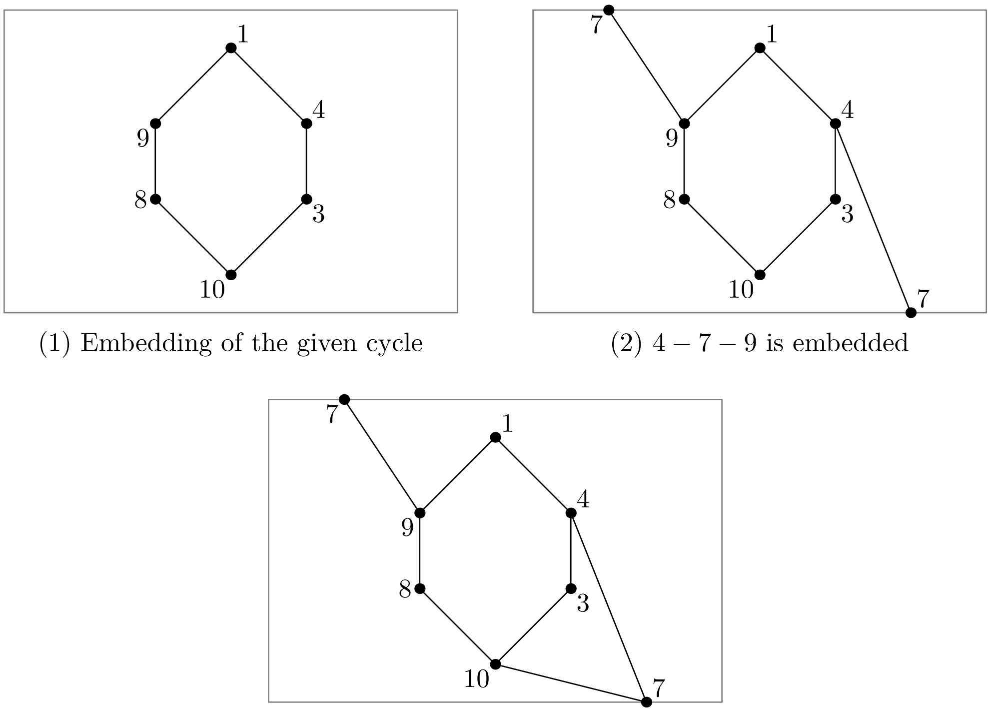


(5) $4 - 7$ is embedded

Figure 4.67: Embedding of Case 1

Figure 4.67 shows that an embedding, with the given restrictions, is not possible, as vertex 6 cannot be embedded.

Embedding of Case 2:



(1) Embedding of the given cycle

(2) $4 - 7 - 9$ is embedded

(3) $7 - 10$ is embedded

Figure 4.68: Embedding of Case 2

Figure 4.68 shows that an embedding, with the given restrictions, is not possible, as vertex 2 cannot be embedded. □

Lemma 4.32
The graph, where the vertices v_1, v_2 and v_3 of $K_5 - (v_1, v_2) - (v_1, v_3) - (v_2, v_3)$ are identified with triples of vertices from orbit B of the graph E_{27}, is not an element of $M_4(\mathcal{S}_2)$.

Proof
Regarding the graph $E_{27} - (5, 7)$ the only cycle we have to consider, using Lemma 1.11 and isomorphisms, is:

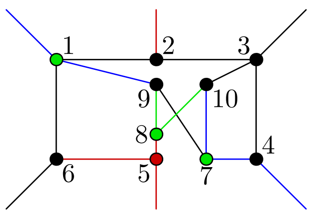
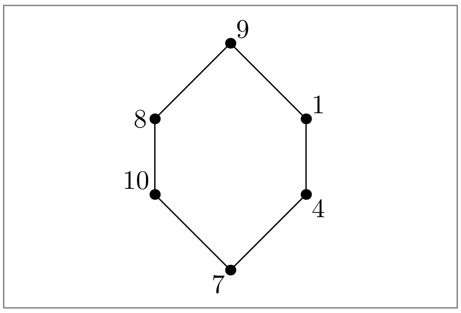
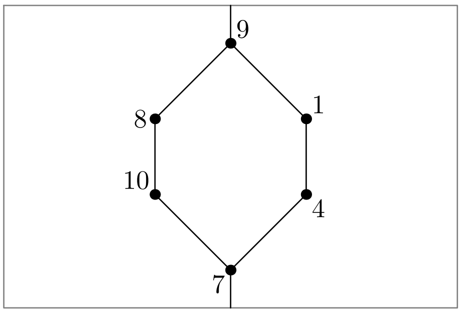


Figure 4.69: Cycle in $E_{27} - (5, 7)$, including vertices $1, 7$ and 8

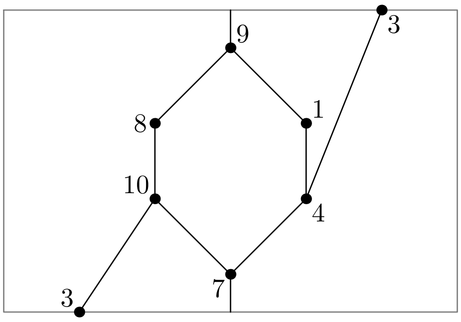
Trying to embed $E_{27} - (5,7)$ with this cycle as the boundary of a face on the Möbius strip does not work:



(1) Embedding of the given cycle



(2) $7-9$ is embedded



(3) $4-3-10$ is embedded

Figure 4.70: Embedding of $E_{27} - (5,7)$

Figure 4.70 shows that an embedding, with the given restrictions, is not possible, as the path $8-5-6-1$ cannot be embedded. □

The graph F_4

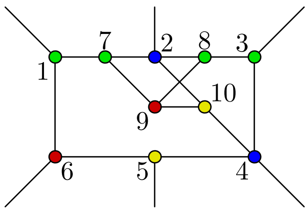


Figure 4.71: F_4 and its vertex-orbits

The only orbit of triples of vertices we have to consider for F_4 is:

$$A = \Big\{\{2,6,9\},\{4,6,9\}\Big\}.$$

Lemma 4.33
The construction of a graph with vertices v_1, v_2 and v_3 of $K_5 - (v_1,v_2) - (v_1,v_3) - (v_2,v_3)$ identified with triples of vertices from orbit A of the graph F_4, does not result in an element of $M_4(\mathcal{S}_2)$.

Proof

Regarding the graph $F_4 - (5,6)$ the only cycle including vertices $4, 6$ and 9 we have to consider, using Lemma 1.11 and isomorphisms, is:

