OC TM

Multiple Representations Help Teachers and Students Understand a Geometry Problem

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This narrative account begins in a high school classroom as we describe how students were mostly unengaged with a "Problem of the Week." As observers in this setting, we sat in the back of the classroom and attempted to solve the problem: Choose any three vertices of a cube at random. What is the probability that any three vertices will form a right triangle? Because of our different answers to the problem and the struggles we experienced as we attempted to visualize a cube with triangles on the faces and in the interior space we later created concrete and virtual manipulatives. Additionally, we posed this problem in a mathematics methods course with preservice high school teachers and then discussed the use of enactive (concrete), iconic (pictorial), and symbolic representations (Bruner, 1966). The significance of using concrete manipulatives for some mathematics problems cannot be overstated.

Introduction

The significance of using concrete manipulatives for some mathematics problems cannot be overstated. Using concrete materials is helpful for understanding abstract concepts (McNeil & Jarvin, 2007; Vinson, 2001). Hatfield (1994) found students' use of manipulatives decreased as the grade level increased from kindergarten to sixth grade. Although the National Council of Teachers of Mathematics (NCTM) recommends their use at all grade levels, manipulatives are frequently used by students in grades K-8 but not in high school courses (Hartshorn & Boren, 1990).

We were visiting a high school classroom and geometry students were supposed to be working individually on a "Weekly Journal Problem," written on the whiteboard. Each week the students were required to solve a problem in their journals and they were given 5-10 minutes at the beginning of class to work on the problem. However on this day, most chit-chatted with each other, sat quietly, or raised their hands for help from the teacher. They were not using concrete materials or manipulatives. Before you read the remainder of this article, we encourage you to solve the "Weekly Journal Problem" that the students were tackling.

Each week the students were required to solve a problem in their journals and they were given 5-10 minutes at the beginning of class to work on the problem.

Problem: **Choose any three vertices of a cube at random. What is the probability that any three vertices will form a right triangle?**

How did you approach the problem? What strategies did you use? How confident are you with your answer?

Our Solution Attempts

We posed these questions because when we first attempted to solve the problem we had different solutions. Erol pictured cubes and triangles and used his knowledge of permutations to find the probability. Shelly drew a picture of a cube, labeled the vertices, attempted to imagine triangles on the faces and within the cube, and then created patterns. When the two of us sat in the back of the classroom and quietly compared our different answers we were not confident that we had accounted for all possible triangles in the cube and, additionally, we were uncertain as to which triangles inside the cube were right.

If someone had videotaped us as we discussed the problem, the images would have shown the two of us first moving our arms and pencils to form right triangles (outlining the faces and edges of a cube) and then utilizing a tissue box from the teacher's desk as a concrete representation of a prism. We pointed to the lines and vertices formed by the walls and ceiling of the classroom, "drawing" imaginary lines on the surfaces and spaces. Employing these methods, we were able to visualize some triangles as two-dimensional shapes on and within a three-dimensional space.

We moved from these initial imaginings and attempted to use symbolic representations, the "Binomial Coefficient," to find all of the different ways to arrange three vertices out of the eight vertices on a cube. Next, we needed to find which of the 56 (8!/[3!(8-3)!]) possible triangles were right. These maneuvers helped, but we left the high school classroom and continued to think about the "Weekly Journal Problem" after our initial deliberations.

In fact, it was only after we built manipulatives or models (see Figure 1) and reflected on the problem that we were convinced of a correct solution (see Figures 2a and 2b).

We pointed to the lines and vertices formed by the walls and ceiling of the classroom, "drawing" imaginary lines on the surfaces and spaces.