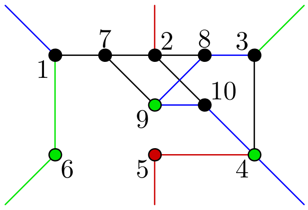
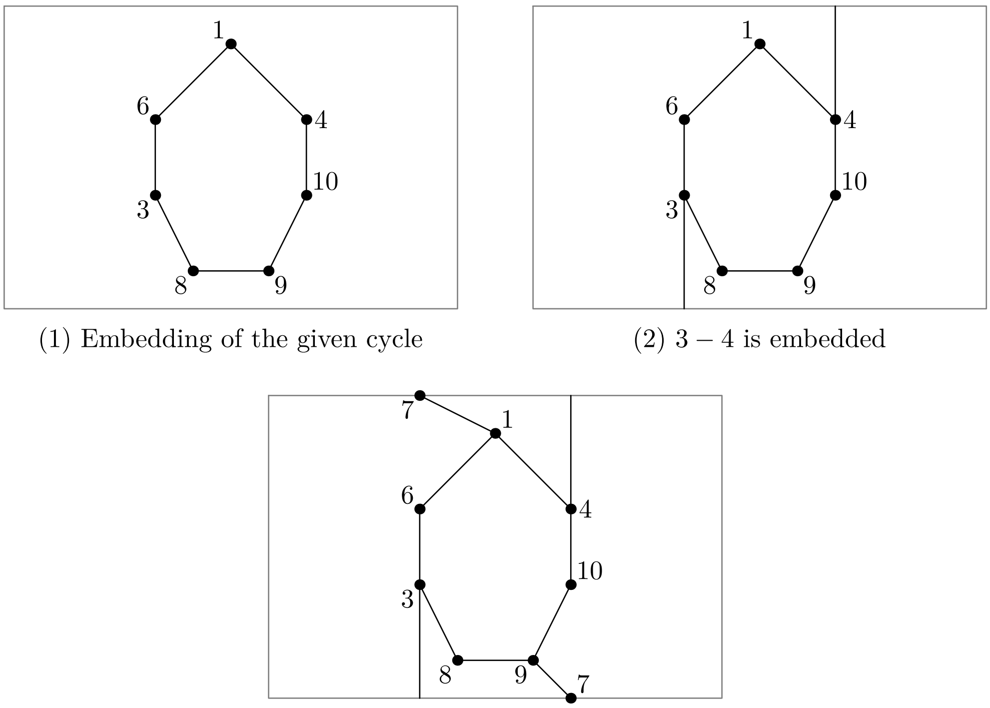


Figure 4.72: Cycle in $F_4 - (5,6)$, including vertices $4, 6$ and 9

Trying to embed $F_4 - (5,6)$ with this cycle as the boundary of a face on the Möbius strip does not work:



(1) Embedding of the given cycle

(2) $3 - 4$ is embedded

(3) $1 - 7 - 9$ is embedded

Figure 4.73: Embedding of $F_4 - (5,6)$

Figure 4.73 shows that an embedding, with the given restrictions, is not possible, as the path $8 - 2 - 10$ cannot be embedded. $\square$

5 Resume

In this thesis, we have constructed 83 graphs which are irreducible for the Klein surface and which are elements of $M_4(\mathcal{S}_2)$. This already is a big number, but we only constructed one class of graphs and we can be sure that the minimal basis $M_4(\mathcal{S}_2)$ has more than 83 elements. The following theorem shows that the question about irreducible graphs in general becomes very extensive for surfaces of higher genus:

Theorem 5.1
The number of irreducible graphs in $M_4(\mathcal{S}_p)$, $\mathcal{S}_p$ a non-orientable surface of genus p, grows exponentially with p.

Proof
Let $\mathcal{S}_p$ be a non-orientable surface of genus p. The non-orientable surface $\mathcal{S}_{p+1}$ then has an additional crosscap. On this crosscap an additional K_5 can be embedded. Using the knowledge about the method of relative components, we know that we can construct an irreducible graph for $M_4(\mathcal{S}_{p+1})$ by identifying one vertex of a graph from $M_4(\mathcal{S}_p)$ and one vertex of a K_5, which is the relative component.

Let us now take one arbitrary graph $G \in M_4(\mathcal{S}_p)$, $G \neq K_5$. For the surface $\mathcal{S}_{p+1}$ we can identify one vertex of the K_5 and one arbitrary vertex of G, and we get an irreducible graph $G' \in M_4(\mathcal{S}_{p+1})$. In the next step, we want to construct graphs which are elements of $M_4(\mathcal{S}_{p+2})$. Again, we want to identify one vertex of the relative component K_5 with one vertex of the graph $G' \in M_4(\mathcal{S}_{p+1})$. We now have the choice between two different vertices as base-points. We can either attach the K_5 to the same base point the K_5 in the previous step was attched to, or we can attach it to an arbitrary vertex of the previously attched K_5. This way we can construct two non-isomorphic graphs in $M_4(\mathcal{S}_{p+2})$. And with this construction-method, the number of graphs grows exponentially to the basis two.

We can even increase the basis of the exponential growth:

If we consider the block-graphs of the graphs we can construct as described above, we get rooted-trees. The number of vertices in these rooted trees corresponds to the genus of the surface. The number of rooted trees growths exponentially to a basis greater than 2,9 as can be found in [HP73]. $\square$

One example for the construction-method described in the proof of Theorem 5.1 could be:

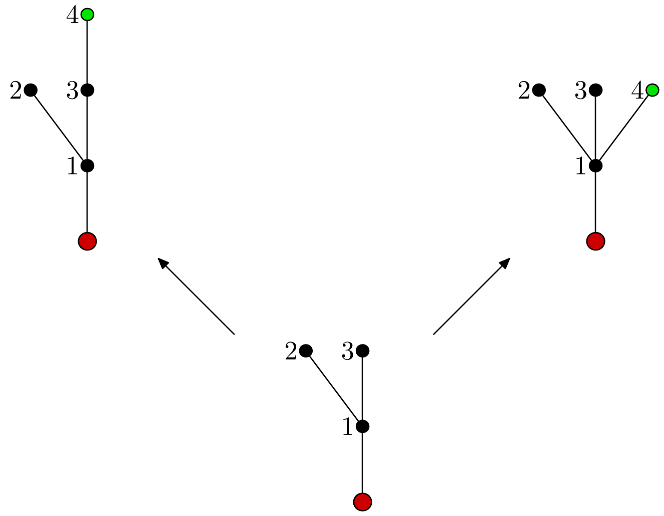


Figure 5.1: An example for the construction-method

In this case, we have one rooted-graph given. Vertices 1, 2 and 3 are already attached in an arbitrary way. If we want to attach vertex 4, we can either attach it to vertex 1, or we can attach it to vertex 3.

This approximation obviously is far smaller than the actual growth rate of graphs in $M_4(\mathcal{S})$. In the proof of Theorem 5.1 we only considered graphs with connectivity $\kappa = 1$ and we only started with one graph in $M_4(\mathcal{S}_p)$ and only attached the first K_5 to one vertex. In reality we already have 11 connected graphs in $M_4(\mathcal{S}_1)$ and a total number of 29 vertex-orbits. We also cannot only attach the additional K_5 to graphs in $M_4(\mathcal{S}_p)$, there might be a possibility to generate graphs like that starting with another graph from $M_2(\mathcal{S}_p)$. We also know that we can construct one disconnected graph in $M_4(\mathcal{S}_{p+1})$ for each graph in $M_4(\mathcal{S}_p)$ and we will always be able to construct a number of graphs with connectivity $\kappa \geq 2$. In general even these additional information are only about attachment of a relative component to graphs in $M_3(\mathcal{S}_p)$ without even considering adding vertices and edges to a graph in $M_2(\mathcal{S}_p)$ and thus being able to attach another relative component than the K_5.

Theorem 5.2
The number of irreducible graphs in $M_4(\widetilde{\mathcal{S}}_q)$, $\widetilde{\mathcal{S}}_q$ an orientable surface of genus q, grows exponentially with q.

Proof
For orientable surfaces the same arguments as for non-orientable surfaces applies, only that a handle instead of a crosscap is added when increasing the genus by one. Each additional handle can take an additional K_5 and thus the same construction-method applies. □

Knowing that the number of irreducible graphs grows at a huge rate, we can say that the question about irreducible graphs for surfaces of higher genus becomes extensive and that even the construction of the complete minimal basis $M_4(\mathcal{S}_2)$ of the Klein surface is very complex.

Appendix

For the computation of the vertex- and edge-orbits, we used [GAP08] using the packages [Teu10] and[Soi06]. The adjacency-lists of graphs in $M_4(\mathcal{S}_2)$ are the results of this thesis.