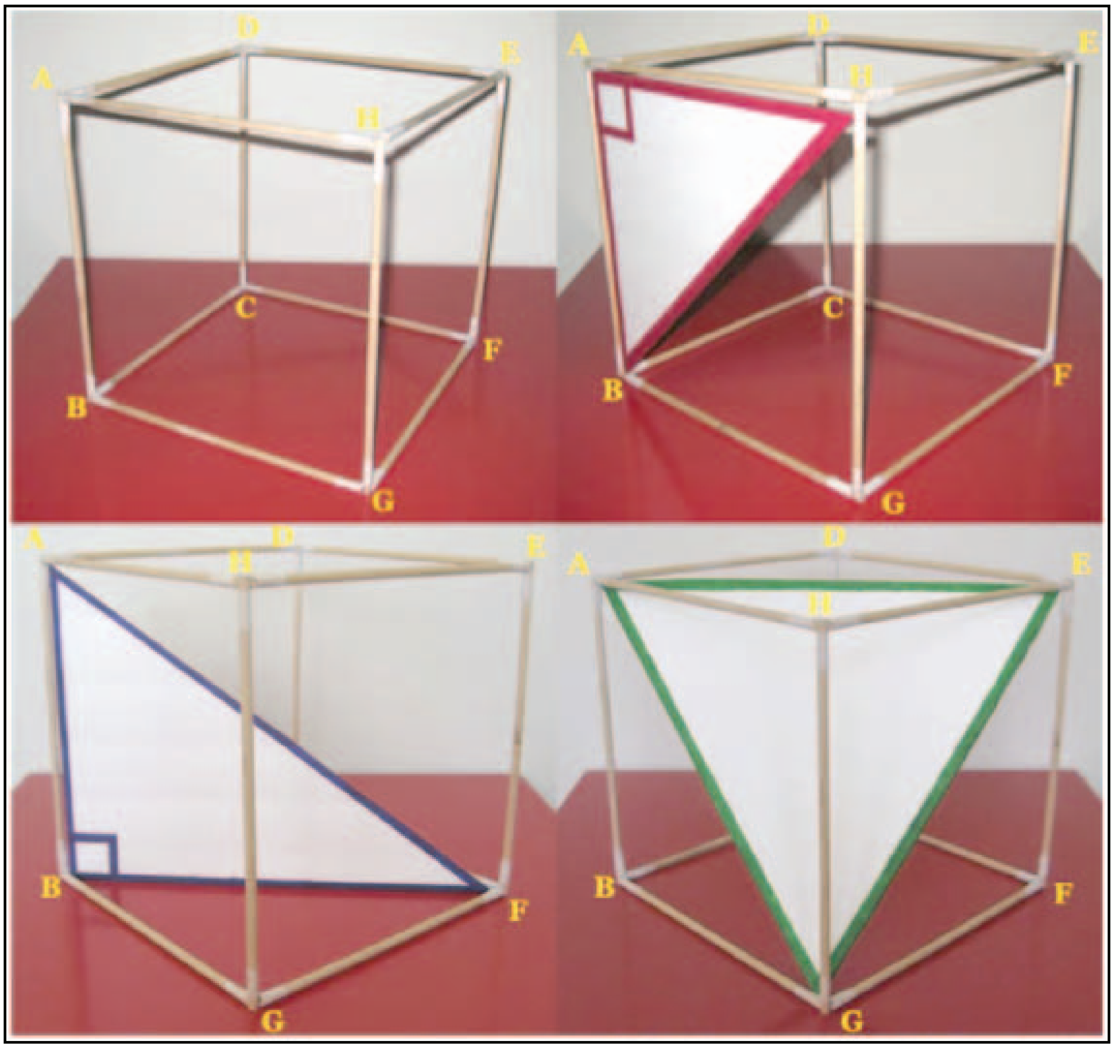

Fig 1 Concrete manipulatives

Two edges and a face diagonal of the cube				An edge, a face diagonal, and a space diagonal of the cube			
ABC	ADH	BGH	DEH	ABF	BCE	CFA	EHB
ABG	AHG	BGF	DEF	ABE	BCH	CFH	EHC
ABD	AHE	CDE	EFG	ABF	BGD	DEB	FGA
ABH	BCD	CDF	EFH	ADG	BGE	DEG	FGD
ADC	BCF	CFG	EHG	AHC	CDG	EFA	GHC
ADE	BCG	GFE	FGH	AHF	CDH	EFB	GHD

24 right triangles

24 right triangles

The total number of right triangles: 24 + 24 = 48

The total number of triangles: 8*7*6/3*2*1=56

The chances of forming a right triangle by randomly choosing any three vertices of a cube: 48/56 = 6/7

Fig 2a Solution to weekly journal problem

Drawn equilateral triangles with three face diagonals of cube

Three face diagonals			
AEG	DHB	BFD	CGE
AEC	DHF	BFH	CGA

8 equilateral triangles

The total number of equilateral triangles: 8

The total number of triangles: 8*7*6/3*2*1 = 56

The total number of right triangles: 56 - 8 = 48

The chances of forming a right triangle by randomly choosing any three vertices of a cube: 48/56 = 6/7

Fig 2b Solution to weekly journal problem

The significance of using concrete manipulatives for some mathematics problems cannot be overstated. We needed tools that we could touch and move; these supported our efforts to visualize the problem.

Furthermore, as mathematics educators we pondered pedagogical implications based on our struggles. We thought about the students in the high school classroom and considered the following questions: Were their struggles the same as ours? What representations were they using? How might concrete manipulatives have helped them make sense of the problem?

Hence, we reemphasize our initial sentence in this article: The significance of using concrete manipulatives for some mathematics problems cannot be overstated. We needed tools that we could touch and move; these supported our efforts to visualize the problem. From those visualizations we were able to create patterns in a symbolic representation (see Figure 2). Indeed, Bruner (1966) suggested that when we attempt to learn new mathematical ideas, it is valuable to follow a sequence through three modes of representation: enactive (action-based or concrete) to iconic (image-based or pictorial) to symbolic (language-based or letters and numerals). This is true for adult learners as well as for young children (Bruner, 1966).

Using the Problem in a Preservice Teachers' Methods Course

Because of our fascination with this problem, we decided to pose it in a methods course for preservice high school teachers in order to investigate how they attempted to solve it. We also wanted them to reflect upon the implications of allowing their future students to use or not use manipulatives for solving problems. Please note these students were seniors who were earning dual majors in mathematics and education. Rather than working individually (as was the case in the high school classroom which we observed), the preservice teachers worked in groups of three. Similar to the enactment in the high school classroom, we gave the preservice teachers no manipulatives and posed the problem. We supplied them with chart paper and markers and walked around the classroom in order to listen to their strategies. All groups of preservice teachers drew cubes on their chart paper but none of them labeled the vertices in order to keep track of the triangles in an organized way. They seemed to move directly from their sketches of cubes to using symbolic representations with numerals, factorials, and ratios.

After about 15 minutes of allowing the groups to explore the problem, we brought the class together as a whole group and asked them to share their thinking about the problem. Needless to say, during this conversation they told us they "wanted" and "needed" concrete manipulatives – cubes and triangles – in order to be more confident about their strategies. Following this conversation, we gave the groups cubes (similar to the ones shown in Figure 1), scissors, and foam board. For a second time, they explored the problem in their groups.

Using the foam board, all groups created right triangles with legs that were the same dimensions as the edges of the cubes and with hypotenuses that were the same dimensions as the face diagonals of the cube. One group of preservice teachers used the Pythagorean Theorem to calculate whether or not the triangles formed within the cube were right; they measured three edge lengths of the triangles they created and checked those with the theorem.

At the end of the exploration, all groups shared their findings and solution methods with the entire class. They correctly found the number of possible triangles (56) by using the concrete cube and the number of right triangles formed on the faces of the cube (24). However, they still had difficulty determining which triangles inside the cube were right (24) and which were equilateral (8). None of the groups found the correct answer (48/56 or 6/7).

With this lesson, the preservice teachers gained experience with manipulatives and they worked with concrete objects to see the triangles in different perspectives. At the end of the lesson, they shared their thoughts about using manipulatives and they indicated that even though their solutions were incorrect, the manipulatives helped them to explore the problem. This led to a conversation about teaching and learning within the context of Bruner's (1966) three modes of representation.