Vertexorbits of graphs in $M_2(\mathcal{S}_1)$

$$
\begin{aligned}
A_1 &= \Big\{\{1,2,6,3,7,4,8,9\},\{5\}\Big\}\\
A_2 &= \Big\{\{1,2,3,5,6,7\},\{4\}\Big\}\\
A_5 &= \Big\{\{1,2,6,3,7,4,8,5,9,10\}\Big\}\\
B_1 &= \Big\{\{1,2,6,7\},\{3,4,5\}\Big\}\\
B_3 &= \Big\{\{1,2,6,3,7,8\},\{4,5\}\Big\}\\
B_7 &= \Big\{\{1,3,5\},\{2,4,6\},\{7\},\{8\}\Big\}\\
C_1 &= \Big\{\{1,2,3,4\},\{5\},\{6,8,10\},\{7,9\}\Big\}\\
C_2 &= \Big\{\{1,3\},\{2,4,6\},\{5\},\{7,8,9\}\Big\}\\
C_3 &= \Big\{\{1,3,4,6\},\{2,8\},\{5\},\{7\},\{9\}\Big\}\\
C_4 &= \Big\{\{1,3,5,6,2,4\},\{7,8\},\{9\}\Big\}\\
C_7 &= \Big\{\{1,3,7,8\},\{2,5\},\{4,6\}\Big\}\\
C_{11} &= \Big\{\{1,2,3,4,5\},\{6,7,9,11,8,10\}\Big\}\\
D_1 &= \Big\{\{1,3\},\{2,4,7,6,9,10\},\{5,8\}\Big\}\\
D_2 &= \Big\{\{1\},\{2,8,7\},\{3,9,5\},\{4,6,10\}\Big\}\\
D_3 &= \Big\{\{1,3\},\{2,6\},\{4\},\{5\},\{7,8\}\Big\}\\
D_4 &= \Big\{\{1,4,6,5\},\{2,3\},\{7,8,9\}\Big\}\\
D_9 &= \Big\{\{1,3,9,10\},\{2,5\},\{4,6\},\{7,8\}\Big\}\\
D_{12} &= \Big\{\{1,3\},\{2,4\},\{5\},\{6\},\{7\},\{8,9\}\Big\}\\
D_{17} &= \Big\{\{1,2,5,3,6,4,7,8\}\Big\}\\
E_1 &= \Big\{\{1,3,7,5,9,11\},\{2,4,8,10\},\{6\}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= \Big\{\{1\},\{2,4,6,11,8,10\},\{3,9,5,7\}\Big\}\\
E_3 &= \Big\{\{1,3,5,7,8\},\{2,4,6\}\Big\}\\
E_5 &= \Big\{\{1,3\},\{2,6,7,8\},\{4\},\{5,9\}\Big\}\\
E_6 &= \Big\{\{1,4,6,5\},\{2,3\},\{7,9,10\},\{8\}\Big\}\\
E_{11} &= \Big\{\{1,3\},\{2,6\},\{4\},\{5\},\{7\},\{8\},\{9,10\}\Big\}\\
E_{18} &= \Big\{\{1,3,2,5,4,6\},\{7,8\}\Big\}\\
E_{19} &= \Big\{\{1,3\},\{2,9\},\{4,6\},\{5\},\{7,8\}\Big\}\\
E_{20} &= \Big\{\{1,3,5\},\{2,8,7\},\{4\},\{6\},\{9\}\Big\}\\
E_{22} &= \Big\{\{1,5,8,9\},\{2,4,6,7\},\{3\}\Big\}\\
E_{27} &= \Big\{\{1,3\},\{2,6\},\{4\},\{5\},\{7\},\{8\},\{9,10\}\Big\}\\
E_{42} &= \Big\{\{1,2,7,4,8,6,3,10,5,12,9,11\}\Big\}\\
F_1 &= \Big\{\{1\},\{2,5\},\{3,9\},\{4,6,7,8\}\Big\}\\
F_4 &= \Big\{\{1,3,7,8\},\{2,4\},\{5,10\},\{6,9\}\Big\}\\
F_6 &= \Big\{\{1,4,7,6,8,9,5,10\},\{2,3\}\Big\}\\
G &= \Big\{\{1,3,7,5,8,9\},\{2,10,4,6\}\Big\}
\end{aligned}
$$

Edge-orbits of graphs in $M_2(\mathcal{S}_1)$

$$
\begin{aligned}
A_1 = \Big\{&\{(1,2),(1,3),(6,7),(1,4),(2,3),(6,8),(2,4),(6,9),(7,8),(3,4),(7,9),(8,9)\},\\
&\{(1,5),(2,5),(5,6),(3,5),(5,7),(4,5),(5,8),(5,9)\}\Big\}\\
A_2 = \Big\{&\{(1,2),(1,3),(1,5),(2,3),(1,6),(2,5),(2,7),(3,6),(3,7),(5,6),(5,7),(6,7)\},\\
&\{(1,4),(2,4),(3,4),(4,5),(4,6),(4,7)\}\Big\}\\
A_5 = \Big\{&\{(1,2),(1,3),(6,7),(1,4),(2,3),(6,8),(1,5),(2,4),(6,9),(7,8),(2,5),(6,10),(3,4),\\
&(7,9),(3,5),(7,10),(8,9),(4,5),(8,10),(9,10)\}\Big\}\\
B_1 = \Big\{&\{(1,2),(6,7)\},\{(1,3),(1,4),(2,3),(3,6),(1,5),(2,4),(4,6),(3,7),(2,5),(5,6),\\
&(4,7),(5,7)\},\{(3,4),(3,5),(4,5)\}\Big\}\\
B_3 = \Big\{&\{(1,2),(1,3),(6,7),(2,3),(6,8),(7,8)\},\{(1,4),(1,5),(2,4),(4,6),(2,5),(5,6),\\
&(3,4),(4,7),(3,5),(5,7),(4,8),(5,8)\}\Big\}\\
B_7 = \Big\{&\{(1,2),(2,3),(4,5),(3,4),(5,6),(1,6)\},\{(1,3),(3,5),(1,5)\},\{(1,8),(3,8),(5,8)\},\\
&\{(2,7),(4,7),(6,7)\},\{(2,8),(4,8),(6,8)\}\Big\}\\
C_1 = \Big\{&\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\},\{(1,5),(2,5),(3,5),(4,5)\},\{(5,6),(5,8),\\
&(5,10)\},\{(6,7),(6,9),(7,8),(8,9),(7,10),(9,10)\}\Big\}
\end{aligned}
$$

$$C_2 = \Big\{\{(1,2),(2,3),(1,4),(3,4),(1,6),(3,6)\},\{(1,7),(3,7),(1,8),(3,8),(1,9),(3,9)\},$$
$$\{(2,5),(4,5),(5,6)\},\{(7,8),(7,9),(8,9)\}\Big\}$$

$$C_3 = \Big\{\{(1,2),(2,3),(4,8),(6,8)\},\{(1,5),(3,5),(4,5),(5,6)\},\{(1,6),(3,4)\},\{(1,9),(3,9),$$
$$(4,9),(6,9)\},\{(2,7),(7,8)\},\{(5,9)\},\{(7,9)\}\Big\}$$

$$C_4 = \Big\{\{(1,2),(2,3),(4,5),(1,6),(3,4),(5,6)\},\{(1,8),(3,8),(5,8),(6,7),(2,7),(4,7)\},$$
$$\{(1,9),(3,9),(5,9),(6,9),(2,9),(4,9)\}\Big\}$$

$$C_7 = \Big\{\{(1,2),(2,3),(5,7),(5,8)\},\{(1,3),(7,8)\},\{(1,4),(3,4),(6,7),(6,8)\},\{(1,6),(3,6),$$
$$(4,7),(4,8)\},\{(2,5)\},\{(2,6),(4,5)\}\Big\}$$