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Extending the Lesson with Technology

The authors of NCTM's Principles and Standards for School Mathematics (2000) note, "Technology is an essential tool for teaching and learning mathematics effectively; it extends the mathematics that can be taught and enhances students' learning" (p. 24). Thus, we decided to extend our own exploration with the "Weekly Journal Problem" and Erol created a virtual manipulative with GeoGebra (see Figure 3), an open source dynamic mathematics software for learning and teaching geometry, algebra, and calculus concepts. The sketch and directions on its use is available from the MathWorld (http://the-math-world.blogspot.com/2011/04/geogebra-5.html). A portion of the sketch is illustrated in Figure 3. Students use the sketch as they write down triangles they find on the datasheet provided at the end of this article.

To use the GeoGebra applet, students complete the following steps.

1. Download GeoGebra 5.0 3D Beta Version (http://www.geogebra.org/forum/viewtopic.php?f=52&t=19846) for Virtual Manipulatives
2. Save Right Triangle/Cube Virtual Manipulatives & Activity Sheet (http://www.geogebra.org/en/upload/index.php?&direction=0&order=&directory=erol) to your computer
3. Launch GeoGebra 5.0
4. Click File/Open/Select the saved file
5. Manipulate sliders (in left pane) and rotate tool (in right pane) to explore possible solutions

The virtual manipulative includes instructions and sliders so that students can select three vertices on the cube. They can rotate, turn, and zoom-in and zoom-out of the cube with mouse clicks.

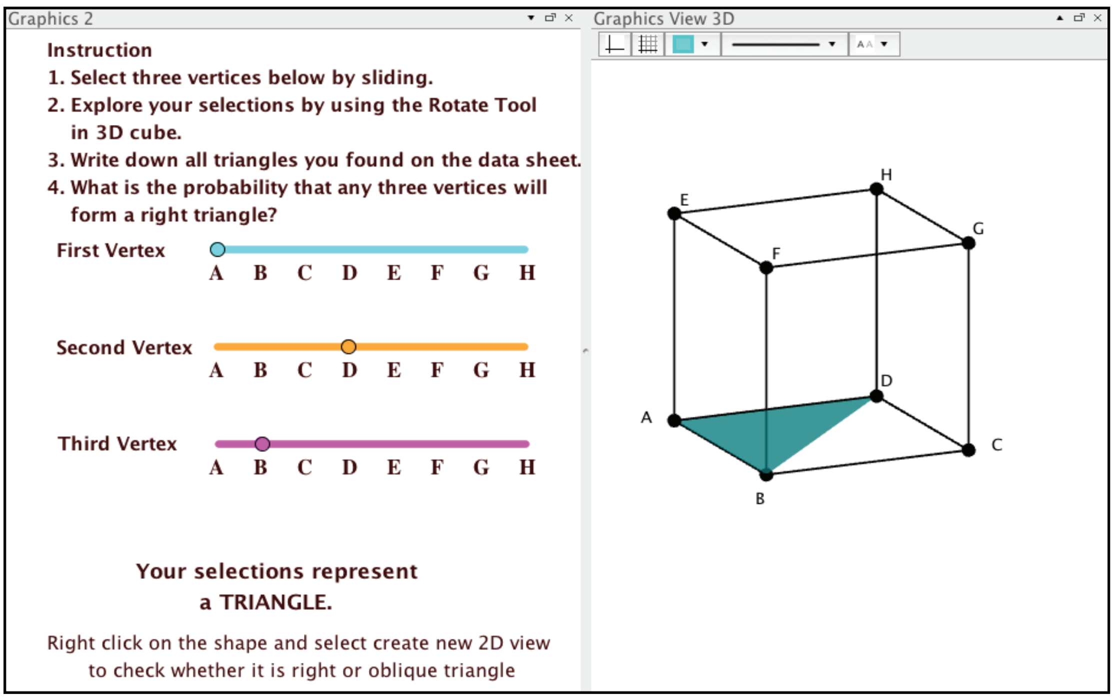

Fig 3 GeoGebra 5.0 sketch of weekly journal problem (Available on-line at http://the-math-world.blogspot.com/2011/04/geogebra-5.html)

Recall that when the preservice teachers were working on the problem with the manipulatives, they did not use letters or numbers to name the vertices of the cube. The lack of vertex labeling may have prevented them from seeing the patterns in the symbolic representation and caused them to make mistakes when they were calculating the number of right and non-right (equilateral) triangles. Therefore, we labeled the vertices of the dynamic cube in the virtual manipulative with letters.