$$C_{11} = \Big\{\{(1,2),(1,3),(1,4),(2,3),(1,5),(2,4),(2,5),(3,4),(3,5),(4,5)\},\{(6,7),(6,9),$$
$$(6,11),(7,8),(7,10),(8,9),(9,10),(8,11),(10,11)\}\Big\}$$

$$D_1 = \Big\{\{(1,2),(2,3),(1,4),(1,7),(3,4),(3,7),(1,6),(1,9),(3,6),(3,9),(1,10),(3,10)\},$$
$$\{(2,5),(4,5),(7,8),(5,6),(8,9),(8,10)\}\Big\}$$

$$D_2 = \Big\{\{(1,2),(1,8),(1,7)\},\{(1,4),(1,6),(1,10)\},\{(2,5),(8,9),(3,7)\},\{(2,7),(2,8),(7,8)\},$$
$$\{(3,4),(4,9),(3,6),(5,6),(9,10),(5,10)\}\Big\}$$

$$D_3 = \Big\{\{(1,2),(2,3),(1,6),(3,6)\},\{(1,4),(3,4)\},\{(1,7),(3,7),(1,8),(3,8)\},\{(2,5),(5,6)\},$$
$$\{(4,5)\},\{(4,7),(4,8)\},\{(7,8)\}\Big\}$$

$$D_4 = \Big\{\{(1,2),(3,4),(3,6),(2,5)\},\{(1,4),(1,6),(4,5),(5,6)\},\{(2,7),(2,8),(3,7),(2,9),$$
$$(3,8),(3,9)\},\{(7,8),(7,9),(8,9)\}\Big\}$$

$$D_9 = \Big\{\{(1,2),(2,3),(5,9),(5,10)\},\{(1,4),(1,6),(3,4),(4,9),(3,6),(6,9),(4,10),(6,10)\},$$
$$\{(2,7),(2,8),(5,7),(5,8)\},\{(4,7),(6,8)\}\Big\}$$

$$D_{12} = \Big\{\{(1,2),(2,3),(1,4),(3,4)\},\{(1,6),(3,6)\},\{(1,8),(3,9)\},\{(2,5),(4,5)\},\{(5,7)\},$$
$$\{(5,8),(5,9)\},\{(6,7)\},\{(7,8),(7,9)\},\{(8,9)\}\Big\}$$

$$D_{17} = \Big\{\{(1,2),(1,3),(5,6),(1,4),(2,3),(5,7),(2,4),(5,8),(6,7),(3,4),(6,8),(7,8)\},$$
$$\{(1,5),(2,6),(3,7),(4,8)\}\Big\}$$

$$E_1 = \Big\{\{(1,2),(1,4),(2,3),(7,8),(3,4),(7,10),(2,5),(8,9),(4,5),(9,10),(8,11),(10,11)\},$$
$$\{(1,6),(3,6),(6,7),(5,6),(6,9),(6,11)\}\Big\}$$

$$E_2 = \Big\{\{(1,2),(1,4),(1,6),(1,11),(1,8),(1,10)\},\{(2,5),(2,7),(3,4),(5,6),(4,9),(3,6),$$
$$(7,11),(7,8),(5,10),(9,11),(3,8),(9,10)\}\Big\}$$

$$E_3 = \Big\{\{(1,2),(1,4),(2,3),(1,6),(3,4),(2,5),(3,6),(4,5),(2,7),(5,6),(4,7),(2,8),(6,7),$$
$$(4,8),(6,8)\}\Big\}$$

$$E_5 = \Big\{\{(1,2),(2,3),(1,6),(1,7),(3,6),(3,7),(1,8),(3,8)\},\{(1,4),(3,4)\},\{(2,5),(5,6),$$
$$(7,9),(8,9)\},\{(4,5),(4,9)\}\Big\}$$

$$E_6 = \Big\{\{(1,2),(3,4),(3,6),(2,5)\},\{(1,4),(1,6),(4,5),(5,6)\},\{(2,7),(2,9),(3,7),(2,10),(3,9),(3,10)\},\{(7,8),(8,9),(8,10)\}\Big\}$$

$$E_{11} = \Big\{\{(1,2),(2,3),(1,6),(3,6)\},\{(1,4),(3,4)\},\{(1,10),(3,9)\},\{(2,5),(5,6)\},\{(4,7)\},\{(5,7)\},\{(5,9),(5,10)\},\{(7,8)\},\{(8,9),(8,10)\}\Big\}$$

$$E_{18} = \Big\{\{(1,2),(1,4),(2,3),(1,6),(3,4),(2,5),(3,6),(4,5),(5,6)\},\{(1,7),(3,7),(2,8),(5,7),(4,8),(6,8)\}\Big\}$$

$$E_{19} = \Big\{\{(1,4),(3,4),(1,6),(3,6)\},\{(1,7),(3,8)\},\{(2,5),(5,9)\},\{(2,7),(7,9),(2,8),(8,9)\},\{(2,9)\},\{(4,5),(5,6)\},\{(7,8)\}\Big\}$$

$$E_{20} = \Big\{\{(1,4),(3,4),(4,5)\},\{(1,6),(3,6),(5,6)\},\{(1,7),(3,8),(2,5)\},\{(2,7),(7,8),(2,8)\},\{(2,9),(8,9),(7,9)\},\{(4,9)\}\Big\}$$

$$E_{22} = \Big\{\{(1,2),(1,4),(2,5),(1,6),(4,5),(2,8),(5,7),(6,8),(4,9),(7,8),(6,9),(7,9)\},\{(2,3),(3,4),(3,6),(3,7)\}\Big\}$$

$$E_{27} = \Big\{\{(1,2),(1,6),(2,3),(3,6)\},\{(1,4),(3,4)\},\{(1,9),(3,10)\},\{(2,5),(5,6)\},\{(4,7)\},\{(5,7)\},\{(5,8)\},\{(7,9),(7,10)\},\{(8,9),(8,10)\}\Big\}$$

$$E_{42} = \Big\{\{(1,2),(1,4),(7,8),(1,6),(2,3),(7,10),(2,5),(7,12),(3,4),(8,9),(4,5),(8,11),(3,6),(9,10),(5,6),(10,11),(9,12),(11,12)\}\Big\}$$

$$F_1 = \Big\{\{(1,4),(1,6),(1,7),(1,8)\},\{(2,3),(5,9)\},\{(2,5)\},\{(2,7),(2,8),(4,5),(5,6)\},\{(3,4),(3,6),(7,9),(8,9)\}\Big\}$$

$$F_4 = \Big\{\{(1,4),(3,4),(2,7),(2,8)\},\{(1,6),(3,6),(7,9),(8,9)\},\{(1,7),(3,8)\},\{(2,5),(4,10)\},\{(2,10),(4,5)\},\{(5,6),(9,10)\}\Big\}$$

$$F_6 = \Big\{\{(1,2),(3,4),(2,7),(3,6),(3,8),(2,9),(2,5),(3,10)\},\{(1,4),(1,6),(7,8),(4,5),(7,10),(8,9),(5,6),(9,10)\}\Big\}$$

$$G = \Big\{\{(1,4),(1,6),(3,4),(2,7),(3,6),(7,10),(4,5),(2,8),(5,6),(8,10),(2,9),(9,10)\},\{(1,7),(3,8),(5,9)\}\Big\}$$

Adjacency lists of graphs in $M_4(\mathcal{S}_2)$

$$G_1 := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\{1,2,3,4,6,7,8,9\},\{5,7,8,9\},\{5,6,8,9\},\{5,6,7,9\},\{5,6,7,8\},\{11,12,13,14\},\{10,12,13,14\},\{10,11,13,14\},\{10,11,12,14\},\{10,11,12,13\}\Big\}$$

$$G_2 := \Big\{\{2,3,4,5,6\},\{1,3,4,5,7\},\{1,2,4,6,7\},\{1,2,3,5,6,7\},\{1,2,4,6,7\},\{1,3,4,5,7\},\{2,3,4,5,6\},\{9,10,11,12\},\{8,10,11,12\},\{8,9,11,12\},\{8,9,10,12\},\{8,9,10,11\}\Big\}$$

$$G_3 := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\{1,2,3,4\},\{7,8,9,10\},\{6,8,9,10\},$$
$$\{6,7,9,10\},\{6,7,8,10\},\{6,7,8,9\},\{12,13,14,15\},\{11,13,14,15\},\{11,12,14,15\},$$
$$\{11,12,13,15\},\{11,12,13,14\}\Big\}$$

$$G_4 := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5,6,7\},\{1,2,3,5,6,7\},\{1,2,3,4,6,7\},\{3,4,5,7\},$$
$$\{3,4,5,6\},\{9,10,11,12\},\{8,10,11,12\},\{8,9,11,12\},\{8,9,10,12\},\{8,9,10,11\}\Big\}$$

$$G_5 := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,6,7,8\},\{1,2,3,6,7,8\},\{4,5,7,8\},\{4,5,6,8\},$$
$$\{4,5,6,7\},\{10,11,12,13\},\{9,11,12,13\},\{9,10,12,13\},\{9,10,11,13\},\{9,10,11,12\}\Big\}$$

$$G_6 := \Big\{\{2,3,4,6\},\{1,3,5,6\},\{1,2,4,6\},\{1,3,5,7,8\},\{2,4,7,8\},\{1,2,3,7,8\},\{4,5,6,8\},$$
$$\{4,5,6,7\},\{10,11,12,13\},\{9,11,12,13\},\{9,10,12,13\},\{9,10,11,13\},\{9,10,11,12\}\Big\}$$

$$G_7 := \Big\{\{2,4,6\},\{1,3,7,8\},\{2,4,6\},\{1,3,7,9,10\},\{7,8,9,10\},\{1,3,8,9,10\},\{2,4,5\},$$
$$\{2,5,6\},\{4,5,6\},\{4,5,6\},\{12,13,14,15\},\{11,13,14,15\},\{11,12,14,15\},$$
$$\{11,12,13,15\},\{11,12,13,14\}\Big\}$$

$$G_8 := \Big\{\{2,4,6,8\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7\},\{5,6,8,9\},\{1,5,7,9\},$$
$$\{3,5,7,8\},\{11,12,13,14\},\{10,12,13,14\},\{10,11,13,14\},\{10,11,12,14\},$$
$$\{10,11,12,13\}\Big\}$$

$$G_9 := \Big\{\{2,3,4,5\},\{1,3,4,6\},\{1,2,4,7\},\{1,2,3,8\},\{1,6,7,8\},\{2,5,7,8\},\{3,5,6,8\},\{4,5,6,7\},$$
$$\{10,11,12,13\},\{9,11,12,13\},\{9,10,12,13\},\{9,10,11,13\},\{9,10,11,12\}\Big\}$$

$$G_{10} := \Big\{\{2,4,6,7,8\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5\},\{1,3,5\},\{1,3,5\},$$
$$\{10,11,12,13\},\{9,11,12,13\},\{9,10,12,13\},\{9,10,11,13\},\{9,10,11,12\}\Big\}$$

$$G_{11} := \Big\{\{2,4,6,7\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{1,3,5\},\{2,4,6\},$$
$$\{10,11,12,13\},\{9,11,12,13\},\{9,10,12,13\},\{9,10,11,13\},\{9,10,11,12\}\Big\}$$

$$G_{12} := \Big\{\{2,4,6\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,9\},\{2,4,7\},\{1,3,8,9\},\{3,5,8,9\},\{2,6,7\},$$
$$\{4,6,7\},\{11,12,13,14\},\{10,12,13,14\},\{10,11,13,14\},\{10,11,12,14\},$$
$$\{10,11,12,13\}\Big\}$$

$$G_{13} := \Big\{\{2,3,4,5,10,11,12,13\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\{1,2,3,4,6,7,8,9\},$$
$$\{5,7,8,9\},\{5,6,8,9\},\{5,6,7,9\},\{5,6,7,8\},\{1,11,12,13\},\{1,10,12,13\},\{1,10,11,13\},$$
$$\{1,10,11,12\}\Big\}$$

$$G_{14} := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\{1,2,3,4,6,7,8,9,10,11,12,13\},$$
$$\{5,7,8,9\},\{5,6,8,9\},\{5,6,7,9\},\{5,6,7,8\},\{5,11,12,13\},\{5,10,12,13\},\{5,10,11,13\},$$
$$\{5,10,11,12\}\Big\}$$

$$G_{15} := \Big\{\{2,3,4,5,6,8,9,10,11\},\{1,3,4,5,7\},\{1,2,4,6,7\},\{1,2,3,5,6,7\},\{1,2,4,6,7\},$$
$$\{1,3,4,5,7\},\{2,3,4,5,6\},\{1,9,10,11\},\{1,8,10,11\},\{1,8,9,11\},\{1,8,9,10\}\Big\}$$

$$G_{16} := \Big\{\{2,3,4,5,6\},\{1,3,4,5,7\},\{1,2,4,6,7\},\{1,2,3,5,6,7,8,9,10,11\},\{1,2,4,6,7\},\\ \{1,3,4,5,7\},\{2,3,4,5,6\},\{4,9,10,11\},\{4,8,10,11\},\{4,8,9,11\},\{4,8,9,10\}\Big\}$$

$$G_{17} := \Big\{\{2,3,4,5,8,9,10,11\},\{1,3,4,5\},\{1,2,4,5,6,7\},\{1,2,3,5,6,7\},\{1,2,3,4,6,7\},\\ \{3,4,5,7\},\{3,4,5,6\},\{1,9,10,11\},\{1,8,10,11\},\{1,8,9,11\},\{1,8,9,10\}\Big\}$$

$$G_{18} := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5,6,7,8,9,10,11\},\{1,2,3,5,6,7\},\{1,2,3,4,6,7\},\\ \{3,4,5,7\},\{3,4,5,6\},\{3,9,10,11\},\{3,8,10,11\},\{3,8,9,11\},\{3,8,9,10\}\Big\}$$

$$G_{19} := \Big\{\{2,3,4,5,9,10,11,12\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,6,7,8\},\{1,2,3,6,7,8\},\\ \{4,5,7,8\},\{4,5,6,8\},\{4,5,6,7\},\{1,10,11,12\},\{1,9,11,12\},\{1,9,10,12\},\{1,9,10,11\}\Big\}$$

$$G_{20} := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,6,7,8,9,10,11,12\},\{1,2,3,6,7,8\},\\ \{4,5,7,8\},\{4,5,6,8\},\{4,5,6,7\},\{4,10,11,12\},\{4,9,11,12\},\{4,9,10,12\},\{4,9,10,11\}\Big\}$$

$$G_{21} := \Big\{\{2,3,4,6,9,10,11,12\},\{1,3,5,6\},\{1,2,4,6\},\{1,3,5,7,8\},\{2,4,7,8\},\{1,2,3,7,8\},\\ \{4,5,6,8\},\{4,5,6,7\},\{1,10,11,12\},\{1,9,11,12\},\{1,9,10,12\},\{1,9,10,11\}\Big\}$$