The virtual manipulative includes instructions and sliders so that students can select three vertices on the cube. They can rotate, turn, and zoom-in and zoom-out of the cube with mouse clicks. Additionally, the virtual manipulative allows students to recognize that one triangle can be named in six different ways. To be specific, when three vertices are selected in different orders, all selected vertices represent the same triangle and this triangle is shown with the same color in the virtual manipulative (see Figures 4 and 5).

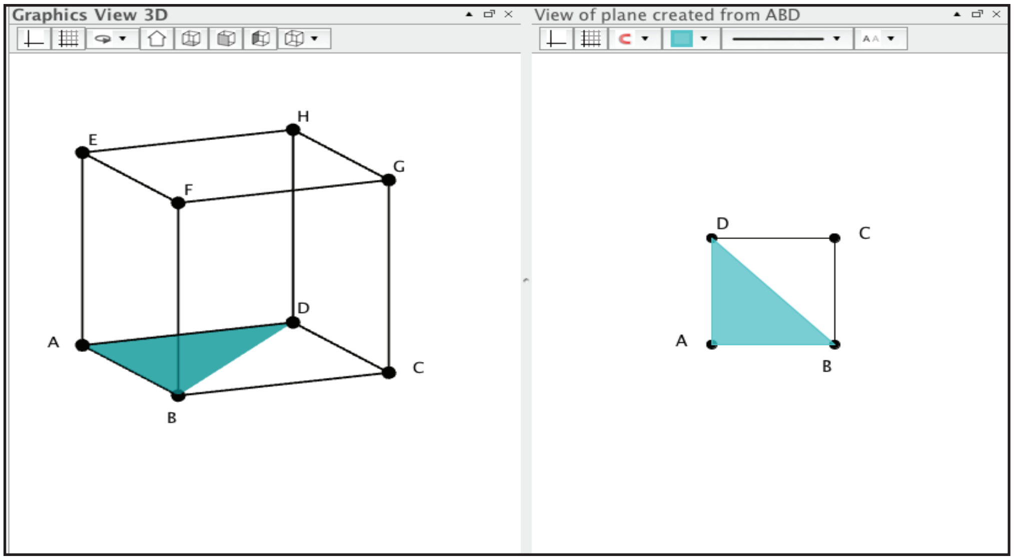

Fig 4 Using plane view in GeoGebra 5.0 to informally verify that triangle ABD is right

First vertex A

ABC	ACB	ADB	AEB	AFB	AGB	AHB
ABD	ACD	ADC	AEC	AFC	AGC	AHD
ABE	ACE	ADE	AED	AFD	AGD	ACB
ABF	ACF	ADF	AEF	AFE	AGE	AHE
ABG	ACG	ADG	AEG	AFG	AGF	AHF
ABH	ACH	ADH	AEH	AFH	AGH	AHG

First vertex E

EAB	EBA	ECA	EDA	EFA	EGA	EHA
EAC	EBC	ECB	EDB	EFB	EGB	EHB
EAD	EBD	ECD	EDC	EFC	EGC	EHC
EAF	EBF	ECF	EDF	EFD	EGD	EHD
EAG	EBG	ECG	EDG	EFG	EGF	EHF
EAH	EBH	ECH	EDH	EFH	EGH	EHG

First vertex B

BAC	BCA	BDA	BEA	BFA	BGA	BHA
BAD	BCD	BDC	BEC	BFC	BGC	BHC
BAE	BCE	BDE	BED	BFD	BGD	BHD
BAF	BCF	BDF	BEF	BFE	BGE	BHE
BAG	BCG	BDG	BEG	BFG	BGF	BHF
BAH	BCH	BDH	BEH	BFH	BGH	BHG