$$G_{22} := \Big\{\{2,3,4,6\},\{1,3,5,6,9,10,11,12\},\{1,2,4,6\},\{1,3,5,7,8\},\{2,4,7,8\},\{1,2,3,7,8\},\\ \{4,5,6,8\},\{4,5,6,7\},\{2,10,11,12\},\{2,9,11,12\},\{2,9,10,12\},\{2,9,10,11\}\Big\}$$

$$G_{23} := \Big\{\{2,3,4,6\},\{1,3,5,6\},\{1,2,4,6,9,10,11,12\},\{1,3,5,7,8\},\{2,4,7,8\},\{1,2,3,7,8\},\\ \{4,5,6,8\},\{4,5,6,7\},\{3,10,11,12\},\{3,9,11,12\},\{3,9,10,12\},\{3,9,10,11\}\Big\}$$

$$G_{24} := \Big\{\{2,4,6,11,12,13,14\},\{1,3,7,8\},\{2,4,6\},\{1,3,7,9,10\},\{7,8,9,10\},\{1,3,8,9,10\},\\ \{2,4,5\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{1,12,13,14\},\{1,11,13,14\},\{1,11,12,14\},\\ \{1,11,12,13\}\Big\}$$

$$G_{25} := \Big\{\{2,4,6\},\{1,3,7,8,11,12,13,14\},\{2,4,6\},\{1,3,7,9,10\},\{7,8,9,10\},\{1,3,8,9,10\},\\ \{2,4,5\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{2,12,13,14\},\{2,11,13,14\},\{2,11,12,14\},\\ \{2,11,12,13\}\Big\}$$

$$G_{26} := \Big\{\{2,4,6\},\{1,3,7,8\},\{2,4,6\},\{1,3,7,9,10,11,12,13,14\},\{7,8,9,10\},\{1,3,8,9,10\},\\ \{2,4,5\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{4,12,13,14\},\{4,11,13,14\},\{4,11,12,14\},\\ \{4,11,12,13\}\Big\}$$

$$G_{27} := \Big\{\{2,4,6\},\{1,3,7,8\},\{2,4,6\},\{1,3,7,9,10\},\{7,8,9,10\},\{1,3,8,9,10\},\\ \{2,4,5,11,12,13,14\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{7,12,13,14\},\{7,11,13,14\},\\ \{7,11,12,14\},\{7,11,12,13\}\Big\}$$

$$G_{28} := \Big\{\{2,4,6,8,10,11,12,13\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7\},\{5,6,8,9\},\\ \{1,5,7,9\},\{3,5,7,8\},\{1,11,12,13\},\{1,10,12,13\},\{1,10,11,13\},\{1,10,11,12\}\Big\}$$

$$G_{29} := \Big\{\{2,4,6,8\},\{1,3,5,10,11,12,13\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7\},\{5,6,8,9\},\\ \{1,5,7,9\},\{3,5,7,8\},\{2,11,12,13\},\{2,10,12,13\},\{2,10,11,13\},\{2,10,11,12\}\Big\}$$

$$G_{30} := \Big\{\{2,4,6,8\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9,10,11,12,13\},\{1,3,7\},\{5,6,8,9\},\{1,5,7,9\},\{3,5,7,8\},\{5,11,12,13\},\{5,10,12,13\},\{5,10,11,13\},\{5,10,11,12\}\Big\}$$

$$G_{31} := \Big\{\{2,4,6,8\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7,10,11,12,13\},\{5,6,8,9\},\{1,5,7,9\},\{3,5,7,8\},\{6,11,12,13\},\{6,10,12,13\},\{6,10,11,13\},\{6,10,11,12\}\Big\}$$

$$G_{32} := \Big\{\{2,4,6,8\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7\},\{5,6,8,9,10,11,12,13\},\{1,5,7,9\},\{3,5,7,8\},\{7,11,12,13\},\{7,10,12,13\},\{7,10,11,13\},\{7,10,11,12\}\Big\}$$

$$G_{33} := \Big\{\{2,4,6,8\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7\},\{5,6,8,9\},\{1,5,7,9,10,11,12,13\},\{3,5,7,8\},\{8,11,12,13\},\{8,10,12,13\},\{8,10,11,13\},\{8,10,11,12\}\Big\}$$

$$G_{34} := \Big\{\{2,3,4,5,9,10,11,12\},\{1,3,4,6\},\{1,2,4,7\},\{1,2,3,8\},\{1,6,7,8\},\{2,5,7,8\},\{3,5,6,8\},\{4,5,6,7\},\{1,10,11,12\},\{1,9,11,12\},\{1,9,10,12\},\{1,9,10,11\}\Big\}$$

$$G_{35} := \Big\{\{2,4,6,7,8,9,10,11,12\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5\},\{1,3,5\},\{1,3,5\},\{1,10,11,12\},\{1,9,11,12\},\{1,9,10,12\},\{1,9,10,11\}\Big\}$$

$$G_{36} := \Big\{\{2,4,6,7,8\},\{1,3,5,9,10,11,12\},\{2,4,6,7,8\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5\},\{1,3,5\},\{1,3,5\},\{2,10,11,12\},\{2,9,11,12\},\{2,9,10,12\},\{2,9,10,11\}\Big\}$$

$$G_{37} := \Big\{\{2,4,6,7,9,10,11,12\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{1,3,5\},\{2,4,6\},\{1,10,11,12\},\{1,9,11,12\},\{1,9,10,12\},\{1,9,10,11\}\Big\}$$

$$G_{38} := \Big\{\{2,4,6,7\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8,9,10,11,12\},\{1,3,5\},\{2,4,6\},\{6,10,11,12\},\{6,9,11,12\},\{6,9,10,12\},\{6,9,10,11\}\Big\}$$

$$G_{39} := \Big\{\{2,4,6,10,11,12,13\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,9\},\{2,4,7\},\{1,3,8,9\},\{3,5,8,9\},\{2,6,7\},\{4,6,7\},\{1,11,12,13\},\{1,10,12,13\},\{1,10,11,13\},\{1,10,11,12\}\Big\}$$

$$G_{40} := \Big\{\{2,4,6\},\{1,3,5,8,10,11,12,13\},\{2,4,6,7\},\{1,3,5,9\},\{2,4,7\},\{1,3,8,9\},\{3,5,8,9\},\{2,6,7\},\{4,6,7\},\{2,11,12,13\},\{2,10,12,13\},\{2,10,11,13\},\{2,10,11,12\}\Big\}$$

$$G_{41} := \Big\{\{2,4,6\},\{1,3,5,8\},\{2,4,6,7,10,11,12,13\},\{1,3,5,9\},\{2,4,7\},\{1,3,8,9\},\{3,5,8,9\},\{2,6,7\},\{4,6,7\},\{3,11,12,13\},\{3,10,12,13\},\{3,10,11,13\},\{3,10,11,12\}\Big\}$$

$$G_{42} := \Big\{\{2,3,4,5,11,12,13\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\{1,2,3,4\},\{7,8,9,10,11,12,13\},\{6,8,9,10\},\{6,7,9,10\},\{6,7,8,10\},\{6,7,8,9\},\{1,6,12,13\},\{1,6,11,13\},\{1,6,11,12\}\Big\}$$

$$G_{43} := \Big\{\{2,3,4,5,8,9,10\},\{1,3,4,5\},\{1,2,4,5,6,7\},\{1,2,3,5,6,7\},\{1,2,3,4,6,7\},\{3,4,5,7,8,9,10\},\{3,4,5,6\},\{1,6,9,10\},\{1,6,8,10\},\{1,6,8,9\}\Big\}$$

$$G_{44} := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,6,7,8,9,10,11\},\{1,2,3,6,7,8,9,10,11\},\{4,5,7,8\},\{4,5,6,8\},\{4,5,6,7\},\{4,5,10,11\},\{4,5,9,11\},\{4,5,9,10\}\Big\}$$

$$G_{45} := \Big\{\{2,3,4,6,9,10,11\},\{1,3,5,6\},\{1,2,4,6\},\{1,3,5,7,8\},\{2,4,7,8\},\{1,2,3,7,8\},$$
$$\{4,5,6,8,9,10,11\},\{4,5,6,7\},\{1,7,10,11\},\{1,7,9,11\},\{1,7,9,10\}\Big\}$$
$$G_{46} := \Big\{\{2,4,6\},\{1,3,7,8,11,12,13\},\{2,4,6\},\{1,3,7,9,10,11,12,13\},\{7,8,9,10\},$$
$$\{1,3,8,9,10\},\{2,4,5\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{2,4,12,13\},\{2,4,11,13\},$$
$$\{2,4,11,12\}\Big\}$$
$$G_{47} := \Big\{\{2,4,6\},\{1,3,7,8,11,12,13\},\{2,4,6\},\{1,3,7,9,10\},\{7,8,9,10,11,12,13\},$$
$$\{1,3,8,9,10\},\{2,4,5\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{2,5,12,13\},\{2,5,11,13\},$$
$$\{2,5,11,12\}\Big\}$$
$$G_{48} := \Big\{\{2,4,6\},\{1,3,7,8\},\{2,4,6\},\{1,3,7,9,10,11,12,13\},\{7,8,9,10\},$$
$$\{1,3,8,9,10,11,12,13\},\{2,4,5\},\{2,5,6\},\{4,5,6\},\{4,5,6\},\{4,6,12,13\},\{4,6,11,13\},$$
$$\{4,6,11,12\}\Big\}$$
$$G_{49} := \Big\{\{2,4,6,8,10,11,12\},\{1,3,5\},\{2,4,6,9,10,11,12\},\{1,3,5\},\{2,4,7,8,9\},\{1,3,7\},$$
$$\{5,6,8,9\},\{1,5,7,9\},\{3,5,7,8\},\{1,3,11,12\},\{1,3,10,12\},\{1,3,10,11\}\Big\}$$
$$G_{50} := \Big\{\{2,4,6,8,10,11,12\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9,10,11,12\},\{1,3,7\},$$
$$\{5,6,8,9\},\{1,5,7,9\},\{3,5,7,8\},\{1,5,11,12\},\{1,5,10,12\},\{1,5,10,11\}\Big\}$$
$$G_{51} := \Big\{\{2,4,6,8\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{2,4,7,8,9,10,11,12\},\{1,3,7,10,11,12\},$$
$$\{5,6,8,9\},\{1,5,7,9\},\{3,5,7,8\},\{5,6,11,12\},\{5,6,10,12\},\{5,6,10,11\}\Big\}$$
$$G_{52} := \Big\{\{2,3,4,5,9,10,11\},\{1,3,4,6\},\{1,2,4,7\},\{1,2,3,8\},\{1,6,7,8\},\{2,5,7,8,9,10,11\},$$
$$\{3,5,6,8\},\{4,5,6,7\},\{1,6,10,11\},\{1,6,9,11\},\{1,6,9,10\}\Big\}$$
$$G_{53} := \Big\{\{2,4,6,7,8,9,10,11\},\{1,3,5\},\{2,4,6,7,8,9,10,11\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5\},$$
$$\{1,3,5\},\{1,3,5\},\{1,3,10,11\},\{1,3,9,11\},\{1,3,9,10\}\Big\}$$
$$G_{54} := \Big\{\{2,4,6,7,8\},\{1,3,5,9,10,11\},\{2,4,6,7,8\},\{1,3,5,9,10,11\},\{2,4,6,7,8\},\{1,3,5\},$$
$$\{1,3,5\},\{1,3,5\},\{2,4,10,11\},\{2,4,9,11\},\{2,4,9,10\}\Big\}$$
$$G_{55} := \Big\{\{2,4,6,7,9,10,11\},\{1,3,5,8\},\{2,4,6,7,9,10,11\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},$$
$$\{1,3,5\},\{2,4,6\},\{1,3,10,11\},\{1,3,9,11\},\{1,3,9,10\}\Big\}$$
$$G_{56} := \Big\{\{2,4,6,7,9,10,11\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,8\},\{1,3,5\},$$
$$\{2,4,6,9,10,11\},\{1,8,10,11\},\{1,8,9,11\},\{1,8,9,10\}\Big\}$$
$$G_{57} := \Big\{\{2,4,6,10,11,12\},\{1,3,5,8\},\{2,4,6,7,10,11,12\},\{1,3,5,9\},\{2,4,7\},\{1,3,8,9\},$$
$$\{3,5,8,9\},\{2,6,7\},\{4,6,7\},\{1,3,11,12\},\{1,3,10,12\},\{1,3,10,11\}\Big\}$$
$$G_{58} := \Big\{\{2,4,6,10,11,12\},\{1,3,5,8\},\{2,4,6,7\},\{1,3,5,9\},\{2,4,7,10,11,12\},\{1,3,8,9\},$$
$$\{3,5,8,9\},\{2,6,7\},\{4,6,7\},\{1,5,11,12\},\{1,5,10,12\},\{1,5,10,11\}\Big\}$$

$$G_{59} := \Big\{\{2,4,6\},\{1,3,5,8,10,11,12\},\{2,4,6,7\},\{1,3,5,9,10,11,12\},\{2,4,7\},\{1,3,8,9\},\{3,5,8,9\},\{2,6,7\},\{4,6,7\},\{2,4,11,12\},\{2,4,10,12\},\{2,4,10,11\}\Big\}$$

$$G_{60} := \Big\{\{2,4,6,7,8\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5,7,8\},\{2,4,6\},\{1,3,5\},\{1,3,4,8,9,10,11,12\},\{1,3,4,7\},\{7,10,11,12\},\{7,9,11,12\},\{7,9,10,12\},\{7,9,10,11\}\Big\}$$

$$G_{61} := \Big\{\{2,3,4,5\},\{1,3,4,5\},\{1,2,4,5\},\{1,2,3,5\},\{1,2,3,4,6,8,10\},\{5,7,9\},\{6,8,10,11,12,13\},\{5,7,9\},\{6,8,10,11,12,13\},\{5,7,9\},\{7,9,12,13\},\{7,9,11,13\},\{7,9,11,12\}\Big\}$$

$$G_{62} := \Big\{\{2,4,6,7,8,9,10,11\},\{1,3,5\},\{2,4,6,7,8\},\{1,3,5,7,8\},\{2,4,6,9,10,11\},\{1,3,5\},\{1,3,4,8\},\{1,3,4,7\},\{1,5,10,11\},\{1,5,9,11\},\{1,5,9,10\}\Big\}$$

$$G_{63} := \Big\{\{2,4,6,7,8\},\{1,3,5,9,10,11\},\{2,4,6,7,8\},\{1,3,5,7,8\},\{2,4,6\},\{1,3,5,9,10,11\},\{1,3,4,8\},\{1,3,4,7\},\{2,6,10,11\},\{2,6,9,11\},\{2,6,9,10\}\Big\}$$

$$G_{64} := \Big\{\{2,4,6,10,11,12\},\{1,5,7,8,9\},\{4,6,7,8,9\},\{1,3,5\},\{2,4,6,10,11,12\},\{1,3,5\},\{2,3,8,9\},\{2,3,7,9\},\{2,3,7,8\},\{1,5,11,12\},\{1,5,10,12\},\{1,5,10,11\}\Big\}$$

$$G_{65} := \Big\{\{4,6,7,10,11,12\},\{5,7,8,9\},\{4,6,8,10,11,12\},\{1,3,5\},\{2,4,6,9\},\{1,3,5\},\{1,2,8,9\},\{2,3,7,9\},\{2,5,7,8\},\{1,3,11,12\},\{1,3,10,12\},\{1,3,10,11\}\Big\}$$