First vertex F

FAB	FBA	FCA	FDA	FEA	FGA	FHA
FAC	FBC	FCB	FDB	FEB	FGB	FHB
FAD	FBD	FCD	FDC	FEC	FGC	FHC
FAE	FBE	FCE	FDE	FED	FGD	FHD
FAG	FBG	FCG	FDG	FEG	FGE	FHE
FAH	FBH	FCH	FDH	FEH	FGH	FHG

First vertex C

CAB	CBA	CDA	CEA	CFA	CGA	CHA
CAD	CBD	CDB	CEB	CFB	CGB	CHB
CAE	CBE	CDE	CED	CFD	CGD	CHD
CAF	CBF	CDF	CEF	CFE	CGE	CHE
CAG	CBG	CDG	CEG	CFG	CGF	CHF
CAH	CBH	CDH	CEH	CFH	CGH	CHG

First vertex G

GAB	GBA	GCA	GDA	GEA	GFA	GHA
GAC	GBC	GCB	GDB	GEB	GFB	GHB
GAD	GBD	GCD	GDC	GEC	GFC	GHC
GAE	GBE	GCE	GDE	GED	GFD	GHD
GAF	GBF	GCF	GDF	GEF	GFE	GHE
GAH	GBH	GCH	GDH	GEH	GFH	GHF

First vertex D

DAB	DBA	DCA	DEA	DFA	DGA	DHA
DAC	DBC	DCB	DEB	DFB	DGB	DHB
DAE	DBE	DCE	DEC	DFC	DGC	DHC
DAF	DBF	DCF	DEF	DFE	DGE	DHE
DAG	DBG	DCG	DEG	DFG	DGF	DHF
DAH	DBH	DCH	DEH	DFH	DGH	DHG

First vertex H

HAB	HBA	HCA	HDA	HEA	HFA	HGA
HAC	HBC	HCB	HDB	HEB	HFB	HGB
HAD	HBD	HCD	HDC	HEC	HFC	HGC
HAE	HBE	HCE	HDE	HED	HFD	HGD
HAF	HBF	HCF	HDF	HEF	HFE	HGE
HAG	HBG	HCG	HDG	HEG	HFG	HGF

Fig 5 Patterns of color-coding

Another advantage that the virtual manipulative offers is that students have a chance not only to see the selected triangle on the three-dimensional cube but also investigate the same triangle in the two- dimensional view by right clicking on the cube. Additionally, students can explore whether or not the selected triangle is right by measuring its edges (and then using the Pythagorean Theorem). Although we did not use this virtual manipulative with the preservice teachers, we feel as though it could provide students with the capability to use iconic (image-based or pictorial) representations to solve the problem. Additionally, the color-coding in Figures 4 and 5 could lead to interesting discussions about combinations and permutations.

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Conclusion: Concrete, Pictorial, and Symbolic Representations

When students are learning new content with concrete manipulatives, they will experience mathematics in an enactive mode. They can move to another mode of representation, iconic, by drawing pictures or using virtual manipulatives. These concrete and iconic representations can ameliorate students' use of symbolic representations via patterns, symbols, formulas, and tables. As a result, we believe that experiences with both physical and virtual manipulatives can impact teaching and learning mathematics because students who need these supports (much like we did as we sat in the high school classroom and worked on the "Problem of the Week") can move through Burner's (1966) modes of representation to the symbolic or abstract. Therefore, the importance of allowing high school students opportunities to use concrete manipulatives to solve some mathematics problems cannot be overstated.

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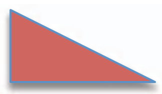
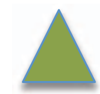
Right Triangle/Cube

NAME ________________________

Problem: Choose any three corners of a cube at random. What are the chances that those points will form a right triangle?

1. Write down your finding under the appropriate category.

Right Triangles 	***Oblique Triangles*** 

2. Number of triangles : _______________

3. Number of right triangles : _______________

4. The probability of the event : _______________