$$G_{66} := \Big\{\{4,6,7\},\{5,7,8,9\},\{4,6,8\},\{1,3,5,10,11,12\},\{2,4,6,9\},\{1,3,5,10,11,12\},\{1,2,8,9\},\{2,3,7,9\},\{2,5,7,8\},\{4,6,11,12\},\{4,6,10,12\},\{4,6,10,11\}\Big\}$$

$$G_{67} := \Big\{\{2,3,5,6,8\},\{1,3,7,8\},\{1,2,4,5,8\},\{3,5,7,8\},\{1,3,4,6,8\},\{1,5,7,8\},\{2,4,6,9,10,11,12\},\{1,2,3,4,5,6\},\{7,10,11,12\},\{7,9,11,12\},\{7,9,10,12\},\{7,9,10,11\}\Big\}$$

$$G_{68} := \Big\{\{2,4,6,7,8,9\},\{1,3,5\},\{2,4,6,7,8,9\},\{1,3,5\},\{2,4,6,10,11,12,13\},\{1,3,5\},\{1,3,8,9\},\{1,3,7,9\},\{1,3,7,8\},\{5,11,12,13\},\{5,10,12,13\},\{5,10,11,13\},\{5,10,11,12\}\Big\}$$

$$G_{69} := \Big\{\{2,4,6\},\{1,5,7,9,10\},\{4,6,7,9,10\},\{1,3,5\},\{2,4,6\},\{1,3,5\},\{2,3,8\},\{7,9,10,11,12,13,14\},\{2,3,8\},\{2,3,8\},\{8,12,13,14\},\{8,11,13,14\},\{8,11,12,14\},\{8,11,12,13\}\Big\}$$

$$G_{70} := \Big\{\{2,4,6,10\},\{1,3,5\},\{2,4,6,9\},\{1,3,7\},\{2,6,7,9,10\},\{1,3,5\},\{4,5,8\},\{7,9,10,11,12,13,14\},\{3,5,8\},\{1,5,8\},\{8,12,13,14\},\{8,11,13,14\},\{8,11,12,14\},\{8,11,12,13\}\Big\}$$

$$G_{71} := \Big\{\{4,6,7\},\{5,7,8,9\},\{4,6,8\},\{1,3,5,9\},\{2,4,6\},\{1,3,5,10,11,12,13\},\{1,2,8,9\},\{2,3,7,9\},\{2,4,7,8\},\{6,11,12,13\},\{6,10,12,13\},\{6,10,11,13\},\{6,10,11,12\}\Big\}$$

$$G_{72} := \Big\{\{2,3,5,6,8\},\{1,3,7,8\},\{1,2,4,5,8\},\{3,5,7,8\},\{1,3,4,6,8\},\{1,5,7,8\},$$
$$\{2,4,6,9,10,11\},\{1,2,3,4,5,6,9,10,11\},\{7,8,10,11\},\{7,8,9,11\},\{7,8,9,10\}\Big\}$$

$$G_{73} := \Big\{\{2,4,6,7,8,9,10,11,12\},\{1,3,5\},\{2,4,6,7,8,9\},\{1,3,5\},\{2,4,6,10,11,12\},\{1,3,5\},$$
$$\{1,3,8,9\},\{1,3,7,9\},\{1,3,7,8\},\{1,5,11,12\},\{1,5,10,12\},\{1,5,10,11\}\Big\}$$

$$G_{74} := \Big\{\{2,5,6,9\},\{1,3,7,10,11,12\},\{2,4,5,9\},\{3,5,8,9\},\{1,3,4,6,9\},\{1,5,8,9\},\{2,8,9\},$$
$$\{4,6,7,10,11,12\},\{1,3,4,5,6,7\},\{2,8,11,12\},\{2,8,10,12\},\{2,8,10,11\}\Big\}$$

$$G_{75} := \Big\{\{2,4,6,11,12,13\},\{1,5,7,9,10\},\{4,6,7,9,10\},\{1,3,5\},\{2,4,6\},\{1,3,5\},\{2,3,8\},$$
$$\{7,9,10,11,12,13\},\{2,3,8\},\{2,3,8\},\{1,8,12,13\},\{1,8,11,13\},\{1,8,11,12\}\Big\}$$

$$G_{76} := \Big\{\{2,4,6,10,11,12,13\},\{1,3,5\},\{2,4,6,9\},\{1,3,7\},\{2,6,7,9,10\},\{1,3,5\},\{4,5,8\},$$
$$\{7,9,10,11,12,13\},\{3,5,8\},\{1,5,8\},\{1,8,12,13\},\{1,8,11,13\},\{1,8,11,12\}\Big\}$$

$$G_{77} := \Big\{\{2,4,6,10\},\{1,3,5\},\{2,4,6,9\},\{1,3,7,11,12,13\},\{2,6,7,9,10\},\{1,3,5\},\{4,5,8\},$$
$$\{7,9,10,11,12,13\},\{3,5,8\},\{1,5,8\},\{4,8,12,13\},\{4,8,11,13\},\{4,8,11,12\}\Big\}$$

$$G_{78} := \Big\{\{2,4,6,10\},\{1,3,5\},\{2,4,6,9\},\{1,3,7\},\{2,6,7,9,10,11,12,13\},\{1,3,5\},\{4,5,8\},$$
$$\{7,9,10,11,12,13\},\{3,5,8\},\{1,5,8\},\{5,8,12,13\},\{5,8,11,13\},\{5,8,11,12\}\Big\}$$

$$G_{79} := \Big\{\{4,6,7\},\{5,7,8,9,10,11,12\},\{4,6,8\},\{1,3,5,9\},\{2,4,6\},\{1,3,5,10,11,12\},$$
$$\{1,2,8,9\},\{2,3,7,9\},\{2,4,7,8\},\{2,6,11,12\},\{2,6,10,12\},\{2,6,10,11\}\Big\}$$

$$G_{80} := \Big\{\{4,6,7\},\{5,7,8,9\},\{4,6,8\},\{1,3,5,9,10,11,12\},\{2,4,6\},\{1,3,5,10,11,12\},$$
$$\{1,2,8,9\},\{2,3,7,9\},\{2,4,7,8\},\{4,6,11,12\},\{4,6,10,12\},\{4,6,10,11\}\Big\}$$

$$G_{81} := \Big\{\{2,4,6,9,11,12,13\},\{1,3,5\},\{2,4,6,10\},\{1,3,7\},\{2,6,7,8\},\{1,3,5\},\{4,5,9,10\},$$
$$\{5,9,10,11,12,13\},\{1,7,8\},\{3,7,8\},\{1,8,12,13\},\{1,8,11,13\},\{1,8,11,12\}\Big\}$$

$$G_{82} := \Big\{\{2,4,6,9\},\{1,3,5\},\{2,4,6,10\},\{1,3,7\},\{2,6,7,8\},\{1,3,5\},\{4,5,9,10,11,12,13\},$$
$$\{5,9,10,11,12,13\},\{1,7,8\},\{3,7,8\},\{7,8,12,13\},\{7,8,11,13\},\{7,8,11,12\}\Big\}$$

$$G_{83} := \Big\{\{4,6,7\},\{5,7,8,10\},\{4,6,8\},\{1,3,5,10\},\{2,4,6\},\{1,3,5,11,12,13\},\{1,2,9\},\{2,3,9\},$$
$$\{7,8,10,11,12,13\},\{2,4,9\},\{6,9,12,13\},\{6,9,11,13\},\{6,9,11,12\}\Big\}$$

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